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ON THE FACTORIZATION OF UNIVERSAL POLES IN A THEORY OF GRAVITATING POINT PARTICLES

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A theory is considered in which point-like particles scatter only gravitationally and electromagnetically but no other exchanges are taken into account. The two-particle amplitude at high s, low t, as computed before, has universal poles at s values whose imaginary parts are integer positive numbers times the Planck mass squared. In this paper, the three-particle amplitude at high s, low t, is computed, and found to yield half-integer values. All these calculations only use general relativity and quantum mechanics as an input. Some speculations upon the relevance of these poles for the quantization problem of gravitation are given in the end.

1. Introduction

In a previous paper we described the gravitational interaction between two light particles whose center-of-mass energy s is of the order of the Planck mass squared or beyond, while the momentum exchange t remains small [1]. Let us summarize the way in which this amplitude was found.

Particle (1) is put in the rest frame and particle (2) moves to the left along the z-axis, with energy

$$p_0^{(2)} = \mathrm{O}\left(1/Gm^{(1)}\right). \tag{1.1}$$

where G is Newton's constant in units in which $\hbar = c = 1$. Its gravitational field then takes the form of a "shock wave" [2]. An ingoing wave function

$$\psi_{\text{in}}^{(1)} = \mathrm{e}^{ip^{(1)}x}, \tag{1.2}$$

for particle (1) will be converted into [1]

$$\psi_{\text{out}}^{(1)} = \int A(\boldsymbol{k})\, \mathrm{d}^3\boldsymbol{k}\, \mathrm{e}^{ikx}, \tag{1.3}$$

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with

$$A(\boldsymbol{k}) = \frac{k_+}{4\pi k_0}\delta\left(k_+ - p_+^{(1)}\right)\frac{\Gamma(1-iB)}{\Gamma(iB)}\left[\frac{4}{\left(\tilde{k}-\tilde{p}^{(1)}\right)^2}\right]^{1-iB}. \tag{1.4}$$

Here

$$p_\pm = p_0 \pm p_3, \qquad \tilde{p} = \begin{bmatrix} p_x \\ p_y \end{bmatrix},$$

$$B = Gs - e^{(1)}e^{(2)}/4\pi, \qquad s = -2\left(p^{(1)}\cdot p^{(2)}\right); \tag{1.5}$$

$e(1)$ and $e(2)$ are the electric charges of our colliding particles. To get a covariant expression for $A(\boldsymbol{k})$ we can rewrite

$$\frac{k_+}{k_0} = -\frac{\left(k\cdot p^{(2)}\right)}{k_0 p_0^{(2)}}, \tag{1.6}$$

and because $k_+^{(1)} = p_+^{(1)}$ we can write

$$\left(\tilde{k}^{(1)} - \tilde{p}^{(1)}\right)^2 = \left(k^{(1)} - p^{(1)}\right)^2 = -t;$$

s and t are the usual Mandelstam variables.

Our next step is an apparently insignificant one; we will assume that the amplitude of the entire scattering process, two particles in – two particles out, is generated by eq. (1.4). The change in momentum of particle (1) can easily be absorbed by particle (2), which would be the only way to restore momentum conservation. Thus, using $k^{(2)} \cong p^{(2)}$, we get

$$_{\text{out}}\langle k^{(1)}, k^{(2)}|p^{(1)}, p^{(2)}\rangle_{\text{in}} = \frac{-\left(p^{(1)}\cdot p^{(2)}\right)}{4\pi k_0^{(1)}k_0^{(2)}}\delta^4\left(\sum k - \sum p\right)\frac{\Gamma(1-iB)}{\Gamma(iB)}\left[\frac{4}{-t}\right]^{1-iB}. \tag{1.7}$$

The only approximation made here was that both particles were considered to be point-like. Form factors would ruin the simple Newtonian form of the gravitational field of one particle, and make the other particle scatter in a more complicated way. This is why we call our theory a theory of point particles. Also, the rest masses were assumed to be small, otherwise one would have to replace the Newtonian field by the much more complicated Schwarzschild metric. All of this implies that the parameter t in eqs. (1.6) and (1.7) should be kept small.

Hence, our approximation is the same as the so-called eikonal approximation [3]. Indeed, if G is put equal to zero we get familiar expressions.

What we found remarkable in this result is that a "universal" bound-state spectrum emerges, with s values

$$s = M_{\text{Planck}}^2(-Ni + k\alpha), \tag{1.8}$$

where α is the fine-structure constant, N is a positive integer, and k is any integer, determined by the electric charges $e^{(1)}$ and $e^{(2)}$. The t-dependence of the residues of the poles (1.8) can be expressed as polynomials in t with degree $N-1$. Hence, the largest spins of these "bound stsates" are $N-1$.

