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LATTICE REGULARIZATION OF GAUGE THEORIES WITHOUT LOSS OF CHIRAL SYMMETRY

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Abstract

A lattice regularization procedure for gauge theories is proposed in which fermions are given a special treatment such that all chiral flavor symmetries that are free of Adler-Bell-Jackiw anomalies are kept intact. There is no doubling of fermionic degrees of freedom. A price paid for this feature is that the number of fermionic degrees of freedom per unit cell is still infinite, although finiteness of the complete functional integrals can be proven (details are outlined in an Appendix). Therefore, although perhaps of limited usefulness for numerical simulations, our scheme can be applied for studying aspects such as analytic convergence questions, spontaneous symmetry breakdown and baryon number violation in non-Abelian gauge theories.

1. Introduction.

Conservation of chiral symmetries in lattice gauge theories is often considered to be an important problem. One would like to study for instance the spontaneous breakdown of chiral $SU(2) \times SU(2)$ in QCD but this study is complicated by the fact that in the Wilson action [1] such symmetries are also explicitly broken [2]. Remedies of this problem were often sought in alternative versions of this action[3], but then it was found that either other subgroups of $SU(2) \times SU(2)$ were sacrificed or else extra fermionic degrees of freedom emerged as unwanted modes of the fermionic action.

The origin of this difficulty was not difficult to identify. Chiral symmetries are offset by anomalies so that the actual symmetries of the system are smaller than the apparent symmetries of the Lagrangian. Hence, the fact that the symmetries have to be smaller than that of the unregularized Lagrangian should not be a surprise. The problem that remains however is that the chiral symmetries are broken more than necessary. The anomaly only removes the chiral $U(1)$ part of this symmetry whereas most lattice actions remove the chiral part of $U(2) \times U(2)$ altogether.

No-go theorems were then formulated [4] stipulating that this is a nuisance that will stay with us forever. But these theorems used certain assumptions concerning the fermionic part of the action, and, as is often the case with such theorems, its conclusion is no worse than the assumption. All that has to be done is be more imaginative in the fermion sector.

In this paper, our essential trick consists of observing that the fermions need to know how the instanton winding numbers are defined on the lattice plaquettes and hypercubes. If one would add a tiny instanton somewhere within one hypercube this would add a fermionic zero mode, or equivalently, an other chiral fermionic degree of freedom inside that hypercube. We will fix these winding numbers simply by extending the gauge fields, originally only defined on the lattice links, in the smoothest possible way to all space-time points within as well as on the edges of each hypercube. This procedure has no effect on the gauge degrees of freedom, which will always be represented just by the connections on the links, but it will affect the interactions with the fermions.

The fermionic part of the action is now defined by having the fermions live in this continuous space-time. We then regularize the fermions separately, using for instance a Pauli-Villars procedure (although any other regularization procedure with the appropriate adjustments of the finite subtraction terms will do). What makes this procedure attractive is that the fermionic functional integral is merely a functional determinant which in principle is easy to compute. The prescription is thus to regularize this determinant first. An apparent difficulty is that the fermions live in a “background gauge field” (which is generated by the gauge link variables $U_\mu(x)$ and is to be integrated over later), and this

field may be fairly strong. We now *prove* that in spite of this the regularization procedure is completely convergent for all U_μ configurations. This proof is outlined in the Appendix.

2. The gauge field in between lattice sites.

The gauge field part of our lattice theory is as usual. On a cubic lattice the gauge field degrees of freedom are chosen to be elements $U_\mu(x)$ of the gauge group G , situated on the unit links connecting the space-time points x and $x + e_\mu$. The gauge field action is also chosen to be the usual thing:

$$S_{YM} = \frac{1}{g^2} \sum_{x,\mu,\nu} \text{tr} \left(U_\mu(x) U_\nu(x + e_\mu) U_\mu^\dagger(x + e_\nu) U_\nu^\dagger(x) \right). \quad (1)$$

Now before being able to introduce the fermion fields we must extend our definition of the gauge field to all other space-time points. Thus, although in Eq. (1) the points x are integers denoting the lattice sites, we now choose points $x \in \mathbb{R}^4$ to define a vector connection field $A_\mu(x)$ in all these points.

