

**QUANTIZATION OF POINT PARTICLES
IN 2+1 DIMENSIONAL GRAVITY
AND SPACE-TIME DISCRETENESS**

G. 't Hooft

Institute for Theoretical Physics
Utrecht University , P.O.Box 80 006
3508 TA Utrecht, the Netherlands

ABSTRACT

By investigating the canonical commutation rules for gravitating quantized particles in a 2+1 dimensional world it is found that these particles live on a space-time lattice. The space-time lattice points can be characterized by three integers. Various representations are possible, the details depending on the topology chosen for energy-momentum space. We find that an $S_2 \times S_1$ topology yields a physically most interesting lattice within which first quantization of Dirac particles is possible. An S_3 topology also gives a lattice, but does not allow first quantized particles.

1. INTROUCTION

Reducing the number of space dimensions from three to two in quantum gravity is a quite severe simplification that drastically changes the physics one attempts to describe. Due to the absence of local dynamical degrees of freedom Einstein's equations for gravity become almost trivial: in the absence of matter space-time is featureless (flat, if there is no cosmological constant). Hence pure gravity is a topological theory in this case, and it has been investigated by a number of authors¹.

Pure 2+1 dimensional gravity is of interest if one wishes to contemplate cosmologies with highly non-trivial boundary conditions in the space-time coordinates. However if one wishes to gain insights in local aspects of gravity these boundary conditions are not so important. The theory then becomes non-trivial only if one adds matter. One of the simplest forms of matter a particle physicist can imagine is a single, non-interacting scalar field with the following Lagrangian:

$$\mathcal{L}_{\text{matter}} = \sqrt{-g}(-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2). \quad (1.1)$$

In the absense of gravity the quantum version of this theory generates a Hilbert space spanned by all N particle states, where each particle has a given momentum, and the particles do not interact.

What gravity does to *classical* point particles in 2+1 dimensions is also clear²: each particle cuts a cone in 2-space, as seen from its own Lorentz rest frame. Since the surrounding space-time is completely flat this system is still exactly solvable, although in the N particle case the book-keeping of the transitions after a long stretch of time can become infinitely complicated and chaotic phenomena can easily occur³.

Does this suggest that the quantum mechanical version of the gravitating point particle theory should also be soluble? Many classically integrable systems are also quantum mechanically integrable. But it appears that we can compare the classical theory best with that of classical particles in a triangular box with arbitrary angles, which is classically integrable only for finite time intervals, but at large times chaotic, and quantum mechanically not exactly solvable.

Actually, the situation for our toy model is much worse than for the particle in a triangular box: we do not really know how to *define* the corresponding quantum model. Is it a functional integral? If we try to define the theory from that angle we must face the fact that it is not renormalizable, so that one must seriously doubt whether the functional integral can be at all meaningfully defined. Should we build a Hilbert space from states

with N (moving) punctures in space and time?⁴ But that is a *first quantized* theory. If we want a theory that in the absence of matter would approach a scalar field theory we want something that is *second* quantized. This would require the introduction of creation and annihilation operators. Creating or annihilating punctures is tricky if we want to use coordinates to localize these operators, since they backfire on the coordinates themselves.

A promising avenue may be first to construct a Hilbert space containing N Dirac like gravitating particles, obeying Fermi-Dirac statistics or at least some sort of exclusion principle. This would be the first quantized model. Then one might hope to be able to construct states where the Dirac sea of negative-energy states is filled, and introduce second quantization that way. This programme has not yet been carried out completely. A problem is that the negative-energy particles in the Dirac sea would generate negative curvature which should somehow have to be compensated, perhaps by a cosmological constant.

In spite of all these obstacles it is of importance to try to find our way in this jungle. If a meaningful model can be constructed it might show us on the one hand how gravity takes care of its own renormalization problems in the ultraviolet region, and on the other hand, in the infrared region, how to handle the system as a *quantum cosmology*, which requires a definition of cosmological time, and a way to handle the measurement problem.

It has been noted before that the *time* coordinate must probably be chosen to be a discrete multiple of a Planckian time unit⁵. This important observation follows from the fact that the Hamiltonian is an angular variable, well defined only *modulo* 2π (in conveniently chosen Planck units). Time is also directly linked to angular momentum. But what about the spacelike coordinates? This author's first attempts to identify appropriate variables for the N -particle Hilbert space did not lead to space quantization⁵. This was because we used the so-called polygon representation^{3,5,6,7}. Natural parameters to characterize an N -particle state appeared to be the lengths L_i of the edges of polygons used to tessellate a two-dimensional Cauchy surface. The angles of the polygons are fixed as soon as the Lorentz boost parameters are given; that is, the Lorentz boosts η_i relating the coordinate frame of one polygon to that of a neighboring polygon to which it is attached. At each L_i we have one η_i , and it was discovered that the L_i and η_i are conjugated variables:

$$\{2\eta_i, L_j\} = \delta_{ij}. \quad (1.2)$$

But the “momenta” η_i are not angles, they are *hyperbolic* angles. The equations of motion relate the trigonometric functions of the Hamiltonian, $\sin H$ and $\cos H$, to the hyperbolic sines and cosines of the η parameters. If anything, this would suggest that the

imaginary parts, not the real parts, of the parameters L_i are to be chosen discrete. If the L_i were to represent the space coordinates, they still appeared to live in a continuum.

But using the polygon representation to formulate Hilbert space leads to other problems as well. We have very complicated boundary conditions in the form of monodromies as soon as particles make a full swing around each other. Also there are constraints: each polygon must be a closed polygon, which appears to correspond to the constraint that the state must be invariant under Lorentz transformations of the coordinate frames inside each polygon. The fact that the exterior angles of each polygon must add up to 2π corresponds to the constraint of invariance with respect to time boosts for the frame inside each polygon. We found that these constraints are easier to deal with if instead of the L_i we use coordinates $\{x_i, y_i\}$ to localize each corner of each polygon in its own frame. This required to deduce the equations of motion and the Poisson brackets from scratch once again.

It is here that we hit upon a pleasant surprise: the momenta conjugated to these x_i and y_i are true angles once again, as if space itself were limited to discrete numbers as well. Is space as well as time a lattice? If this is so it will be hard to recover rotational and Lorentz invariance. Our first attempts were to replace the wave equations by lattice equations which reduce Poincaré invariance to one of its discrete subgroups. But this seriously jeopardized the entire structure of the theory. If our monodromies were to be restricted to lattice group elements it would become impossible to consider continuum limits. This wasn't the theory we were looking for.

