

**QUANTIZATION OF SPACE AND TIME  
IN 3 AND IN 4 SPACE-TIME DIMENSIONS\***

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e-mail: g.thoofth@fys.ruu.nl**Abstract**

The fact that in Minkowski space, space and time are both quantized does not have to be introduced as a new postulate in physics, but can actually be derived by combining certain features of General Relativity and Quantum Mechanics. This is demonstrated first in a model where particles behave as point defects in 2 space dimensions and 1 time, and then in the real world having 3+1 dimensions. The mechanisms in these two cases are quite different, but the outcomes are similar: space and time form a (non-commutative) lattice.

These notes are short since most of the material discussed in these lectures is based on two earlier papers by the same author (gr-qc/9601014 and gr-qc/9607022), but the exposition given in the end is new.

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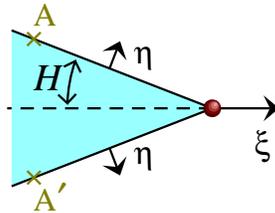
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## 1. IN 2+1 DIMENSIONS

If we remove one space-dimension, Einstein's theory of gravity becomes a beautiful and simple theory. In the absence of a cosmological constant, space-time is locally flat, and the simplest matter sources, point particles, form conical singularities in 2-space. When at rest, they cause no curvature in the time direction<sup>†</sup> Space-time surrounding moving point particles is understood by performing Lorentz transformations. Quite generally, space-time can be described by sewing together flat 3-simplexes<sup>1</sup>.

The rich structure of this apparently very simple model emerges when one attempts to construct sequences of Cauchy surfaces. It is convenient to choose these Cauchy surfaces also to consist of simplexes (polygons) sewn together. At the seams, the surface thus obtained may be curved, but of course the Riemann curvature of 3-space is still required to vanish at these seams; it is only non-vanishing at the location of the point particles.

Within each simplex of the Cauchy surface there is a preferred Lorentz frame (with the time axis orthogonal to the surface). By choosing time to run equally fast on all simplexes we define a simple series of Cauchy surfaces. The polygons glued together evolve according to well defined rules. Polygons may even split in two, or disappear, and in each of these cases the further evolution of the Cauchy surface is uniquely defined<sup>2</sup>. It can be simulated on a computer<sup>3</sup>.



Wedge cut out by a moving particle (dot).  $\xi$  is the boost parameter for the velocity of the particle;  $\eta$  is that for the velocity of the wedge. The Hamiltonian  $H$  is one-half the wedge angle.

The rules for the evolution of a Cauchy surface have been derived in Refs<sup>2, 3</sup>. Where there is a particle there is a cusp (Fig. 1), where the points  $A$  and  $A'$  must be identified. When the particle is at rest we identify (one-half of) the opening angle of the cusp with the mass  $\mu$  of the particle. If the particle moves, the cusp must be oriented in such a way that the direction on the velocity coincides with the bisectrix of the cusp angle, so as to avoid any time jump across the cut. The Lorentz contraction formula gives the new

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<sup>†</sup> If the particle has *spin* however, the monodromies on curves surrounding them show a constant jump in time.

angle  $H$ , and plain geometry relates the velocity  $\tanh \eta$  of the cusp's edges to the velocity  $\tanh \xi$  of the particle:

$$\tan H = \cosh \xi \tan \mu, \quad (1.1)$$

$$\tanh \eta = \sin H \tanh \xi. \quad (1.2)$$

Algebraically, one derives from this:

$$\cos \mu = \cos H \cosh \eta, \quad (1.3)$$

$$\sinh \eta = \sin \mu \sinh \xi. \quad (1.4)$$

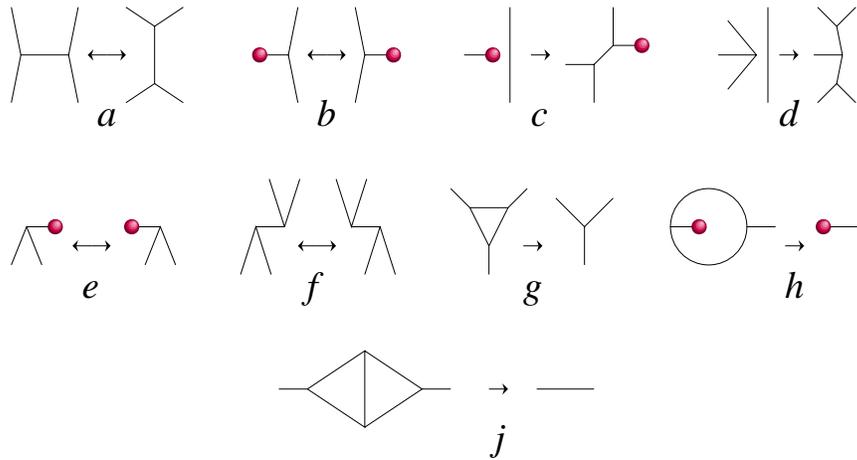
These equations are to be compared with the more familiar properties of particles in flat space-time:

$$H = \mu \cosh \xi, \quad (1.1a)$$

$$p = H \tanh \xi, \quad (1.2a)$$

$$\mu^2 = H^2 - p^2, \quad (1.3a)$$

$$p = \mu \sinh \xi. \quad (1.4a)$$



**2.** The nine distinct transitions that can occur among the polygons, indicated diagrammatically.

At a vertex between three polygons  $I$ ,  $II$ , and  $III$ , one must note that the Lorentz boost from  $I$  to  $III$  can be written as the product of the boost from  $II$  to  $III$  and the one from  $I$  to  $II$ . This gives us relations between the velocities of the edges of the

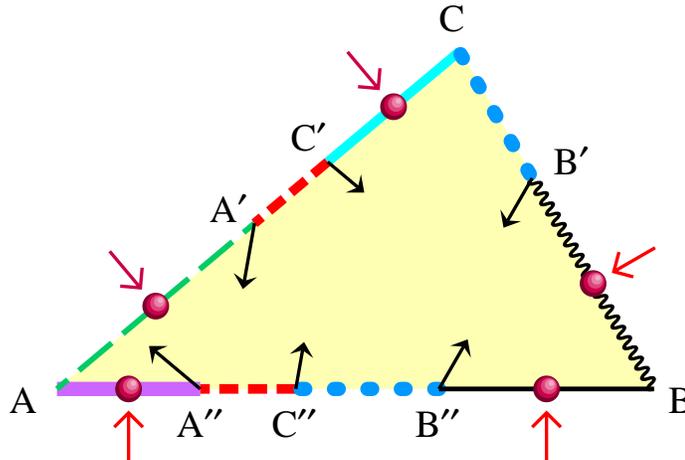
adjacent polygons and their angles<sup>2</sup>. Because the Cauchy surface is not flat, the three angles at one vertex need not add up to  $2\pi$ . We write

$$\alpha_1 + \alpha_2 + \alpha_3 = 2\pi - 2\omega. \quad (1.5)$$

The nine different possible polygon transitions are indicated diagrammatically in Fig. 2. It turned out to be instructive to study the classical cosmological models obtained with a limited number of particles. The space-time topology is typically chosen to be  $S_2 \times R_1$ , but one can take also higher genus surfaces for the spacelike component. Depending on the initial state chosen, the final state of the “universe” is found to be in one of two possible classes:

*i)* an indefinitely expanding universe, in which the edges of all polygons continue forever to increase in length. Eventually, everything goes radially outwards, and no further transitions take place. Or:

*ii)* the universe continues to shrink, faster and faster. There is a natural end point at a time  $t_{\text{end}}$  at which it shrinks to a point. Before that time is reached, however, an infinite number of transitions have taken place, and each particle sees all the other particles pass at ever decreasing impact parameters (transverse separation distances). The speed at which they pass each other, in the center of mass frame, approaches exponentially that of light. A typical final state is depicted in Fig. 3.



Example of a shrinking final state of a universe. The particles have so large  $\xi$  values that all wedges opened up to form angles of practically  $180^\circ$ . They all move inwards, nearly with the speed of light (arrows outside frame). Edges of equal texture in the picture are to be matched.  $A$ ,  $A'$  and  $A''$  are to be identified; similarly  $B$ ,  $B'$ ,  $B''$ , and  $C$ ,  $C'$  and  $C''$ , respectively. The vertex points all appear to move faster than light (see arrows).

Figure 3

In this state the Cauchy surface is a single polygon, such that most of its angles are very close to  $180^\circ$ , so it converges to a triangle (sometimes an other simple shape). The sides move inwards with a velocity exponentially approaching that of light. The particles (dots) every now and then slip over the edges, after which they reappear at one of the other image points of the vertex in question. This boosts them so much that their velocity is much closer to that of light than before, and the process is repeated an infinite number of times before the universe has shrunk to a single point, at which it terminates its existence.

It was found that a  $g = 0$  universe might begin with a Big Bang (the time reverse of the above shrinking process) and either end expanding forever or shrinking forever. This is sketched in Fig. 4a. If  $g = 1$  (a torus), there are only two possibilities: either a Big Bang, or a Big Crunch, but not both (Fig. 4b). We conjecture that at higher genus, also an evolution from a shrinking mode into an expanding mode is possible, but this was not checked explicitly.