In contrast, black holes would presumably give poles at

$$s = M_{\text{Planck}}^2(\mathscr{A} - \mathscr{B}i), \tag{1.9}$$

where $\mathscr{A}$ is large and $\mathscr{B}$ of order one. Our t values must be too small to reproduce such poles.

In this paper we ask whether more can be learned by colliding more than two particles at Planckian energies. Although we will not be able to reach the large t region or the black hole region, we will find new channels with new poles. The result suggests that scattering between *more* than three particles will not give us anything new. The amplitude with three particles in – three particles out will be giving us poles such as those given in eq. (1.8), but with N half-integer.

Sect. 2 of this paper shows how one might calculate this three-particle process. It is essential that we consider the impact parameters to be large. Further, we must assume that the collision takes place in three steps (1 against 2, 1 against 3, and 2 against 3), but within our approximation of large impact parameters this assumption is probably exactly true.

We then consider the pole structure of the resulting amplitude in sect. 3, and find the aforementioned result. Our conclusions are discussed in sect. 4.

2. The three-particle process

The same approximation which gave eq. (1.7) can be applied to the three-particle system. Consider three particles coming in with momenta $p^{(i)}$, $i = 1,2,3$. All $p^{(i)}$ may have vector elements of the order of the Planck mass. They leave the interaction area with momenta $k^{(i)}$. Let us define

$$q^{(i)} = k^{(i)} - p^{(i)}, \tag{2.1}$$

so that

$$\sum q = 0.$$

As before we will assume that the impact parameters are large compared to the Planck length, so

$$|q^{(i)}| \ll M_{\text{Pl}}, \qquad k^{(i)} \cong p^{(i)}. \tag{2.2}$$

The Mandelstam variables will be defined to be

$$s_{ij} = -\left(p^{(i)} + p^{(j)}\right)^2 = -2p^{(i)}p^{(j)};$$

$$t_i = -\left(q^{(i)}\right)^2. \tag{2.3}$$

Furthermore, we will need

$$A_{ij} = q^{(i)} \cdot p^{(j)}, \tag{2.4}$$

with

$$A_{23} = -A_{13}, \qquad A_{21} = -A_{31}, \qquad A_{12} = -A_{32}. \tag{2.5}$$

Consider now a coordinate frame which is roughly at the center of mass. Because of (2.2) all three particles will be seen to go forward, nearly without change of velocity. They will not collide exactly head-on, but will miss each other at certain distances. In general, there will be six distinct instances at which one particle is seen to cross the shock wave of another particle. It is, therefore, natural to assume that the total transition amplitude will be the product of six separate amplitudes. When two particles cross each other's shock wave they together produce an amplitude of the form (1.7). The *order* in which the crossings take place is irrelevant because the momenta $p^{(i)}$ and the impact parameters remain the same at each "collision".

In short, we expect the total transition amplitude to be given by

$$\begin{aligned}&{}_{\text{out}}\langle k^{(1)}, k^{(2)}, k^{(3)} | p^{(1)}, p^{(2)}, p^{(3)}\rangle_{\text{in}} \\ &= \int \mathrm{d}^3\boldsymbol{r}^{(1)}\mathrm{d}^3\boldsymbol{r}^{(2)}\mathrm{d}^3\boldsymbol{r}^{(3)}\langle k^{(1)}, k^{(2)}|r^{(1)}, r^{(2)}\rangle\langle r^{(2)}, k^{(3)}|p^{(2)}, r^{(3)}\rangle\langle r^{(1)}, r^{(3)}|p^{(1)}, p^{(3)}\rangle,\end{aligned} \tag{2.6}$$

where the second line is nothing but the product of three two-particle amplitudes, eq. (1.7). We easily observe that the order in which the particles 1, 2 and 3 were put is irrelevant in our approximation, because apart from (2.2) we also have

$$r^{(i)} \cong p^{(i)}. \tag{2.7}$$

Defining $\delta^+(r) = \theta(r_0)\delta(r^2)$ we have

$$\int \mathrm{d}^3\boldsymbol{r} = \int 2r_0\delta^+(r)\,\mathrm{d}^4r. \tag{2.8}$$