The present paper explains how to set up a more satisfactory scheme. Complete invariance under the continuum Lorentz group needs not be sacrificed even if space and time are restricted to points on a lattice. Everything is derived from the Poisson algebra, which leads one to conclude that the results are inevitable. It is somewhat puzzling to the present author why the lattice structure of space and time had escaped attention from other investigators up till now.

There are different possible representations of the lattice. We first thought that the S_3 topology for energy-momentum space would be the most natural choice. It will turn out however that in the lattice thus obtained no Lorentz-invariant quantum particles can exist unless negative probabilities are accepted. Even a Dirac square root of the partial difference equations for the wave functions does not allow for a positive probability interpretation. We will demonstrate that this disease can be removed if we use an $S_2 \times S_1$ topology instead. The lattice looks only slightly different, and the Dirac equation becomes a beautiful first order difference equation.

2. THE EQUATIONS FOR ONE PARTICLE

For a particle at rest, surrounding space is a cone with a deficit angle β that can be identified with the mass m . When the particle moves it is convenient to consider the relation between *half* the deficit angles at motion and at rest. Therefore we will use units such that the deficit angle, which is easily seen to be additively conserved, is identified with twice the Hamiltonian H of the system. We will use the symbol μ for half the deficit angle at rest, being a parameter that corresponds to mass of a particle. For the time being we will assume that $\mu < \pi/2$. Let the particle move with velocity $v = \tanh \xi$, where ξ is the particle's Lorentz boost parameter. The geometry is then sketched in Figure 1.

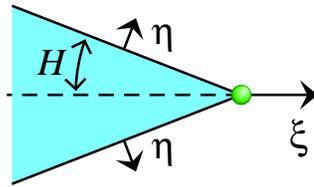


Fig. 1. Wedge cut out by moving particle (dot). ξ is the boost parameter for the velocity of the particle, η the one for the velocity of one edge of the wedge. The hamiltonian H is half the wedge angle.

By choosing the wedge such that the velocity vector lies in the direction of its bisectrix one achieves that the jump across the wedge is purely spacelike, and gluing the edges together provides with a Cauchy surface. Lorentz contraction causes the wedge of a particle to widen when it moves. The velocity of the edges, $\tanh \eta$, easily follows from the geometry. The result is^{2,3,5,6}:

$$\tan H = \cosh \xi \tan \mu; \quad (2.1)$$

$$\tanh \eta = \sin H \tanh \xi; \quad (2.2)$$

$$\cos \mu = \cos H \cosh \eta; \quad (2.3)$$

$$\sinh \eta = \sin \mu \sinh \xi. \quad (2.4)$$

As explained in the introduction, since η is canonically conjugated to the length L of the wedge, the hyperbolic sines and cosines suggest a discretization of the imaginary part of L , a somewhat mysterious feature.

When more particles are being considered the total Hamiltonian will be the sum of the individual contributions, but there will also be “gravitational” contributions from the vertex points between triples of polygons. This is because the Cauchy surface is curved

at these points, so that the angles there do not add up to 2π . These contributions are completely determined by the η 's, as in Eq. (2.3), but we will not discuss them further in this paper.

Now consider the coordinates (x, y) of the particle. First let's have the particle move in the x -direction. Let p_x be the canonical momentum associated to x . What is p_x ? One must require

$$\frac{d}{dt}x = \tanh \xi = \frac{\partial H}{\partial p_x}. \quad (2.5)$$

This is a partial derivative where μ is kept fixed. The Hamiltonian has been postulated to be half the wedge angle. We use Eq. (2.3) to derive

$$\begin{aligned} d(\cos H \cosh \eta) &= 0 && \rightarrow \\ \tan H dH &= \tanh \eta d\eta && \rightarrow \\ dp_x &= \frac{dH}{\tanh \xi} = \frac{\sin H dH}{\tanh \eta} = && (2.6) \\ &= \cos H d\eta = \frac{\cos \mu}{\cosh \eta} d\eta = (\cos \mu) d \arctan(\sinh \eta), \end{aligned}$$

where also Eq. (2.2) was used.

This we can write as

$$p_x = \theta \cos \mu ; \quad \tan \theta = \sinh \eta, \quad (2.7)$$

which is synonymous to

$$\sin \theta = \tanh \eta ; \quad (2.8)$$

$$\cos \theta \cosh \eta = 1 ; \quad (2.9)$$

$$\tan(\frac{1}{2}\theta) = \tanh(\frac{1}{2}\eta), \quad (2.10)$$

provided we keep

$$|\theta| \leq \pi/2 \quad (2.11)$$

(a condition that we will ignore later). Thus we find that, apart from a factor $\cos \mu$, the canonical momentum p_x is an angle θ .

3. ENERGY-MOMENTUM SPACE, AND THE SPACE-TIME LATTICE

Our first impression was that therefore the coordinate x must be quantized,

$$x \stackrel{?}{=} n_1 / \cos \mu, \tag{3.1}$$

and similarly the coordinate y . But then one hits two problems: first, space would become a rectangular lattice, so that invariance under continuous rotations would be lost, and furthermore, if the particle does *not* move in the x -direction, Eqs. (2.6)—(2.11) would hold only for the absolute value $|p|$, not for p_x and p_y separately. We have to look at momentum space more carefully. There are two angles: the value $\theta = p / \cos \mu$, and the angle φ describing the direction in which the particle moves, which must be the direction of the vector \mathbf{p} . This, of course, is the property of line segments lying on an S_2 sphere. Let us therefore write

$$\begin{aligned} \tan(p_x / \cos \mu) &= Q_1 / Q_3; \\ \tan(p_y / \cos \mu) &= Q_2 / Q_3. \end{aligned} \tag{3.2}$$

Then (Q_1, Q_2, Q_3) is a three-vector whose length is immaterial, whose angle with the 3-axis corresponds to the total momentum, and whose angle in the 1-2 plane is the angle of motion of the particle.

The space coordinates x and y should canonically conjugated[†] to these variables p_x and p_y . Since p_x and p_y together form the points of an S_2 sphere it is obvious what these conjugated variables are: they are the quantum numbers (ℓ, m) of the spherical harmonics. These indeed form a two-dimensional lattice, but one not quite as simple as a rectangular one. Note that they allow for the full rotation group $U(1)$ as well as discrete translations – the latter being obtained by multiplying a spherical harmonic with another spherical harmonic, using the Clebsch-Gordan coefficients for the addition of angular momenta. How does this lattice have to be combined with the timelike lattice, and how do we perform Lorentz transformations?

The relation between the Hamiltonian and the momentum (that is, the Schrödinger equation), can be read off from Eqs. (2.9) and (2.3):

$$\cos H = \cos \mu \cos \theta. \tag{3.3}$$

[†] Note however the remark at the end of Sect. 3.