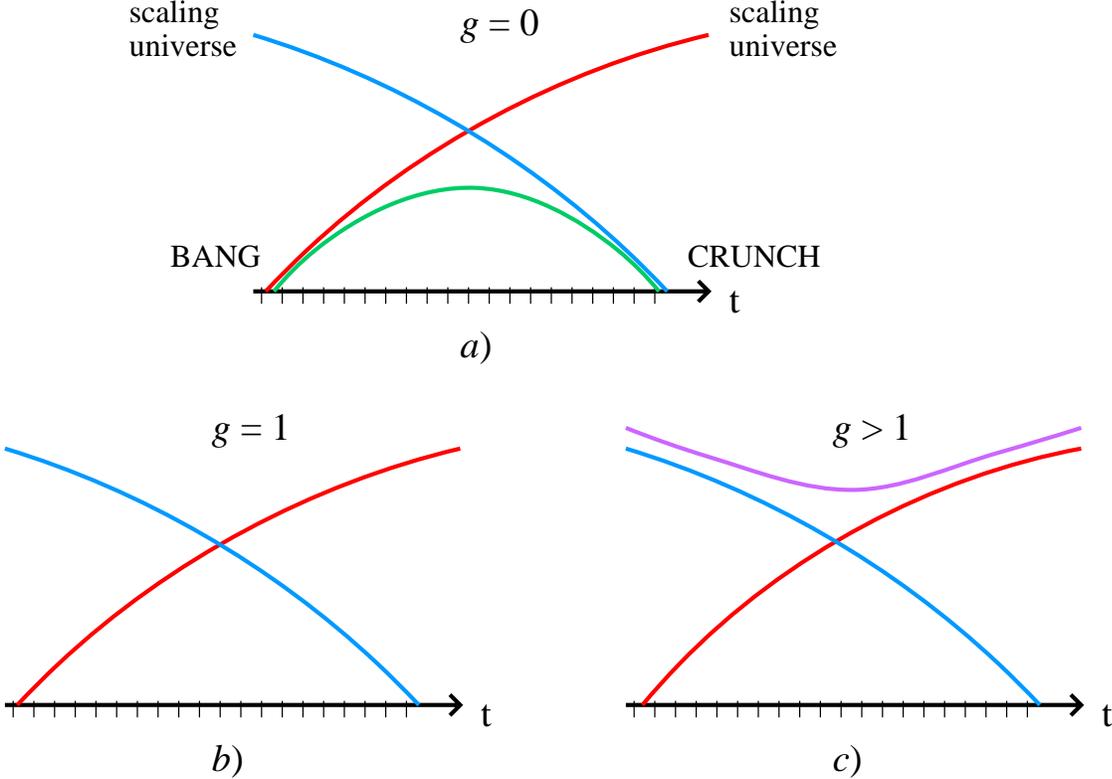


Figure 4: Evolving universes. *a)* an  $S_2 \times R_1$  topology; *b)* if the topology is  $S_1 \times S_1 \times R_1$ ; *c)* for higher topology (conjectured).

As for the quantization of this model, there exist various opinions and procedures. The Chern-Simons procedure as advocated by Carlip<sup>4</sup> and Witten<sup>5</sup> does not indicate any discreteness in space and/or time. Waelbroeck<sup>6</sup> claims that there are inequivalent

quantization procedures. In this author's opinion it is still not obvious whether any of these procedures at all is completely consistent. Certainly one would like to perform second quantization, so that in a limit where the gravitational constant vanishes an ordinary scalar (or Dirac) field theory emerges. This has never been demonstrated, and indeed, we find that Hilbert spaces with transitions between states with different particle numbers are difficult to construct. From Fig. 4, one suspects that the evolution near a big Bang or a Big Crunc might violate unitarity because there might not be acceptable states to evolve to or from.

In the polygon representation, the most natural dynamical degrees of freedom are the *lengths*  $L_i$  of the edges of all polygons, and their canonically conjugated variables, the Lorentz boost parameters  $\eta_i$  of Eqs. (1.2)–(1.4). If the Hamiltonian is taken to be

$$H_{\text{tot}} = \sum_{\text{particles } i} H_i + \sum_{\text{vertices } j} \omega_j, \quad (1.6)$$

with  $H_i$  as described in (1.1)–(1.3) and  $\omega_j$  as in (1.5), then the Poisson brackets

$$\{L_i, \eta_j\} = \delta_{ij}, \quad (1.7)$$

give the correct equations of motion:

$$\dot{L}_i = \{L_i, H\}. \quad (1.8)$$

The fact that this gives time quantization<sup>7</sup> is then read off directly from Eqs. (1.1)–(1.5), since the Hamiltonian consists exclusively of angles. The relevant operator one can construct directly is not  $H$  but the time step operator  $e^{\pm iH}$ . In contrast, the lengths  $L_i$  are *not* quantized, since their canonically conjugated variables are hyperbolic angles, not real angles. If anything there is quantized, it is the *imaginary parts* of  $L_i$ .

