Monopoles, Instantons and Confinement

Gerard 't Hooft, University Utrecht

Saalburg, September 1999

notes written by Falk Bruckmann, University Jena

(version July 9, 2004)

Contents

1 Solitons in 1+1 Dimensions **1**
- 1.1 Definition of the Models 1
- 1.2 Soliton Solutions 3
- 1.3 Chiral Fermions 6
- 1.4 Outlook to Higher Dimensions 9

2 The Abrikosov-Nielsen-Olesen-Zumino Vortex **11**
- 2.1 Approach to the Vortex Solution 11
- 2.2 Introduction of the Gauge Field 13
- 2.3 Bogomol'nyi Bound for the Energy 15
- 2.4 Gauge Topology Description 16

3 Magnetic Monopoles **22**
- 3.1 Electric and Magnetic Charges and the Dirac Condition 22
- 3.2 Construction of Monopole Solutions 24
- 3.3 Existence of Monopoles 27
- 3.4 Bogomol'nyi Bound and BPS States 27
- 3.5 Orbital Angular Momentum for **qg** Bound States 29
- 3.6 Jackiw-Rebbi States at a Magnetic Monopole 32

4 Instantons **35**
- 4.1 Topological Gauge Transformations 35
- 4.2 Semiclassical Approximation for Tunnelling 38
- 4.3 Action for a Topological Transition, Explicit Instanton Solutions 39
- 4.4 Bogomol'nyi Bound and Selfdual Fields 42
- 4.5 Intermezzo: Massless Fermions in a Gauge Theory 44
- 4.6 Jackiw-Rebbi States at an Instanton 46
- 4.7 Estimate of the Flip Amplitude 48

4.8 Influence of the Instanton Angle 50

5 Permanent Quark Confinement **52**
5.1 The Abelian Projection 52
5.2 Phases of the Abelian Theory 57
5.3 A QCD-inspired Theory for the Electroweak Force 59
5.4 Spontaneous Chiral Symmetry Breaking in QCD 61

6 Effective Lagrangians for Theories with Confinement **63**

7 Exercises **66**

References **69**

Chapter 1

Solitons in 1+1 Dimensions

As an introduction we consider in this chapter the easiest field theoretic examples for solitons. These are real scalar field theories in $1+1$ dimensions with a quartic and a sine-Gordon potential, respectively. We will concentrate on physical aspects which are relevant also in higher dimensions and more complicated theories like QCD.

1.1 Definition of the Models

We investigate the theory of a single real scalar field $\phi(t,x)$ in one time and one space dimension. The usual Lagrangian (density) is,

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\left(\partial_x \phi\right)^2 - V(\phi) \tag{1.1}$$

We consider two cases for the potential. Case (a) refers to the 'Mexican-hat' potential, well-known from spontaneous symmetry breaking,

$$\text{case (a):} \quad V(\phi) = \frac{\lambda}{4!}\left(\phi^2 - F^2\right)^2 \tag{1.2}$$

while case (b) is the sine-Gordon model,

$$\text{case (b):} \quad V(\phi) = A\left(1 - \cos\frac{2\pi\phi}{F}\right) \tag{1.3}$$

which is an exactly solvable system. The important point about these models is that the vacuum is degenerate. In case (a) it is two-fold degenerate,

$$\phi = \pm F$$

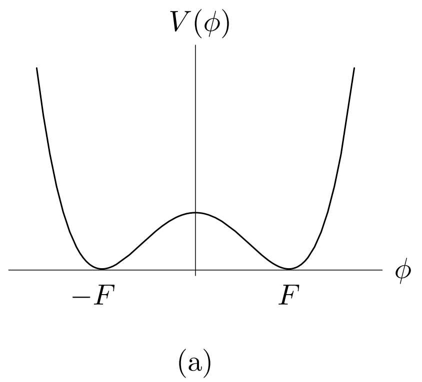


Figure 1.1: Potentials under consideration: the Mexican-hat (case (a), see (1.2)) and the sine-Gordon model (case (b), see (1.3)). Both have degenerate vacua and allow for non-trivial solutions.

while in case (b) we have an infinite number of vacua,

$$\phi = nF, \qquad n \in \mathbb{Z}$$

In standard perturbation theory one considers small fluctuations η around the vacua,

$$\begin{array}{rcl} \text{case (a):} \quad \phi & = & F + \eta \\ \text{case (b):} \quad \phi & = & 0 + \eta, \quad (\eta \equiv \phi) \end{array}$$

and expands the potential. Here we get the usual mass term together with three and four point interactions,

$$\begin{array}{rcl} \text{case (a):} \quad V(\eta) & = & \frac{1}{2}m^2\eta^2 + \frac{g}{3!}\eta^3 + \frac{\lambda}{4!}\eta^4 \\ & & m^2 \equiv \lambda F^2/3, \quad g \equiv \lambda F \\ \text{case (b):} \quad V(\eta) & = & \frac{1}{2}m^2\eta^2 - \frac{\lambda}{4!}\eta^4 + \ldots \\ & & \left.\begin{array}{l} m \equiv 2\pi\sqrt{A}/F \\ \lambda \equiv 16\pi^4 A/F^4 \end{array}\right\} \left\{\begin{array}{l} A \equiv m^4/\lambda \\ F \equiv 2\pi m/\sqrt{\lambda} \end{array}\right. \end{array}$$

m is the mass of the particles of the theory (we use $\hbar = 1$) and g and λ are defined such that the three and four point vertices are proportional to them. Usually one has small λ and large F such that the mass $\sqrt{\lambda}F$ is fixed. Notice that the mass square would be negative on the maxima 0 and $n + \frac{1}{2}$, respectively, these so-called tachyons would render the theory unstable.

1.2 Soliton Solutions

The degeneracy of the vacuum results in the fact that these models possess non-trivial static solutions, which interpolate (in space) between the vacua. We call them *kinks* or *solitons*. Their existence and shape are given by the Euler-Lagrange equation[1] derived from $\mathcal{L}$ in (1.1),

$$\ddot{\phi} = \partial_x^2 \phi - \frac{\partial V}{\partial \phi} = 0$$

If we think of $\phi(x)$ as $x(t)$, this is the equation of motion of a non-relativistic particle, $\ddot{x} = -\frac{\partial V}{\partial x}$, but in a potential $-V$. Like the energy in ordinary mechanics, we find a first integral,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}(\partial_x\phi)^2 - V(\phi)\right) &= \partial_x\phi\left(\partial_x^2\phi - \frac{\partial V}{\partial \phi}\right) = 0 \\ \frac{1}{2}(\partial_x\phi)^2 - V(\phi) &= \text{const.} \qquad (1.4) \end{aligned}$$

Since we want solutions with finite energy, we have to demand that the energy density,

$$E = \int_{-\infty}^{\infty}\left[\frac{1}{2}(\partial_x\phi)^2 + V(\phi)\right]\mathrm{d}x \qquad (1.5)$$

vanishes at spatial infinity,

$$|x| \to \infty: \quad \partial_x\phi \to 0,\ V(\phi) \to 0$$

i.e. the above constant is zero. The remaining first order differential equation can easily be solved,

$$x(\phi) = \int \frac{\mathrm{d}\phi}{\sqrt{2V(\phi)}} \qquad (1.6)$$

In our models we can write down the solutions exactly,

$$\begin{aligned} \text{case (a):} \quad \phi(x) &= F \tanh \frac{1}{2} m(x - x_0) \\ \text{case (b):} \quad \phi(x) &= \frac{2F}{\pi} \arctan\left(e^{m(x-x_0)}\right) \end{aligned}$$

[1] For time-independent solutions we could also work in the Hamiltonian formalism. Moreover, any of the following static solutions can be transformed into a steadily moving one by a Lorentz transfomation.

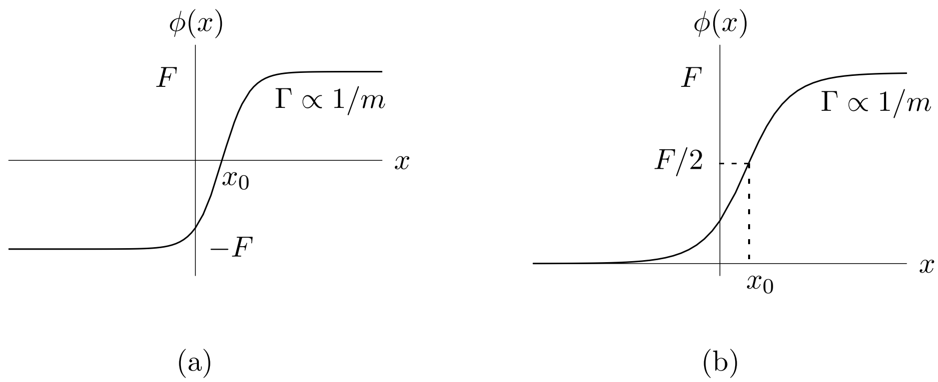


Figure 1.2: Solitonic solutions in both potentials have nearly identical shape: The transition from one vacuum value to the next one takes place around an arbitrary position x_0. It decays with a rate that is proportional to the inverse mass of the (light) particles of the theory.

x_0 is an arbitrary constant (of integration) due to translational invariance. Their shapes are very similiar (cf Fig. 1.2) and show the typical behaviour: (i) Solitons interpolate between two neighbouring vacua:

$$\begin{aligned} \text{case (a):} \quad \phi(x \to \pm\infty) &= \pm F \\ \text{case (b):} \quad \phi(x \to -\infty) &= 0, \\ \phi(x \to +\infty) &= F \end{aligned}$$

(ii) The solutions are (nearly) identical to a vacuum value everywhere except a transition region around an arbitrary point x_0. The shape of the solution there is given by the shape of the potential between the vacua. Near the vacua the solution is decaying exponentially with a rate $\Gamma \propto 1/m$. Thus ϕ can be approximated by a step function, for instance in case (a),

$$\phi(x) = F \,\mathrm{sgn}\,(x - x_0) \qquad \text{for } |x - x_0| \gg 1/m$$

In terms of mechanics one could think of a particle which passes the bottom of a valley at some time t_0. It has just the energy to climb up the hill and stay there. Actually this will take infinitely long, but after a short time it has already reached a position very near the top. Of course, the particle must have been on top of the opposite hill in the infinite past. Whenever both tops have the same height such a solution exists, no matter which form V has inbetween.

The energy of the solution (its mass) can be computed from (1.5) and (1.4),

$$\begin{aligned} E &= 2\int_{-\infty}^{\infty} V(\phi)\,\mathrm{d}x = \int_{\phi_{-\infty}}^{\phi_\infty} \partial_x\phi\,\mathrm{d}\phi \\ &= \begin{cases} 2m^3/\lambda & \text{for case (a),} \\ 8m^3/\lambda & \text{for case (b).} \end{cases} \end{aligned}$$

Alternatively we can use the saturation of the Bogomol'nyi bound (cf Exercise (i)),

$$E = \int \Big[(0)^2 + \text{total derivative}\Big] = W(\phi)\big|_{\phi_{-\infty}}^{\phi_\infty} \tag{1.7}$$

In both cases the mass of the soliton is given by the cube of the mass of the elementary particles divided by λ (which has dimension (mass)^2). The soliton is very massive in the perturbative limit. That means we are dealing with a theory which describes both, light fluctuations, the elementary particles, and heavy solutions, the solitons. This mass gap supports the validity of perturbation theory (for example for tunneling).

Up to now we have only considered one-soliton solutions which interpolate between neighbouring vacua. Due to the ambiguity of the square root in (1.6) there are also solutions interpolating backwards, anti-solitons. Now one could immediately imagine solutions consisting of whole sequences of solitons and

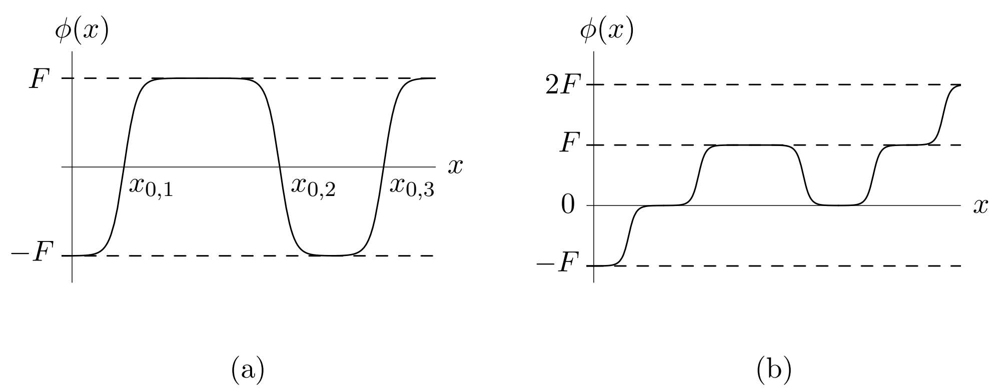


Figure 1.3: Multi-Solitons are approximate solutions built of solitons and anti-solitons at arbitrary positions $x_{0,i}$. The sequences are strongly constrained in case (a), while in case (b) they are arbitrary.

anti-solitons (see Fig. 1.3). As discussed in Exercise (ii), these approximate solutions are only valid for widely separated objects $|x_{0,1} - x_{0,2}| \gg 1/m$, such that we have a 'dilute gas'.

Here the Mexican-hat and the sine-Gordon model differ slightly. Solitons and anti-solitons have to alternate in the first model. From a particle point of view the anti-soliton is really the anti-particle of the soliton. If the potential is symmetric, we cannot distinguish between them. The situation is like in a real scalar field theory.

In the latter model we can arrive at any vacuum by choosing the right difference between the number of solitons and anti-solitons. Now solitons and anti-solitons are distinguishable and the analog is a complex scalar field theory.

1.3 Chiral Fermions

Now we investigate the (still 1+1 dimensional) system,

$$\mathcal{L}_\psi = -\bar{\psi}\left(\gamma^\mu \partial_\mu + g\phi(x)\right)\psi$$

where ψ is a Dirac field, γ_μ are the (Euclidean) Dirac matrices, which we can choose to be the Pauli matrices,

$$\gamma^1 = \sigma_1, \quad \gamma^4 = \sigma_3, \quad \gamma_5 = \sigma_2$$

and ϕ is a solution from above[2]. We could say we put fermions in a soliton background or we study the consequences to the solution if we couple fermions to it.

The interaction is provided by the usual Yukawa coupling. The field ϕ acts like a (space-dependent) mass. If ϕ takes the vacuum value F everywhere, then we simply have a theory with massive fermions,

$$\mathcal{L}_\psi = -\bar{\psi}\left(\gamma^\mu \partial_\mu + m_\psi\right)\psi, \qquad m_\psi = gF$$

The energies, i.e. the eigenvalues of the (hermitean) Hamilton-Operator,[3]

$$\mathcal{H} = -i\partial_t = i\sigma_2 \partial_x + \sigma_3 m_\psi$$

[2] In particular we take ϕ from case (a), since with the sine-Gordon model we would be forced to use a cos-interaction which is not normalisable in 4 dimensions.

[3] We use the complex notation $\partial_4 = -i\partial_t$.

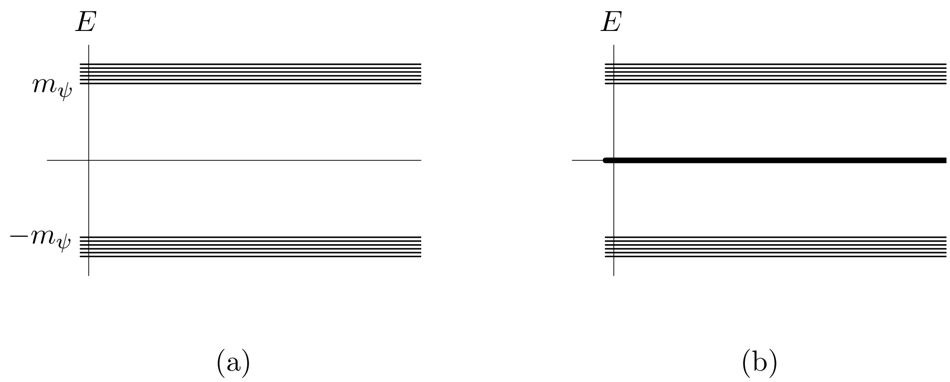


Figure 1.4: The usual spectrum of massive fermions (a) is produced by a constant vacuum solution $\phi \equiv F$. For a soliton there is *one* additional zero mode (b).

come in pairs $\pm E$ with $|E| \geq m$. The fields with the opposite energies are generated by $\gamma^1 = \sigma_1$,

$$\{\sigma_1, \sigma_2\} = \{\sigma_1, \sigma_3\} = 0 \Rightarrow \mathcal{H}\psi = E\psi \rightleftharpoons \mathcal{H}(\gamma^1\psi) = -E(\gamma^1\psi)$$

In the spirit of Dirac we define the vacuum to be filled with negative energy states, the particles to be excitations with $E > 0$ and the anti-particles to be holes in $E < 0$ (cf Fig. 1.4). γ^1 is the fermion number conjugation.

As well-known the chiral symmetry generated by γ_5,

$$\psi \to \gamma_5\psi, \quad \bar{\psi} \to -\bar{\psi}\gamma_5$$

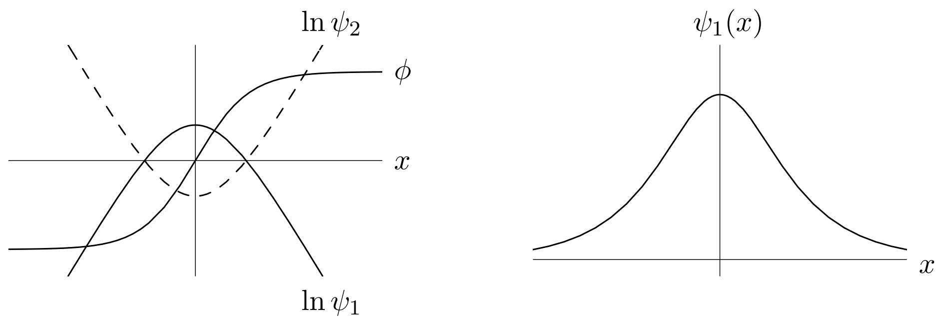
is broken by the mass term. But as an interaction this term can be made invariant by

$$\phi \to -\phi$$

which is again a solution. That means γ_5 generates a state with the same energy but with ϕ in the opposite vacuum.