First we define $A_\mu(x)$ for the points x on the lattice links (x, μ) , simply by

$$U_\mu(x) \equiv e^{iaA_\mu(x)}, \quad (2)$$

where a is the lattice length, and A_μ is given as an element of the Lie algebra of G . It will turn out to be important to observe that Eq. (2) only defines $A_\mu(x)$ uniquely if we insist that the eigenvalues of $aA_\mu(x)$ lie in the interval $(-\pi, \pi]$. We will call these elements of the Lie algebra *minimal*.

Next, the gauge field on the elementary plaquettes (2-simplexes) can also be defined in a straightforward manner. For a given plaquette (x, μ, ν) first pick the gauge in which

$$U_\mu(x) = U_\nu(x) = U_\mu(x + e_\nu) = \mathbb{I}, \quad U_\nu(x + e_\mu) = e^{iaA^1}, \quad (3)$$

where again aA^1 is chosen to be minimal. Then in that gauge

$$A_\nu(x + \alpha e_\mu + \beta e_\nu) \equiv \frac{\alpha}{a} A^1, \quad A_\mu(x + \alpha e_\mu + \beta e_\nu) \equiv 0, \quad \alpha, \beta \in [0, a]. \quad (4)$$

With this choice the gauge field connection on the plaquette is constant:

$$F_{\mu\nu}(x + \alpha e_\mu + \beta e_\nu) = \frac{A^1}{a}, \quad (5)$$

and it is not hard to verify that gauge transformations Ω of the form

$$\Omega(x + \alpha e_\mu + \beta e_\nu) = e^{if(\alpha,\beta)A^1} \quad (6)$$

enable us to permute the four links of the plaquette in the gauge condition (3). Thus the prescription (4) (including the minimality condition) is a symmetric one.

Another way to see that this prescription is symmetric is by observing that it obeys the field equation in 2-space:

$$\sum_{i \text{ in plaquette 2-space}} D_i F_{ij}(x) = 0, \quad (7)$$

where D_i stands for the covariant derivative, and it is the unique solution to the requirement that the 2-dimensional action on the plaquette, $\int d^2x \sum_{i,j \in \text{plaquette}} F_{ij} F_{ij}$ has the smallest possible value, given the boundary conditions (3) on the surrounding links.

Extending the gauge field inside a cube (3-simplex) T^3 is slightly less straightforward. There are in principle many possibilities, and the exact choice made is of lesser importance apart from one crucial condition: we must again impose a minimality condition on the one hand, and on the other we insist that the fields be connected continuously. A good example is the following prescription:

Choose the fields in a three-simplex to obey the sourceless field equations of that three space:

$$\sum_{a \text{ in space of 3-simplex}} D_a F_{ab} = 0, \quad (8)$$

and of all possible solutions choose the one that minimises the Euclidean value of $\int d^3x \sum_{a,b \text{ in space of 3-simplex}} F_{ab} F_{ab}$, under the boundary condition (4) on the surrounding plaquettes.

This is the most natural generalisation for the 3-simplex of our prescription on the 2-simplex, and now also the field continuation onto the 4-simplexes T^4 is evident:

Choose the fields in a four-simplex to obey the usual sourceless field equations

$$\sum_{\mu=1}^4 D_\mu F_{\mu\nu} = 0, \quad (9)$$

and again of all possible solutions the one that minimises the Euclidean value of $\int d^4x \sum_{\mu\nu} F_{\mu\nu} F_{\mu\nu}$, with the fields constructed earlier on the surrounding 3-simplexes as boundary conditions.

Indeed the conditions (7), (8) and (9) ensure that our gauge fields are extended in the smoothest possible way to all space-time points. Of course the derivatives of (some of) the field components are discontinuous on the lattice links, plaquettes and 3-simplexes, but

not more than needed to obey the other requirement, *i.e.* that the lattice link variables $U_\mu(x)$ as independent integration variables in the Euclidean lattice path integral coincide with $\exp(\int_{\xi \in \text{link}(x,\mu)} iA_\mu(\xi)d\xi^\mu)$. Therefore (7), (8) and (9) do not hold on the lower-dimensional edges where they are corrected by Dirac delta distributions.