This situation changes radically if we use a different representation of the particle system. It should be stressed that this is a change in representation, not in the physical contents of the theory. We introduce a *reference point*, the origin  $\mathcal{O}$  of a coordinate frame in 2-space, where the Lorentz frame will be kept fixed. Particles can be reached from  $\mathcal{O}$  via various different geodesics. For each particle  $i$ , at given time  $t$ , we take the shortest geodesic to that particle, and use the coordinates  $(x_i, y_i)$  of the particle seen over this geodesic. Again, our 2-surface at given time is used as a Cauchy surface, and we study its evolution. The same Hamiltonian is used as before. Now we ask what the momentum variables are, conjugated to  $x_i$  and  $y_i$ . They form a vector  $(p_{i,x}, p_{i,y})$ . The length  $p$  of this vector is found to be given by<sup>8</sup>

$$p = \theta \cos \mu; \quad \tan \theta = \sinh \eta. \quad (1.9)$$

This is an angle! Consequently, the coordinates  $x_i$  and  $y_i$  are quantized. Time remains quantized as it was before, since we did not change our Hamiltonian. Eq. (1.3) turns into

$$\cos H = \cos \mu \cos \theta. \quad (1.10)$$

We now refer to Ref <sup>8</sup> for a much more detailed exhibition of the resulting lattice in 2+1 dimensional Minkowski space. A quick summary is as follows. The angle  $\theta$ , together with the orientation  $\varphi$  of the momentum vector, form a compact 2-sphere. The space coordinates are generated from the spherical harmonics on this 2-sphere, hence they are represented by two integers  $\ell$  and  $m$ . The *mass shell condition*, Eq. (1.10), is now a *difference equation* on this lattice. If  $L_1, L_2, L_3$  are the usual angular momentum operators on our spherical momentum space, the coordinates of one particle can be identified as

$$x = \frac{L_2}{\cos \mu}; \quad y = \frac{-L_1}{\cos \mu}; \quad L = L_3. \quad (1.11)$$

Here,  $L$  is the ordinary angular momentum in 2-space, and  $\mu$  is the particle mass. These could be seen as “quantum coordinates”:

$$\begin{aligned} [x, y] &= \frac{i}{\cos^2 \mu} L, \\ [L, x] &= iy, \\ [L, y] &= -ix. \end{aligned} \quad (1.12)$$

The difference equations for the wave function, as resulting from Eq. (1.10), is still second order in time. One can turn our wave equation into a *Dirac equation* which is first order in time. The Dirac particle has spin  $1/2$ . Second quantization should be performed by filling the Dirac sea, but a difficulty encountered is that there will be two Fermi levels, of which one carries negative energy particles. We have no resolution of the resulting problems at hand.

## 2. BLACK HOLE PHYSICS

A direct generalization of the results of the previous chapter to 3+1 dimensions would lead to deceptive results. In 3+1 dimensions space-time outside the matter sources is not flat; this would only be if the matter sources could be taken to be stretches of rigid string pieces. It would be highly preferable if we could derive certain features concerning Planckian physics from facts out of everyday life, without relying on any drastic assumptions.

We now report that such a thing might well be possible. One well-known fact in general relativity is that the gravitational force appears to be unstable. given sufficient

amounts of matter, gravitational attraction can become so strong that collapse takes place, and no classical variety of matter can withstand such a collapse. Indeed, if the quantity of matter is large enough then during the collapse the situation as seen by local observers may be quite normal and peaceful; matter densities and temperatures could be those of ordinary water. According to the outside world however, a black hole is formed. As long as one adheres to the formalisms of classical, that is, unquantized, laws of physics, there is no contradiction anywhere. A black hole is an interesting object, but we do not learn much from it about local laws of physics.

Yet in a quantum theory what happens during gravitational collapse turns out to be much more problematic and controversial. First of all it is found that black holes will emit particles<sup>9</sup>, and thereby loose mass-energy. Then one discovers that the laws of quantum field theory at the local scale appear to be in conflict with the laws of quantum mechanics for the black hole entire. Now we do not know whether the black hole entire will obey ordinary laws of quantum mechanics, but if it is allowed to decay into very tiny black holes that may pervade the quantum vacuum state, we may arrive at a self-consistency problem. Is or is not the small distance limit of our world quantum mechanical? If not, how do we understand energy-momentum conservation and the stability (and apparent uniqueness) of our vacuum?

The present author is investigating the train of thought following the assumption that collapsing objects are still in complete agreement with ordinary quantum mechanics (in particular there is no communication with “other universes” which would be tantamount to violation of ordinary quantum determinism). The procedure has recently been laid down precisely in our review paper<sup>10</sup>, which we advise to be used in conjunction with this paper. Here we will explain how “quantization of space and time” may follow from these considerations.