Now we really want to insert a non-trivial ϕ and look for its spectrum. Still σ_1 generates opposite energy solutions. It comes out that again they are displaced by the soliton. What about the special case of solutions of zero energy? Acting on them, $\mathcal{H}$ and σ_1 commute, and we choose the zero modes ψ to be eigenfunctions of σ_1,

$$\sigma_1\psi_\pm = \pm\psi_\pm, \quad \psi_+ = \frac{\psi_1(x)}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \quad \psi_- = \frac{\psi_2(x)}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$



(a) (b)

Figure 1.5: Zero energy solutions in a soliton background: $\ln\psi_{1,2} \propto \pm \int \phi \mathrm{d}x$ (cf (1.8)). Only one of the solutions is normalisable, the one with the full line in (a). It is localised at the soliton position which we have chosen to be 0 here (b). A similiar picture applies for the anti-soliton.

Now we have to solve

$$(\sigma_1 \partial_x + g\phi)\psi_\pm = \sigma_3 E \psi_\pm = 0$$

which becomes

$$\partial_x \psi_{1,2} = \mp g\phi\psi_{1,2}, \qquad \ln\psi_{1,2} = \mp g \int \phi \, \mathrm{d}x \tag{1.8}$$

Knowing the general shape of the soliton ϕ we see immediately that the solution with the lower sign is non-normalisable, while the one which the upper sign fulfills every decent boundary condition, since it drops exponentially. Notice that $\psi_1\psi_2 = $ const. as a general property. Thus, if we add the continuum, there is an 'odd' (but infinite) number of solutions.

For completeness we give the formula for the normalisable zero mode,

$$\psi_+ = \mathrm{const} \left(\cosh \frac{m}{2}(x - x_0) \right)^{-2m/m_\psi} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

It is strongly localized at the position x_0 of the soliton (cf Fig. 1.5). For the anti-soliton $-\phi$ the solution with the lower sign is normalisable,

$$\psi_- = \mathrm{const} \left(\cosh \frac{m}{2}(x - x_0) \right)^{-2m/m_\psi} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

This of course agrees with the chiral transformed ψ_+,

$$\begin{aligned}\psi_+ \quad \to \quad \gamma_5\psi_+ &= \text{const}\left(\cosh\frac{m}{2}(x-x_0)\right)^{-2m/m_\psi}\sigma_2\begin{pmatrix}1\\1\end{pmatrix}\\ &= \text{const}\left(\cosh\frac{m}{2}(x-x_0)\right)^{-2m/m_\psi}\begin{pmatrix}i\\-i\end{pmatrix}\propto\psi_-\end{aligned}$$

These zero modes are called Jackiw-Rebbi modes [1]. We view them as soliton-fermion bound states indistinguishable from the original true soliton[4].

When we quantise the theory, ψ becomes an operator with Fermi-Dirac statistics/anti-commutation relations, especially $\hat{\psi}^2 = 0$. $\hat{\psi}_+$ and $\hat{\psi}_-$ commute with the Hamiltonian: $[\hat{\psi}_\pm, \hat{H}] = 0$. In general this commutator involves the energy, but here we have $E = 0$. Whether these states are filled or empty has no effect on the energy, they are 'somewhere inbetween fermions and anti-fermions'. As we have seen they are related by γ_5. In fact Jackiw and Rebbi [1] have shown that the soliton has two states with

$$\boxed{\text{fermion number: } n = \pm 1/2\text{, no spin, no Fermi-Dirac statistics!}}$$

n is the expectation value of the conserved charge $\hat{Q}_0 = \int \mathrm{d}x : \hat{\bar{\psi}}\gamma^0\hat{\psi} :$ in these states.

1.4 Outlook to Higher Dimensions

In higher dimensions the field will still like to sit in a vacuum for most of the space-time. The kinks will now be substituted by (moving) *domain walls*, i.e. transition regions between domains with different ϕ-values. Their shape will depend on the model. In any case passing domain walls will have huge physical consequences. For example in case (a) passing from ϕ to $-\phi$ means transforming (by chiral symmetry) matter into anti-matter.

The domains themselves are related through a discrete global symmetry, namely Z_2 in case (a) and $\mathbb{Z}$ in case (b). In other words we have two sorts of domains in case (a) and infinitely many in case (b), respectively.

Concerning the chiral fermions, they are massive inside the domains and massless on the walls. The latter are localised in the direction perpendicular

[4]The counting of states depends on this interpretation (cf [1]).

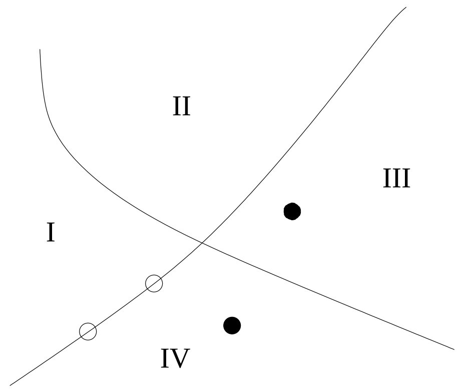


Figure 1.6: The picture we expect in higher dimensions: In the domains I to IV ϕ sits in different vacua, which are related by a discrete symmetry. The transition takes place on domain walls. Inside the domains fermions are massive ($\bullet$), on the walls they are massless ($\circ$) and localised perpendicular to the walls.

to the domain wall. Along the domain wall we can view them as lower-dimensional Dirac fermions. This scenario has become helpful for studying fermions on the lattice.

Chapter 2

The Abrikosov-Nielsen-Olesen-Zumino Vortex

2.1 Approach to the Vortex Solution

We try to find the analogue of domain walls in 2+1 dimensions. They will come out as *vortices* or *strings*. We have to take a complex (or two component real) scalar field,

$$\phi = \phi_1 + i\phi_2, \;\; \vec{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{2.1}$$

We use the generalization of (1.1) with a global U(1) invariance,

$$\mathcal{L} = -\partial_\mu \phi^* \partial_\mu \phi - \frac{\lambda}{2} \left(\phi^* \phi - F^2 \right)^2 \tag{2.2}$$

as our starting point. Notice that the vacuum manifold is now a circle $|\phi| = F$. For the desired soliton solution we combine it with the directions in space at spatial infinity,

$$|x| \to \infty : \;\; \vec{\phi} \to F \frac{\vec{x}}{|x|}, \;\; \phi \to F e^{i\varphi} \tag{2.3}$$

where φ is the polar angle in coordinate space. One such solution is depicted in Fig. 2.1. The solution could also be a deformation of this, but should go a full circle around the boundary. Since the map on the boundary is non-trivial, ϕ must have a zero inside.

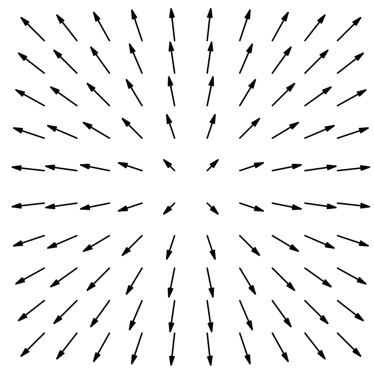


Figure 2.1: A typical vortex solution with isospace vectors $\vec{\phi}$ depicted in (2 dimensional) coordinate space $\vec{x}$. The field 'winds around once' at spatial infinity as a general feature. Angle and length of $\vec{\phi}$ inside, especially the position of the zero, are still arbitrary. All these configurations are specified by the winding number 1.

But this non-trivial map at spatial infinity has the effect that the energy,

$$E = \int \mathrm{d}^2 x \left(\vec{\partial}\phi^* \vec{\partial}\phi + V(\phi, \phi^*) \right) \tag{2.4}$$

is divergent, since the rotation of ϕ enters the kinetic energy,

$$\begin{aligned}
|x| \to \infty : \quad \partial_i \phi_j \quad &\to \quad \frac{F}{|x|} \left(\delta_{ij} - \frac{x_i x_j}{|x|^2} \right) \\
\sum_{i,j=1}^{2} (\partial_i \phi_j)^2 \quad &\to \quad \frac{F^2}{|x|^2} (2 - 2 + 1) = \frac{F^2}{|x|^2} \\
\int \mathrm{d}^2 x \, \vec{\partial}\phi^* \vec{\partial}\phi \quad &\to \quad 2\pi \int_0^\infty \mathrm{d}|x| \, \frac{F^2}{|x|} \ldots \text{log. divergent}
\end{aligned}$$

Thus in a theory with *global* $U(1)$ invariance, there exists a vortex, but its energy (per time unit in three dimension) is logarithmically divergent!

Derrick's Theorem [2] states that this divergence is unavoidable for time-independent solutions in $d \geq 2$. Since it is only a mild divergence, the solution still plays a role in phase transitions in statistical mechanics.

2.2 Introduction of the Gauge Field

Now we cure the above divergence by making the U(1) invariance *local* in the standard manner. We add a gauge field A_μ and replace the partial derivative in (2.2) by the covariant one,

$$\partial_\mu \phi \to \mathrm{D}_\mu \phi = \left(\partial_\mu - ieA_\mu\right)\phi \tag{2.5}$$

This gives $\vec{\mathrm{D}}\phi$ the chance to converge better than $\vec{\partial}\phi$ (we still deal with static solutions). In other words the divergence is absorbed in $\vec{A}$. Since asymptotically ϕ depends only on the angle φ, $\vec{A}$ will only have a component in this direction

Asymptotically, ϕ is real at the x-axis,

$$\phi \to Fe^{i\varphi}|_{\varphi=0} = F$$

and the gradient has only a y component,

$$\vec{\partial}\phi \to \begin{pmatrix}\partial_x\phi\\ \partial_y\phi\end{pmatrix}\Bigg|_{\varphi=0} = \begin{pmatrix}\partial_r\phi\\ \frac{1}{r}\partial_\varphi\phi\end{pmatrix}\Bigg|_{\varphi=0} = \begin{pmatrix}0\\ iF/r\end{pmatrix}$$

We can read off $\vec{A}$ from the demand of vanishing covariant derivative,

$$\vec{A} \to \frac{1}{ie}\phi^{-1}\vec{\partial}\phi, \qquad A_x \to 0, \qquad A_y \to \frac{1}{er}$$

For the general case (at any point (x, y)) we perform a trick, namely we can rotate ϕ locally to be real,

$$\phi \to \Omega F \qquad \text{with } \Omega(\vec{x}) = e^{i\varphi}$$

thus,

$$\vec{A} \to -\frac{1}{ie}\Omega\vec{\partial}\Omega^{-1}$$

In fact the covariant derivative vanishes asymptotically,

$$\vec{\mathrm{D}}\phi \to \left(\vec{\partial}\Omega + \Omega(\vec{\partial}\Omega^{-1})\Omega\right)F = \Omega\vec{\partial}\left(\Omega^{-1}\Omega\right)F = 0$$

The general form of $\vec{A}$ is,

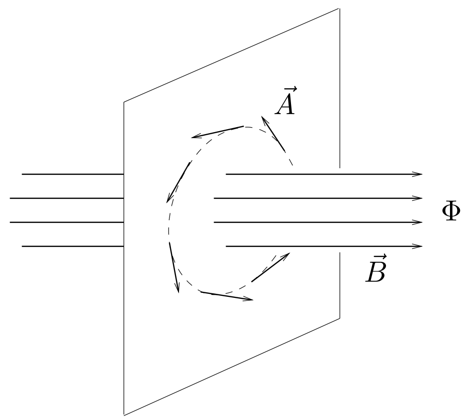


Figure 2.2: The introduction of a circular gauge field $\vec{A}$ (cf (2.6)) leads to a vortex with quantised magnetic flux $\Phi = n\frac{2\pi}{e}$.

$$A_i \longrightarrow -\frac{1}{e}\epsilon_{ij}\frac{x_j}{r^2} \tag{2.6}$$

As we expected it has only a φ-component,

$$A_r \longrightarrow 0, \qquad A_\varphi \longrightarrow \frac{1}{er}$$

Furthermore it is a pure gauge asymptotically and the field strength vanishes,

$$\vec{A} \longrightarrow \frac{1}{e}\vec{\partial}\varphi, \qquad F_{ij} \longrightarrow 0$$

giving a solution with finite energy per unit length.

It can be shown that the choices for ϕ and A are solutions of the Euler-Lagrange equations asymptotically. If we try to extend them naively towards the origin, A runs into a singularity. Instead one could make an ansatz for ϕ and A and try to solve the remaining equations numerically [3]. But already from the asymptotic behaviour we can deduce a *quantised magnetic flux*,

$$\Phi = \int_S \vec{B}\mathrm{d}\vec{\sigma} = \int_{C=\partial S} \vec{A}\,\mathrm{d}\vec{x} = g_m, \qquad g_m = \frac{2\pi}{e}$$

For higher windigs we have analogously,

$$\Phi = n g_m \qquad \text{with } n \in \mathbb{Z}$$

The situaton is very much like in the Ginzburg-Landau theory for the superconductor. In this theory an electromagnetic field interacts with a fundamental scalar field describing Cooper pairs. The latter are bound states of two electrons with opposite momentum and spin. As bosonic objects they can fall into the same quantum state resulting in *one* scalar field ϕ. The potential of the scalar field is of the Mexican-hat form with temperature-dependent coefficients. In the low temperature phase the symmetry is broken and the photons become massive. That means if a magnetic field enters the superconductor at all, it does so in flux tubes. Performing an Aharonov-Bohm gedankenexperiment around such a tube leads to a flux quantum,

$$\Phi_{\mathrm{SC}} = n g_{\mathrm{SC}}$$

The only difference to the above model is a factor 2 from the *pair* of electrons,

$$q_{\mathrm{SC}} = 2e, \qquad g_{\mathrm{SC}} = \frac{2\pi}{q_{\mathrm{SC}}} = \frac{\pi}{e}$$

2.3 Bogomol'nyi Bound for the Energy

Adding the field strength term to (2.2) and (2.5), the complete Lagrangian reads,

$$\mathcal{L} = -\mathrm{D}_\mu \phi^* \mathrm{D}_\mu \phi - \frac{\lambda}{2}\left(\phi^*\phi - F^2\right)^2 - \frac{1}{4} F_{\mu\nu} F_{\mu\nu} \tag{2.7}$$

The energy integral is now,

$$E = \int \mathrm{d}^2 x \left[\mathrm{D}_i \phi^* \mathrm{D}_i \phi + \frac{1}{2} F_{12}^2 + \frac{\lambda}{2}\left(\phi^*\phi - F^2\right)^2 \right]$$

In the gauge where ϕ is real[1] the integrand consists of a sum of squares,

$$E = \int \mathrm{d}^2 x \left[(\partial_i \phi)^2 + e^2 \vec{A}^2 \phi^2 + \frac{1}{2} F_{12}^2 + \frac{\lambda}{2}\left(\phi^*\phi - F^2\right)^2 \right]$$

where the second and the fourth term cannot be zero at the same time. For the Bogomolnyi bound we reduce the number of squares by partial integration

[1] At the origin this gauge $\Omega^{-1} = e^{-i\varphi}$ becomes singular, since φ is ambiguous at this point.

as for the soliton (cf (1.7) and Exercise (i)),

$$\begin{aligned}(\partial_i\phi)^2 + e^2\vec{A}^2\phi^2 &= (\partial_i\phi \pm e\epsilon_{ij}A_j\phi)^2 \pm e\phi^2 F_{12} + \text{total der.}\\ \frac{1}{2}F_{12}^2 + \frac{\lambda}{2}\left(\phi^*\phi - F^2\right)^2 &= \frac{1}{2}\left(F_{12} \pm \sqrt{\lambda}(\phi^2 - F^2)^2\right)^2 \mp \sqrt{\lambda}(\phi^2 - F^2)F_{12}\end{aligned}$$

Notice that the new square in the first equation looks like a covariant derivative, but it is not. The boundary contributions are easily to be calculated.

For the special choice[2] of

$$\boxed{\lambda = e^2} \qquad m_\phi = m_A = \sqrt{2}eF$$

the energy simplifies further,

$$\begin{aligned}E &= \int \mathrm{d}^2x \left[(\partial_i\phi \pm e\epsilon_{ij}A_j\phi)^2 + \frac{1}{2}\left(F_{12} \pm \sqrt{\lambda}(\phi^2 - F^2)^2\right)^2 \pm eF^2 F_{12}\right]\\ &\geq eF^2 \left|\int F_{12}\,\mathrm{d}^2x\right|\end{aligned}$$

For the saturation of the bound the first two equations can be solved numerically, while the rest gives the total magnetic flux,

$$E \geq eF^2\, n\frac{2\pi}{e} = n\frac{\pi m^2}{e^2} \tag{2.8}$$

Again we have found the typical dependence mass^2/coupling for heavy topological objects.

2.4 Gauge Topology Description

For the vortex as well as for the soliton we have seen that the asymptotic behaviour is important, in the sense that the requirement of finite energy forces the configurations to fall into disjoint 'classes'. Interpolating between these classes must include configurations with divergent energy. Now we want to clarify this topological property.

Along the lines of spontaneous symmetry breaking, (2.7) is a $U(1)$ gauge theory coupled to a Higgs field ϕ. The vacuum manifold $|\phi| = F$ is $U(1)$-invariant, but the special choice $\phi = F$ breaks the $U(1)$ down to $\mathbb{1}$: No

[2] This choice corresponds to the type I/type II phase boundary of the superconductor.

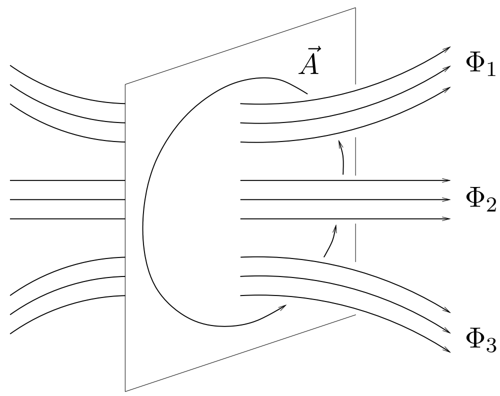


Figure 2.3: A non-trivial configuration of vortices carrying total flux $(n_1+n_2+n_3)\frac{2\pi}{e^2}$. For topological reasons it cannot be continuously shrinked to the trivial vacuum (unless $n_1+n_2+n_3=0$). The total flux also results in a lower bound for the energy (cf (2.8)): $E \geq |n_1+n_2+n_3|\frac{\pi m^2}{e^2}$ for $\lambda = e^2$.

gauge transformation leaves this special value invariant. That is, the gauge transformation leading to this gauge must be a mapping from the boundary of $\mathbb{R}^2$ to $U(1)/\mathbb{1}$,

$$\Omega : S^1 \longrightarrow U(1)/\mathbb{1} \equiv U(1)$$

The identity has been (formally) divided out, since for the general case Ω need not come back to the same group element. It is allowed to differ by another group element belonging to the subgroup which leaves the vacuum choice invariant. We say the Higgs field ϕ transforms under the group $U(1)/\mathbb{1}$.