Note that the exercise carried out in this section has no effect on the pure gauge field action which is still given by Eq. (1), or on the pure gauge field functional integral. The only reason for this exercise is that it will be needed for the fermionic part of the gauge theory.

3. Fermions.

The fermions in our theory are sensitive to the gauge field topology. This is why the Wilson action for the fermions gives problems. The conventional lattice formulation leaves holes between the lattice sites and so the topological winding numbers are ill-defined. In our present formalism this problem is cured. We fixed the topology by specifying what our gauge fields are between the lattice points and links. Our minimality condition in eqs. (7), (8) and (9) implies that usually the topological winding numbers there are kept at a minimum. This does not imply that these winding numbers vanish. If we integrate over several lattice sites we can easily find appreciable cumulation of topological winding numbers, depending on the values of the gauge field integrands $U_\mu(x)$. There will be monopoles and instantons, but their sizes must be larger than the lattice lengths a .

We have not found a discrete lattice version of the fermionic part of the action, such that fermionic degrees of freedom are as discrete as the gauge field variables. The point however that we wish to stress in this paper is that such a discrete fermionic action is not at all necessary for our theory to be properly and completely regularized. We simply propose to keep as our fermionic action

$$S_{\text{fermions}} = - \int d^4x \bar{\psi}(x) (\gamma_\mu (\partial_\mu + iA_\mu(x)) + m) \psi(x), \quad (10)$$

where now $A_\mu(x)$ stands for the extended gauge field constructed in the previous section.

Of course this fermionic action still suffers from an ultraviolet infinity. However, it is merely a one-loop infinity that is quite easy to regularize separately. Just introduce the familiar Pauli-Villars regulator fields, *i.e.* massive spinor fields with masses Λ_i and weights $e_i = \pm 1$. The $i = 0$ component is the physical field with $\Lambda_0 = m$ and $e_0 = 1$, where m may or may not be zero, as dictated by whatever chiral symmetry one would like to impose. Of the other fields the ones with $e_i = -1$ have indefinite metric, but this does not matter

since we let all $\Lambda_i \rightarrow \infty$. Throughout the limiting process one imposes the identities

$$\sum_i e_i \equiv \sum_i e_i \Lambda_i \equiv \sum_i e_i \Lambda_i^2 \dots \equiv 0, \quad (11)$$

where usually no higher powers than the fourth are needed for all diagrams to converge. Furthermore

$$\sum_i e_i \log \Lambda_i \equiv \log \Lambda; \quad \sum_i e_i \Lambda^n \log \Lambda_i \equiv 0, \quad 0 < n \leq 4. \quad (12)$$

If Λ in here would be kept finite then for all field configurations $A_\mu(x)$ the regularized fermionic determinant,

$$\prod_i \left(\det(\gamma D + \Lambda_i) \right)^{e_i}, \quad (13)$$

is known to be absolutely finite *in perturbation theory*. In particular it is well known that in the limit where the Λ_i are sent to infinity all chiral symmetries are restored with the exception of the ones that suffer explicitly from an anomaly. In our appendix we show that the same statements are true if we calculate the regularized fermionic determinant *exactly* by multiplying the eigenvalues of the Dirac operator. The limit $\Lambda_i \rightarrow \infty$ is finite.

4. The lattice gauge action. Conclusion.

From the above it should now be clear how to prescribe rigorously a gauge theory free from infinities and with all anomaly-free chiral symmetries preserved. The combined action from Eqs (1) and (10), $S_{YM} + S_{\text{fermions}}$ is *first* integrated over the continuous fermion variables using the Pauli-Villars regulators. Then we send $\Lambda_i \rightarrow \infty$. This yields a finite fermionic determinant, which however still depends on the link variables $U_\mu(x)$. The remaining integrand depends only on these variables, which form a finite-dimensional space so that standard approximation techniques may be applied to integrate over these. It is important that before integrating over them the Λ_i of the regulator fields must have been sent to infinity so that we are ensured of the required chiral symmetries.