Units are chosen such that

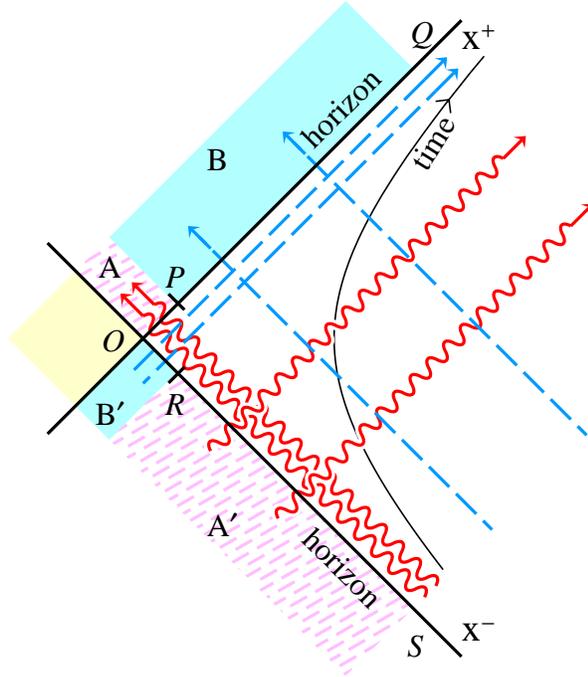
$$\mathcal{G} \stackrel{\text{def}}{=} 8\pi G = 1, \quad (2.1)$$

which gives us new Planck units of length, mass and energy:

$$\begin{aligned} \mathcal{L}_{\text{Planck}} &= \sqrt{\frac{\hbar \mathcal{G}}{c^3}} = 8.102 \times 10^{-33} \text{cm}, \\ \mathcal{M}_{\text{Planck}} &= \sqrt{\frac{\hbar c}{\mathcal{G}}} = 4.35 \mu\text{g}, \\ \mathcal{E}_{\text{Planck}} &= \mathcal{M}_{\text{Planck}} c^2 = \sqrt{\frac{\hbar c^5}{\mathcal{G}}} = 2.39 \times 10^{27} \text{eV}. \end{aligned} \quad (2.2)$$

In its most elementary form, the *S-matrix Ansatz* for the behavior of a black hole stipulates that, barring certain irrelevant infra-red effects, the entire process of black hole

creation and subsequent evaporation can be viewed as a quantum mechanical scattering event, to be described by a scattering matrix. In practice, for a given black hole, it implies that the number of different possible states it can be in is given by the exponent of the entropy  $S = 4\pi GM^2 = 1/2 M^2$ . This could be mimicked by a simple boundary condition near the horizon (the “brick wall”), forcing ingoing radiation to be bounced back at a distance scale of the order of the Planck distance from the horizon.



**Figure 5.** Short distance - large distance duality in the scattering matrix Ansatz. Particles entering a black hole in  $A$  will determine what comes out from  $A'$  (wavy lines); what enters at  $B$  determines radiation from  $B'$  (dashed lines). The fields on the small region  $OP$  are mapped as fields on  $RS$  and fields on  $PQ$  are mapped onto  $OR$ .

In terms of a local Rindler frame near the horizon, see Fig. 5, we expect a mapping. All information passing the line  $OQ$  in Fig. 5 should reemerge as information from the line  $OS$ . This implies that the fields on  $OQ$  determine the fields on  $OS$ . Such a mapping appears not to exist in ordinary field theories in flat space-time. However, one has to realize that the mapping relates distances shorter than the Planck length (trans-Planckian distances) to distances large than the Planck length (cis-Planckian distances). In Fig. 5, fields on the trans-Planckian line  $OP$  are mapped as fields on the cis-Planckian line  $RS$ . Similarly,  $PQ$  maps onto  $OR$ . This may be seen as a long-distance-short distance duality not unlike  $T$ -duality as discussed in string theories.

It is suspected that long the distance – short distance duality constraint should be

imposed in all field theories in approximately flat space-times, regardless whether the point  $O$  (actually a 2-surface) acts as the intersection point of a future horizon and a past horizon, but we will concentrate on the case that there is a real horizon.

In Ref<sup>10</sup> it is explained how *interactions* between in- and outgoing particles may restore a causal relationship that could actually correspond to the mapping just described. The most important interaction here is the gravitational one. An ingoing particle with momentum  $p_{\text{in}}$  causes a shift in the geodesics of outgoing particles. This shift is usually in the inward direction, so it may be that particles that were on their way out are moved back in again by an ingoing particle. If the outgoing particles were represented as usual by a Fock space, information loss would be unavoidable.