And so, using eq. (2.7), eq. (2.6) becomes

$$\langle \text{out} | \text{in} \rangle = \frac{-(p^{(1)}p^{(2)})(p^{(1)}p^{(3)})(p^{(2)}p^{(3)})4^{3-i\Sigma B}}{(2\pi)^3 k_0^{(1)} k_0^{(2)} k_0^{(3)}}$$

$$\times \prod_{i>j} \frac{\Gamma(1-iB_{ij})}{\Gamma(iB_{ij})} \delta^4\left(\sum k - \sum p\right) \times (\text{Int.}), \qquad (2.9)$$

where

$$B_{ij} = Gs_{ij} - e^{(i)}e^{(j)}/4\pi, \qquad (2.10)$$

$$(\text{Int.}) = \int \mathrm{d}^4 q \frac{\delta^+(p^{(1)}+q)\delta^+(q-p^{(3)})\delta^+(q-q^{(1)}-k^{(2)})}{q^{2^{(1-iB_{13})}}(q-q^{(1)})^{2^{(1-iB_{12})}}(q+q^{(3)})^{2^{(1-iB_{23})}}}. \qquad (2.11)$$

Here, we chose as integration variable

$$q = r^{(1)} - p^{(1)} = p^{(3)} - r^{(3)} = k^{(2)} + k^{(1)} - p^{(1)} - r^{(2)}.$$

Now $p^2 = k^2 = r^2 = 0$, whereas all $p_0^{(i)}$, $k_0^{(i)}$, $r_0^{(i)}$ are large and positive. Therefore

$$\delta^+(p^{(1)}+q) = \delta\left((p^{(1)}+q)^2\right) = \delta(2p^{(1)}\cdot q), \qquad (2.12)$$

etc., and the numerator of eq. (2.11) becomes

$$\delta(2p^{(1)}\cdot q)\delta(2p^{(3)}\cdot q)\delta\left(2k^{(2)}\cdot(q-q^{(1)})\right). \qquad (2.13)$$

In a special coordinate frame we may choose

$$p^{(1)} = (0,0,p,p), \qquad p^{(3)} = (0,0,-p,p), \qquad k^{(2)} \cong p^{(2)} = (0,f,g,h), \qquad (2.14)$$

with $f^2+g^2=h^2$. Then eq. (2.13) becomes (see eq. (2.4))

$$\frac{1}{8p^2}\delta(q_0)\delta(q_3)\delta(2fq_2 - 2A_{12}). \qquad (2.15)$$

Clearly, our integral turns into a one-dimensional integral over q_1 only. We will be interested in its pole structure, which of course, in principle, can be read off from the obtained expressions directly. It is more convenient however to convert our integral into one which is two-dimensional using Feynman multiplicators, because the expressions one obtains then are more symmetric and easier to interpret. We write

$$(\text{Int.}) = \frac{\Gamma(3-i\Sigma B)}{\Gamma(1-iB_{13})\Gamma(1-iB_{12})\Gamma(1-iB_{23})} \frac{1}{16p^2 f}(\text{Int.}'), \qquad (2.16)$$

with

$$(\text{Int.}') = \int_{x_{1,2,3}>0} \mathrm{d}x_1\,\mathrm{d}x_2 \int \mathrm{d}q_1\, x_1^{-iB_{23}} x_2^{-iB_{13}} x_3^{-iB_{12}} \big/ D^{3-i\Sigma B}, \tag{2.17}$$

$$x_3 = 1 - x_1 - x_2,$$

$$D = \left(q_1 - x_3 q_1^{(1)} + x_1 q_1^{(3)}\right)^2 + \left(\frac{A_{12}}{f} - x_3 q_2^{(1)} + x_1 q_2^{(3)}\right)^2 + \tilde{q}^{(1)^2} x_3(1-x_3)$$
$$+ \tilde{q}^{(3)^2} x_1(1-x_1) + 2x_1x_3\left(\tilde{q}^{(1)}\cdot\tilde{q}^{(3)}\right). \tag{2.18}$$

The q_1 integral can now be done. Notice that in our approximation, $(q^{(i)}\cdot p^{(i)}) = 0$, so that we can write

$$q^{(1)} = \left(\tilde{q}^{(1)}, Q_1, Q_1\right), \qquad q^{(3)} = \left(\tilde{q}^{(3)}, -Q_3, Q_3\right), \tag{2.19}$$

$$q^{(i)^2} = \tilde{q}^{(i)^2} = -t_i, \qquad \textit{only} \text{ for } i = 1 \text{ or } 3, \tag{2.20}$$

$$-t_2 = q^{(2)^2} = 2\left(\tilde{q}^{(1)}\cdot\tilde{q}^{(3)}\right) - 4Q_1Q_3 - t_1 - t_3. \tag{2.21}$$