The mappings from S^1 into a manifold M themselves form a group, called the *first homotopy group* $\pi_1(M)$. π_1 measures the non-contractibiliy of M, i.e. the 'existence of holes'. For contractible M all mappings are identified and π_1 is simply the identity.

The Lie group $U(1)$ itself is a circle S^1. The first homotopy group of S^1 is well-known to be the group of integers,

$$\pi_1(S^1) = \mathbb{Z}$$

Notice that this group is Abelian.

Thus from topological arguments each vortex carries a quantum number

$$Q \in \pi_1(U(1)/\mathbb{1}) = \mathbb{Z}$$

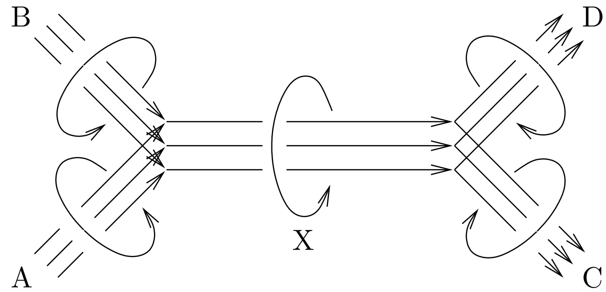


Figure 2.4: In a general Higgs theory $G \to G_1$ each vortex represents an element of the group $G_2 = \pi_1(G/G_1)$. The fusion rules are governed by this group, which might be non-Abelian. For the depicted case we have $g_{\rm A} g_{\rm B} = g_{\rm X} = g_{\rm C} g_{\rm D}$.

which can be identified with the total flux number n. We have found an abstract reasoning for the quantisation of this physical quantity. There are infinitely many $U(1)/\mathbb{1}$ vortices and they are *additively stable*.

Other situations may occur. Let a group G be spontaneously broken down to a subgroup G_1,

$$G \overset{\text{Higgs}}{\longrightarrow} G_1$$

Then in the same spirit

$$G_2 = \pi_1(G/G_1)$$

is the group of vortex quantum numbers. Whenever G_2 is non-trivial $G_2 \neq \mathbb{1}$, there are stable vortices. Their fusion rules are given by the composition law of the group G_2, which in general might be non-Abelian. Then the quantum number of the vortex is not additive, and one vortex cannot 'go through the other one' without leaving a third vortex (cf Fig. 2.4, 2.5 and Exercise (iii)). This situation plays a role in the theory of crystal defects, where the vortices go under the name of 'Alice strings'. Generically it does not occur in the Standard Model of elementary particle physics.

But let us consider a 'double Higgs theory',

$$SU(2) \underset{I=1}{\overset{\text{Higgs}}{\longrightarrow}} U(1) \underset{I=1}{\overset{\text{Higgs}}{\longrightarrow}} Z_2$$

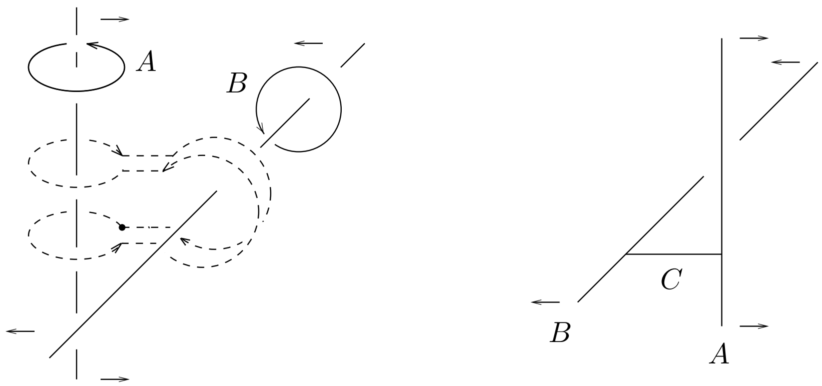


Figure 2.5: Alice strings: When a non-Abelian group G_2 is associated to the vortices, the hitting of two of them (A and B) will lead to a third one $C = ABA^{-1}B^{-1}$ as indicated by the dashed contour.

The $SU(2)$ theory is broken by a Higgs field in the adjoint ($I = 1$) representation down to the maximal Abelian subgroup $U(1)$. This $U(1) \cong SO(2)$ corresponds to the residual rotations around the preferred vacuum direction. Afterwards the theory is broken down further to Z_2 by another adjoint Higgs field. Z_2 as the center of $SU(2)$ is mapped onto the identity in the adjoint representation (cf (2.10)) and thus acts trivially on the Higgs field.

To be explicit we parametrise $SU(2)$ as a three-sphere (cf Fig. 2.6),

$$SU(2) \ni \Omega = a_0 \mathbb{1} + i a_i \sigma_i, \quad a_\mu \text{ real}, \quad a_0^2 + \vec{a}^2 = 1 \tag{2.9}$$

The center Z_2 sits on the poles $a_0 = \pm 1$, $\vec{a} = 0$, $\Omega = \pm \mathbb{1}$. Clearly it sends an $I = 1$ field ϕ back to itself,

$$\phi \rightarrow {}^{\Omega}\phi = \Omega^\dagger \phi \, \Omega = (\pm)^2 \mathbb{1} \phi \mathbb{1} = \phi \tag{2.10}$$

It is just the identification of opposite points that leads to the group $SO(3)$,

$$SU(2)/Z_2 \cong SO(3)$$

In addition non-contractible closed paths are created, namely those which connect two opposite points. The first homotopy group of $SO(3)$ is non-trivial,

$$\pi_1(SO(3)) = Z_2$$

Therefore, a $SU(2)/Z_2$ vortex carries a *multiplicative* quantum number ± 1. $+1$ stands for the contractible situation, which is homotopic to the trivial

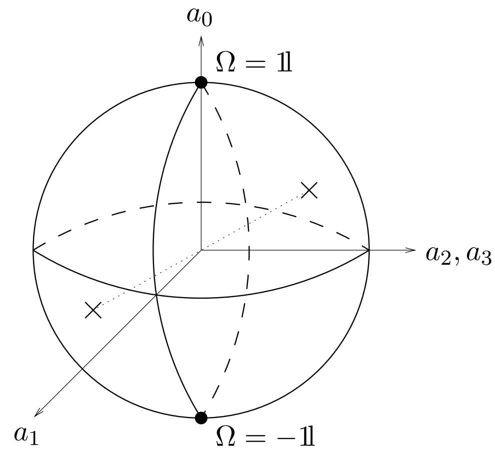


Figure 2.6: The group $SU(2)$ parametrised as a three-sphere $a_0^2 + \sum_{i=1}^3 a_i^2 = 1$. The center $Z_2 = \pm 1\!\!1$ sits on the poles, and every closed path can be contracted to a point: $\pi_1(SU(2)) = 1\!\!1$. After identification of opposite points ($\times$) one arrives at the group $SO(3)$. Every path connecting two opposite points is now closed, but not contractible: $\pi_1(SO(3)) = Z_2$.

vacuum. Unlike the case above there is only a finite number of different vortices, namely two.

Another significant difference to the $U(1)/1\!\!1$ case is the *orientability*: One could try to label the quantum numbers of the vortices by arrows. But as Fig. 2.7 indicates, these arrows are unstable in the $SU(2)/Z_2$ case.

Altogether we have that

the $U(1) \to 1\!\!1$ vortex has an additive quantum number $n \in \mathbb{Z}$ and is orientable.

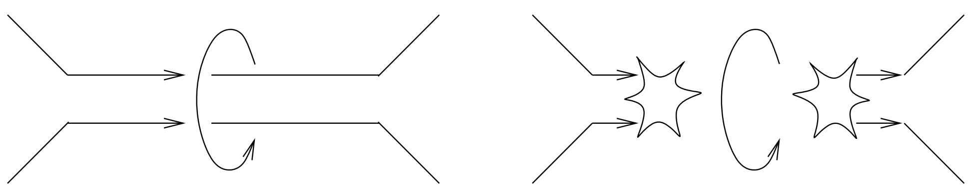
while

the $SU(2) \to U(1) \to Z_2$ vortex has a multiplicative quantum number $\pm 1 \in Z_2$ and is non-orientable.

This statement has a very interesting physical consequence (cf Fig. 2.8). Imagine two $U(1) \to 1\!\!1$ vortices with flux $2\pi/e$, respectively. The total flux is $4\pi/e$. But seen as $SU(2) \to U(1) \to Z_2$ vortices the intermediate vortex is equivalent to the vacuum with flux zero. The vortices have snapped creating

Figure 2.7: A graphical proof that the $SU(2)/Z_2$ vortex is non-orientable: Two incoming vortices with quantum number -1 produce a vortex with quantum number $+1$. It is equivalent to the vacuum, and the arrows become inconsistent.

a pair of something that carries magnetic charge. We conclude there must be *magnetic monopoles* with magnetic charge $4\pi/e$ (or an integer multiple of it). We will analyse these magnetic monopoles in the next chapter.



$U(1) \rightarrow Z_2$ $\qquad$ $SU(2) \rightarrow U(1) \rightarrow Z_2$

Figure 2.8: The snapping of vortices (see text).

Chapter 3

Magnetic Monopoles

3.1 Electric and Magnetic Charges and the Dirac Condition

Studying the vortices of Chapter 2 automatically revealed the existence of pure magnetic charges in non-Abelian gauge theories $G \to U(1)$. As worked out,

$$SU(2) \stackrel{I=1}{\to} U(1)$$

produces magnetic monopoles with magnetic flux $\pm 4\pi/e = g_m$. The minimally allowed electric charge is $q = e/2$ for $I = 1/2$ doublets. Indeed the *Dirac condition*,

$$\boxed{q g_m = 2\pi n} \qquad n \in \mathbb{Z}$$

is exactly obeyed.

We remind the reader of its origin. In Maxwell's theory isolated magnetic sources are excluded and the magnetic field is the curl of a smooth gauge field. Thus for a monopole the Maxwell field has to be singular at the so-called Dirac string. This is a curve which extends from the monopole to infinity[1] and carries a magnetic flux g_m. Physically the magnetic monopole is the endpoint of a tight magnetic solenoid which is too thin to detect. The string itself can be moved to a different position by a singular gauge transformation.

[1] It could also extend to a second monopole with inverse charge such that the net flux through a surface including both vanishes.

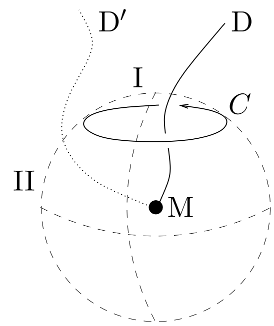


Figure 3.1: For introducing a magnetic monopole (M) into an Abelian theory, a Dirac string (D) is needed. When moving around the string on a circle C the wave function picks up a phase. This phase is proportional to the magnetic flux carried by the string, and the Dirac condition follows. The Dirac string can be put on a different position (D′) by a singular gauge transformation. In the fibre bundle construction the circle C is the overlap region of two patches (I,II).

Now we consider a matter field in the presence of that string. The vector potential enters the Schrödinger equation via the conjugate momentum,

$$H = H(\vec{p} - q\vec{A}, V)$$

When we go around the string the wave function picks up a phase,

$$q \int_C \vec{A}\,\mathrm{d}\vec{r} = q g_m$$

In the Aharonov-Bohm effect the same consideration leads to a phase shift of two electron beams. Since the wave function has to be single-valued and no AB effect shall take place, we have the restriction that $qg_m = 2\pi n$. The existence of one monopole quantises all electric charges.[2]

One can avoid the singularities by a fibre bundle construction: Every two-sphere around the monopole consists of two patches on which the gauge fields are regular, respectively. The patches overlap on some circle C around the string. There a gauge transformation ('transition function') $\Omega = e^{ie\Lambda}$ connects the fields,

$$\vec{A}^{(\mathrm{II})} = \vec{A}^{(\mathrm{I})} + \vec{\nabla}\Lambda, \qquad \psi^{(\mathrm{II})} = \psi^{(\mathrm{I})} e^{ie\Lambda}$$

[2] Introducing the magnetic charge $q_{\mathrm{mag}} = g_m/4\pi$ the Dirac condition reads $qq_{\mathrm{mag}} = n/2$.

For Ω to be single-valued Λ has to fulfil,

$$q[\Lambda(\varphi = 2\pi) - \Lambda(\varphi = 0)] = 2\pi n$$

The functions Λ fall into disjoint classes, the simplest representatives of which are just proportional to the angle φ around the string,

$$\Lambda = \frac{n}{q}\varphi$$

On the other hand the magnetic flux is given by the A-integral on the boundary. Here it is the $\vec{\nabla}\Lambda$-integral on that circle,

$$g_m = \int_C \vec{\nabla}\Lambda \mathrm{d}\vec{r} = \Lambda|_0^{2\pi} = 2\pi n/q \tag{3.1}$$

The Dirac condition has a topological meaning: The transition function $\Omega : S^1 \to U(1)$ has a winding number and (3.1) is how to compute it.

3.2 Construction of Monopole Solutions

After the excursions through lower dimensions we present in this section a 3+1 dimensional theory. No surprise, it is a non-Abelian gauge theory with gauge group $SU(2)$ and a Higgs field ϕ in the $I = 1$ representation,

$$\mathcal{L} = -\frac{1}{2}\left(\mathrm{D}_\mu\phi_a\right)^2 - \frac{\lambda}{8}\left(\phi_a^2 - F^2\right)^2 - \frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} \tag{3.2}$$

Both ϕ and A are elements of the Lie algebra $su(2) \cong \mathbb{R}^3$,

$$\phi = \phi^a\tau_a, \qquad A_\mu = A^a_\mu\tau_a, \qquad \tau_a = \sigma_a/2$$

and the non-Abelian definition of the covariant derivative and the field strength includes commutator terms,

$$\mathrm{D}_\mu\phi_a = \partial_\mu\phi_a + \epsilon_{abc}A^b_\mu\phi^c, \qquad F^a_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \epsilon_{abc}A^b_\mu A^c_\nu$$

We look for static solutions of the field equations. Repeating the arguments from the previous chapter we expect ϕ to live on a sphere with radius F asymptotically: $\phi^a\phi^a = F^2$. Topologically it is a mapping from S^2 (as the boundary of the coordinate space) to another S^2 (of algebra elements with

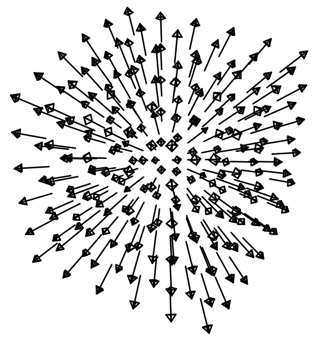


Figure 3.2: The Higgs field of a monopole configuration shows a 'hedgehog' behaviour. It points in the same direction (in isospace) like its argument (in coordinate space) and has winding number 1.

fixed length). The degree of this mapping is an integer. Alternatively one can see immediately that ϕ transforms under $(SU(2)_{I=1} \equiv SO(3))/U(1)$. Its second homotopy group[3] is the group of (even) integers. The one to one mapping,

$$\phi^a(x) \to F\hat{x}^a, \qquad \hat{x}^i = x^i/|\vec{x}|,\ |\vec{x}| = \sqrt{x^i x^i},\ \ i = 1, 2, 3$$

is the first non-trivial mapping. On the boundary the same things happen as before, ϕ 'winds around once'. Notice that the isospace structure (indices a) is mixed with the space-time structure (indices i). Inside, ϕ is of the same form,

$$\phi^a(x) = \phi(|\vec{x}|)\hat{x}^a \tag{3.3}$$

with a regular function $\phi(|\vec{x}|)$. The solution is depicted in Fig. 3.2. It is called a 'hedgehog' and has a zero inside.

Corresponding to our previous discussions we make the natural ansatz:

$$A_0 = 0, \qquad A_i^a = \epsilon_{iaj}\hat{x}_j A(|\vec{x}|) \tag{3.4}$$

The first condition means that there is no electric field, the second one is the analogue of the circular gauge field in (2.6). Again it exploits the mixing of isospace and coordinate space indices. The magnetic field at spatial infinity looks like if there were a magnetic charge inside: $B_i \propto x_i/|\vec{x}|^3$.

[3] Analogously to π_1, the second homotopy group π_2 is the group of mappings from S^2 into the given manifold.

What happens after spontaneous symmetry breaking? To extract the physical content one usually makes use of the local gauge symmetry. We diagonalise ϕ, i.e. force it to have only a third component. The corresponding gauge is called 'unitary gauge',

$$\phi \to \begin{pmatrix} 0 \\ 0 \\ F \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \eta \end{pmatrix} \tag{3.5}$$

F and η are the vacuum expectation value and the fluctuations of the Higgs field, respectively. The third component of the Higgs field gets a mass,

$$M_\eta \equiv M_H = F\sqrt{\lambda}$$

For the gauge fields it is the other way round. A_μ^1 and A_μ^2 are massive vector bosons, A_μ^3 is the massless photon referring to the unbroken $U(1)$ in the third direction,

$$M_{A^{1,2}} \equiv M_{W^\pm} = eF, \qquad M_{A^3} = 0$$

The gauge field coupling is denoted by 'e', since this is also the charge unit with respect to the residual Maxwell potential A_μ^3.

It can be shown that under spontaneous symmetry breaking the hedgehog configuration turns into a Dirac monopole. It resides in the origin, while the Dirac string is placed along the negative z-axis. The last point is not difficult to explain (see also Exercise (iv)): The gauge transformation has to rotate ϕ onto the positive z-axis in the algebra. It can be written in terms of the spherical coordinates θ and φ. The latter becomes ambiguous on the z-axis. This does not matter at its positive part, since ϕ is already of the desired form there. But on the negative part it points just in the opposite direction and there are a lot of rotation matrices. Whatever direction we choose for the spontaneous symmetry breaking in (3.5), a singularity occurs: the unitary gauge changes the asymptotic behaviour of ϕ from the hedgehog to the trivial one. Like in the explicit case the singularity is always situated on the opposite part of the chosen axis. We have found again that the existence of the Dirac string is gauge invariant, its position is gauge dependent.

