NONCONVERGENCE OF THE 1/N EXPANSION FOR SU(N) GAUGE FIELDS ON A LATTICE

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Received 1 May 1977

We present specific examples that demonstrate the non-convergence of the $1/N$ expansion for the lattice theory of SU(N) gauge fields.

$1/N$ expansions in field theories with N or N^2 field components are a useful device for simplification and/or bookkeeping purposes of Feynman diagrams [1]. In the conventional perturbation expansion of SU(N) gauge theories one may consider the limit $N \to \infty$ keeping g^2N fixed, g being the coupling constant. One then finds at each order of g^2N a finite polynomial in $1/N$ with coefficients that are related in a precise manner to the topology of the corresponding diagrams as twodimensional surfaces [2]. In particular the leading term consists of planar Feynman diagrams only, which suggests that in the limit $N \to \infty$ one obtains hadrons that are essentially non-interacting. The $1/N$ expansion then corresponds to an expansion with respect to the coupling strength between the hadrons. Our general experience with coupling-constant expansions in field theories then suggests that the $1/N$ expansion will diverge at a fixed *value* for g^2N, even though the series is finite and therefore converges at fixed *order* in g^2N. We think that the probable formal divergence of the $1/N$ expansion is not a sufficient argument to reject $1/N$ expansions altogether, first because in the physically interesting case of SU(3) the effective coupling strength of 1/3 may be small enough so that the spectrum obtained in the $N \to \infty$ limit will still resemble the physical spectrum, and secondly because fundamental problems such as the quark-confinement mechanism are likely to be independent of N, and understanding of such mechanisms in the $N \to \infty$ limit could be of great significance.

Thus we were motivated to study the $1/N$ expansion further, but now in the SU(N) gauge theory on a lattice. Here the usual expansion is made with respect to $1/g^2$ and $1/m_q$ where m_q are the masses of the quarks [3]. Alternatively, one may expand with respect to $1/g^2N$ and $1/N$, keeping m_q fixed and arbitrary [4]. Again we look at fixed order in $1/g^2N$ and this time we find that the series in $1/N$ does not only continue up to infinity as an essentially geometric series, but, more annoyingly, fails to produce the correct answer at finite N when summed. To be precise: we find for N larger than a few units pure rational functions of N, but when $N = 1$, 2 or 3 is substituted in here we find incorrect or even infinite answers. The critical value of N above which the rational function is valid and below which it fails depends on the order of $1/g^2N$ considered. We interpret this result as an aspect of the formal divergence of the $1/N$ expansion, but it must be kept in mind that also in this case we are unable to interchange the limits $g^2N \to \infty$ and $N \to \infty$.

To demonstrate the aforementioned properties of the $1/N$ expansion is the purpose of this note. The action for gauge fields and quarks on an infinite Euclidean lattice is given by [3]

$$S[\bar{\psi}_q, \psi_q, U^\dagger, U]$$

$$= \sum_{x,q} \bar{\psi}_q(x) \{\tfrac{1}{2} \sum_{\mu} (1+\gamma_\mu) U(x, \hat{\mu}) \psi_q(x+\hat{\mu}) +$$

$$+\tfrac{1}{2}\sum_{\mu}(1-\gamma_\mu)U^\dagger(x-\hat\mu,\hat\mu)\psi_q(x-\hat\mu)-M_q\psi_q(x)\}$$

$$+\frac{a^{d-4}}{g^2}\sum_{x,\mu\neq\nu}\mathrm{Tr}\{U(x,\hat\mu)$$

$$\times U(x+\hat\mu,\hat\nu)U^\dagger(x+\hat\nu,\hat\mu)U^\dagger(x,\hat\nu)\}\,. \qquad (1)$$

Here a is the lattice length, x labels the lattice sites, and $M_q = am_q + d$; further, d is the dimensionality of space-time ($d \to 4$), and $\hat\mu$ and $\hat\nu$ are unit vectors with length a in the direction μ and ν. The quark fields ψ_q are elementary representations of SU(N) and $U(x, \hat\mu)$ are $N \times N$ unitary matrices, which are related to the gauge vector fields in the continuum limit as follows:

$$U(x,\hat\mu) \to \exp\left(\tfrac{1}{2}\, i a g A_\mu^\alpha(x)\lambda_\alpha\right). \qquad (2)$$