However, Fock space may have to be replaced by something else when it comes to trans-Planckian (or near-Planckian) distance scales. Two particles that enter the horizon at the same angular position  $\tilde{x} = (\theta, \varphi)$  may have to be considered inseparable. Indicating the coordinates of an outgoing particle as  $(x^-, \tilde{x})$ , we propose to replace their Fock space by the set of observables  $u^-(\tilde{x})$ , defined as

$$u^-(\tilde{x}) \stackrel{\text{def}}{=} \left\langle x_i^-(\tilde{x}) \right\rangle_{\text{Average over all particles } i} - x^-(\tilde{x}) \Big|_{\text{Horizon}}. \quad (2.3)$$

This is *one* observable at each transverse position  $\tilde{x}$ . Since there will always be particles at our side of the horizon, this observable will continue to be observable regardless the amount of the shift. Similarly, we have the observables  $x^+(\tilde{x})$ , referring to the ingoing particles.

Being related to the actual position of the horizon, one might refer to the operators  $x^\mu(\tilde{x}) = (u^+(\tilde{x}), u^-(\tilde{x}), \tilde{x})$  as “the shape of the horizon”, more precisely, “of the intersection between past and future horizon.” Later, we will replace the independent coordinates  $\tilde{x}$  by a set of arbitrary coordinates  $\tilde{\sigma}$ , so that one has a sheet described as  $x^\mu(\tilde{\sigma})$ .

According to the  $S$ -matrix Ansatz,  $x^\mu(\tilde{\sigma})$  contains *all* information there is about the ingoing and outgoing states. Now, in the conventional theory, this information is contained by the fields in the first quadrant. Thus we arrive at the important conclusion that these fields can be *replaced* by the single (vector) function  $x^\mu(\tilde{\sigma})$ . This is what may be called *black hole complementarity*<sup>11</sup>, or, since we seem to have some sort of projection of information in 3-space onto a two-dimensional surface<sup>12</sup>, the *holographic principle*. It must be stressed, however, that approximations were made; all non-gravitational forces were neglected. Adding the electromagnetic force, for instance, yields an additional component  $x^5(\tilde{\sigma})$ .

## QUANTIZATION OF SPACE AND TIME IN 3+1 DIMENSIONS

The shift  $\delta x^-$  among the outgoing particles at transverse coordinates  $\tilde{x}$  is proportional to the momentum  $p_{\text{in}}$  of the ingoing particles at  $\tilde{x}'$ :

$$\delta x^-(\tilde{x}) = \int d^2\tilde{x}' f(\tilde{x} - \tilde{x}') p_{\text{in}}(\tilde{x}'), \quad (3.1)$$

where  $f$  is a Green function obeying

$$\tilde{\partial}^2 f(\tilde{x}) = -\delta^2(\tilde{x}). \quad (3.2)$$

If  $x^+$  is the operator canonically conjugated to  $p_{\text{in}} = p_+$ , one would be tempted to write

$$\begin{aligned} [x^-(\tilde{x}), x^+(\tilde{x}')] &= f(\tilde{x} - \tilde{x}') [p_+(\tilde{x}'), x^+(\tilde{x})] \\ &= -\hbar i f(\tilde{x} - \tilde{x}'). \end{aligned} \quad (3.3)$$

In case of many particles, labeled by indices  $i, j$ :

$$[x_i^-(\tilde{x}), x_j^+(\tilde{x}')] \stackrel{?}{=} -\hbar i f(\tilde{x} - \tilde{x}') \delta_{ij}. \quad (3.4)$$

One then would have a “quantum space-time”, with beautifully non-commuting coordinates. But this of course would be incorrect. Since *all* ingoing particles interact gravitationally with *all* outgoing ones, the Kronecker delta,  $\delta_{ij}$ , should not be there. If we had two ingoing particles, 1 and 2, that happen to be at the same transverse position  $\tilde{x}$ , then  $x_1^+(\tilde{x}) - x_2^+(\tilde{x})$  would be an operator that commutes with everything, so that this “observable” would truly get lost in the black hole. We have to drop this observable, as explained in the previous section, and we should work exclusively with the horizon shape operator  $x^\mu(\tilde{\sigma})$  defined there.

It is these operators that obey the commutation rule<sup>10, 13</sup>

$$[x^-(\tilde{x}), x^+(\tilde{x}')] = -i f(\tilde{x} - \tilde{x}'). \quad (3.5)$$

Now this equation has been derived for the case when one may neglect the transverse components of the gravitational force. But if we define<sup>10, 14</sup> the surface orientation 2-form  $W^{\mu\nu} = dx^\mu \wedge dx^\nu$ , or

$$W^{\mu\nu}(\tilde{\sigma}) = -W^{\nu\mu}(\tilde{\sigma}) = \varepsilon^{ab} \frac{\partial x^\mu}{\partial \sigma^a} \frac{\partial x^\nu}{\partial \sigma^b}, \quad (3.6)$$

we have, in the same approximation (where  $\tilde{\sigma} = \tilde{x}$ ),

$$\sum_{\mu} [W^{\mu\alpha}(\tilde{\sigma}), W^{\mu\beta}(\tilde{\sigma}')] = \frac{1}{2} \delta^2(\tilde{\sigma} - \tilde{\sigma}') \sum_{\mu\nu} \varepsilon^{\alpha\beta\mu\nu} W^{\mu\nu}(\tilde{\sigma}). \quad (3.7)$$

It is then argued that this equation, being Lorentz-invariant, should continue to hold regardless the orientation of the gravitational shift.