Furthermore we have

$$\tan \theta = \sin \mu \sinh \xi ; \quad (3.4)$$

$$\sin \theta = \sin H \tanh \xi . \quad (3.5)$$

These equations are to be compared with the usual relations for non-gravitating relativistic particles (corresponding to the $\mu \rightarrow 0$ limit):

$$H^2 = \mu^2 + p^2 ; \quad (3.3a)$$

$$p = \frac{\mu v}{\sqrt{1 - v^2}} ; \quad (3.4a)$$

$$p = H v . \quad (3.5a)$$

Let us now investigate how a Lorentz transformation in the x direction affects p_x and H . Let us increase the boost parameter ξ by a small Lorentz boost ε :

$$\begin{aligned} \xi &\rightarrow \xi + \varepsilon ; \\ \sinh \xi &\rightarrow \sinh \xi + \varepsilon \cosh \xi . \end{aligned} \quad (3.6)$$

We find from Eq. (2.7):

$$\begin{aligned} \tan \theta &\rightarrow \tan \theta + \varepsilon \sin \mu \cosh \xi \\ &= \tan \theta + \varepsilon (\cos \mu) \tan H , \end{aligned} \quad (3.7)$$

and, using Eq. (2.1),

$$\begin{aligned} \tan H &\rightarrow \tan H + \varepsilon \tan \mu \sinh \xi \\ &= \tan H + \frac{\varepsilon}{\cos \mu} \tan \theta . \end{aligned} \quad (3.8)$$

Clearly, the couple

$$\begin{pmatrix} \tan \theta \\ \cos \mu \tan H \end{pmatrix} , \quad (3.9)$$

which we can also write as

$$\sin \mu \begin{pmatrix} \sinh \xi \\ \cosh \xi \end{pmatrix}, \quad (3.9a)$$

transforms as a Lorentz vector. One may write

$$\tan H = \frac{Q_4}{Q_3}, \quad (3.10)$$

where Q_4 is again a *real* quantity, so that we have invariance under the Lorentz transformation

$$\begin{pmatrix} Q_4 \cos \mu \\ Q_1 \\ Q_2 \end{pmatrix} \rightarrow L \begin{pmatrix} Q_4 \cos \mu \\ Q_1 \\ Q_2 \end{pmatrix}, \quad (3.11)$$

where L is an ordinary $SO(2,1)$ matrix. Q_3 is *not* involved in Lorentz transformations. There now are topologically distinct ways to represent energy-momentum space. The length of the Q vector can be chosen freely. Since we are dealing with angles, not hyperbolic angles, it is not advised to use hyperbolic spaces but rather compact spaces. If we take $Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2 = 1$, this space forms a Euclidean S_3 sphere. The $SO(4)$ spherical harmonics then generate a lattice in space and time. Our first investigations involved this lattice, and the field equations on it. We now defer further discussion of this lattice to the Appendix, because it was discovered that the field equations one then obtains are difference equations but they link *five* consecutive time layers, which prohibits the construction of any quantity that can serve as a conserved, non-negative probability density. The square roots of these equations, a discrete analogy of the Dirac equation, still links *three* consecutive time layers, which is still physically not acceptable. We explain this situation further in the Appendix.

In our search for a physically most appealing theory we found that the S_3 topology for energy-momentum space must be abandoned, replacing it by the $S_2 \times S_1$ torus. Thus we keep the S_2 sphere for momentum space and put the Hamiltonian on S_1 . In this energy-momentum space Lorentz transformations are still possible. We keep the 3-vector

$$\begin{pmatrix} \cos \mu \tan H \\ \tan \theta \cos \varphi \\ \tan \theta \sin \varphi \end{pmatrix} \quad (3.12)$$

(where φ is the angle in which the particle moves) as being the one that transforms as usual under Lorentz transformations, but now define points on the space-time lattice to

correspond to harmonic functions on this torus:

$$(\ell, m, t) \leftrightarrow Y_{\ell m}(\theta, \varphi)e^{-itH}, \quad (3.13)$$

where t is an integer denoting time. It is clear that here ℓ roughly corresponds to the distance from the origin[‡], whereas m exactly corresponds to angular momentum in the 2-plane. A discrete Fourier transformation provides something like an angular position with respect to the origin, so that the picture that emerges is as given in Fig. 2a. Depicted there is the case when ℓ takes integral values. Of course we also expect the possibility of half-integral values, in which case the space lattice is as sketched in Fig. 2b. Since then angular momentum has half-odd-integral values we expect this case to be appropriate for fermions. We have not contemplated the possibility of having ‘anyons’ in this theory (see some remarks in the Discussion Section).

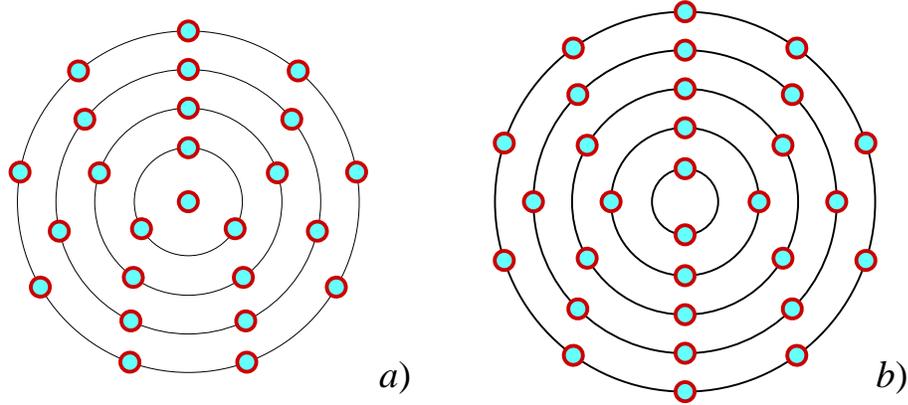


Fig. 2. Lattice in 2-space, a) the bosonic case, b) for fermions.

It should be emphasized that a Lorentz transformation will not correspond to a simple point transformation on our lattice. Lorentz transformations do transform the harmonic functions into linear superpositions of such functions. This implies that a Lorentz transformation is not a local transformation in this theory. It is of course a point transformation in energy-momentum space.

A second remark is that the coordinates x and y most closely correspond to the angular momentum operators L_2 and L_1 on the 2-sphere:

$$x = \frac{L_2}{\cos \mu}; \quad y = \frac{-L_1}{\cos \mu}; \quad L = L_3. \quad (3.14)$$

[‡] More precisely: $\ell(\ell + 1) = (x^2 + y^2)/\cos^2 \mu + m^2$.