In computing the functional integrals with this Lagrangian we only keep the mass term,

$$\bar\psi_q M_q \psi_q\,,$$

in the exponential and expand the exponential of the remaining part of the total action. Subsequently one first performs the integration over the U-variables (without imposing a gauge condition, as was emphasized by Wilson). We then encounter group-invariant integrals of the type

$$\int \mathcal{D}U(x,\hat\mu)\,U_{\underline{j}}^{i}(x,\hat\mu)\,U_{\underline{l}}^{k}(x,\hat\mu)\ \cdot\ U_{q}^{\dagger\underline{p}}(x,\hat\mu)\,U_{s}^{\dagger\underline{r}}(x,\hat\mu) \qquad (3)$$

which are only nonzero if both the underlined indices and the non-underlined indices can be grouped into invariant tensors. In particular,

$$\int \mathcal{D}U(x,\hat\mu)U(x,\hat\mu) = 0 \qquad (4)$$

(unless $N = 1$). It is this latter identity that prevents quark lines to occur separately and therefore assures confinement.

The total amplitudes can be split into "diagrams" that consist of assemblies of "gauge squares" (coming from the last term of eq. (1)) and "quark line units" (coming from the first term of eq. (1)) which have to fit together because of eq. (4). By working out the integrals of the type (3) one can find the general N-dependence of such diagrams. It is very satisfying that, as in the continuum theory, we have found that the limit $N \to \infty$, g^2N fixed, exists and that only planar constructions survive in this limit. At fixed order in $1/N$ and $1/g^2N$ one can easily sum the $1/M_q$ expansion and then one obtains a "meson" propagator. In fact, it can be seen that the procedure that lead Wilson to compute such propagators corresponds exactly to neglecting certain $1/N$ corrections.

It was our intention to set up a systematic $1/g^2N$ and $1/N$ expansion and formulate the corresponding "Feynman rules". We have encountered two types of difficulties:

1) The combinatorics turn out to become extremely cumbersome ‡, so although this expansion is of formal importance, we doubt its usefulness when applied to actual calculations in a lattice theory.

2) The phenomenon mentioned before: non-summability of the expansion. It is this second point that we wish to demonstrate by considering more closely several example diagrams.

Let us start by discussing the diagrams (a)–(d) of fig. 1. They consist of certain simple quark-line configurations, which may be contained in a more general diagram, up to the first non-trivial order in $1/g^2N$. To calculate these diagrams for general values of N one can make use of explicit expressions for the group-invariant integrals (3). For instance, to calculate diagram (d) we use, at each side of the square,

$$\int \mathcal{D}\, U_{i_1}^{j_1} U_{i_2}^{j_2} U_{i_3}^{j_3} U_{k_1}^{\dagger l_1} U_{k_2}^{\dagger l_2} U_{k_3}^{\dagger l_3}$$

$$= \frac{1}{N(N^2-1)(N^2-4)}$$

$$\times\Big\{(N^2-2)\sum_{6\ \mathrm{perm}} \delta_{i_1}^{l_1}\delta_{i_2}^{l_2}\delta_{i_3}^{l_3}\delta_{k_1}^{j_1}\delta_{k_1}^{j_2}\delta_{k_3}^{j_3} \qquad (5)$$

$$-N\sum_{18\ \mathrm{perm}} \delta_{i_1}^{l_1}\delta_{i_2}^{l_2}\delta_{i_3}^{l_3}\delta_{k_1}^{j_2}\delta_{k_2}^{j_1}\delta_{k_3}^{j_3}$$

$$+2\sum_{12\ \mathrm{perm}} \delta_{i_1}^{l_1}\delta_{i_2}^{l_2}\delta_{i_3}^{l_3}\delta_{k_1}^{j_3}\delta_{k_2}^{j_1}\delta_{k_3}^{j_2}\Big\}\,.$$

‡ A "preliminary" preprint by us on this subject contains some errors on this point.

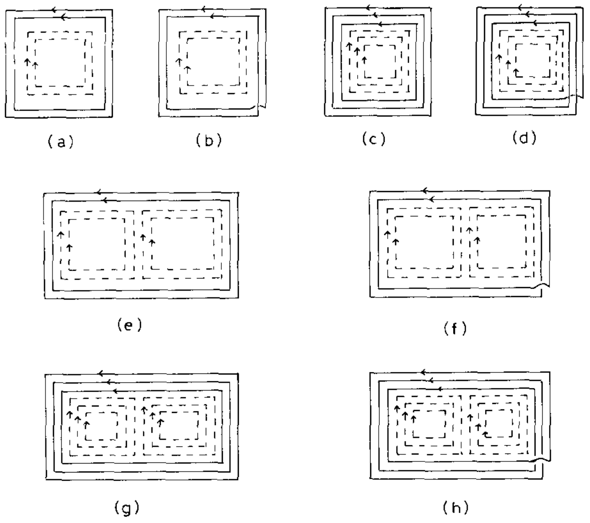

Fig. 1. Diagrams considered in the text The dashed lines represent the oriented gauge squares. The solid lines denote certain quark-line configurations, which may be part of a larger configuration. For instance, the lines could be disjunct at the corners with external quark-antiquark lines attached to it, as long as we keep the sum over the color index at that vertex. Therefore we ignore possible minus signs for closed fermion loops.