3.3 Existence of Monopoles

What is the general feature of theories which allow for monopole solutions? The $U(1)_{em}$ must be embedded as a subgroup[4] in a larger non-Abelian group G, and

$$\pi_1(G) < \mathbb{Z}$$

For the winding number to be finite, G must have a *compact* covering group[5]. This is the topological reason for the statement, that there are **no magnetic monopoles in the electroweak sector of the Standard Model**

$$SU(2)_I \times U(1)_Y \rightarrow U(1)_{em}$$

$U(1)$ has a non-compact covering group, namely $\mathbb{R}^+$. Thus $\pi_1(SU(2)_I \times U(1)_Y)$ is still $\mathbb{Z}$ and vortices refuse to snap.

In Grand Unified Theories (GUT) the Standard Model is embedded in a larger group like $SU(5)$. Then monopole solutions become possible again and have magnetic flux $2\pi/e$. As we will see in the next section, its mass is bigger than the mass of the massive vector bosons W of the theory. The GUT scale is 10^{16}GeV and the monopole mass is of the order of m_{Planck}.

So far no experiment has detected magnetic monopoles. Perhaps they exist somewhere in the universe. Not only GUT's but also cosmological models predict their existence.

The generalization to monopoles with added *electric* charge was introduced by Julia and Zee [4]. These particles are called 'dyons'. It is easy to imagine that one can add multiples of $A_\pm$ to a monopole,

$$g_m = \frac{4\pi}{e}, \qquad q = ne$$

3.4 Bogomol'nyi Bound and BPS States

For estimating the monopole mass we again use the Bogomol'nyi trick,

$$E = \int d^3x \left(\frac{1}{2}(\vec{\mathrm{D}}\phi_a)^2 + \frac{\lambda}{8}(\phi_a^2 - F^2)^2 + \frac{1}{2}\vec{B}_a^2 \right)$$

[4] It is also possible to embed a product of $U(1)$'s into G which will be the case for the Abelian Projection of $SU(3)$ and higher groups in chapter 5.

[5] The covering group of a given Lie group is constructed from the same Lie algebra, but is simply connected.

Note that the homogeneous field equation for non-Abelian theories read,

$$\mathrm{D}_i B_i^a = \frac{1}{2}\epsilon_{ijk}\mathrm{D}_i F_{jk}^a = 0$$

It is the usual Bianchi identity that allows for the introduction of the A-field. We use it to reduce the number of squares,

$$(\vec{\mathrm{D}}\phi_a)^2 + \vec{B}_a^2 = (\vec{\mathrm{D}}\phi_a \pm \vec{B}_a)^2 \mp 2\vec{B}_a\vec{\mathrm{D}}\phi_a$$

We rewrite the last term in a total derivative,

$$\vec{B}_a\vec{\mathrm{D}}\phi_a = \vec{\partial}(\vec{B}_a\phi_a)$$

Its contribution to the energy is gauge invariant, and we compute it in the unitary gauge,

$$\int \mathrm{d}^3x\vec{\partial}(\vec{B}_a\phi_a) = F\int_{S^2_\infty}\vec{B}_3\vec{n} = \frac{4\pi}{e}F$$

Thus the energy of the monopole is bounded from below by the mass of the W-boson,

$$E = \int \mathrm{d}^3x\left[\frac{1}{2}(\vec{\mathrm{D}}\phi_a \pm \vec{B}_a)^2 + \frac{\lambda}{8}(\phi_a^2 - F^2)^2\right] + \frac{4\pi}{e^2}M_W \tag{3.6}$$

The bound is saturated for vanishing potential, $\lambda = 0$. The exact solution to the remaining equations,

$$\vec{\mathrm{D}}\phi_a \pm \vec{B}_a = 0, \qquad |\phi| \to F$$

was given by Sommerfield and Prasad. These so-called BPS states have a mass,

$$M_{\mathrm{mon}} = \frac{4\pi}{e^2}M_W$$

and are important for supersymmetric theories.

3.5 Orbital Angular Momentum for qg Bound States

In this section we look for electrically charged particles bound to the monopole. All particles with $U(1)$ charge originate from $SU(2)$ representations,

$$\begin{aligned} I = \text{integer} \quad &\longrightarrow \quad q_{U(1)} = ne \\ I = \text{integer} + \frac{1}{2} \quad &\longrightarrow \quad q_{U(1)} = (n + \frac{1}{2})e \end{aligned}$$

consistent with the Dirac condition,

$$g_{\text{mon}} = \frac{4\pi}{e} \longrightarrow q g_{\text{mon}} = 2\pi n$$

Let us take a minimal charge $q = \frac{1}{2}e$, i.e. a field ψ in the defining representation of $SU(2)$,

$$I = \frac{1}{2}: \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

and consider ψ near a monopole. The wave equation reads:

$$\mathrm{D}^2\psi + \mu^2\psi \to 0 \qquad \text{or} \qquad (\gamma_\nu \mathrm{D}_\nu + \mu)\,\psi \to 0$$

μ plays the role of a binding potential. In the regular description the monopole solution has $\phi^a(x) = \phi(|x|)\hat{x}^a$. Its rotational symmetry can only be exploited if we rotate ϕ^a together with $\vec{x}$. That is spacial $SO(3)$ rotations must be *coupled* to isospin $SU(2)$ rotations,

$$SU(2)_{\text{space}} \times SU(2)_{\text{isospin}} \longrightarrow SU(2)_{\text{diag}}$$

where $SU(2)_{\text{diag}}$ is the invariance group of the monopole. The representation of our ψ in this $SU(2)_{\text{diag}}$ is,

$$\begin{aligned} L_{\text{tot}} &= L_{\text{space}} + L_{\text{isospin}} \\ l_{\text{tot}} &= l_{\text{space}} \pm \frac{1}{2} \end{aligned}$$

ψ may be a scalar under spacial rotations but carries now half spin! Similar things happen, when we give ψ an ordinary spin $\pm\frac{1}{2}$: the angular momentum

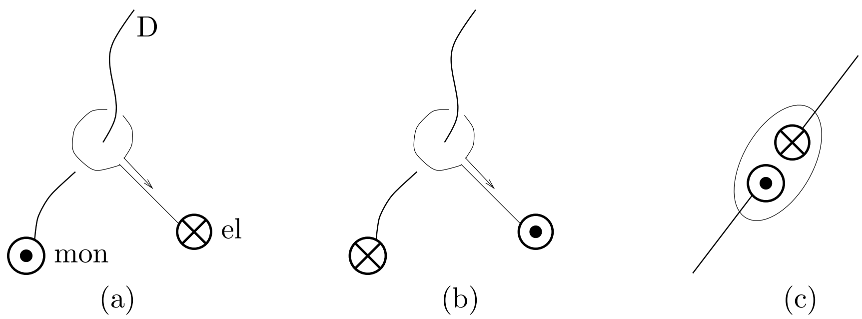


Figure 3.3: An electrically charged particle (el) feels the Dirac string (D) of a magnetic monopole (mon). When moving around the string, $\psi_{\rm el}$ picks up a phase factor (a). This is equivalent to move the monopole around the electric charge with its electric string (b). For bound states (c) we choose both strings to point in opposite directions.

becomes an integer! We have found that [5]

$$\boxed{\begin{array}{c} q=\frac{1}{2}e \text{ particles will bind to a magnetic monopole} \\ \text{with } g_m=\frac{4\pi}{e}=\frac{2\pi}{q} \text{ in such a way that} \\ \text{the orbital angular momentum is integer} + \frac{1}{2}. \end{array}}$$

The spin becomes half-odd integer, although the monopole is a spin 0 object. The *anomalous spin addition theorem* for qg bound states with $q \cdot g = 2\pi$ reads:

$$\boxed{\text{integer} + \text{integer} \longrightarrow \text{integer}+\tfrac{1}{2}}$$

etc. Something like this was never seen in quantum field theory before.

The reasoning heavily relies on the existence of Dirac strings. Imagine an electric charge in the fundamental representation and a monopole like in Fig. 3.3. The wave function of the electric charge $\psi_{\rm el}$ feels the string coming from the monopole. The Maxwell equations allow us to interprete the resulting phase shift also after interchanging electric and magnetic charges. Then $\psi_{\rm mon}$ feels the string coming from the electric charge. Accordingly, the eg bound state has two strings. When they are oppositely oriented, the bound state looks like if it has a string running from $-\infty$ to ∞. We remind the reader that one part of the string is only felt by the magnetic monopole wave function, while the other part only by the electric charge wave function.

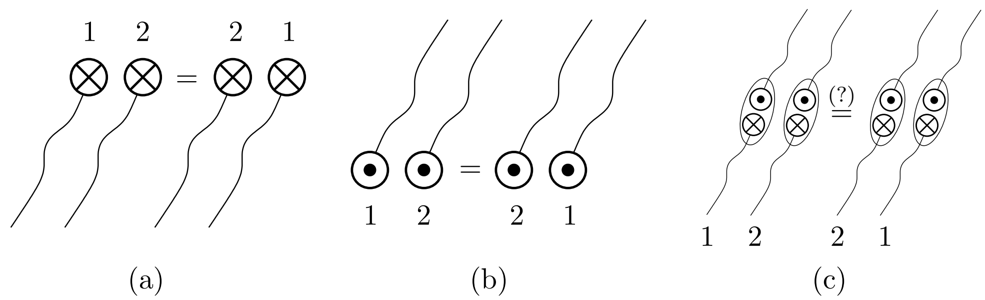


Figure 3.4: The two-particle-wave function of monopoles (a) and electric charges (b) is symmetric due to the fact that the charges do not feel string of their own kind. What will happen for bound states (c) is discussed in the text.

Now consider two identical *bosonic* monopoles and two identical *bosonic* electric charges. Since the charges do not feel strings of its own kind, they can be moved around freely (cf Fig. 3.4(a) and 3.4(b)). In the same way combine the charges into *identical* bound states. What is their statistics? What happens to the wave function when we interchange two of these?

They are the states of Fig. 3.4(c), and, considering the wave functions of these objects, *with all strings attached*, there will be no anomalous sign switch if we interchange the two objects.

However, we may now observe that, as long as the objects remain tightly bound, each as a whole feels a string that runs from $-\infty$ to $+\infty$: since they carry both electric and magnetic charge, they each feel the combination of the strings from Fig. 3.4(a) and 3.4(b). To be precise: if $\vec{r}_1$ is the center of mass of bound state 1 and $\vec{r}_2$ is the center of mass of bound state 2, the wave function is,

$$\psi_{12}(\vec{r}_1, \vec{r}_2) = \psi_{\mathrm{cm}}(\frac{\vec{r}_1 + \vec{r}_2}{2})\psi_{\mathrm{rel}}(\vec{r}_1 - \vec{r}_2)$$

and it is $\psi_{\mathrm{rel}}(\vec{r}_1 - \vec{r}_2)$ that feels a Dirac string running through the origin from $z = -\infty$ to $z = +\infty$.

The point is now that we may *remove* this Dirac string by multiplying ψ_{rel} with

$$e^{i\varphi(\vec{r}_1 - \vec{r}_2)}$$

This produces a minus sign under the interchange $\vec{r}_1 \leftrightarrow \vec{r}_2$. The bound states obey Fermi-Dirac statistics [6]! After the Dirac string is removed, the system of two identical bound states is treated as an ordinary system of particles such as molecules.

3.6 Jackiw-Rebbi States at a Magnetic Monopole

In the first chapter we have seen that there are chiral fermions in the background of a kink. We briefly discuss this effect for the monopole. We introduce fermions ψ transforming under some representation of the gauge group $SU(2)$ given by the generators $(T^a)_{ij}$,

$$\mathcal{L} = \mathcal{L}_{\text{mon}} - \bar{\psi}\gamma \mathrm{D}\psi - G\bar{\psi}_i \phi_a T^a_{ij} \psi_j$$

The first part $\mathcal{L}_{\text{mon}}$ is the theory (3.2) we have discussed so far. Since the fermion couples to the Higgs field it gets a mass,

$$\text{unitary gauge: } \langle \phi_a \rangle \to \begin{pmatrix} 0 \\ 0 \\ F \end{pmatrix} : \quad m_\psi = C \cdot GF$$

where C is a coefficient depending on the representation.

The energies are again the eigenvalues of the Hamilton operator,

$$\gamma_4 \frac{\partial}{i\partial t} \to \gamma_4 E$$

We use an off-diagonal representation for the matrices $\vec{\alpha}$ and β,

$$\gamma_4 \vec{\gamma} = -i\vec{\alpha}, \quad \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \quad \gamma_4 = \beta, \quad \beta = -i \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$$

The energy equation reads

$$\left[\vec{\alpha}(\vec{p} + gT^a \vec{A}_a) + \beta G T^a \phi_a\right] \psi = E\psi$$

We split ψ into its chirality components $\psi = \binom{\chi^+}{\chi^-}$ and insert the magnetic monopole in its regular form ((3.3),(3.4)),

$$[\vec{\sigma}(\vec{p} + gA(|\vec{x}|)T^a\,(\vec{\sigma} \wedge \vec{r})_a) \pm iG\phi(|\vec{x}|)\,T^a \hat{x}_a]\,\chi^\pm = E\chi^\mp$$

which for $E = 0$ separates into equations for χ^+ and χ^-, respectively.

We have already discussed the symmetry of this equation. Invariant rotations are generated by the total angular momentum $\vec{J}$,

$$\vec{J} = \vec{L} + \vec{S} + \vec{T}$$

where $\vec{L}$, $\vec{S}$ and $\vec{T}$ are the ordinary angular momentum, the spin and the isospin, respectively. Let us work in the defining representation $t = 1/2$ ($q = \pm e/2$) and look for the simplest solutions with $\vec{J} = \vec{L} = 0$,

$$\vec{S} + \vec{T} = 0$$

For this case Jackiw and Rebbi found *one* solution,

$$E = 0, \qquad (j = l = 0).$$

Note that $j = 0$ inspite of $s = 1/2$.

As for the kink, the Jackiw-Rebbi state lies inbetween the fermion and anti-fermion eigenstates. Whether it is full or empty does not change the energy of the system (Fig. 3.5). In most cases the baryon number is just a conserved charge, the monopole (anti-monopole) contributes to since it has:

$$\begin{aligned} \text{baryon number} &= \pm 1/2 \\ \text{electric charge} &= \pm e/4 \end{aligned}$$

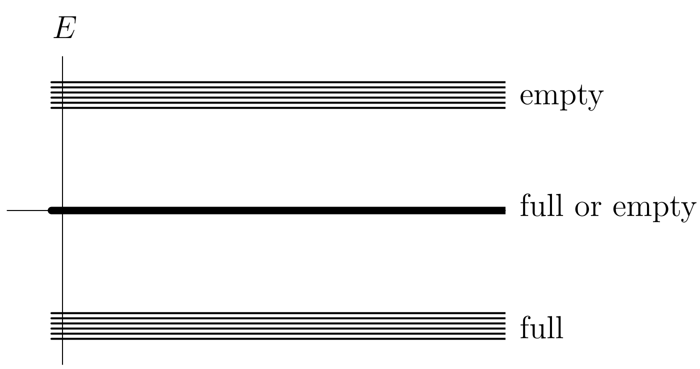


Figure 3.5: The spectrum of fermions in the background of a magnetic monopole. There are $E = 0$ Jackiw-Rebbi states, the degeneracy of which depends on the total angular momentum j.

Much more is to be said about the electric charges. Here we only state that the bound states behave like particles or anti-particles under $U(1)_{\text{charge}}$.

For the adjoint representation $t = 1$ we have $j = 1/2$. There are now two solutions with $E = 0$ and $j_z = \pm 1/2$ and we get a 2^2-fold degeneracy.

Notice that the Jackiw-Rebbi solution is a *chiral* wave function: χ^+ and χ^- are eigenstates of γ_5, which is block-diagonal in the chosen representation.

Chapter 4

Instantons

New topological objects, the so-called instantons, arise in pure non-Abelian gauge (Yang Mills) theories in four dimensions. We approach the topic by investigating the structure of gauge transformations.

4.1 Topological Gauge Transformations

Let us work in the Weyl (=temporal) gauge $A_0 = 0$, where the theory reduces to

$$F_{0i}^a = \partial_0 A_i^a, \qquad \mathcal{L} = \frac{1}{2}(\partial_0 A_i^a)^2 - \frac{1}{4}F_{ij}^a F_{ij}^a = \frac{1}{2}\left(\vec{E}_a^2 - \vec{B}_a^2\right).$$

The Lagrangian density is nothing but the difference of kinetic and potential energy in a Yang Mills sense. The action of a gauge transformation Ω on a gauge field A is,

$$A_\mu \to \Omega(x)(\frac{1}{ie}\partial_\mu + A_\mu)\Omega^{-1}(\vec{x})$$

Obviously the surviving invariance of the gauge $A_0 = 0$ consists of time-independent gauge transformations,

$$\partial_t \Omega = 0 \Rightarrow \Omega(\vec{x}, t) = \Omega(\vec{x})$$

just like a global symmetry in time. The Hamiltonian of the theory,

$$H = \int \mathrm{d}^3\vec{x}(E_i^a \partial_0 A_i^a - \mathcal{L}) = \frac{1}{2}\int \mathrm{d}^3\vec{x}(\vec{E}_a^2 + \vec{B}_a^2)$$

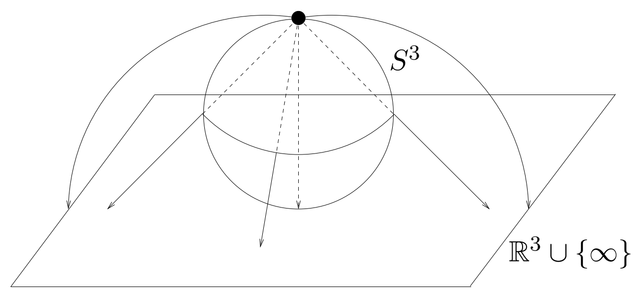


Figure 4.1: The stereographic projection identifies the three-sphere with the three-space compactified at spacial infinity, which is the image of the north pole.

is the sum of the kinetic and potential energy, and commutes with these gauge transformations,

$$[H, \Omega(\vec{x})] = 0$$

We can diagonalise both operators simultaneously,

$$H|\psi\rangle = E|\psi\rangle, \qquad \Omega(\vec{x})|\psi\rangle = \omega(\vec{x})|\psi\rangle$$

The eigenvalues ω are constants of motion. Now infinitesimal gauge transformations give rise to eigenvalues λ,

$$\Omega(\vec{x}) = \mathbb{1} + i\epsilon\Lambda(\vec{x}), \qquad \Lambda(\vec{x})|\psi\rangle = \lambda(\vec{x})|\psi\rangle$$

The only values of λ consistent with the unbroken spacial Lorentz transformations is

$$\boxed{\lambda(\vec{x}) = 0}$$

However, *a class of* $\Omega(\vec{x})$ *exists that cannot be obtained from infinitesimal gauge rotations* $\Lambda(\vec{x})$. We remind the reader of the stereographic projection, which identifies the three-space $\mathbb{R}^3$ compactified at spacial infinity with the three-sphere S^3 (Fig. 4.1). If Ω has the same limit when going to spacial infinity in any direction, it can be regarded as a function on $\mathbb{R}^3 \cup \{\infty\} \cong S^3$. Since $SU(2)$ is again a three-sphere we have,

$$\Omega : \, S^3 \to S^3$$

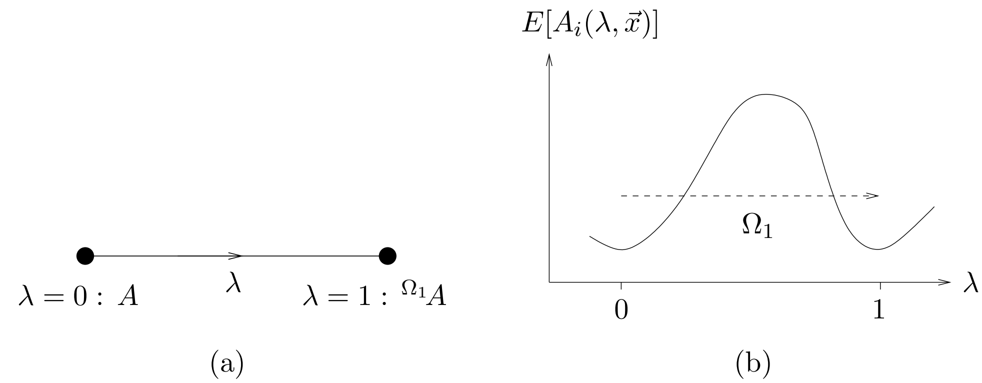


Figure 4.2: A continuous line of gauge fields connects two gauge equivalent configurations (a). Since the intermediate points are physically different, they have different energies (b) and tunnelling is expected.