Consequently, we have the commutation rules

$$\begin{aligned}
[x, y] &= \frac{i}{\cos^2 \mu} L, \\
[L, x] &= iy, \\
[L, y] &= -ix.
\end{aligned}
\tag{3.15}$$

Here we reinserted the factors $\cos \mu$ from Eq. (2.7).

A third remark: the commutation rules between \mathbf{x} and \mathbf{p} appear to deviate from the standard expressions. This is because the momenta have become angles. The commutation rules are now the ones generated by the spherical harmonics $Y_{\ell m}$ in the usual way. This is a delicate feature since we used Eq. (2.5) as a starting point. It may be verified that Eq. (2.5) holds in the limit of large distances and large times, which is what one really must require for the classical limit.

4. A FIELD EQUATION

Let us consider the 2-sphere

$$Q_1^2 + Q_2^2 + Q_3^2 = 1, \tag{4.1}$$

and define the momenta as in Eq. (3.2). The mass shell condition, Eq. (3.3), can then be rewritten as

$$e^{iH} + e^{-iH} = (2 \cos \mu) Q_3. \tag{4.2}$$

It is not difficult to see what this equation implies for functions defined on our space-time lattice. First consider *any* function $\psi(\ell, m, t)$. Since the functions H and Q_i are defined on its ‘Fourier transform’

$$\hat{\psi}(\theta, \varphi, H) = \sum_{\ell, m, t} Y_{\ell m}^*(\theta, \varphi) e^{iHt} \psi(\ell, m, t), \tag{4.3}$$

the functions $e^{\pm iH}$ and Q_i act as operators on ψ . The first cause a shift in time by one step down and up, respectively. The action of the operators Q_i is found as follows:

$$\begin{aligned} (Q_1 + iQ_2)|\ell, m\rangle &= C(\ell)\sqrt{(\ell+m+1)(\ell+m+2)}|\ell+1, m+1\rangle \\ &\quad - C(\ell-1)\sqrt{(\ell-m)(\ell-m-1)}|\ell-1, m+1\rangle, \end{aligned} \quad (4.4)$$

$$\begin{aligned} Q_3|\ell, m\rangle &= C(\ell)\sqrt{(\ell+m+1)(\ell-m+1)}|\ell+1, m\rangle \\ &\quad + C(\ell-1)\sqrt{(\ell+m)(\ell-m)}|\ell-1, m\rangle, \end{aligned} \quad (4.5)$$

$$\begin{aligned} (Q_1 - iQ_2)|\ell, m\rangle &= -C(\ell)\sqrt{(\ell-m+1)(\ell-m+2)}|\ell+1, m-1\rangle \\ &\quad + C(\ell-1)\sqrt{(\ell+m)(\ell+m-1)}|\ell-1, m-1\rangle. \end{aligned} \quad (4.6)$$

where $C(\ell)$ is a normalization factor, depending only on ℓ . These equations are derived using standard Clebsch-Gordan techniques: one considers the Q functions in these equations as being proportional to the states $|1, 1\rangle$, $|1, 0\rangle$ and $|1, -1\rangle$, respectively. The coefficients are then found by considering the operators L_{\pm} on these states. The normalization factor $C(\ell)$ is found, after a series of algebraic manipulations, by inserting Eq. (4.1):

$$C(\ell) = \left((2\ell+1)(2\ell+3) \right)^{-1/2}. \quad (4.7)$$

Inserting Eq. (4.5) into Eq. (4.2) now yields a difference equation which in the continuum limit should reproduce the Klein-Gordon equation. Notice that this difference equation only involves nearest neighbors in the $\{\ell, m, t\}$ lattice:

$$\begin{aligned} \psi(\ell, m, t+1) + \psi(\ell, m, t-1) &= (2\cos\mu) \left[\sqrt{\frac{(\ell+m+1)(\ell-m+1)}{(2\ell+1)(2\ell+3)}} \psi(\ell+1, m, t) \right. \\ &\quad \left. + \sqrt{\frac{(\ell+m)(\ell-m)}{(2\ell-1)(2\ell+1)}} \psi(\ell-1, m, t) \right]. \end{aligned} \quad (4.8)$$

For large ℓ , small m and small μ this approaches to:

$$\begin{aligned} \psi(\ell, m, t+1) - 2\psi(\ell, m, t) + \psi(\ell, m, t-1) &= \frac{2-\mu^2}{2} \left[1 - \frac{m^2}{2\ell^2} + \frac{1}{8\ell^2} \right] \times \\ &\quad \left[\psi(\ell+1, m, t) - 2\psi(\ell, m, t) + \psi(\ell-1, m, t) \right] - \left[\mu^2 + \frac{m^2 - \frac{1}{4}}{\ell^2} \right] \psi(\ell, m, t). \end{aligned} \quad (4.9)$$

Identifying here m with angular momentum and ℓ with $r = \sqrt{x^2 + y^2}$, we recognise the Klein-Gordon equation in polar coordinates.

The fact that Eq. (4.8) emerges as a difference equation involving only nearest neighbors is the primary physical reason for introducing the $\{\ell, m, t\}$ lattice as proposed in this paper.

5. THE LATTICE DIRAC EQUATION

Since Eq. (4.8) connects three time slices it is not suitable for serving as a quantum wave function. We believe that a promising route towards a useful quantized model for gravitating particles in 2+1 dimensions is to follow Dirac's philosophy: find an acceptable wave function for fermions, which allows for an interpretation of $|\psi|^2$ as a probability distribution, after which we handle the negative energy solutions by filling all these levels. This paper will not go that far (there are various important problems that would yet have to be addressed), but the first step, a Lorentz covariant equation with a non negative preserved probability distribution, can now be made.

Instead of Eq. (4.2) we need an equation containing e^{-iH} only, so that an evolution operator would be obtained connecting exactly two time slices. To do this, we need not only an expression for $\cos H$, as in Eq. (3.3), but also one for $\sin H$. We have, on the sphere (4.1):

$$\begin{aligned}\sin^2 H &= \sin^2 \mu + \cos^2 \mu \sin^2 \theta = \\ &= \sin^2 \mu + \cos^2 \mu (Q_1^2 + Q_2^2).\end{aligned}\tag{5.1}$$

Therefore

$$\begin{aligned}e^{-iH} &= \cos \mu Q_3 - i\sqrt{\sin^2 \mu + \cos^2 \mu (Q_1^2 + Q_2^2)} = \\ &= \cos \mu Q_3 - i\sqrt{Q_1^2 + Q_2^2 + \sin^2 \mu Q_3^2}.\end{aligned}\tag{5.2}$$

To have the equation homogeneous in the Q 's will be of importance later. Clearly, the square root in this equation would turn it into a non-local one on the lattice. Following Dirac we replace this equation by a linear one:

$$e^{-iH} \psi = [\cos \mu Q_3 - i\alpha_1 Q_1 - i\alpha_2 Q_2 - i\beta \sin \mu Q_3] \psi,\tag{5.3}$$

where α_i and β obey the usual Dirac anticommutation rules, so that the Eigenvalues of the operator in (5.3) correspond to the field equation (4.8).