It is important to realise that the extrapolated fields $A_\mu(x)$ play an essential role in this construction. To see this, take a field configuration where the U variables are all close to the identity, but still form an instanton configuration, spread over many lattice sites. Of course such a configuration is allowed, because instantons scaled to large sizes have only relatively weak field values at any specific lattice site. This configuration generates a fermionic zero mode so that the determinant (13) vanishes unless an external source is added. Now scale the instanton gently towards smaller sizes. The U fields will become stronger, but by the time the instanton slips between the confines of only one lattice

hypercube the U variables will return towards small values again. Halfway this process, the topological winding number must suddenly have disappeared. Thus, although the U fields changed continuously towards their new values, the fermionic determinant must have made a jump. This is because the extrapolated fields $A_\mu(x)$ made a jump! The discontinuity occurred when two solutions of the equation (9) emerged for which the values of the action $\int d^4x F_{\mu\nu} F_{\mu\nu}$ coincided. The extra requirement that we choose the solution for which this Euclidean action is minimal forced us to jump from one solution to the other.

The fact that instantons and other topological features of our lattice theory are constrained to be larger than the lattice length a needs not bother us. It is a lattice artifact that does not affect any of the important symmetries, and compares well with the actual physical situation where small instantons are suppressed because of asymptotic freedom.

Whether this prescription can be used in practical lattice calculations is quite a different question. A perturbative calculation of the logarithm of the determinant (in terms of one-loop diagrams, as one usually does) will often not converge because of vanishing eigenvalues. The determinant itself will converge in principle, but maybe only at very high orders. Thus for large U fields this calculation may well become too cumbersome in practice for fast Monte-Carlo simulations. One may ask whether the determinant can be approximated by a discrete determinant and the Pauli-Villars regulators may perhaps be replaced by some lattice scheme with a similar effect. We have to remember however that the number of fermionic modes may vary discontinuously as a function of the U fields, and this will pose problems in practice. But the fundamental question whether gauge theories can be made absolutely finite while keeping maximal (*i.e.* the anomaly free part of the) chiral symmetry has been answered. The approach advocated in this paper may well be useful for numerical approaches towards understanding the baryon number violation phenomenon within the Standard model, since here it is essential that the anomaly must be the *only* cause for a minute symmetry breakdown, and the question how effective this symmetry breaking will be at extremely high energies has not yet been answered convincingly.

There is a relationship between the regularization proposed here and a continuum regularization procedure proposed by Slavnov and Faddeev [5]. These authors propose to regularize the continuum gauge theory by the addition of extra derivatives in the Lagrangian. This has the effect that all irreducible Feynman diagrams with more than one closed loop are made convergent, except for the one-loop subdivergences, which must be regularized first by some other method. One-loop divergences are easy to regularize, although in their case there are also one-loop graphs with gauge photons going around the loops and then a simple-minded Pauli-Villars trick leads to incorrect results. There one

can use for instance the 5-dimensional regularization scheme proposed in Ref. [6].

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APPENDIX: Convergence of the regularization procedure for the fermions.

We here address the following problem. Given a gauge field configuration $A_\mu(x)$, generated by our link variables $U_\mu(x)$: compute the regularised determinant of the Dirac operator

$$D \equiv \gamma_m (\partial_\mu + iA_\mu(x) + i\gamma_5 A_\mu^5(x)) + m, \quad (A.1)$$

where m is a mass term that may or may not be present, $A_\mu(x)$ is a (Abelian or non-Abelian) vector potential field and $A_\mu^5(x)$ an axial vector potential field. The construction sketched in this paper that resulted in the gauge fields A_μ and A_μ^5 can easily be seen to always yield *bounded* values for these fields. It must be noted that the bounds do not depend on the size of our lattice box, by construction. Our problem is to outline a construction procedure for this determinant, to prove that it is well-defined and finite, and to prove that if the anomalies cancel in the one-loop 2-, 3- and 4-point diagrams, the (chiral) flavor symmetries will be completely restored.