Unfortunately, Eq. (3.7) does not contain sufficient information to find a representation of this algebra, since, at the left hand side, there is still an index  $\mu$  that is summed over (without the summation one gets non-local commutators). On the other hand, the  $W^{\mu\nu}$  operators overdetermine the surface  $x^\mu(\tilde{x})$ . We therefore restrict ourselves to its self-dual part  $K_a(\tilde{\sigma})$ ,  $a = 1, 2, 3$ . Defining  $K$ , and the anti-self-dual part  $\bar{K}$ , as

$$\begin{aligned} K_1(\tilde{\sigma}) &= iW^{23} + W^{10}, & \bar{K}_1(\tilde{\sigma}) &= -iW^{23} + W^{10}, \\ K_2(\tilde{\sigma}) &= iW^{31} + W^{20}, & \bar{K}_2(\tilde{\sigma}) &= -iW^{31} + W^{20}, \\ K_3(\tilde{\sigma}) &= iW^{12} + W^{30}, & \bar{K}_3(\tilde{\sigma}) &= -iW^{12} + W^{30}. \end{aligned} \quad (3.8)$$

we find the commutation rules

$$[K_a(\tilde{\sigma}), K_b(\tilde{\sigma}')] = i\varepsilon_{abc}K_c(\tilde{\sigma})\delta^2(\tilde{\sigma} - \tilde{\sigma}'), \quad (3.9)$$

and similarly for the  $\bar{K}$ . Mixed commutators of  $K$  and  $\bar{K}$  are non-local however.

The operators  $W$ ,  $K$  and  $\bar{K}$  are distributions, so we want to convolute them with test functions. It is convenient to take a test function  $\varrho(\tilde{\sigma})$  with the property  $\varrho^2 = \varrho$ , which means that  $\varrho = 1$  within some region in  $\tilde{\sigma}$  space and  $\varrho = 0$  in its complement. Let  $D$  be the domain where  $\varrho = 1$ . Then

$$W^{\mu\nu}(D) \stackrel{\text{def}}{=} \int d^2\tilde{\sigma} g(\tilde{\sigma})W^{\mu\nu}(\tilde{\sigma}) = \int_D d^2\tilde{\sigma} W^{\mu\nu}(\tilde{\sigma}) = \oint_{\delta D} x^\mu dx^\nu. \quad (3.10)$$

Defining  $L_a(D) = \int_D K_a(\tilde{\sigma})d^2\tilde{\sigma}$ , we find that these obey the commutation rules of angular momenta:

$$[L_a(D), L_b(D)] = i\varepsilon_{abc}L_c(D). \quad (3.11)$$

Thus, if we divide the  $\tilde{\sigma}$ -plane up in domains  $D$ , then we have a *discrete representation* of our algebra, formed by the quantum numbers  $\{\ell_D, m_D\}$  for all domains. If two domains are combined into one then  $L(D_1 + D_2) = L(D_1) + L(D_2)$ , according to the familiar rules of adding angular momenta.

It appears that the states

$$|\{\ell_D, m_D\}\rangle, \quad (3.12)$$

with

$$\ell = 0, 1/2, 1, 3/2, \dots, \quad m = -\ell, -\ell + 1, \dots, \ell. \quad (3.13)$$

have the kind of degeneracy one would expect for a black hole with entropy proportional to its surface area. The  $S$ -matrix Ansatz would demand a degeneracy not much worse

than this. It must be stressed, however, that (3.12), (3.13) is not the only representation of our algebra. The operators  $L_a$  are not hermitean. Instead, we have

$$L_a^\dagger = \bar{L}_a, \quad (3.14)$$

and consequently one cannot derive the usual properties (3.13) of the quantum numbers  $\ell$  and  $m$ . We do have, from the definition (3.6),

$$\varepsilon_{\mu\nu\alpha\beta} W^{\mu\nu} W^{\alpha\beta} = 0. \quad (3.15)$$

from which it follows that

$$K^2(\tilde{\sigma}) = \bar{K}^2(\tilde{\sigma}), \quad (3.16)$$

but this does not directly lead to constraints on  $\ell_D$  and  $m_D$ ; for instance,  $\ell_D$  could easily be negative. One does have the orthonormality property

$$\langle \{\bar{\ell}_D, \bar{m}_D\} | \{\ell'_D, m'_D\} \rangle = \prod_D \delta_{\bar{\ell}_D, \ell'_D} \delta_{\bar{m}_D, m'_D}, \quad (3.17)$$

where  $\bar{\ell}_D$  and  $\bar{m}_D$  refer to the representations of  $\bar{L}_a(D)$ .