To check Lorentz invariance we consider the Lorentz vector (3.11), (3.12). Q_4 , as defined in (3.10), obeys

$$Q_3^2 + Q_4^2 = 1/\cos^2\mu, \quad (5.4)$$

so that

$$e^{-iH} = \cos\mu(Q_3 - iQ_4), \quad (5.5)$$

and Eq. (5.3) can be rewritten as

$$\cos\mu Q_4 = \alpha_1 Q_1 + \alpha_2 Q_2 + \beta \sin\mu Q_3, \quad (5.6)$$

or, renaming β as γ_3 and $\beta\alpha_i$ by $i\gamma_i$, we get

$$\cos\mu \gamma_3 Q_4 = i\gamma_1 Q_1 + i\gamma_2 Q_2 + \sin\mu Q_3; \quad \{\gamma_i, \gamma_j\} = 2\delta_{ij}. \quad (5.7)$$

This equation of course has the required Lorentz covariance if (3.11) transforms as a Lorentz vector. Note that without the second step in Eq. (5.2) we would have obtained a different equation that does not appear to have the required invariance.

Let us rewrite Eq. (5.3) as

$$e^{-iH}\psi = [e^{-i\beta\mu}Q_3 - i\alpha_i Q_i]\psi. \quad (5.8)$$

On the lattice it becomes

$$\begin{aligned} \psi(\ell, m, t+1) &= e^{-i\gamma_3\mu} [F_{\ell,m}^1 \psi(\ell+1, m, t) + F_{\ell,m}^2 \psi(\ell-1, m, t)] + \\ &+ \gamma_3 \frac{\gamma_1 - i\gamma_2}{2} [F_{\ell,m}^3 \psi(\ell+1, m+1, t) - F_{\ell,m}^4 \psi(\ell-1, m+1, t)] + \\ &+ \gamma_3 \frac{\gamma_1 + i\gamma_2}{2} [-F_{\ell,m}^5 \psi(\ell+1, m-1, t) + F_{\ell,m}^6 \psi(\ell-1, m-1, t)]. \end{aligned} \quad (5.9)$$

where the coefficients can be read off from Eqs (4.4)–(4.7):

$$\begin{aligned}
F_{\ell,m}^1 &= \sqrt{\frac{(\ell+m+1)(\ell-m+1)}{(2\ell+1)(2\ell+3)}}, & F_{\ell,m}^2 &= \sqrt{\frac{(\ell+m)(\ell-m)}{(2\ell-1)(2\ell+1)}}, \\
F_{\ell,m}^3 &= \sqrt{\frac{(\ell+m+1)(\ell+m+2)}{(2\ell+1)(2\ell+3)}}, & F_{\ell,m}^4 &= \sqrt{\frac{(\ell-m)(\ell-m-1)}{(2\ell-1)(2\ell+1)}}, \\
F_{\ell,m}^5 &= \sqrt{\frac{(\ell-m+1)(\ell-m+2)}{(2\ell+1)(2\ell+3)}}, & F_{\ell,m}^6 &= \sqrt{\frac{(\ell+m)(\ell+m-1)}{(2\ell-1)(2\ell+1)}}.
\end{aligned} \tag{5.10}$$

By construction, our equation (5.8) produces Eigenvalues with absolute values equal to one, and hence we immediately see that the norm of ψ is preserved:

$$\|\psi(t+1)\| = \|\psi(t)\|. \tag{5.11}$$

This equation can be used as an evolution equation for a quantum state ψ . Needless to say, our Hamiltonian H has as many positive Eigenvalues as negative ones, and so an eventual physically viable theory should have the negative states filled up like a Dirac sea, or else it will be impossible to perform thermodynamical calculations. As stated before, this would go beyond the scope of the present paper.

6. DISCUSSION

The introduction of lattice versions of gravity has been proposed before⁸. This particular lattice may also have been considered before. Here we emphasize that the lattice structure has been *derived* from the mathematical expressions for the relation between momenta and the Hamiltonian, rather than postulated. Only with the lattice spacing precisely defined by the Planck length do the field equations take the form of difference equations involving only nearest neighbors on the lattice.

It should be kept in mind that the lattice structure is more complicated than the familiar rectangular or triangular lattices. For a given value of ℓ , the angular momentum is limited by the inequality

$$|m| \leq \ell. \tag{6.1}$$

The distance to the origin, r , is quantized as

$$r = \frac{1}{\cos \mu} \sqrt{\ell(\ell + 1) - m^2}, \quad (6.1)$$

Concerning the quantum of angular momentum, the reader might have wondered why it should be taken as integers (or half-odd-integers) at all. First of all, one could consider “anyon” statistics. Now this appears to be difficult in our approach. If energy-momentum space is indeed represented by an S_2 sphere we will be forced to restrict ourselves to the $SO(3)$ quantum numbers ℓ and m , but it is conceivable that more complex theories can be constructed; we leave this to other investigators.

Secondly however, one might argue that we also have to deal with the non-trivial holonomies generated by the cusps around the particles. It may seem that angular momentum ought to be quantized in units $\pi/(\pi - H)$ rather than 1. We deliberately postponed this complication. The reason is that this topological aspect of space emerges if *one particle circumnavigates another particle*, in other words, it becomes relevant only if we consider two or more particles. In the present work only single particle states were considered. The center of our coordinate frame is not a particle. It cannot be taken as an infinitely heavy particle because such a thing would distort the surrounding space too much. As yet we treat the origin as a more abstract reference point in our coordinates. This situation is clearly not completely satisfactory. In order to construct a more complete theory one will have to address the many particle situation.[†] One then can restrict oneself to describing distances between particles only, while avoiding abstract reference points altogether. This however will be a major exercise, complicated by the fact that in the polygon representation (which we insist on using since it involves proper Cauchy surfaces) shows extra contributions to the Hamiltonian from the vertex points between the polygons. We suspect that such a treatment will also further clarify the role of the factors $\cos \mu$ in our expressions (see Eq. 6.1), which presently seem to be rather ugly.