Further, we compute

$$4p^2 = s_{13}, \tag{2.22}$$

$$2p(h-g) = s_{12}, \tag{2.23}$$

$$2p(h+g) = s_{23}, \tag{2.24}$$

$$f^2 = h^2 - g^2 = \frac{s_{12}s_{23}}{s_{13}}, \tag{2.25}$$

$$A_{12} = -A_{32} = fq_2^{(1)} + Q_1(g-h) = -fq_2^{(3)} + Q_3(g+h), \tag{2.26}$$

$$2pQ_1 = A_{23} = -A_{13}, \tag{2.27}$$

$$2pQ_3 = A_{21} = -A_{31}, \tag{2.28}$$

so that

$$fq_2^{(1)} = A_{12} + \frac{s_{12}}{s_{13}}A_{23}, \tag{2.29}$$

$$fq_2^{(3)} = -A_{12} - \frac{s_{23}}{s_{13}}A_{31}, \tag{2.30}$$

$$16p^2 f = 4\sqrt{s_{12}s_{23}s_{13}}\,. \tag{2.31}$$

Our integral (2.17) becomes

$$(\mathrm{Int.}') = \int \mathrm{d}x_1\,\mathrm{d}x_2 \frac{\sqrt{\pi}\,\Gamma\left(2\tfrac{1}{2} - i\Sigma B\right)/\Gamma(3 - i\Sigma B)}{D'^{2\frac{1}{2} - i\Sigma B}} x_1^{-iB_{23}} x_2^{-iB_{13}} x_3^{-iB_{12}}, \qquad (2.32)$$

with

$$D' = \left[A_{12}/f - x_3 q_2^{(1)} + x_1 q_2^{(3)}\right]^2 + \tilde{q}^{(1)^2} x_2 x_3 + \tilde{q}^{(2)^2} x_1 x_3 + \tilde{q}^{(3)^2} x_1 x_2$$

$$= -t_1 x_2 x_3 - t_2 x_1 x_3 - t_3 x_1 x_2 - \frac{\Delta\left(x_1 s_{23} A_{31},\, x_2 s_{31} A_{12},\, x_3 s_{12} A_{23}\right)}{s_{12} s_{23} s_{13}}, \qquad (2.33)$$

where we defined

$$\Delta(x, y, z) = 2(xy + yz + zx) - x^2 - y^2 - z^2. \qquad (2.34)$$

All taken together our amplitude becomes

$$\langle \mathrm{out} | \mathrm{in} \rangle = \frac{4^{2 - i\Sigma B}\sqrt{\pi}\,\Gamma\left(2\tfrac{1}{2} - i\Sigma B\right)\delta^4(\Sigma k - \Sigma p)}{(2\pi)^3 \prod_{i>j} \Gamma\left(iB_{ij}\right) k_0^{(1)} k_0^{(2)} k_0^{(3)} \sqrt{s_{12} s_{23} s_{13}}} \int \frac{x_1^{-iB_{23}} x_2^{-iB_{13}} x_3^{-iB_{12}}\,\mathrm{d}^2 x}{D'^{2\frac{1}{2} - i\Sigma B}}$$

(2.35)

where D' is given by eqs. (2.33) and (2.34). In this notation the symmetry under exchange of 1, 2 and 3 is evident.

3. The "bound-state spectrum"

The singularities in eq. (2.35) are now not difficult to find. We have

$$-\left(p^{(1)} + p^{(2)} + p^{(3)}\right)^2 = s_{12} + s_{13} + s_{23}, \qquad (3.1)$$

$$\sum B = G \sum s - \sum_{i>j} e^{(i)} e^{(j)}/4\pi. \qquad (3.2)$$

Apparently, the Γ function in the numerator generates poles at

$$\sum s = M_{\mathrm{P}}^2 (k\alpha - N_{123} i), \qquad N_{123} = 2\tfrac{1}{2}, 3\tfrac{1}{2}, \ldots, \quad k = \text{integer}. \qquad (3.3)$$

There will now be various sets of additional poles, on top of these. They arise when the x-integrations diverge at the various edges of the integration domain.