An essential complication is that D is not Hermitean, nor can it easily be made Hermitean by multiplying it with simple operators such as γ_5 . We do have that $\gamma_5 D$ only gets a non-Hermitean contribution from the axial gauge field. We will concentrate on computing the *regularised* determinant of $F = iD$. As regulators one may use the Pauli-Villars regulator of Eqs (11) and (12). Our argument will now go in two steps: first we give a rigorous (non-perturbative) construction of the regularised determinant at finite values of the regulator masses Λ_j ; secondly we prove that the limit $\Lambda_j \rightarrow \infty$ exists and is approached uniformly for all A_μ configurations within our bound.

Of course F may have zero modes; if these are not disturbed by anything the determinant simply vanishes, and that would be the end of the calculation. In the other case, for doing the first computation, we need all eigenvalues of F . Write

$$F \equiv K + A \quad , \quad K \equiv i\gamma_\mu \partial_\mu, \quad (A.2)$$

where K can be diagonalized exactly giving eigenvalues (in Euclidean space) $\lambda_n = \pm|k_n|$, and A is strictly bounded:

$$\|A\| \leq \mathcal{A}. \quad (A.3)$$

We now employ an important theorem concerning matrices:

Theorem: Given an $N \times N$ matrix $F = K + A$ such that K is Hermitean and A is bounded by \mathcal{A} as in Eq. (A.3). Let κ_i be the eigenvalues of K . Then the complete set of eigenvalues λ_i of F (including possible degeneracies) obeys

$$|\lambda_i - \kappa_i| < C \mathcal{A}, \quad (\text{A.4})$$

where the coefficient C may grow only logarithmically with the dimension N of the matrices.

In fact, if A is also Hermitean C is one, and furthermore the imaginary parts of λ_i are bounded by \mathcal{A} itself.

The proof of this theorem is somewhat lengthy. First we handle the case that A is Hermitean. This is easy, since we can write $F(x) = K + xA$, $x \in [0, 1]$ and differentiate with respect to x . To handle the general case we now split off the hermitean part of A , so that what remains to be considered is the case that A is purely anti-hermitean. So from now on A is anti-hermitean. One now chooses the orthonormal basis that brings F in semi-diagonal form:

$$F = \begin{pmatrix} \lambda_1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ & \lambda_2 & \cdot & \cdot & \cdot & \cdot \\ & & \lambda_3 & \cdot & \cdot & \cdot \\ & & & * & \cdot & \cdot \\ & 0 & & & * & \cdot \\ & & & & & \lambda_N \end{pmatrix}, \quad (\text{A.5})$$

where λ_i are now the eigenvalues of F . Let P be the off-diagonal part of F . Write

$$\begin{aligned} \lambda_i &= R_i + iS_i; \quad R_i, S_i \text{ real,} \\ K &= \text{diag}\{R_i\} + \frac{1}{2}(P + P^\dagger), \\ A &= \text{diag}\{iS_i\} + \frac{1}{2}(P - P^\dagger), \\ P_{k,\ell} &= 0 \quad \text{if } \ell \leq k. \end{aligned} \quad (\text{A.6})$$

The condition (A.3) implies that

$$|S_i| \leq \mathcal{A}, \quad (\text{A.7})$$

so the imaginary parts of the eigenvalues λ_i obey our bounds. Next we deduce from (A.7) that for all wave functions $|\psi\rangle$,

$$|\langle \psi | (P - P^\dagger) | \psi \rangle| \leq 4\mathcal{A}, \quad (\text{A.8})$$

and what remains to be done is to derive from that a bound on $|\langle \psi | (P + P^\dagger) | \psi \rangle|$. This one does as follows. Take a wave function $|\psi^0\rangle$. Consider then a class of normalized wave functions

$$\psi(\omega)_k = \psi_k^0 e^{i\omega k}, \quad (\text{A.9})$$

where k is the index in the above basis representation. Define the function

$$f(\ell) = \sum_k \psi_k^{0*} \psi_{k+\ell}^0 P_{k,k+\ell}, \quad (\text{A.10})$$