The equations on our lattice are local in the sense that only nearest neighbors interact, and the Cauchy surfaces are well-defined. In contrast, Poincaré transformations are in general non-local. Related to this is the fact that we have some freedom in choosing the details of our lattice. Instead of $S_2 \times S_1$ we could have picked S_3 . What happens then is explained in the Appendix. The equations also involve nearest neighbors, except that in the time direction next-to-nearest neighbors then also appear, and this makes this lattice less suitable. The arguments of Sect. 5, in particular Eq. (5.11), do not work in that case.

The Poincaré group in our model is different from the usual one. We still have the

[†] An approach towards handling this problem using conformal gauges for space and time is advocated by Menotti, Seminara⁹ and Welling¹⁰.

complete Lorentz group, but the translations are limited to the procedures of adding angular momenta using the familiar Clebsch-Gordan coefficients.

We seem to have hit upon a new terrain of lattice theories where much remains to be explored.

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NOTE ADDED

Just before this manuscript was mailed a paper by Waelbroeck and Zapata appeared in the electronic archives¹¹ in which also a space-time lattice is discussed in connection with 2+1 dimensional gravity. His lattice however is the lattice formed by the imaginary parts of the link variables L_i as discussed in the Introduction. It is quite distinct from what we propose in this paper.

APPENDIX: THE LATTICE GENERATED BY S_3 .

The algebra of the spherical harmonics on S_3 is generated by first considering rotation operators L_a and M_a , $a = 1, 2, 3$, in a four-dimensional space with coordinates (q_1, \dots, q_4) :

$$L_a = -i\varepsilon_{abc}q_b\partial_c ; \quad M_a = i(q_4\partial_a - q_a\partial_4). \quad (A.1)$$

They obey the commutation rules

$$[L_a, L_b] = i\varepsilon_{abc}L_c ; \quad (A.2a)$$

$$[L_a, M_b] = i\varepsilon_{abc}M_c ; \quad (A.2b)$$

$$[M_a, M_b] = i\varepsilon_{abc}L_c . \quad (A.2c)$$

Construct the self-dual and the anti-self-dual combinations,

$$L_a^L = \frac{1}{2}(L_a + M_a) ; \quad L_a^R = \frac{1}{2}(L_a - M_a). \quad (A.3)$$

We have

$$[L_a^L, L_b^L] = i\varepsilon_{abc}L_c^L ; \quad [L_a^R, L_b^R] = i\varepsilon_{abc}L_c^R ; \quad [L_a^L, L_b^R] = 0. \quad (\text{A.4})$$

Since $\mathbf{L} \cdot \mathbf{M} = 0$ we find

$$\begin{aligned} \mathbf{L}^{L^2} = \mathbf{L}^{R^2} &= \frac{1}{4}(\mathbf{L}^2 + \mathbf{M}^2) = \\ &= -q_\mu^2 \partial_\nu^2 + (q_\mu \partial_\mu)^2 + 2q_\mu \partial_\mu = \ell(\ell + 1). \end{aligned} \quad (\text{A.5})$$

The simplest non-trivial function is $f_\mu(q) = q_\mu$, which has

$$\partial_\mu^2 f_\nu = 0 ; \quad (q_\mu \partial_\mu) f_\nu = f_\nu, \quad (\text{A.6})$$

so that we read off from Eq. (A.5) that it has $\ell = \frac{1}{2}$. Its four components are represented by $m_L = \pm \frac{1}{2}$, $m_R = \pm \frac{1}{2}$. These functions f_μ are then to be considered on the sphere $|q| = 1$. By multiplying these together we can construct functions with all other (integral or half integral) ℓ values, whereas on the other hand it is easy to convince oneself that every set of numbers (ℓ, m_L, m_R) with $|m_L| \leq \ell$ and $|m_R| \leq \ell$ corresponds to one and only one spherical harmonic function (up to a phase factor). These numbers ℓ , m_L and m_R therefore form a complete set of coordinates.

We have a freedom in our choice for orienting the coordinates Q_μ in the q_μ space. How one does this turns out to make little difference; in all cases the ℓ , m_L , m_R do not directly correspond to all space-time coordinates. A convenient choice is:

$$Q_1 = q_1, \quad Q_2 = q_2, \quad Q_3 = q_4, \quad Q_4 = q_3. \quad (\text{A.7})$$

Then the variable conjugated to the Hamiltonian (3.10) is

$$t = -M_3 = m_R - m_L. \quad (\text{A.8})$$

and from Eqs. (3.2) and (A.1) we have

$$x \cos \mu = L_2 ; \quad y \cos \mu = -L_1, \quad (\text{A.9})$$

Consider an *arbitrary* function of space-time, i.e., one that does not obey any equation of motion such as a mass shell condition. For such a function the three coordinates x , y , and t can all three be regarded as operators. They are quantized according to

$$x_i = n_i / \cos \mu ; \quad t = n_3 / \cos \mu , \quad (\text{A.10})$$

however, these three numbers n_i are non-commuting. We have

$$\begin{aligned} [x, y] &= i(\cos^2 \mu) L_3 , \\ [y, t] &= -i(\cos \mu) M_2 , \\ [x, t] &= -i(\cos \mu) M_1 . \end{aligned} \quad (\text{A.11})$$

The operators $L_{1,2,3}$ are the usual generators of *Euclidean* rotations among the M_i . Thus only the operators t , L_3 , and ℓ are diagonalized simultaneously. ℓ roughly corresponds to ‘total Euclidean distance from the origin’. If we would limit ourselves to integral ℓ we find that at even time $m_R - m_L$ is even, therefore $m_R + m_L$ is even. This would imply that at even time *parity* of all functions $f(x, y)$ is even: $f(x, y) = f(-x, -y)$, and at odd times parity is odd. If one wants to avoid that, half-odd-integer values of ℓ should be admitted together with the integral ones.

The L_i represent Euclidean rotations in space-time. Lorentz invariance is not a property of the coordinates but of the mass shell condition (3.3). Lorentz transformations are defined in terms of Eq. (3.11) in energy-momentum space. They map the sphere onto itself, after which decomposition in spherical harmonics will give the new coordinates. Since the L ’s and M ’s of Eqs (A.2) are the generators of Euclidean rotations they do not transform into combinations of each other under Lorentz transformations. Therefore the Lorentz group will act in a fairly complicated way on our coordinates; this statement holds both for the S_3 case and for the $S_2 \times S_1$ case treated in the text.