As was mentioned before such integrals are indeed decompositions into the various invariant tensors. We now wish to point out two related aspects of decompositions such as (5). First we have implicitly assumed here that the number of U's and $U^\dagger$'s in the integrand is smaller than N, since we have not taken into account the fully antisymmetric, invariant tensor $\epsilon_{i_1 i_2 \cdot \cdot i_N}$, which carries N indices. In addition the various invariant tensors, expressed by products of Kronecker deltas, are no longer independent as soon as the number of U's and/or $U^\dagger$'s exceeds N. To illustrate this we consider the integral (5) with $N = 2$. A direct calculation at $N = 2$ leads to a similar decomposition as for general N, but now with coefficients 1/6, −1/24, 0, i.e. the last invariant tensor no longer occurs, simply because it is linearly dependent on the first two.

This leads directly to the second more disturbing aspect, namely that the result for $N = 2$ cannot be obtained by substituting $N = 2$ into the general result (5). Indeed the coefficients in these decompositions are generally rational functions of N which may become infinite at certain integer values of N. This property is, apparently, less disturbing if one realizes that to calculate a diagram one must make a full contraction over all color indices. This will lead to new N-dependent terms in the numerator, which may cancel the pole terms.

And indeed, after evaluating the diagrams (a)–(d) (see table 1), these poles disappear. The N dependence is regular, and precisely given by the topology of the corresponding diagrams. Only the coefficients change for small values of N, because, as we have indicated above, the order of the diagram becomes comparable to N.

It turns out that this is more or less the general rule for simple diagrams, namely that the contractions over the color indices precisely cancel the disturbing poles in the decompositions like (5). However, by systematically analysing the N-dependence of general diagrams we have been able to construct examples where this is

Table 1

Numerical results for the diagrams of fig. 1, calculated for $N = 1, 2$ and general N. n gives the order of the diagram in $1/g^2$, i.e. the number of gauge squares. χ represents the Euler characteristic of the diagram, a measure of its topological structure. For large N the diagrams should behave like $N^{\chi}(g^2 N)^{-n}$, which agrees with the entries in the fourth column. (The contributions of the γ-matrices in the Fermion lines are of course N-independent and have not been included here.)

Diagram	$N = 1$	$N = 2$	General N		
a	1/4	1/2	1/2	2	2
b	1/4	0	0	2	1
c	1/36	5/36	1/6	3	3
d	1/18	−1/18	0	3	1
e	1/8	1/6	$\frac{1}{2}(N^2-1)^{-1}$	4	2
f	1/8	−1/12	$-\frac{1}{2}N^{-1}(N^2-1)^{-1}$	4	1
g	1/216	17/864	$\frac{1}{6}(N^2-2)N^{-1}(N^2-1)^{-1}(N^2-4)^{-1}$	6	3
h	1/108	−7/432	$\frac{4}{3}N^{-1}(N^2-1)^{-1}(N^2-4)^{-4}$	6	1

clearly no longer the case. Examples of such diagrams are given in fig. 1 (e)–(h). As one can see in table 1, the corresponding results for general N still lead to infinite answers for $N = 1$, and/or 2, whereas direct computations at these values yield finite answers as it should. And it is only asymptotically, for large N, that the diagrams behave according to their topological structure.

These results indicate very clearly that the $1/N$ expansion will not converge to the right answer. For larger diagrams we expect to find similar singularities at increasing integer values of N, so that the $1/N$ expansion for the full theory is probably not summable for any value of N. Two remarks are of order. First, this problem must be added to the occurrence of "baryons", which, at finite N, are composed of N quarks. Baryonic states are not see in the $1/N$ expansion but clearly give finite contributions at finite N. The anomalies we found above cannot be explained in terms of these baryonic contributions. Secondly, we can easily get rid of the baryonic effects if we wish, by turning SU(N) into $U(N)$, using the same Lagrangian as before. The irregularity of the $1/N$ expansion that we observed remains unchanged.

References

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