Similiarly to vortices and monopoles, these mappings are classified by the third homotopy group, which for $SU(2)$ is an integer,

$$\pi_3(SU(2)) = \pi_3(S^3) = \mathbb{Z}$$

Again the one to one mapping Ω_1 is distinct from the trivial mapping $\Omega_0(\vec{x}) \equiv \mathbb{1}$ and has winding number one. Representatives of higher windings are delivered by raising this function to the nth power,

$$\Omega_n(\vec{x}) = (\Omega_1(\vec{x}))^n$$

Still these operators can be diagonalised together with the Hamiltonian. Since they are unitary, their contants of motion are characterised by an angle θ,

$$\Omega_1(\vec{x})|\psi\rangle = e^{i\theta}|\psi\rangle, \qquad \Omega_n(\vec{x})|\psi\rangle = e^{in\theta}|\psi\rangle, \qquad \theta \in [0, 2\pi) \tag{4.1}$$

θ is a Lorentz invariant. It is called the *instanton angle*. It is a fundamental parameter of the theory, which could be measured in principle[1].

Although $\Omega_n(\vec{x})$ form topologically distinct gauge transformations, they act on the space $\{A_i(\vec{x})\}$ which is topologically trivial. Consider a continuous line of gauge fields connecting two gauge equivalent A's (Fig. 4.2(a)),

$$A_i(\vec{x}) \to A_i(\lambda, \vec{x}), \qquad A_i(1, \vec{x}) = {}^{\Omega_1}A_i(0, \vec{x})$$

[1]The experimental evidence that there is little CP violation in QCD indicates that θ must be very small or zero.

But at fractional λ this is *not* a gauge transformation. These gauge fields lie on different orbits, i.e. are physically different! So do their energies, i.e. the expectation value of H in these configurations,

$$E[A_i(\lambda, \vec{x})] = \langle \{A_i(\lambda, \vec{x})\} | H | \{A_i(\lambda, \vec{x})\} \rangle$$

If $\lambda = 0$ and $\lambda = 1$ are vacua, the energy is higher inbetween as drawn in Fig. 4.2(b). The system may *tunnel* through the gauge transformation Ω_1. How one actually computes the tunnelling rate and how the action enters this calculation will be explained in the next section.

4.2 Semiclassical Approximation for Tunnelling

For the eigenfunctions ψ of the Hamiltonian H of an ordinary one dimensional quantum mechanical system

$$H\psi = E\psi, \qquad H = \frac{1}{2}p^2 + V(x) \qquad (\hbar = m = 1)$$

we write formally,

$$p\psi = -i\frac{\partial}{\partial x}\psi = \sqrt{2(E - V(x))}\psi$$

Thus

$$\psi \propto \exp(i \int \sqrt{2(E - V(x))}\, \mathrm{d}x)$$

is an approximate solution, i.e. describes the leading effects (in $\hbar$). In the classically allowed regions $E > V(x)$ the wave function just oscillates, while in the forbidden regions there is an exponential suppression,

$$E < V(x): \quad \psi \propto \exp(-\int \sqrt{2(V(x) - E)}\, \mathrm{d}x)$$

We deduce that the following quantity approximates the tunnelling amplitude,

$$\exp(-\int_A^B \sqrt{2(V(x) - E)}\, \mathrm{d}x)$$

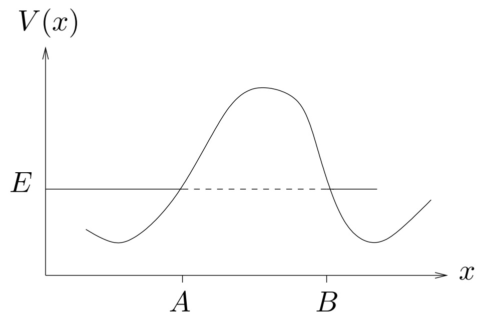


Figure 4.3: The semiclassical situation for tunnelling through a potential barriere (see text).

where A and B are the boundary points of the forbidden region $V(A) = V(B) = E$.

The sign switch $V - E \to E - V$ is equivalent to $p \to ip$, $p^2 \to -p^2$ or to,

$$t \to it = \tau, \quad E \to iE, \quad V \to iV$$

The first replacement means that we can interchange the meaning of 'allowed' and 'forbidden' by going to an imaginary time. For field theories one passes from Minkowski to *Euclidean* space, accordingly.

Moreover, the integral can be rewritten as the action for imaginary times,

$$\int_A^B \sqrt{2(V(x)-E)}\,\mathrm{d}x = \int_{t_A}^{t_B} p\,\dot{x}\,\mathrm{d}t = \int_{\tau_A}^{\tau_B} \mathcal{L}(\tau)\,\mathrm{d}\tau = S_{\mathrm{tot}} \qquad (\text{if } E = 0)$$

Thus the dominant contribution to a tunnelling transition is obtained by computing the *action* of a classical motion in *Eulidean* space, and write,

$$e^{-|S_{\mathrm{tot}}|} \tag{4.2}$$

For tunnelling in the space of gauge fields we are automatically driven to the following topic.

4.3 Action for a Topological Transition, Explicit Instanton Solutions

Let us seek for a tunnelling configuration along the lines of Fig. 4.4. In the infinite (Euclidean) past the gauge field is trivial $A = 0$. Then it evolves

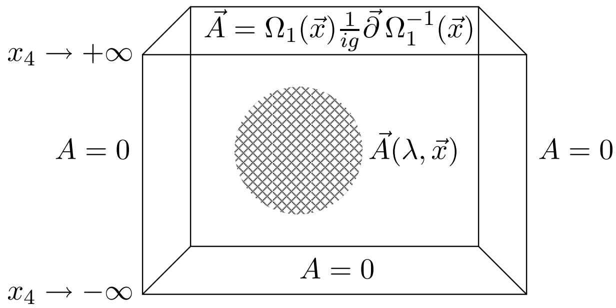


Figure 4.4: A tunnelling process in Yang Mills theory. A trivial vacuum at $x_4 \to -\infty$ evolves into a vacuum with winding number 1 at $x_4 \to +\infty$.

somehow and arrives at the first non-trivial vacuum $\vec{A} = \Omega_1(\vec{x})\frac{1}{ig}\vec{\partial}\,\Omega_1^{-1}(\vec{x})$ in the infinite future. During the whole process A should vanish at the spacial boundary. For $x_4 \to +\infty$ we already know this, since $\Omega_1(\vec{x})$ becomes constant there. But now we can write A as a pure gauge on the whole boundary of $\mathbb{R}^4$,

$$\vec{A} \to \Omega_1(x)\frac{1}{ig}\vec{\partial}\,\Omega_1^{-1}(x) \quad \text{with } \Omega_1(x) = \begin{cases} \Omega_1(\vec{x}) & \text{at } x_4 \to +\infty, \\ \text{const.} & \text{elsewhere.} \end{cases}$$

A gauge equivalent (now we leave $A_4 = 0$), but more symmetric way is to choose

$$A_\mu \to \Omega_1(x)\frac{1}{ig}\partial_\mu\,\Omega_1^{-1}(x) \tag{4.3}$$

with

$$\Omega_1(x) \to \frac{x_4 \mathbb{1} + i x_i \tau_i}{|x|}, \qquad |x| = \sqrt{x_\mu x_\mu}$$

Notice that Ω_1 lives on the boundary of $\mathbb{R}^4$ which is a three-sphere. It has the same degree as discussed above and mixes coordinate space and isospace.

The last point will be crucial for finding explicit instanton solutions. The problem becomes simpler due to the higher symmetry. The action of Ω_1 on a *fundamental spinor* is,

$$\Omega_1(x)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_4 + i x_3 \\ -x_2 + i x_1 \end{pmatrix}\frac{1}{|x|} \tag{4.4}$$

and covers the whole sphere. The symmetry is such that an $SO(4)$ rotation in Euclidean space is linked to isospin $SU(2)$ rotations,

$$\begin{array}{ccccc} SO(4) & \cong & SU(2)_{\rm L} & \otimes & SU(2)_{\rm R} \\ SU(2) & \to & SU(2) & \otimes & 1\!\!1 \end{array}$$

Obviously the Lie algebra $so(4)$ is 6=3+3 dimensional. For a matrix $\alpha \in so(4)$,

$$\alpha_{\mu\nu} \in \mathbb{R}, \quad \alpha_{\mu\nu} = -\alpha_{\nu\mu}$$

we define the 'dual transform' $\tilde{\alpha}$ as,

$$\tilde{\alpha}_{\mu\nu} := \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\alpha_{\rho\sigma}$$

The six degrees of freedom can be divided as follows,

$$\begin{array}{ccccc} \alpha_{\mu\nu} & = & \frac{1}{2}(\alpha+\tilde{\alpha})_{\mu\nu} & + & \frac{1}{2}(\alpha-\tilde{\alpha})_{\mu\nu} \\ 6 & = & 3 & + & 3 \end{array}$$

The terms on the right handside are selfdual and anti-selfdual ($\tilde{\tilde{\alpha}} \equiv \alpha$) and correspond to representations of $su(2)_L$ and $su(2)_R$, respectively.

A field ψ^a transforming as an $I = 1$ representation under $SU(2)_{\rm L}$ can be written as,

$$\psi^a = \eta^a_{\mu\nu} a_{\mu\nu}$$

The coefficients are denoted by the tensor η. It is selfdual,

$$\eta^a_{\mu\nu} = \tilde{\eta}^a_{\mu\nu}$$

and of course anti-symmetric in (μ, ν), as easily seen from the explicit representation,

$$\boxed{\eta^a_{ij} = \epsilon^{aij}, \quad \eta^a_{i4} = \delta^a_i, \quad \eta^a_{4i} = -\delta^a_i}$$

As ϵ_{iaj} in three dimensions it provides the mixing of coordinate space and isospace.

The η-tensor can now be used to describe the *vector* field A^a_μ in the adjoint representation. One finds from (4.3) and (4.4) that, asymptotically,

$$A_\mu \to \Omega_1 \frac{1}{ig}\partial_\mu \, \Omega_1^{-1} \equiv 2\eta^a_{\mu\nu}\frac{x_\nu}{|x|^2}\tau_a$$

It becomes singular when approaching the origin. Which *smoothened* connection near the origin *minimises* the action? With our knowledge we try,

$$A^a_\mu = \eta^a_{\mu\nu} x_\nu A(|x|)$$

Indeed, the profile,

$$A(|x|) = \frac{2}{|x|^2 + \rho^2} \tag{4.5}$$

makes the action minimal,

$$S = \frac{1}{4} \int F^a_{\mu\nu} F^a_{\mu\nu} = -\frac{8\pi^2}{g^2} \tag{4.6}$$

This number has to be exponentiated in (4.2). If $|g|$ is small, the resulting rate is very very small. Furthermore since its expansion for small g gives zero in all orders, tunnelling processes will not be seen in pertubation theory.

The new length ρ is the width of the profile. Since these configurations are local events in space *and* time, they are called *instantons* or *pseudo-particles*. The action is independent of ρ, i.e. we have found a whole manifold of instanton solutions. This so-called moduli space also contains the position z_μ of the center of the instanton which was chosen to be at the origin in above.

4.4 Bogomol'nyi Bound and Selfdual Fields

The instanton fulfills a Bogomol'nyi bound. We write

$$-S = \frac{1}{8} \int (F^a_{\mu\nu} - \tilde{F}^a_{\mu\nu})^2 + \frac{1}{4} \int F^a_{\mu\nu} \tilde{F}^a_{\mu\nu} \tag{4.7}$$

The number of squares has reduced from $3 \cdot 6$ in (4.6) to $3 \cdot 3$. To see that the second term is a total derivative needs some effort,

$$\begin{aligned}
\frac{1}{4} F^a_{\mu\nu} \tilde{F}^a_{\mu\nu} &= \frac{1}{8} \epsilon_{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma} \\
&= \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} (\partial_\mu A^a_\nu + \frac{g}{2} \epsilon^{abc} A^b_\mu A^c_\nu)(\partial_\rho A^a_\sigma + \frac{g}{2} \epsilon^{ade} A^d_\rho A^e_\sigma) \\
&= \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} (\partial_\mu A^a_\nu \partial_\rho A^a_\sigma + g \epsilon^{ade} \partial_\mu A^a_\nu A^d_\rho A^e_\sigma + \frac{g^2}{4} \epsilon^{abc} \epsilon^{ade} A^b_\mu A^c_\nu A^d_\rho A^e_\sigma)
\end{aligned}$$

The g^2 term vanishes because of the symmetry of the δ's in (b, c, d, e) together with the anti-symmetry of ϵ in (μ, ν, ρ, σ). Similar symmetry arguments give the following result[2],

$$\frac{1}{4} F^a_{\mu\nu} \tilde{F}^a_{\mu\nu} = \frac{8\pi^2}{g^2} \partial_\mu K_\mu \tag{4.8}$$

with the Chern-Simons current

$$K_\mu = \frac{g^2}{16\pi^2} \epsilon_{\mu\nu\rho\sigma} (A^a_\nu \partial_\rho A^a_\sigma + \frac{g}{3} \epsilon^{abc} A^a_\nu A^b_\rho A^e_\sigma) \tag{4.9}$$

being a gauge *variant* quantity. The asymptotic behaviour of this current,

$$|x| \to \infty : \quad K_\mu \to \frac{1}{2\pi^2} \frac{x_\mu}{|x|^4}$$

gives the following surface integral

$$-S = \frac{8\pi^2}{g^2} \int_{S^3_\infty} \mathrm{d}^3\sigma K_\perp = \frac{8\pi^2}{g^2} |x|^3 \mathrm{area}(S^3_1) \frac{1}{2\pi^2} \frac{1}{|x|^3} = \frac{8\pi^2}{g^2} \tag{4.10}$$

The vanishing of the square in (4.7) means that the field strength is selfdual. From (4.5) we compute

$$F^a_{\mu\nu} = \tilde{F}^a_{\mu\nu} = -\frac{4}{g} \eta^a_{\mu\nu} \frac{\rho^2}{(|x|^2 + \rho^2)^2}$$

and indeed,

$$\mathrm{D}_\mu F_{\mu\nu} (\equiv \mathrm{D}_\mu \tilde{F}_{\mu\nu}) = 0.$$

In general, the Bogomol'nyi bound is a useful tool to solve the Yang-Mills equations. After having introduced the A-field, one needs to solve $\mathrm{D}_\mu F_{\mu\nu} = 0$. This equation corresponds to the inhomogeneous Maxwell equation and therefore is *second order* in A. The demand for selfdual fields $F_{\mu\nu} = \tilde{F}_{\mu\nu}$ is only *first order* in A. Now the Yang-Mills equation is automatically fulfilled because of the Bianchi identity $\mathrm{D}_\mu \tilde{F}_{\mu\nu} = 0$.

For *all* configurations the second term in (4.7) is a multiple of $-\frac{8\pi^2}{g^2}$. It is a topological quantity, called the *Pontryagin index*. Since the integral can be reduced to the surface, it corresponds to the winding number $\Omega_1 : S^3 \to S^3$ discussed above.

[2] For readers familiar with differential forms we give the following equivalent equation: $\mathrm{tr} F \wedge F \propto \mathrm{d}\, \mathrm{tr}(A \wedge \mathrm{d}A - \frac{2ig}{3} A \wedge A \wedge A)$ with a proper definition of the wedge product for algebra elements

4.5 Intermezzo: Massless Fermions in a Gauge Theory

The coupling of fermions to the gauge field is done in a standard way by the *vector current*,

$$J_\mu = \bar{\psi}\gamma_\mu\psi$$

where we dropped the isospace structure. For massless fermions, the *axial current* is conserved, too,

$$J^5_\mu = \bar{\psi}\gamma_\mu\gamma_5\psi, \qquad \partial_\mu J_\mu = \partial_\mu J^5_\mu = 0$$

Denoting by $J^{\rm L}_\mu$ and $J^{\rm R}_\mu$ the projections onto γ_5 eigenstates, we can write,

$$J_\mu = J^{\rm L}_\mu + J^{\rm R}_\mu, \qquad J^5_\mu = J^{\rm L}_\mu - J^{\rm R}_\mu$$

Thus the total number of fermions as well as the difference of left-handed and right-handed fermions are classically conserved.