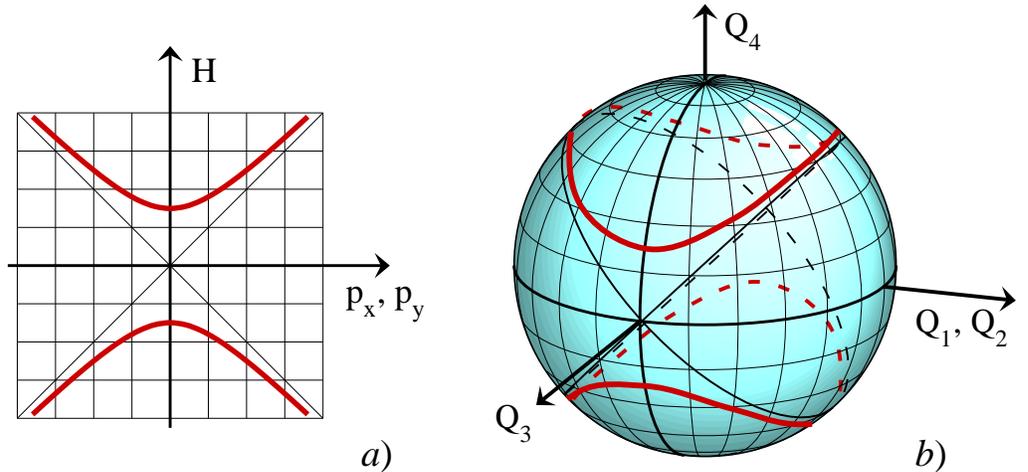


Fig. 3. The mass shell: a) in ordinary 2+1 dimensional particle theory, b) in 2+1 dimensional gravity for the S_3 case. Energy and momentum live on an S_3 sphere. In both cases the two space axes are drawn as one.

Now consider the mass shell condition, Eq. (3.3), which in the $S_2 \times S_1$ case had to be rewritten as Eq. (4.2). In order to be able to process it in S_3 we have to rewrite it as

$$Q_4^2 \cos^2 \mu = Q_1^2 + Q_2^2 + Q_3^2 \sin^2 \mu \quad (\text{A.12})$$

(compare Eq. (5.2)). In ordinary flat space it is replaced by Eq. (3.3a), which corresponds to the Klein-Gordon equation in coordinate space. In Fig. 3 we show how the mass shell looks on the S_3 sphere. What is the equation generated by (A.12) in our space-time lattice? The functions $Q_\mu Q_\nu$ form a representation of $SO(4)$ obtained from the symmetrized product of two $\ell = \frac{1}{2}$ representations Q_μ :

$$4 \times 5 / 2 = 10 = 9 + 1, \quad (\text{A.13})$$

where the 1 is of course the scalar Q_μ^2 , which is fixed to be 1 on the sphere. The 9 is formed from the states $\ell = 1, |m_L| \leq 1, |m_R| \leq 1$. Multiplying a function at the point ℓ, m_L, m_R with this representation leads one to a neighboring point $\ell + \Delta\ell, m_L + \Delta m_L, m_R + \Delta m_R$, with $|\Delta\ell| \leq 1, |\Delta m_L| \leq 1$ and $|\Delta m_R| \leq 1$. We see that, as in Sect. 4, the Klein-Gordon equation is replaced by a difference equation on the lattice. We also see that only points differing by integral amounts of ℓ, m_L and m_R are connected, so that the space-time lattice breaks up into two mutually non-interacting systems. This corresponds to the conservation of parity.

Let us consider the action of the Q_μ operators (or equivalently the q_μ) in somewhat more detail. From Eqs. (A.1) and (A.3) we have

$$L_3^{\frac{L}{R}} = \frac{1}{2}i(q_2\partial_1 - q_1\partial_2 \pm q_4\partial_3 \mp q_3\partial_4). \quad (\text{A.14})$$

In order to expand a “state” $\sum_\mu \alpha_\mu q_\mu$ in terms of eigenstates $|\frac{1}{2}, m_L, m_R\rangle$, we solve the equations

$$\begin{aligned} L_3^{\frac{L}{R}}(\alpha \cdot q) &= \frac{1}{2}i(q_2\alpha_1 - q_1\alpha_2 \pm q_4\alpha_3 \mp q_3\alpha_4) \\ &= m_{\frac{L}{R}}(\alpha_1q_1 + \alpha_2q_2 + \alpha_3q_3 + \alpha_4q_4), \end{aligned} \quad (\text{A.15})$$

Indicating the states $|\frac{1}{2}, m_L, m_R\rangle$ (where of course all m values are $\pm\frac{1}{2}$) by the short-hand notation $|\sigma_L, \sigma_R\rangle$, where the σ 's denote the signs of the m 's, we find:

$$\begin{array}{lcl}
\sqrt{2}|++\rangle & \rightarrow & \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \\
\sqrt{2}|+-\rangle & \rightarrow & 1 \quad i \quad 0 \quad 0 \\
\sqrt{2}|-+\rangle & \rightarrow & 0 \quad 0 \quad -1 \quad -i \\
\sqrt{2}|--\rangle & \rightarrow & 0 \quad 0 \quad -1 \quad i \\
& & -1 \quad i \quad 0 \quad 0
\end{array}
\qquad
\begin{array}{lcl}
\sqrt{2}q_1 & = & |++\rangle - |--\rangle \\
\sqrt{2}q_2 & = & -i|++\rangle - i|--\rangle \\
\sqrt{2}q_3 & = & -|+-\rangle - |-+\rangle \\
\sqrt{2}q_4 & = & i|+-\rangle - i|-+\rangle
\end{array}
\tag{A.16}$$

The relative signs here have been fixed by considering the (left and right) angular momentum raising and lowering operators.

Now that we know their left and right quantum numbers we are in a position to calculate the action of the operators q_μ on the states $|\ell, m_L, m_R\rangle$. The m quantum numbers just add up, whereas the total angular momentum numbers ℓ change by one half unit. Thus we can state that

$$\begin{aligned}
|\sigma_L, \sigma_R\rangle|\ell, m_L, m_R\rangle &= \alpha_{m_L, m_R}^{\sigma_L, \sigma_R} |\ell + \frac{1}{2}, m_L + \frac{1}{2}\sigma_L, m_R + \frac{1}{2}\sigma_R\rangle + \\
&+ \beta_{m_L, m_R}^{\sigma_L, \sigma_R} |\ell - \frac{1}{2}, m_L + \frac{1}{2}\sigma_L, m_R + \frac{1}{2}\sigma_R\rangle,
\end{aligned}
\tag{A.17}$$

where α and β are coefficients that can now be calculated. The technique to be used is a standard manipulation with Clebsch-Gordan coefficients, similar to what we did in Sect. 4, but we will not repeat it here.