Take, for instance, the case that x_1 becomes small

$$D' \to -t_1 x_2 x_3 + \frac{\left(x_2 s_{13} A_{12} + x_3 s_{12} A_{23}\right)^2}{s_{12} s_{23} s_{13}}. \tag{3.4}$$

The x_1 integration, tending to

$$\int \mathrm{d}x_1\, x_1^{-iB_{23}}, \tag{3.5}$$

when extended analytically for complex B_{23}, generates poles at

$$s_{23} = M_{\mathrm{P}}^2\left(e^{(2)}e^{(3)}/4\pi - iN_{23}\right), \qquad N_{23} = 1, 2, \dots. \tag{3.6}$$

We then still have the x_2 integration, which diverges at $x_2 \to 0$ and at $x_3 \to 0$. When $x_2 \to 0$ we have

$$D' \to \frac{s_{12} A_{23}^2}{s_{23} s_{13}}, \tag{3.7}$$

and the x_2 integral behaves as

$$\int \mathrm{d}x_2\, x_2^{-iB_{13}}, \tag{3.8}$$

generating poles at

$$s_{13} = M_{\mathrm{P}}^2\left(e^{(1)}e^{(3)}/4\pi - iN_{13}\right), \qquad N_{13} = 1, 2, \dots. \tag{3.9}$$

The residues of the combined poles are easily seen to be polynomials in t_i and A_{ij}, whose degrees are limited by the numbers N_{123}, N_{23} and N_{13}. We suggest that the poles found at the points given by eqs. (3.4)–(3.9) can be interpreted as coming from a cascade scattering process, illustrated in fig. 1.

The solid double and triple lines are intermediate states. The location of their poles in the complex s-plane is given by the numbers N_{23}, N_{123} and N_{13}. There are two diagrams of this type giving the same pole structure (obtained from each other by interchanging the in- and out-lines).

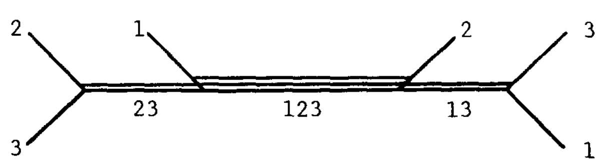

Fig. 1. Cascade scattering process.

4. Discussion

The positions of the poles in eqs. (1.8) and (2.34) only depend on the electric charges of the scattered particles, and are all at

$$s = M_P^2(k\alpha - Ni), \qquad N = 1, 2, 2\tfrac{1}{2}, 3, 3\tfrac{1}{2}, \ldots, \quad k = \text{integer}, \tag{4.1}$$

where α is the fine-structure constant. We suspect that no other N values will arise when more than three particles scatter. This is because the half-integer values came from Γ functions with half integers in them, and they arose because we had to integrate over an odd number of variables (the one-dimensional integration in eq. (2.11). Therefore, it seems that these poles will be universal.

It is not obvious that all these poles are really due to physical particles or resonances. After all, our expressions are valid only for low (nonplanckian) t values, and in principle it could be that they simply disappear when the full amplitude is considered. Calculations done in more sophisticated but more ambitious string models [4], summing diagrams also at high s, low t, do not reproduce our poles. It has been argued [5] that string models essentially "soften" the gravitational forces at very small impact parameters, by modifying the potential $\log|\tilde{x}|$ into a smoother analytic function at very small $|\tilde{x}|$. We stress, however, that our results are exact in the low t limit. Whatever the true scattering amplitudes are, they must show the tendency to either reproduce these poles or other (earlier) singularities. The claim is [5] that the superstring amplitudes are entirely analytic so at low t they should mimic our poles.

One might suspect (but we are not insisting on it) that in contrast to the result of ref. [4], which relied on string theories, the poles found in our calculations have a more physical interpretation. After all, it is quite striking that their positions are t-independent, and their residues are polynomials in t, whose degrees increase with N, suggesting well-defined spins for these "objects".

Unfortunately, some of these nice features disappear when we try to "improve" our calculation. For instance, it would be natural to suspect that α in eq. (4.1) should be replaced by a running coupling constant $\alpha(t)$

$$\alpha(t) \cong \alpha(0)/(1 - C\alpha(0)\log t), \tag{4.2}$$

where the constant C depends on certain details of the theory. But obviously this would ruin the t-independence of the pole position. If we let α run with s then we have to accept the resulting complex logarithm. Moreover, this does not seem to be correct at low t values, where we think our expressions should be correct as they were given. Most likely, replacing α simply by a running α is a too simplistic representation of the complicated field-theoretic corrections due to photon exchange at high t. The poles, if they are indeed resonance states, should be at t-independent positions.

Our proposal is that amplitudes such as our eqs. (1.7) and (2.35) should be incorporated in attempts at quantizing gravity such as the superstring theories. We get a spectrum of states for free. They are not the states postulated in string theory because the mass values are complex.

How degenerate these states are is another important question. In principle they seem to carry as many quantum numbers as the particles used in the scattering process. However, there are reasons to suspect that black holes, at least, cannot preserve any continuous global symmetry; hence, maybe also the objects found in this paper violate conservation laws, so that their degeneracy could be much lower than would be necessary otherwise.

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