Then

$$\langle \psi(\omega) | P | \psi(\omega) \rangle = \sum_{\ell > 0} f(\ell) e^{i\omega\ell} \equiv g(\omega). \quad (\text{A.11})$$

From Eq. (A.8) we know that for all ω there is a bound on the imaginary part of $g(\omega)$:

$$|\text{Im } g(\omega)| \leq 2\mathcal{A}, \quad (\text{A.12})$$

whereas from (A.6) we see that our theorem requires a bound on $\text{Re } g(\omega)$. Now since according to (A.11) $g(\omega)$ only has positive Fourier coefficients it obeys a dispersion relation:

$$\text{Re } g(\omega_1) = -\frac{1}{\pi} \mathcal{P} \int d\omega \frac{\text{Im } g(\omega)}{\omega - \omega_1}, \quad (\text{A.13})$$

where \mathcal{P} stands for principal value. This integral is only logarithmically divergent, and that only if $\text{Im } g(\omega)$ makes a θ jump near ω_1 . From this we derive the bound (A.4) with a coefficient C that can diverge only logarithmically if the dimension N of our matrices becomes large.

Consider now the regularised determinant of (A.1), assuming it has no vanishing eigenvalues. Its logarithm is

$$\Gamma = \sum_n \sum_j e_j \log(\lambda_n + i\Lambda_j), \quad (\text{A.14})$$

where the first summand behaves for large λ_n as

$$\sum_j e_j (\log \lambda_n + \sum_{r=1}^4 \frac{-1}{r} \left(\frac{-i}{\lambda_n}\right)^r \Lambda_j^r + \mathcal{O}(\Lambda_j^5 \lambda_n^{-5})). \quad (\text{A.15})$$

According to Eqs. (11) the logarithm term and the first four terms in the sum can be taken to be zero. As for the remainder we may put

$$|\lambda_n(A) - \kappa_n| < C\mathcal{A}, \quad (\text{A.16})$$

with $C = \mathcal{O}(\log N)$, where κ_n are the Eigenvalues of K in (A.2). For N we take the number of points on a lattice that is sufficiently fine mazed to enable us to reproduce the eigenfunction with eigenvalue λ_n with reasonable accuracy. One may therefore safely say

that \mathcal{C} will be of order $4 \log n$, when the n^{th} eigenvalue is considered. One may conclude that the sum over eigenvalues λ_n in (A.15) converges uniformly.

Next we address the question of the limit where the Λ_j become large. This is actually easy. We split the logarithm Γ up into the first four expansion terms for small values of the A_μ fields (the diagrams with at most four external lines) and the remainder. The diagrams with four lines or less have been calculated many times, and we know that there are many models for which the anomalies in these diagrams can be made to cancel out. It is these models that we are interested in. The remainder is the set of ultraviolet convergent diagrams. The sum over these diagrams in general does not converge (because they correspond to the expansion of the logarithm of the determinant, which blows up whenever there is a vanishing eigenvalue), but the sum over the *regulator contributions* converges rapidly. For sufficiently large Λ_j all regulator contributions for the diagrams with k external lines are bounded by expressions of the form

$$\frac{1}{k} \mathcal{A}^k \Lambda_j^{4-k}, \quad (\text{A.16})$$

as one can easily convince oneself. Thus, if we sum over the k values larger than 4, (A.16) sums up to a finite bound, and we see that this contribution vanishes in the limit $\Lambda_j \rightarrow \infty$. So not only do these diagrams converge, the regulator contributions for all *convergent* diagrams converge uniformly to zero. We conclude that the regulator limit of the determinant exists and obeys all wanted flavor symmetries.

Notes added:

This appendix should not be regarded as a ‘first proof’ of finiteness of the regulated chiral determinant in an arbitrary gauge field. I thank R.D. Ball for pointing out references to earlier treatments of this problem [7].

That infinitely many fermion fields would be needed on a lattice was observed in [8]. In our paper we have one chiral field, but living on a continuum, which in practice amounts to the same thing. Narayanan and Neuberger also explain in their paper the possible relevance of a good regularization scheme for the study of baryon non-conservation in the Standard Model [9].

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