We see confirmed that the equation (A.12), in which each term corresponds to acting twice with operations as described in Eq. (A.17), amounts to a difference equation on the (ℓ, m_L, m_R) lattice. In general this equation takes the form:

$$\sum_{\sigma_\ell, \sigma_L, \sigma_R} F(\ell, m_L, m_R, \sigma_\ell, \sigma_L, \sigma_R) \psi(\ell + \sigma_\ell, m_L + \sigma_L, m_R + \sigma_R) = 0,
\tag{A.18}$$

where all variables $\sigma_{\ell, L, R}$ take the values ± 1 only. Now, in view of Eq. (A.8), we see that to describe the complete space of solutions to this equation one needs the Cauchy data on several consecutive ‘‘Cauchy planes’’: $t = t_1 + 1, t_1, \dots, t_1 - 2$, after which the values at $t = t_1 + 2$ follow.

The Dirac equation again takes the form

$$i\gamma_1 Q_1 + i\gamma_2 Q_2 + \gamma_3 Q_4 \cos \mu + Q_3 \sin \mu = 0,
\tag{A.19}$$

where $\gamma_{1,2,3}$ can be taken to be the three Pauli matrices. Multiplying the equation with

$$i\gamma_1 Q_1 + i\gamma_2 Q_2 + \gamma_3 Q_4 \cos \mu - Q_3 \sin \mu, \quad (\text{A.20})$$

reproduces (A.12). Note that Lorentz covariance follows from the usual arguments, using the transformation law (3.11).

Consider a solution of the difference equation (A.19). In the notation of Eq. (A.16) our equation (A.19) takes the form

$$(\gamma_2 + i\gamma_1)|++\rangle + |\gamma_2 - i\gamma_1|--\rangle + i\gamma_3 e^{i\gamma_3\mu}|+-\rangle - i\gamma_3 e^{-i\gamma_3\mu}|-+\rangle = 0. \quad (\text{A.21})$$

Since $t = m_R - m_L$ (Eq. A.8), the last term advances time by one unit and the third term retards by one unit. The first two terms stay on the same time sheet. One can never have a positive and conserved probability density for such a wave equation.

In conclusion, besides giving mathematically more cumbersome difference equations, the S_3 lattice is also physically less appealing than $S_2 \times S_1$. Ultimately, however, both physical systems are mathematically equivalent, since in momentum space the field equations are identical.

REFERENCES

1. J.R. Gott and M. Alpert, *Gen. Rel. Grav.* **16** (1984) 243;
S. Giddings, J. Abbot and K. Kuchar, *Gen. Rel. Grav.* **16** (1984) 751;
J.E. Nelson and T. Regge, *Quantisation of 2+1 gravity for genus 2*, Torino prepr. DFTT 54/93, gr-qc/9311029
2. A. Staruszkiewicz, *Acta Phys. Polon.* **24** (1963) 734;
J.R. Gott, and M. Alpert, *Gen. Rel. Grav.* **16** (1984) 243;
S. Giddings, J. Abbot and K. Kuchar, *Gen. Rel. Grav.* **16** (1984) 751;
S. Deser, R. Jackiw and G. 't Hooft, *Ann. Phys.* **152** (1984) 220; *ibid.*, *Phys. Rev. Lett.* **68** (1992) 267.
3. S.M. Carroll, E. Farhi and A.H. Guth, *Phys. Rev. Lett.* **68** (1992) 263,
C. Cutler, *Phys. Rev. D* **45** (1992) 487, see also A. Ori, *Phys. Rev. D* **44** (1991) R2214;
G. 't Hooft, *Class. Quantum Grav.* **10** (1993) 1023, *ibid.* **10** (1993) S79; *Nucl. Phys. B* **30** (Proc. Suppl.) (1993) 200.

4. E. Witten, *Nucl. Phys.* **B311** (1988) 46;
 S. Carlip, *Nucl. Phys.* **B324** (1989) 106; and in: “*Physics, Geometry and Topology*”, NATO ASI series B, Physics, Vol. **238**, H.C. Lee ed., Plenum 1990, p. 541;
 S. Carlip, *Six ways to quantize (2+1)-dimensional gravity*, Davis Preprint UCD-93-15, gr-qc/9305020.
 G. Grignani, *2+1-dimensional gravity as a gauge theory of the Poincaré group*, Scuola Normale Superiore, Perugia, Thesis 1992-1993;
 G. 't Hooft, *Commun. Math. Phys.* **117** (1988) 685; S. Deser and R. Jackiw, *Comm. Math. Phys.* **118** (1988) 495.
5. G. 't Hooft, *Class. Quantum Grav.* **10** (1993) 1653 (gr-qc/9305008).
 See also: A.P. Balachandran and L. Chandar, *Nucl. Phys.* **B 428** (1994) 435.
6. G. 't Hooft, *Class. Quantum Grav.* **9** (1992) 1335.
7. H. Waelbroeck, *Class. Quantum Grav.* **7** (1990) 751; *Phys. Rev. Lett.* **64** (1990) 2222; *Nucl. Phys.* **B 364** (1991) 475; H. Waelbroeck and F. Zertuche, *Phys. Rev.* **D50** (1994) 4966
8. See for instance: T. Regge, *Nuovo Cimento* **19** (1961) 558;
 J. Ambjørn and J. Jurkiewicz, *Phys. Lett.* **B 278** (92) 42;
 J. Ambjørn, J. Jurkiewicz, and C.F. Kristjansen, *Nucl. Phys.* **B 393** (93) 601;
 M.E. Agishtein and A.A. Migdal, *Mod. Phys. Lett.* **A7** (92) 1039; *Nucl. Phys.* **B385** (92) 395.
 G. 't Hooft, in “*Recent Developments in Gravitation*”, Cargèse 1978, ed. M. Lévy and S. Deser, Plenum, New York/London, p. 323;
 B.V. de Bakker, and J. Smit, *Phys. Lett.* **B334** (1994) 304; *Nucl. Phys.* **B439** (1995) 239.
9. P. Menotti and D. Seminara, *Fermi-Walker gauge in 2+1 dimensional gravity*, Pisa/MIT preprint MIT-CTP#2400, IFUP-TH-1/95, January 1995.
10. M. Welling, *Gravity in 2+1 Dimensions as a Riemann-Hilbert Problem*, Utrecht prepr. THU 95/24 (hep-th/9510060); id, *Some Approach to 2+1 Dimensional Gravity Coupled to Point Particles*, Utrecht prepr. THU 95/33 (hep-th/9511211).
11. H. Waelbroeck and J.A. Zapata, *2+1 Covariant Lattice Theory and 't Hooft's Formulation*, Pennsylvania State Univ. prepr. CGPG-95/8 (gr-qc/9601011).