In order to look whether these statements survive the quantisation of the theory, consider the matrix element

$$\langle 0|J^5_\mu|gg\rangle$$

g are the gauge photons (gluons) which couple to J_μ, not to J^5_μ. The corresponding lowest order Feynman diagram is a one-loop graph depicted in Fig. (4.5). We do not want to go into the details of the calculation, but rather sketch the Dirac matrix structure,

$$\Gamma_{\mu\alpha\beta}(k,p,q) \propto \mathrm{Tr}\,\gamma_\mu\gamma_5\frac{(\gamma,k_1)}{k_1^2}\gamma_\alpha\frac{(\gamma,k_2)}{k_2^2}\gamma_\beta\frac{(\gamma,k_3)}{k_3^2} \tag{4.11}$$

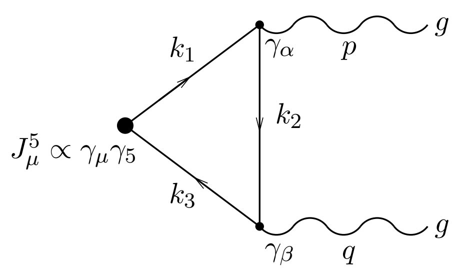


Figure 4.5: The lowest order Feynman graph leading to the chiral anomaly.

k_i and (p,q) are the momenta of the fermions and the gauge photons, respectively ($k+p+q=0$). The diagram is totally symmetric in the sense that in (4.11) we can put γ_5 also after γ_α or γ_β due to the anti-commutation relations. But the diagram is linearly divergent, and the infinity must be regularised. We prefer the introduction of Pauli-Villars mass terms, but the result will be independent of the regularisation method,

$$\Gamma^{\rm PV}_{\mu\alpha\beta}(k,p,q) \propto \mathrm{Tr}\,\gamma_\mu\gamma_5 \frac{M-i(\gamma,k_1)}{k_1^2+M^2}\gamma_\alpha \frac{M-i(\gamma,k_2)}{k_2^2+M^2}\gamma_\beta \frac{M-i(\gamma,k_3)}{k_3^2+M^2}$$

The symmetry is lost by renormalisation, namely *the finite part of the diagram will depend on where one puts* γ_5. The ambiguity in γ_5 is removed by the following choice,

$$\begin{aligned} p_\alpha\Gamma_{\mu\alpha\beta}(k,p,q) = q_\beta\Gamma_{\mu\alpha\beta}(k,p,q) &= 0 \\ k_\mu\Gamma_{\mu\alpha\beta}(k,p,q) \propto \epsilon_{\alpha\beta\gamma\delta}p_\gamma q_\delta &\neq 0 \end{aligned}$$

The gauge invariance due to the two gauge photons has survived, but J^5_μ is *not* conserved anymore,

$$\begin{aligned} \partial_\mu\langle 0|J_\mu(x)|gg\rangle &= 0 \\ \partial_\mu\langle 0|J^5_\mu(x)|gg\rangle &= \frac{g^2}{16\pi^2}\langle 0|F^a_{\mu\nu}\tilde{F}^a_{\mu\nu}|gg\rangle \end{aligned}$$

The last identity is the non-Abelian version of the Adler-Bell-Jackiw anomaly[3]. The topological density enters here, remember that $\int \mathrm{d}^4x F^a_{\mu\nu}\tilde{F}^a_{\mu\nu} = \frac{32\pi^2}{g^2}$ for an instanton. It effects the charges $Q_5 = \int \mathrm{d}^3x J^5_0(x)$ in the way that the charge 'after the instanton' (at $x_4 \to -\infty$) differs by two from the charge 'before the instanton' (at $x_4 \to +\infty$),

$$\int \mathrm{d}^4x\,\partial_\mu J^5_\mu = Q_5^{\rm after} - Q_5^{\rm before} = 2$$

One fermion has flipped its helicity from right to left. In other words, the instanton adds a left-handed particle and removes a right-handed anti-particle (the other way round for right-handed particles).

[3] The Adler-Bardeen theorem guaranteees that there are no effects in higher order pertubation theory.

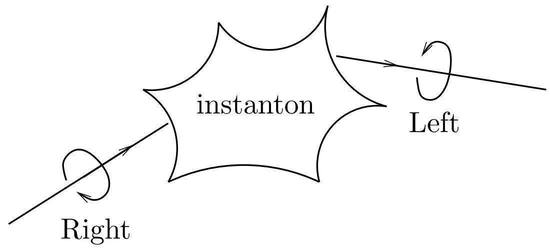


Figure 4.6: The (interaction with the) instanton flips the helicity of the fermion from right to left as shown in the text.

4.6 Jackiw-Rebbi States at an Instanton

How to understand the fact, that the interaction with an instanton flips the helicity of the fermion (Fig. 4.6)?

Let us investigate the gauge group $SU(2)$ with fundamental fermions ($I = 1/2$). As we know the spinorial group $SU(2)_{\mathrm{L}} \otimes SU(2)_{\mathrm{R}}$ couples to the gauge group $SU(2)_{\mathrm{L}}$,

$$SU(2)_{\mathrm{L}} \otimes (SU(2)_{\mathrm{L}} \otimes SU(2)_{\mathrm{R}})$$

For left-handed and right-handed fermions we have,

$$2_{\mathrm{L}} \times (2_{\mathrm{L}} \times 1_{\mathrm{R}}) = 3_{\mathrm{L}} + 1_{\mathrm{L}}, \quad 2_{\mathrm{L}} \times (1_{\mathrm{L}} \times 2_{\mathrm{R}}) = 2_{\mathrm{L}} \times 2_{\mathrm{R}}$$

respectively. There is one state with $j_{\mathrm{L}} = j_{\mathrm{R}} = 0$ which indeed has a normalisable solution in four-space,

$$\psi = \frac{\text{const.}}{(|x|^2 + \rho^2)^{3/2}}$$

This Jackiw-Rebbi state is a chiral eigenstate and fulfils the (Euclidean) Dirac equation

$$\gamma \mathrm{D} \psi = 0$$

Let us come back to the line $\vec{A}(\lambda, \vec{x})$ connecting two vacua in the gauge $A_4 = 0$ (Fig. 4.2(a)) and choose just x_4 as the parameter of the configuration,

$$\lambda \equiv x_4$$

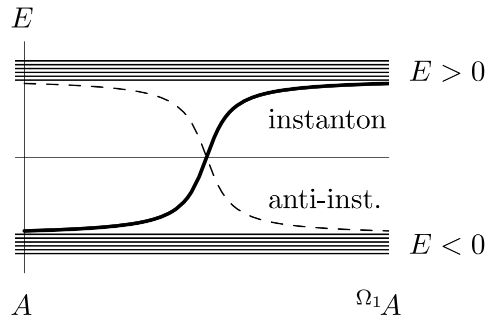


Figure 4.7: The instanton not only changes the winding number by 1, but also creates a (left-handed) particle and removes a (right-handed) anti-particle. For the anti-instanton it is vice versa. Infact the picture is delicate: In the original theory there is no mass and no gap, but we could add a small mass and the considerations still hold.

The operator $\frac{\partial}{\partial\lambda} \equiv \frac{\partial}{\partial x_4}$ enters the Dirac equation and extracts the energy,

$$(\gamma_4\partial_4 + \vec{\gamma}\vec{\mathrm{D}})\psi = 0 = (-\gamma_4 E + \vec{\gamma}\vec{\mathrm{D}})\psi$$

We represent its action on ψ by two functions of λ,

$$\begin{aligned} \frac{\partial}{\partial\lambda}\psi &= +\alpha(\lambda)\psi \quad \lambda \to -\infty \\ \frac{\partial}{\partial\lambda}\psi &= -\beta(\lambda)\psi \quad \lambda \to +\infty \end{aligned}$$

The signs follow from the general shape of normalisable modes. For the case at hand it has a power law behaviour: $\psi(\lambda, \vec{x}) \propto \frac{1}{\lambda^3}$. If we approximate it by an exponential law, α and β become constants and we arrive at the qualitative spectrum shown in Fig. 4.7: The instanton provides a transition from A to ${}^{\Omega_1}A$ during which the number of left-handed particles increases by 1, while the number of right-handed anti-particles drops by 1,

$$\triangle Q^{\mathrm{L}} = -\triangle Q^{\mathrm{R}} = 1$$

accordingly -1 for anti-instantons.

4.7 Estimate of the Flip Amplitude

How to calculate the amplitude for a process with $\triangle Q_5 = 2$? As a device for such a process we add the following term to the Lagrangian,

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} \tilde{F}^a_{\mu\nu} - \bar{\psi}\gamma \mathrm{D}\psi - J\bar{\psi}^{\mathrm{R}}(x)\psi^{\mathrm{L}}(x')$$

It simulates the flip from right to left resulting from an interaction with an instanton. We wrote x and x' allowing J to be non-local. The vacuum-to-vacuum amplitude is given by the usual path integral,

$$\int \mathrm{D}A_\mu \mathrm{D}\bar{\psi}\mathrm{D}\psi \exp[\, i \int \mathcal{L}_A - \bar{\psi}(\gamma\mathrm{D} + J)\psi]$$

The fermionic part gives the determinant of the operator $\gamma\mathrm{D} + J$, therefore we have to solve,

$$\gamma\mathrm{D}\psi(x') + J\psi(x) = \lambda\psi(x'), \qquad \lambda \neq 0$$

The Jackiw-Rebbi mode has $\gamma\mathrm{D}\psi = 0$ and thus $\lambda \propto J$.

Not only $\gamma\mathrm{D}$ but also the fluctuation operator of A_μ has zero modes. We expand around the instanton field,

$$A_\mu = A_\mu^{\mathrm{inst}} + \delta A_\mu$$

Since A_μ^{inst} is a classical solution the change in the action is second order,

$$\delta \int \mathcal{L} = \int \delta A_\mu M_{\mu\nu} \delta A_\nu$$

Zero modes of M are connected with the collective coordinates for the instanton. We already mentioned five of them, namely the width ρ and the position z_μ,

$$A^{\mathrm{inst}}(z_\mu + \delta z_\mu, \rho + \delta\rho) = A^{\mathrm{inst}} + \delta A \Rightarrow \delta \int \mathcal{L}_A = 0$$

There are also three gauge-collective coordinates for $SU(2)$, so in total we have 5+3=8 zero modes.

All other eigenstates of γD and M have non-vanishing eigenvalues λ_i. The path integral (in a semiclassical approximation for the gauge field) is the product,

$$\int \mathrm{D}\delta A_\mu \mathrm{D}\bar{\psi}\mathrm{D}\psi \propto \prod_i \lambda_i$$

If $J = 0$ then the amplitude vanishes, it would have been an amplitude with $\triangle Q_5 = 0$. The term linear in J has

$$\triangle Q_5 = \pm 2$$

and this amplitude is not equal to zero.

We now ask: which *effective* Lagrangian would mimic the instanton effects? It must be a Lagrangian which (a) gives an interaction with $\triangle Q_\mathrm{L} = -\triangle Q_\mathrm{R} = \frac{1}{2}\triangle Q_5 = N_f$ (the number of flavours) and (b) obeys the flavour symmetry $SU(N_f)^\mathrm{L} \times SU(N_f)^\mathrm{R} \times U(1)^\mathrm{vector}$ and violates $U(1)^\mathrm{axial}$.

For one flavour the instanton induces an effective mass term,

$$\triangle \mathcal{L}_\mathrm{eff} \propto e^{-8\pi^2/g^2}\bar{\psi}\psi, \qquad \bar{\psi}\psi \propto \bar{\psi}^\mathrm{R}\psi^\mathrm{L} + \bar{\psi}^\mathrm{L}\psi^\mathrm{R}$$

For N_f flavours $\partial_\mu J_\mu^{5\,ij}$ is diagonal in the flavour indices i and j,

$$\partial_\mu J_\mu^{5\,ij} \propto F_{\mu\nu}^a \tilde{F}_{\mu\nu}^a \delta^{ij}, \quad i, j = 1..N_f$$

but the effective instanton Lagrangian may contain Dirac indices. The original gauge Lagrangian has no mass and thus the following symmetry,

$$U(N_f)^\mathrm{L} \times U(N_f)^\mathrm{R} = SU(N_f)^\mathrm{L} \times SU(N_f)^\mathrm{R} \times \underbrace{U(1)^\mathrm{L} \times U(1)^\mathrm{R}}_{U(1)^\mathrm{V} \times U(1)^\mathrm{A}}$$

The instanton contribution violates $U(1)^\mathrm{A}$ only and the first approximation

$$e^{-8\pi^2/g^2}(\bar{\psi}_1\psi_1) \cdot \ldots \cdot (\bar{\psi}_{N_f}\psi_{N_f})$$

should better be replaced with

$$e^{-8\pi^2/g^2} \det_{ij}(\bar{\psi}_i\psi_j) \tag{4.12}$$

Even more precisely the effective Lagrangian is [7]

$$e^{-8\pi^2/g^2} \sum_{a_i, b_i, j_i} R(a_i, b_i) \epsilon^{j_1 \ldots j_n} (\bar{\psi}_1^{a_1}[1+\gamma_5]\psi_{j_1}^{b_1}) \cdot \ldots \cdot (\bar{\psi}_{N_f}^{a_{N_f}}[1+\gamma_5]\psi_{j_{N_f}}^{b_{N_f}})$$

4.8 Influence of the Instanton Angle

If we allow for θ, the effective Lagrangian includes a multiplication with $e^{i\theta}$ for the instanton ($e^{-i\theta}$ for the anti-instanton),

$$e^{-8\pi^2/g^2+i\theta}\det(\bar{\psi}_{\mathrm{R}}\psi_{\mathrm{L}})$$

To get the same effect on the level of the original Lagrangian, we have to add the topological/surface term from (4.8),

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} + \frac{g^2}{4\cdot 8\pi^2}i\theta F^a_{\mu\nu}\tilde{F}^a_{\mu\nu}$$

Then the instanton action becomes $\exp(-S)\to\exp(-S+i\theta\frac{g^2}{8\pi^2}\int\frac{1}{4}F\tilde{F})$ with $\int\frac{1}{4}F\tilde{F}=\frac{8\pi^2}{g^2}$ for an instanton (cf. (4.10)). In the preferred notation $gA_\mu =: \mathcal{A}_\mu$, $gF_{\mu\nu} =: \mathcal{F}_{\mu\nu}$, where D_μ is free of g, we have,

$$\mathcal{L} = -\frac{1}{g^2}\cdot\frac{1}{4}\mathcal{F}^a_{\mu\nu}\mathcal{F}^a_{\mu\nu} + \frac{i\theta}{8\pi^2}\cdot\frac{1}{4}\mathcal{F}^a_{\mu\nu}\tilde{\mathcal{F}}^a_{\mu\nu}$$

Pure gauge theory has two constants of nature, g and θ, both of which are in principle observable. Especially in SUSY theories they are combined as,

$$z = \frac{1}{g^2} + \frac{i\theta}{8\pi^2}$$

How to observe θ? Let us study the effect of θ on the *electric* charge of a magnetic monopole,

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} + \frac{i\theta g^2}{8\pi^2}F^a_{\mu\nu}\tilde{F}^a_{\mu\nu} = \frac{1}{2}(\vec{E}_a^2-\vec{B}_a^2) - \frac{\theta g^2}{8\pi^2}\vec{E}_a\cdot\vec{B}_a \\ &= \frac{1}{2}(\vec{E}_a - \frac{\theta g^2}{8\pi^2}\vec{B}_a)^2 - \frac{1}{2}(1+\frac{\theta^2 g^4}{(8\pi^2)^2})\vec{B}_a^2 = \frac{1}{2}\vec{E}_a'^{\,2} - \frac{1}{2}\vec{B}_a'^{\,2}\end{aligned}$$

θ *shifts* the electric field and changes the energy of the magnetic field (slightly). This can be interpreted as the energy of a background electric field, which gives small corrections to the mass.

Integrating the E' equation around a monopole gives a relation for the charges,

$$q_e' = q_e - \frac{\theta g^2}{8\pi^2}g_m, \quad q_e = 0, \quad g_m = \frac{4\pi}{g}$$

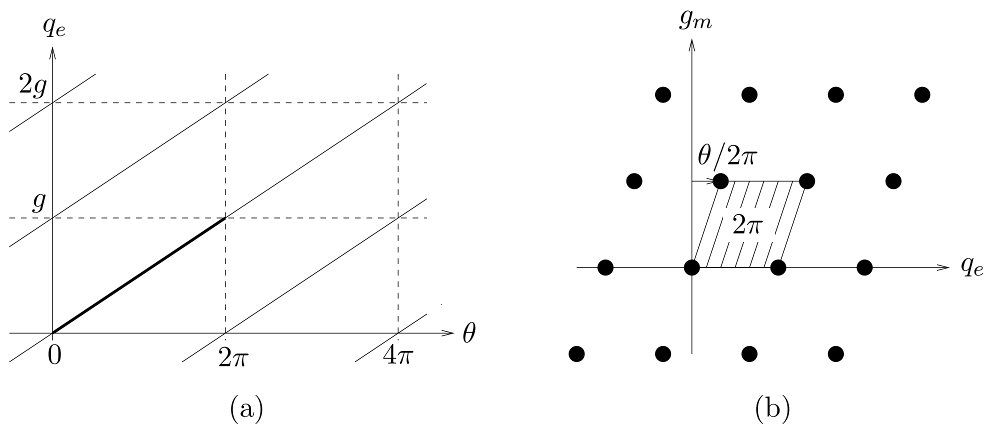


Figure 4.8: The figures show (a) the electric charge q_e of a monopole as a function of the instanton angle θ and (b) the possible values of electric and magnetic charges.

The monopole behaves like it has a *fractional electric* charge,

$$\boxed{q_e = \frac{\theta}{2\pi} g}$$

Notice that we were not allowed to shift the B field, since this would give magnetic charges to electric objects.

The function $q_e(\theta)$ for the monopole is plotted in Fig. 4.8(a). We already know that the monopoles could get an integer electric charge by binding to an electric particle, i.e. the original line in the figure has vertically shifted copies. In this way everything becomes periodic in θ as it should be (from its introduction via $e^{i\theta}$, cf (4.1)). It is helpful to plot the possible electric and magnetic charges like in Fig. 4.8(b).