Since the  $S$ -matrix Ansatz requires a finite degeneracy, we can now ask ourselves what the consequences would be for the  $K$  and  $\bar{K}$  operators if we do restrict ourselves to the representations (3.13). If the operators  $K_a$  differ only infinitesimally from  $\bar{K}_a$ , Eq. (3.13) should still hold. Thus, we want the real parts in Eq. (3.8) to be much bigger than the imaginary parts. If

$$x^0, x^3 \gg x^1, x^2, \quad (3.18)$$

then  $\ell$  is real and  $m \approx \ell \gg 0$ . Note that in this case we have a *timelike* surface, whereas the horizon surface that we started off with was spacelike. We suspect that what (3.13) is really telling us is that the smallest domains must be timelike surface elements, and that the spacelike horizon can be considered to be a globally spacelike patchwork of many such timelike pieces.

However, the constraint (3.13) is not yet fully guaranteed by (3.18). It is better to postulate for each domain  $D$

$$|\delta x^0| \gg |\delta x_i|, \quad i = 1, 2, 3. \quad (3.19)$$

Again, this describes timelike “string worldsheets” joined together to form the horizon. A more precise interpretation is as follows.

We may *choose* the shapes of the domains  $D$ . For instance, we may choose the time intervals  $\delta x^0$ , and draw the domains as rectangles (Fig. 6a). If we choose these to be

an integer multiple of a quantum  $\Delta t$ , then time is quantized. The time quantum  $\Delta t$  is arbitrary, but as for now we choose it to be much bigger than the Planck time.

In this case, for small enough domains,

$$L_a(D) \approx \bar{L}_a(D) \approx \oint_{\delta D} x^a dx^0 = \Delta t \cdot (x_a(2) - x_a(1)), \quad (3.20)$$

where  $x_a(1)$  is the average value of  $x_a$  at one edge of the rectangle and  $x_a(2)$  the average value at the other side. Writing  $\delta x_a = x_a(2) - x_a(1)$ , we find

$$L_a(D) = \Delta t \cdot \delta x_a. \quad (3.21)$$

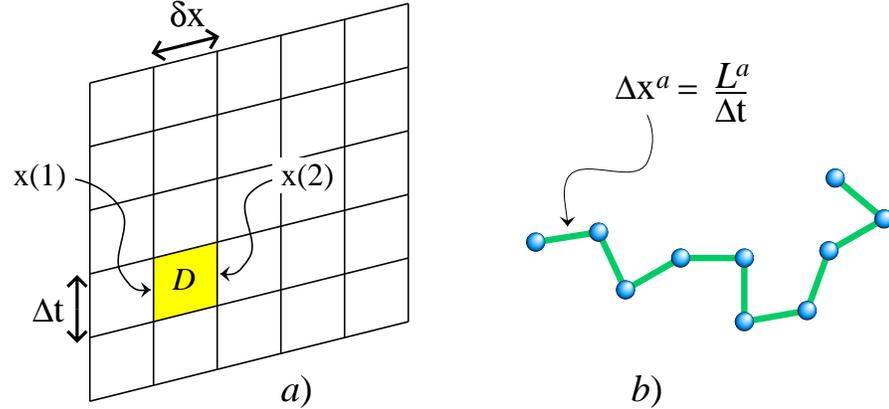
Consequently,  $\delta x_a$  are quantized in multiples of

$$\Delta x = \frac{1}{2} / \Delta t. \quad (3.22)$$

Putting the units back in, we have

$$\Delta t \cdot \Delta x = 4\pi G. \quad (3.23)$$

The resulting “string” is pictured in Fig. 6b. We note that the string bits are vectors in 3-space obeying the quantization rules of angular momenta.



**6.** a) Timelike segment of the horizon, divided into domains  $D$ . b) At given time  $t$  the string consists of pieces quantized in units of  $\Delta x$ , obeying the commutation rules of angular momenta.

Apparently, space-time now forms a lattice. Note that we did *not* derive equations of motion for this string, whose target space appears to be a quantum space-time, much like in the 2+1 dimensional case. Note also that the string bit vector elements at a given

time *commute* with the string bit vector elements at other times, unlike the situation in ordinary field theories. Our space-time quantization rules have much in common with the surface area quantization rules suggested by Bekenstein and Mukhanov<sup>15</sup>, for example, but are more detailed.

An interesting consequence of Eq. (3.23) is that the Hamiltonian will be limited to the region  $0 \leq H < 2\pi/\Delta t = \Delta x/2G$ . Apparently, gravitational disturbances of 3-space then always remain within one space quantum away from flat space. It goes without saying that the question exactly how all this has to be combined in a more comprehensive theory remains to be studied.

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