Since the monopole now has an electric charge the Dirac condition has to be modified as well,

$$\boxed{q_e^1 g_m^2 - q_e^2 g_m^1 = 2\pi n_{12}} \qquad n_{12} \in \mathbb{Z}$$

In a theory with gauge group $U(1)^{N-1}$ like in the next chapter only the sum of all $U(1)$ charges is constrained,

$$\sum_{i=1}^{N-1} q_e^{1\,(i)} g_m^{2\,(i)} - q_e^{2\,(i)} g_m^{1\,(i)} = 2\pi n_{12} \qquad n_{12} \in \mathbb{Z}$$

Geometrically, the area of the elementary cell in Fig. 4.8(b) is 2π, independent of the value of θ.

Chapter 5

Permanent Quark Confinement

The phenomenological observation that quarks cannot be seen in isolation is called quark *confinement*. Although being a basic issue of the strong interaction theory, it remained a mystery before the 70's. The reason is that confinement is not a feature of perturbation theory. There one has to fix the gauge first, then quantise the theory and expand in a small coupling constant. Certain phenomenological aspects are described correctly, but there is no confinement.

To derive this new non-perturbative effect, one has to investigate the theory more precisely. As in the theory of electrodynamics there occur gauge fixing ambiguities when fixing the gauge. These so-called *Gribov ambiguities* may explain confinement.

Here we can give only a qualitative picture of the mechanism. We use a special partial gauge fixing, the *Abelian projection*, where the (local) gauge group is reduced to its (local) Abelian subgroup. Besides quarks and gauge photons the Abelian theory contains magnetic monopoles, which via a (dual) Meissner effect should confine all chromoelectric charges.

5.1 The Abelian Projection

In the following we deal with $SU(N)$ as the prototype of a non-Abelian gauge group. We do not focus on renormalisation (i.e. we do not attempt to account for infinities), because it does not have much to do with confinement.

The principle of the Abelian projection is to fix the gauge 'as locally as possible' (using the gauge field only). How is it done?

$\boxed{\mathbf{1}}$ In the first step we pick a field X (elementary or composite) in the adjoint representation of the gauge group[1]. Such a field is a $N \times N$-matrix which is hermitian,

$$X^\dagger = X$$

and usually also traceless. It transforms as

$$X \to {}^\Omega X = \Omega X \Omega^{-1} \tag{5.1}$$

The gauge field itself is ruled out because of the inhomogeneous term. But there are other candidates containing the field strength like

$$X^{ij} = G_{12}^{ij} \quad \text{or} \quad X^{ij} = (\mathrm{D}_\mu G_{\alpha\beta} \mathrm{D}_\mu G_{\alpha\beta})^{ij} \qquad i, j = 1 \dots N$$

the latter having the advantage of being Lorentz invariant, but it is more complicated. Adding a scalar to the theory in order to fix the gauge would be the easiest, but it would of course change the model.

As (5.1) does not involve derivatives of Ω nor Ω at different points, it is a *local* transformation, which will be important when considering the ghosts.

$\boxed{\mathbf{2}}$ In the next step we use the field X to fix the gauge (partially). We choose the gauge Ω in which X *is diagonal*. Then X is of the form

$$X = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix}$$

We further sort the eigenvalues,

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \tag{5.2}$$

This can always be done, but it does not fix Ω entirely. In technical terms we introduce Lagrange multipliers α for the off-diagonal components,

$$\mathcal{L}^{\text{gauge}} = \sum_{i<j} \alpha_{ij} X_{ij}$$

[1] A field in the fundamental representation would even be better, but in QCD there are only the quarks which as fermions are more difficult to treat.

$\boxed{\mathbf{3}}$ Observe that the previous step leaves a local $U(1)^N/U(1) = U(1)^{N-1}$ invariance. The gauge transformed X equals the original one if and only if X and Ω commute,

$$X = \Omega X \Omega^{-1} \quad \text{iff } [X, \Omega] = 0$$

For diagonal X this is true for (the group of) diagonal Ω's,

$$\begin{aligned} \Omega &= \begin{pmatrix} \omega_1 & & 0 \\ & \ddots & \\ 0 & & \omega_N \end{pmatrix} \qquad \text{with } \prod_i \omega_i = 1 && (5.3) \\ &= \exp i \begin{pmatrix} \Lambda_1 & & 0 \\ & \ddots & \\ 0 & & \Lambda_N \end{pmatrix} \qquad \text{with } \sum_i \Lambda_i = 0 && (5.4) \end{aligned}$$

Every element of the diagonal stands for one $U(1)$ but the condition $\det \Omega = 1$ traces out an overall $U(1)$. For $SU(N)$ the number $N-1$ is just the rank of the group, i.e. the dimension of the Cartan subalgebra.

The stability group of X is bigger if two eigenvalues coincide (see step 7), but generically this is not the case.

The remainder of the gauge group will be fixed in the next step.

$\boxed{\mathbf{4}}$ The residual Abelian gauge might be fixed just as in QED, for instance via the Lorentz gauge,

$$\mathcal{L}^{\text{gauge,Abelian}} = \sum_{i=1}^{N-1} \beta_i \, \partial_\mu A^\mu_{ii}$$

The total gauge fixing Lagrangian reads

$$\mathcal{L}^{\text{gauge}} = \sum_{i<j}^{N} \alpha_{ij} X_{ij} + \sum_{i=1}^{N-1} \beta_i \, \partial_\mu A^\mu_{ii}$$

We have introduced $2\frac{N(N-1)}{2} + N - 1$ real Lagrange multipliers which together give the dimension of the group.

Renormalisation is still to be done, so we expect highly singular Feynman rules.

$\boxed{\mathbf{5}}$ Fixing the gauge always includes a measure factor, the Faddeev Popov determinant, which usuallly is exponentiated with the help of *ghosts*. As we will now show they do not interact in Abelian Projections.

If the gauge is fixed by functions C_k of the fields their change under infinitesimal gauge transformations is

$$C_k \to C_k + m_{k\lambda}\Lambda^\lambda$$

Λ stands for the algebra element of the gauge transformations under consideration $\Omega = \exp(i\Lambda)$. In our case X_{ij} transforms according to the adjoint action of the group, which infinitesimally gives the commutator,

$$X_{ij} \to X_{ij} + i[\Lambda, X]_{ij}$$

$\partial_\mu A^\mu$ transforms like the gauge field itself, namely with the covariant derivative of the gauge parameter,

$$\partial_\mu A^\mu_{ii} \to \partial_\mu A^\mu_{ii} + \partial_\mu(\mathrm{D}^\mu\Lambda)_{ii}$$

The Faddeev Popov determinant is included by adding $-\bar{\eta}_k m_{k\lambda}\eta_\lambda$ to the Lagrangian. η are *anti-commuting* variables, but as *scalars* carry no spin. Violating the spin statistics theorem they are not observable, but integrating them out gives the right measure factor,

$$\int \mathrm{D}\eta\mathrm{D}\bar{\eta}\exp(-\int \mathrm{d}x\bar{\eta}_k m_{k\lambda}\eta_\lambda) \propto \det m$$

For the gauge fixing above we get

$$\begin{aligned}
\mathcal{L}^{\text{ghost}} &= i\sum_{i<j}\bar{\eta}_{ij}[\eta, X]_{ij} + \sum_i \bar{\eta}_{ii}\partial_\mu(\mathrm{D}_\mu\eta)_{ii} \\
&= i\sum_{i<j}\bar{\eta}_{ij}(\lambda_i - \lambda_j)\eta_{ij} + \sum_i \bar{\eta}_{ii}\partial_\mu^2\eta_{ii} - ig\sum_i \bar{\eta}_{ii}\partial_\mu(A_\mu^{ik}\eta_{ki} - A_\mu^{ki}\eta_{ik})
\end{aligned}$$

The third term transforms η only one-way, hence it does not contribute in loops. The second term corresponds to the free theory for the diagonal η's, while the first term is local in the off-diagonal η's, in other words these ghosts have infinite mass. In total, there are **no (harmful) ghosts**.

$\boxed{\mathbf{6}}$ Observe that we have all charcteristics of a $U(1)^{N-1}$ Abelian gauge theory. The residual gauge transformations Ω (5.3) transform A_μ as

$$\begin{aligned} (A_\mu)_{ii} &\rightarrow (A_\mu)_{ii} - \frac{1}{g}\partial_\mu \Lambda_{ii} \\ (A_\mu)_{ij} &\rightarrow \exp(i(\Lambda_i - \Lambda_j))(A_\mu)_{ij} \end{aligned}$$

The diagonal gauge fields $(A_\mu)_{ii}$ are as photons in ($N-1$ independent) QED ('s). All other fields carry $U(1)^{N-1}$ charges Q_i with $\sum_i Q_i = 0$. For the (ij)-component of A_μ we have $Q_i = 1$, $Q_j = -1$. In addition to the electrically charged quarks the theory contains now electrically charged 'off-diagonal photons'.

Moreover, these fields become massive. Since we removed the off-diagonal gauge symmetry, their masses are *not* protected by gauge invariance anymore,

$$\begin{aligned} (\mathrm{D}_\mu X)_{ij} &= \partial_\mu X_{ij} + ig(\lambda_i - \lambda_j)(A_\mu)_{ij} \\ \mathrm{Tr}\,(\mathrm{D}_\mu X)^2 &\rightarrow (\partial_\mu X)^2 + g^2(\lambda_i - \lambda_j)^2 (A_\mu^{ij})^2 \end{aligned}$$

But the theory is not exactly QED^{N-1}, something of the non-Abelian character has to survive.

$\boxed{\mathbf{7}}$ The gauge fixing may lead to *singularities* if $\lambda_i = \lambda_j$, which we argued away so far by handwaving. Near such a point the Higgs field looks like (from the ordering (5.2) it is clear that i and j are neighbours):

$$X = \begin{pmatrix} \ddots & & 0 \\ & \boxed{\begin{matrix} \lambda & 0 \\ 0 & \lambda \end{matrix}} & \\ 0 & & \ddots \end{pmatrix} + \sum_{k=1}^{3} a_k(x) \begin{pmatrix} \ddots & & \cdots \\ & \boxed{\sigma_k} & \\ \cdots & & \ddots \end{pmatrix}$$

We have used the parametrisation (2.9) for the $SU(2)$ subset. The first part is gauge invariant and the second part shall vanish when approaching some subspace,

$$x \rightarrow x_0 : \; a_k(x) \rightarrow 0 \quad k = 1, 2, 3$$

Generically these three conditions rule out three planes crossing at x_0. In three-space this fixes a point, while in four-space it is a (world) line.

Up to little deformations we can set $a_k(x) = (x - x_0)_k$ which is the hedgehog from Chapter 3. At x_0 the residual gauge group is enlarged from $U(1)^{N-1}$ to $U(1)^{N-3} \times U(2)$ and non-Abelian. These are

$$\boxed{\text{magnetic monopoles}}$$

w.r.t. that subgroup. Their magnetic charges are $g_i = (0, \dots, 0, 1, -1, 0, \dots, 0)$ and sum up to zero.

We conclude that in the Abelian projection we get electric and magnetic charges which are all point-like. Note that in a θ-vacuum (with instantons) the magnetic monopoles receive electric charges $\frac{\theta}{2\pi} g$ (cf. Fig. 4.8(b)).

For gauge group $SU(N)$ the Dirac condition reads

$$\sum_{i=1}^{N-1} g_i q_i = 2\pi$$

We expect complicated interactions among charged particles and magnetic monopoles. The latter will acquire a mass, since there is no reason for them to be massless. Hence electro-magnetism provides the only long-ranged fields, $N-1$ $U(1)$ photons. We can now ask what happens to these objects in a Higgs mechanism?

5.2 Phases of the Abelian Theory

In the usual Higgs mechanism $\langle \phi \rangle \neq 0$ the Higgs field ϕ is an 'ordinary' elementary field with electric charges only. This so-called **Higgs phase** is similar to the superconductor. All magnetic charges will be confined by Meissner flux tubes, see Fig. 5.1(a). We do not see weak magnetic monopoles. Pure electric charges can move freely, see Fig. 5.1(b).

The **confinement phase** can be thought of as the *dual transform* of the Higgs phase. The condensed field ϕ is now a magnetic object. Analogously all objects on the tilted line in Fig. 5.2 are free, while gluons are confined, since they are connected by N vortices. Quarks in the fundamental representation have electric charge $\frac{e}{N}$ and are connected by one vortex.

In the **Coulomb phase** no condensation takes place. Hence all charges are free, but there are long-ranged electromagnetic fields. This phase is *self-dual*.

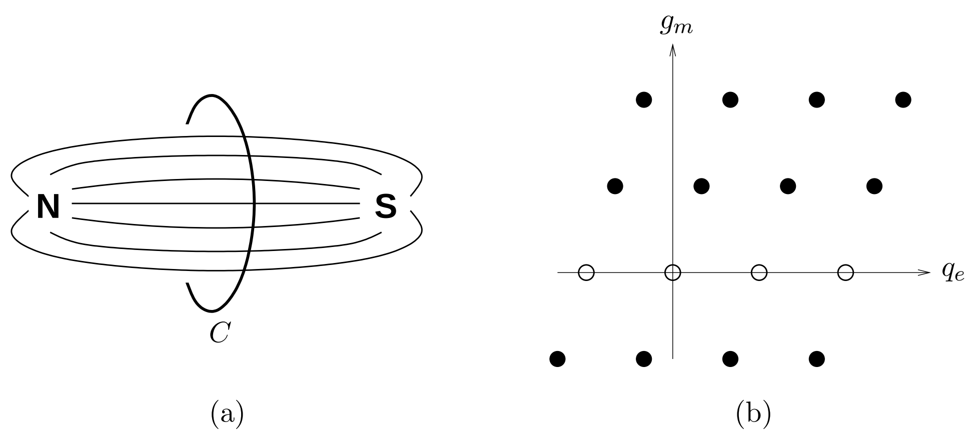


Figure 5.1: In the Higgs phase magnetic charges are connected by Meissner flux tubes. On the circle C the Higgs field makes a full rotation (a). In this phase all magnetic charges (•) are confined, while electric charges (∘) are free/screened (b). This and the following gq diagrams refer to the simplest case of $SU(2)_{\text{color}}$.

If we are in the confinement mode, then as θ runs from 0 to 2π there must be a phase transition: For small θ like in Fig. 5.2(a) the charges along θ are condensed, while those along $2\pi - \theta$ are confined. For θ near 2π it is energetically favourable to have charges condensed along $2\pi - \theta$ and confined along θ like in Fig. 5.2(b).

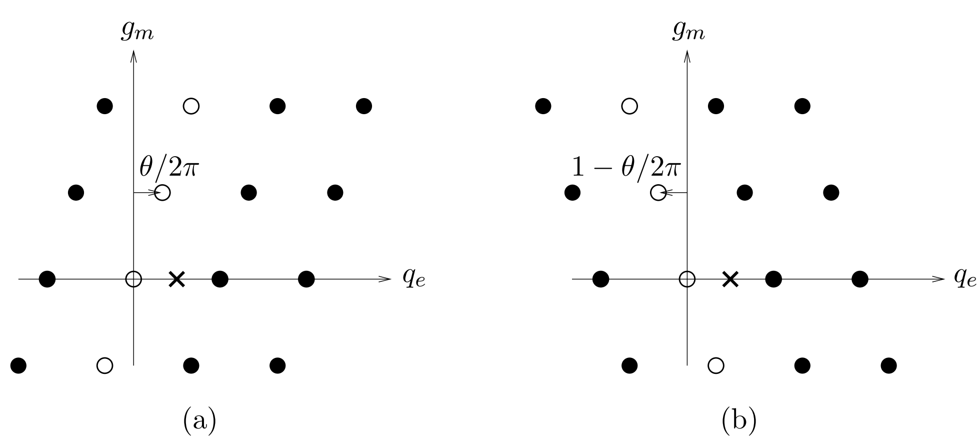


Figure 5.2: In the confinement phase all objects along a particular line in the charge lattice are free (∘). Now electric objects like gluons (•) and quarks (×) are confined by vortices. The diagrams show the confinement phase for small θ (a) and θ near 2π (b). The switch inbetween refers to a phase transition.

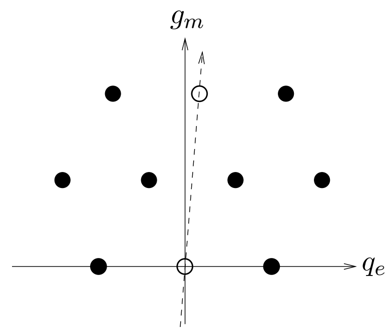


Figure 5.3: Oblique confinement removes the ambiguity for θ near π by chosing a third charge ($\circ$) to be condensed.

Note that physics is periodic in θ, hence *monopoles are not fundamentally distinguishable from dyons.*

The switch mentioned above refers to a phase transition, presumably at $\theta = \pi$. This phase transition is somewhat artificial, because in nature θ is a constant. It may be found in simulations, but the situation is hard to do on the lattice (complex action, instantons on the lattice).

We can imagine a more exotic condensation, *oblique confinement.* Let us have θ close to π. Now Buridan's donkey cannot decide which of the haystacks to choose. Fortunately there is a third choice in front of him (Fig. 5.3): Let the object inbetween (near g_m-axis, small electric charge) be free and the residual charges be confined. This phase will not occur in QCD because there we know that $\theta \simeq 0$, but it has peculiar features.

For some theories, the Higgs mode and the confinement mode are the same thing. As an example let us look at the electroweak theory in the next section.

5.3 A QCD-inspired Theory for the Electroweak Force

The weak interaction is described by the Weinberg-Salam model with gauge group

$$SU(2)_{\text{`color'}} \times U(1)_{\text{`em'}}$$

Besides the gauge fields

$$U(1)\text{-photon: } \gamma, \quad SU(2)\text{-gluon: } g^a$$

it has matter fields in the fundamental representation of $SU(2)$, i.e. color doublets. We call them 'partons' (the physical particles will come out later),

leptonic partons	$l_{i=1,2}$	(spin 1/2, charge $-1/2$)
quark partons	$q_{i=1,2}$	(spin 1/2, $SU(3)$-triplets, charge 1/6)
Higgs partons	$h_{i=1,2}$	(spin 0, charge $-1/2$)

The theory is not fundamentally different from QCD, but the (scalar) Higgs is really there.

This theory has 'mesons' (parton-antiparton), 'baryons' (two partons) and the usual photon γ. It comes out that the bound states without Higgs decay very quickly. The scalar[2] part of the bound states are identified with physical particles as follows

'Mesons'

$\bar{l}l, \bar{l}q, \bar{q}q$	unstable	
$\bar{h}l$	neutrino	(charge 0)
$\bar{h}q$	up-quark	(charge $+2/3$)
$\bar{h}h$	Higgs and Z^0 (orbital momentum)	(charge 0)

'Baryons'

ll, lq, qq	unstable	
hl	electron	(charge -1)
hq	down-quark	(charge $-1/3$)
hh	$W^\pm$	(charge -1)

The only difference from QCD is that one can do a nice *pertubation expansion* and recovers everything from the Standard Model. Like in Section 3.2 we fix the $SU(2)$ gauge first. In the unitary gauge we write the Higgs field h as

$$h = \begin{pmatrix} F + h_1 \\ 0 \end{pmatrix}$$

[2] w.r.t. the 'color' gauge group

Expanding in small fluctuations h_1 the bilinear mesonic field combinations create the up-components,

$$\begin{aligned}\bar{h}l &= Fl_1 + \ldots \\ \bar{h}q &= Fq_1 + \ldots \\ \bar{h}h &= F^2 + 2Fh_1 + \ldots\end{aligned}$$

while the bilinear baryonic field combinations create the down-components,

$$\begin{aligned}\epsilon^{ij}h_i l_j &= Fl_2 + \ldots \\ \epsilon^{ij}h_i q_j &= Fq_2 + \ldots\end{aligned}$$

By virtue of the covariant derivative we get the vector bosons,

$$\begin{aligned}\bar{h}\mathrm{D}_\mu h &= FgW_\mu^3 F + \ldots = gF^2 W_\mu^3 \\ \epsilon^{ij}h_i \mathrm{D}_\mu h_j &= Fg(W_\mu^1 + iW_\mu^2)F + \ldots = gF^2 W_\mu^-\end{aligned}$$

We recovered the physical doublets,

$$\begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = \begin{pmatrix} \nu \\ e \end{pmatrix}, \quad \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} u \\ d \end{pmatrix}, \quad \text{etc.}$$

and all the rest is just the ordinary electroweak theory. In this model physical particles are all confined, yet it trivially coincides with the perturbative sector of the Standard Model, so confinement is no longer a mystery at all.

5.4 Spontaneous Chiral Symmetry Breaking in QCD

Let us take $N_f = 2$ massless flavours (u and d quark). The composite operator $\bar{\psi}_{\mathrm{L}}^a \psi_{\mathrm{R}}^b = \phi^{ab}$ with colour indices a, b will be *coupled* to the agent of the Abelian projection. An effective coupling $\bar{\psi}_{\mathrm{L}} X \psi_{\mathrm{R}}+$ h.c. will produce a *constituent mass* (not algebraic mass) for the quarks.

But then each flavour has a Jackiw-Rebbi zero mode at the monopole singularities. Hence the monopoles have 'fractional' chiral flavour! Although there are no explicit monopole solutions yet, we expect a $2^2 = 4$-fold degeneracy for each monopole. This procedure would suggest that these four states transform as

$$1(\text{no JR mode}) + 2(\text{any flavour}) + 1(\text{both flavours})$$

under flavour $SU(2)$, see also Exercise (x).

But we have flavour $SU(2)_{\rm L} \times SU(2)_{\rm R}$. An alternative possibility then is to attach one Jackiw-Rebbi mode to every *left* quark,

$$\bar{2}_{\rm L} + 2_{\rm R} = 4$$

Then we have $2^4 = 16$ states which transform as follows

$$\begin{array}{ccccccccccc} 16 & = & 1 & + & \bar{2}_{\rm L} + 2_{\rm R} & + & 1 + \boxed{\bar{2}_{\rm L} \times 2_{\rm R}} + 1 & + & \bar{2}_{\rm L} + 2_{\rm R} & + & 1 \\ & & 1 & + & 4 & + & 6 & + & 4 & + & 1 \end{array}$$

If the $\bar{2}_{\rm L} \times 2_{\rm R}$ monopole condenses, i.e. gets a vacuum expectation value, we have spontaneous chiral symmetry breaking, as realised in nature.

Moreover, we know from lattice simulations, that the confinement-deconfinement phase transition and the chiral symmetry breaking take place at the same temperature. We have learned that different mechanisms are responsible for these effects, but both refer to the same objects: magnetic monopoles.

It is interesting to study these mechanisms in supersymmetric theories like the Seiberg-Witten model.

Chapter 6

Effective Lagrangians for Theories with Confinement

The effective mesonic fields ϕ_{ij} basically correspond to the quark-antiquark composite operators,

$$\phi_{ij} = -\bar{q}_j^{\mathrm{R}}\, q_i^{\mathrm{L}}$$

where $i, j = 1..N_f$ are the indices of the chiral $U(N_f)^{\mathrm{L}}$ and $U(N_f)^{\mathrm{R}}$, respectively [7]. This symmetry acts as

$$\phi'_{ij} = U_{ik}^{\mathrm{L}}\, \phi_{kl}\, U_{lj}^{\mathrm{R}\dagger}$$

Decomposing ϕ into its hermitean and anti-hermitean part we get the scalars. Let us study the mesonic spectrum in an effective theory. Its Lagrangian contains besides the kinetic term three potential terms,

$$\begin{aligned}
\mathcal{L}_{\text{eff}} &= -\mathrm{Tr}\, \partial_\mu \phi \partial_\mu \phi^\dagger - V(\phi, \phi^\dagger) \\
V(\phi, \phi^\dagger) &= V_0 + V_m + V_{\text{inst.}}
\end{aligned}$$

which are the potential for spontaneous symmetry breaking, the contribution of the quark masses and instanton contribution from (4.12), respectively,

$$\begin{aligned}
V_0 &= -\mu^2\, \mathrm{Tr}\, \phi^\dagger \phi + A\, (\mathrm{Tr}\, \phi^\dagger \phi)^2 + B\, \mathrm{Tr}\, (\phi^\dagger \phi \phi^\dagger \phi) \\
V_m &= -\sum_i m_i (\phi_{ii} + \phi_{ii}^\star) \\
V_{\text{inst.}} &= -2\kappa\, \mathrm{Re}(e^{i\theta} \det \phi)
\end{aligned}$$

where A, B and μ^2 are free parameters ($\mu^2 > 0$ for spontaneous symmetry breaking), $m_1 = m_u$, $m_2 = m_d$, $m_3 = m_s$, ... are the quark masses and κ contains the standard factor $e^{-8\pi^2/g^2}$. V_0 preserves the $U(N_f)^{\mathrm{L}} \times U(N_f)^{\mathrm{R}}$ symmetry, while V_m and $V_{\mathrm{inst.}}$ break it down to $U(N_f)$ and $SU(N_f)^{\mathrm{L}} \times SU(N_f)^{\mathrm{R}} \times U(1)^{\mathrm{V}}$, respectively.

Let us study the case of **3 flavours** with $m_1 \simeq m_2 \ll m_3$ and keep $\theta = 0$ for simplicity [8]. We expand the 3×3 matrix ϕ around its vacuum expectation values F_i,

$$\phi = \begin{pmatrix} F_1 & & \\ & F_2 & \\ & & F_3 \end{pmatrix} + \tilde{\phi}$$

where the quark masses make the F_i different. Scalar and pseudoscalar particles can be identified by decomposing the fluctuations $\tilde{\phi}$,

$$\tilde{\phi}_{ij} = S_{ij} + iPij = -\frac{1}{2}\bar{q}_j(\mathbb{1} + \gamma_5)q_i, \quad S = S^\dagger,\ P = P^\dagger$$

By construction $\mathcal{L}$ is quadratic in $\tilde{\phi}$,

$$\mathcal{L}(F + \tilde{\phi}) = \mathcal{L}(F) + 0 \cdot \tilde{\phi} + \mathcal{L}_2(S) + \mathcal{L}_2'(P)$$

All scalars acquire masses via V_0 in $\mathcal{L}_2(S)$, the pseudoscalar part becomes,

$$V_2'(P) \propto \kappa F_1 F_2 F_3 \left(\frac{P_{11}}{F_1} + \frac{P_{22}}{F_2} + \frac{P_{33}}{F_3} \right)^2 + \sum_i \frac{m_i}{F_i} P_{ii}^2 + \sum_{i \neq j} \ldots |P_{ij}|^2$$

The instanton effect $\propto \kappa$ produces a mass in the pseudoscalar sector: η, η'. The other diagonal pseudoscalars, π and K, get $M^2 \propto (m_1 + m_2)$, the off-diagonal ones carry masses anyway (through symmetry breaking).

Generalizing the model to **6 flavours**, we make a further simplification,

$$V_0(\phi, \phi^\dagger) \to \delta |\phi\phi^\dagger - \mathbb{1}|^2, \qquad \phi \text{ unitary}$$

Now all $|F_i|^2$ equal 1, and fluctuations of the modulus of ϕ cost infinite energy. Since the scalars seem not to play a role, their masses were sent to infinity. The chiral symmetry,

$$\phi' = U^{\mathrm{L}}\, \phi\, U^{\mathrm{R}\dagger}$$

is unchanged. For the other two potential terms we take the straightforward generalisation of the $N_f = 3$ model,

$$\begin{aligned} V_m &= -\mathrm{Tr}\, m_{ij}\phi_{ji} + \text{h.c.} \\ m_{ij} &= \mathrm{diag}\,(m_u, m_d, m_s, m_c, \dots) \\ V_{\text{inst.}} &= -\kappa \det \phi + \text{h.c.} \end{aligned}$$

Note that the instanton angle may be absorbed in one of the masses,

$$m_u \to m_u e^{i\theta}$$

Expanding ϕ now means

$$\phi = e^{iP}, \qquad P = P^\dagger$$

The second order terms in P are

$$\begin{aligned} V_2'(P) &\propto -\mathrm{Tr}\, m(\mathbb{1} + iP - \frac{1}{2}P^2 + \dots) + \text{h.c.} - \kappa \exp(i\mathrm{Tr}\, P) + \text{h.c.} \\ &\to -2\,\mathrm{Tr}\,(m\cos P) - 2\kappa\cos(\mathrm{Tr}\, P) \\ &\to \mathrm{Tr}\,(mP^2) + \kappa(\mathrm{Tr}\, P)^2 \\ &= \frac{1}{2}\sum_{ij} |P_{ij}|^2(m_i + m_j) + \kappa(P_{11} + P_{22} + \dots + P_{N_f N_f})^2 \end{aligned}$$

These expressions refer to two main phenomenological observations. The first fact is that the meson masses squared are approximately linearly proportional to the quark masses,

$$M^2(\bar{q}_i\gamma_5 q_j) = \text{const}\,(m_i + m_j)$$

From the light pion one now concludes that the up and down quarks are light, too: $m_u \simeq m_d \propto m_\pi^2 \ll \Lambda_{\text{QCD}}$.

The instanton produces a mass (only) for the N_f pseudosinglet η'. This explains the exception to the rule above, namely the mystery of the η and η' masses. The Chern-Simons current K_μ (cf (4.9)) is still conserved, hence its Goldstone boson η could not carry a mass. But K_μ is not gauge-invariant, it is rather like a ghost. Hence K_μ does not protect η' from getting a mass. Indeed, instantons are the only stable onjects with a non-vanishing value of the integral (4.8) and they contribute to the η' mass.

Chapter 7

Exercises

(i) Derive a Bogomol'nyi bound for the kink solution by writing

$$V(\phi) = \frac{1}{2}\left(\frac{\partial W(\phi)}{\partial \phi}\right)^2 , \quad \mathcal{H} = \frac{1}{2}\left(\partial_x \phi + \frac{\partial W(\phi)}{\partial \phi}\right)^2 + \text{ total derivative.}$$

Discuss the function $W(\phi)$ and its extrema for the cases (a) and (b). Can the bound always be saturated?

(ii) In case (a) a soliton and an anti-soliton approach each other. Assume an approximate solution describing a stationary situation.

Why can this 'solution' not be exact?

Show that the two Jackiw-Rebbi zero modes for fermions now mix, and that their energies will no longer vanish. Estimate the amount to which the energy levels split. Discuss the energy spectrum of the two soliton system as a result of this effect.

(iii) Consider an $SU(2)$ gauge theory with an $I = 1$ Higgs field $\phi^{ab} = \phi^{ba}$, $\sum_a \phi^{aa} = 0$. Assume the potential such that

$$\langle \phi^{ab} \rangle_0 = F \begin{pmatrix} 1 & & \\ & 2 & \\ & & -3 \end{pmatrix}$$

Find the (discrete) subgroup of $SU(2)$ that leaves this expectation value invariant.

Hint: First write these elements as $SO(3)$ matrices, then, by exponentiation, as $SU(2)$ matrices.

Show that this is a non-Abelian group. Discuss the fusion rules for the vortices that may occur in this system.

(iv) Consider the three-sphere $\sum_{\mu=1}^{4} x_\mu^2 = 1$ and the $SU(2)$ matrices

$$U(\vec{x}) = x_4 1\!\!1 + i\sum_{a=1}^{3} x_a\sigma_a \qquad x_\mu \text{ real, } \sigma_a \text{ Pauli matrices}$$

Find a gauge transformation such that ${}^\Omega U(\vec{x}) = \Omega U \Omega^{-1}$ is diagonal, ${}^\Omega U = \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix}$. Study the point $\vec{x}^*$ where your $\Omega(\vec{x})$ is singular, and show that one cannot avoid that there is at least one such point on S^3.

(v) In the 1+1 dimensional model case (a) with $V(\phi) = \frac{\lambda}{4!}\left(\phi^2 - F^2\right)^2$ construct an operator $\Omega(x)$ such that

$$\Omega(x_1)\phi(x_2) = \phi(x_2)\Omega(x_1)(-1)^{\theta(x_2-x_1)} \quad \text{if } |x_2 - x_1| > \epsilon > 0$$

Show that $\Omega(x)$ is the operator field that creates or annihilates a soliton at x (or at least some sort of kink).

Find an approximate algorithm (or prescription) to compute the propagator $\langle \mathrm{T}\,\Omega(x_1,\, t_1)\Omega(x_2,\, t_2)\rangle_0$ in *Euclidean* space-time. Find its behaviour at large $|(x_2,\, t_2) - (x_1,\, t_1)|$.

Note: We must assume that a local observer can observe $|\phi(x)|$, but not the *sign* of $\phi(x)$.

(vi) The magnetic monopole mass reads after the Bogomol'nyi trick (3.6)

$$E = \int \mathrm{d}^3x \left[\frac{1}{2}(\vec{\mathrm{D}}\phi_a \pm \vec{B}_a)^2 + \frac{\lambda}{8}(\phi_a^2 - F^2)^2\right] + \frac{4\pi}{e}F$$

Write down the usual ansätze (3.3), (3.4) for the fields,

$$\phi^a(x) = \hat{x}^a\phi(|\vec{x}|), \quad A_i^a(x) = \epsilon_{iaj}\hat{x}_j A(|\vec{x}|), \quad |\vec{x}| = \sqrt{\sum_{i=1}^{3} x_i^2}$$

Find the energy in terms of $\phi(|\vec{x}|)$, $A(|\vec{x}|)$, the field equations and boundary conditions at $|\vec{x}| = 0$, $|\vec{x}| \to \infty$ for these functions.

Give the Bogomol'nyi equations for $\phi(|\vec{x}|)$ and $A(|\vec{x}|)$.

(vii) Consider the candidate instanton solution $A^a_\mu(x) = \eta^a_{\mu\nu} x^\nu A(|x|)$, $|x| = \sum_{\mu=1}^4 x_\mu^2$, where $A(|x|)$ is an arbitrary function of $|x|$. Compute $G^a_{\mu\nu}$ and find the equation for $A(|x|)$ corresponding to

$$\text{(a) } G_{\mu\nu} = \tilde{G}_{\mu\nu} \qquad \text{(b) } G_{\mu\nu} = -\tilde{G}_{\mu\nu}$$

Use

$$\begin{aligned} \epsilon_{abc}\eta^b_{\mu\nu}\eta^c_{\kappa\lambda} &= \delta_{\mu\kappa}\eta^a_{\nu\lambda} - \delta_{\mu\lambda}\eta^a_{\nu\kappa} - \delta_{\nu\kappa}\eta^a_{\mu\lambda} + \delta_{\nu\lambda}\eta^a_{\mu\kappa} \\ \eta^a_{\mu\nu}\eta^b_{\mu\lambda} &= \delta^{ab}\delta_{\nu\kappa} + \epsilon_{abc}\eta^c_{\nu\kappa} \end{aligned}$$

Hint: In case (b) one has $\eta^a_{\mu\nu} G^b_{\mu\nu} = 0$.

Solve these equations.

The two solutions look entirely different: (a) is an instanton, while (b) is an anti-instanton. Explain the situation.

(viii) Consider the real scalar field $\phi(x)$ with $\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)^2 + \frac{\lambda}{4!}\phi^4$, such that λ has the 'wrong sign'.

Show that there is an instanton solution in Euclidean four-space of the form $\phi = \phi(|x|)$, $\phi(|x| \to \infty) \to 0$.

Find the action and give a physical interpretation of this event and the quantity $e^{S_{\text{inst.}}}$.

(ix) Consider a theory with N kinds of Maxwell fields: $G = U(1)^N$. Let there exist objects p with electric charges $(e_1, \ldots, e_N)_p$ and magnetic charges $(g_1^m, \ldots, g_N^m)_p$.

Write down the Dirac condition for two such objects p and q.

(x) Let there be a monopole coupled to three fermion species ψ_1, ψ_2, ψ_3 which are 3-representations of a *global* $SU(3)$-symmetry. Each of them has *one* Jackiw-Rebbi zero mode solution with the monopole. Suppose that the 'completely empty' monopole is an $SU(3)$-singlet. There is an $2^3 = 8$-fold degeneracy.

How do the other states transform under $SU(3)$?

Bibliography

[1] R. Jackiw, C. Rebbi, Solitons with fermion number 1/2, Phys. Rev. D13 (1976) 3398.

[2] G. H. Derrick,J. Math. Phys. 5 (1964) 1252.

[3] H.B. Nielsen, P. Olesen, Vortex-Line Models for Dual Strings, Nucl. Phys. B61 (1973) 45.

[4] B. Julia, A. Zee, Phys. Rev. D11 (1975) 2227.

[5] G. 't Hooft, P. Hasenfratz, Phys. Rev. Lett. 36 (1976) 1119.

[6] A.F. Goldhaber, Phys. Rev. Lett. 36 (1976) 1122.

[7] G. 't Hooft, How instantons solve the $U(1)$ problem, Phys. Rep. 142, 6 (1986) 357.

[8] G. 't Hooft, The Physics of Instantons and Vector Mesons Mixing, hep-th/9903189.