GRAVITON DOMINANCE IN ULTRA-HIGH-ENERGY SCATTERING

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The scattering process of two pointlike particles at CM energies in the order of Planck units or beyond, is very well calculable using known laws of physics, because graviton exchange dominates over all other interaction processes. At energies much higher than the Planck mass black hole production sets in, accompanied by coherent emission of real gravitons.

Much of the physics at the Planck length is very mysterious to us. It is therefore of importance to realize that some processes in that area of physics are well computable using presently known laws of physics. One of these computable processes is the emission of Hawking radiation by black holes [1]. In this paper we point out that other processes are also completely fixed by what we know already, just by using quantum field theory in combination with general relativity.

Consider two (electrically neutral) particles with rest masses $m^{(1)}$, $m^{(2)} \ll M_{\text{Planck}}$. Let us first use a coordinate system in which the ingoing particle (1) is at rest or moves slowly. Let the second particle arrive from the right along the trajectory

$$\begin{bmatrix} x^{(2)} \\ y^{(2)} \end{bmatrix} \underset{\text{def}}{=} \tilde{x}^{(2)} = 0 , \quad z^{(2)} = -t^{(2)} , \tag{1}$$

with energy

$$\tfrac{1}{2} p_{-}^{(2)} = p_0^{(2)} = -p_3^{(2)} = \mathrm{O}(1/Gm^{(1)}) , \tag{2}$$

where G is Newton's constant in units where $\hbar = c = 1$. Since we take $m^{(i)} \ll M_{\text{Planck}}$ the velocity of particle (2) can be regarded to be that of light. The energy (2) is so tremendous that we can no longer ignore the gravitational field of particle (2). This field consists of an impulsive wave (or "shock wave") [2]. The best way to describe this wave is by saying that two flat regions of space–time, $R^4_{(+)} (t > -z)$ and $R^4_{(-)} (t < -z)$, are glued together at the null plane $z = -t$, but such that on this plane

$$\begin{aligned} &\tilde{x}_{(+)} = \tilde{x}_{(-)} , \\ &z_{(+)} = z_{(-)} + 2Gp_0^{(2)} \log(\tilde{x}^2/C) , \\ &t_{(+)} = t_{(-)} - 2Gp_0^{(2)} \log(\tilde{x}^2/C) , \end{aligned} \tag{3}$$

where C is an irrelevant constant. In short,

$$x_{\mu(+)} = x_{\mu(-)} - 2Gp_\mu^{(2)} \log(\tilde{x}^2/C) . \tag{4}$$

That space–time surrounding a fast particle has this shape in the limit where its rest mass vanishes compared to its energy, is easily established by performing a Lorentz boost on a particle at rest.

For simplicity we now take particle (1) to be spinless. In $R^4_{(-)}$ its wave function is

$$\begin{aligned} \psi^{(1)}_{(-)} &= \exp(\mathrm{i}\tilde{p}^{(1)}\tilde{x} + ip_3^{(1)} z - \mathrm{i}p_0^{(1)} t) \\ &= \exp(\mathrm{i}\tilde{p}^{(1)}\tilde{x} - \mathrm{i}p^{(1)}_{(+)} u - \mathrm{i}p^{(1)}_{(-)} v), \end{aligned}$$

where $u = (t-z)/2$ and $v = (t+z)/2$ are light-cone coordinates.

Immediately after the shock wave went by we have the shifted wave function in $\mathrm{R}^4_{(+)}$:

$$\begin{aligned} \psi^{(1)}_{(+)} = \exp\{&\mathrm{i}\tilde{p}^{(1)}\tilde{x} \\ &- \mathrm{i}p_+^{(1)}[u + 2Gp_0^{(2)} \log(\tilde{x}^2/C)]\} , \\ &\text{at } v = 0 , \end{aligned} \tag{6}$$

This we can expand in plane waves,

$$\begin{aligned} \psi^{(1)}_{(+)} = &\int A(k_+, \tilde{k})\, \mathrm{d}k_+\, \mathrm{d}^2\tilde{k} \\ &\times \exp(\mathrm{i}\tilde{k}\tilde{x} - \mathrm{i}k_+ u - \mathrm{i}k_- v) , \end{aligned} \tag{7}$$

with

$$k_- = (\tilde{k}^2 + m^{(1)2})/k_+ \,. \quad (8)$$

Clearly,

$$A(k_+, \tilde{k}) = \delta(k_+ - p_+^{(1)}) \times \frac{1}{(2\pi)^2} \int d^2\tilde{x} \, \exp[(\tilde{p}^{(1)} - \tilde{k}^{(1)})\tilde{x} - 2iGp_+^{(1)} p_0^{(2)} \log(\tilde{x}^2/C)] \,. \quad (9)$$

The integral here is elementary:

$$\int d^2\tilde{x} \, \exp(i\tilde{k}\tilde{x} - iB \log \tilde{x}^2) = \frac{\pi\Gamma(1-iB)}{\Gamma(iB)} \left(\frac{4}{\tilde{k}^2}\right)^{1-iB} \,. \quad (10)$$

In our case (see eq. (2)),

$$B = 2Gp_+^{(1)} p_0^{(2)} = -2G(p^{(1)} \cdot p^{(2)}) = Gs \,,$$

where s is the usual Mandelstam variable. Notice furthermore that

$$dk_+ \, d\tilde{k} = (k_+/k_0) dk_3 d\tilde{k} \,. \quad (11)$$

Thus, concentrating only on particle (1), we get

$$_{\text{out}}\langle \boldsymbol{k}^{(1)} | \boldsymbol{p}^{(1)} \rangle_{\text{in}} = \frac{k_+}{4\pi k_0} \delta(k_+ - p_+) \frac{\Gamma(1-iGs)}{\Gamma(iGs)} \left(\frac{4}{(\tilde{p} - \tilde{k})^2}\right)^{1-iGs} \,. \quad (12)$$

No particle production or bremsstrahlung is seen in any coordinate frame (as long as particle (1) is electrically neutral and $m^{(1)} \ll M_{\text{Planck}}$), so the scattering is elastic. There is an exchange of momentum

$$q = k^{(1)} - p^{(1)} \,. \quad (13)$$

The Dirac delta in (12) is just energy conservation. Defining the Mandelstam variable $t = -q^2$, we find the elastic scattering amplitude to be (apart from the canonical factor $(k_+/k_0)\delta(\Sigma k - \Sigma p)$),

$$U(s, t) = \frac{\Gamma(1-iGs)}{4\pi\Gamma(iGs)} \left(\frac{4}{-t}\right)^{1-iGs} \,, \quad (14)$$

from which the cross section follows,

$$\sigma(\tilde{p}^{(1)} \to \tilde{k}^{(1)}) \, d^2\tilde{k} = \frac{4}{t^2} \left|\frac{\Gamma(1-iGs)}{\Gamma(iGs)}\right|^2 d^2\tilde{k} = 4G^2(s^2/t^2) d^2\tilde{k} \,. \quad (15)$$

This resembles Rutherford scattering, except for the factor $s^2 \approx p_+^{(1)2} p_-^{(2)2}$. Such an extra energy dependence is of course to be expected from a theory of gravity. Apparently the cross section is just as if a single graviton were exchanged, but the amplitude, eq. (14), is more complicated.

We also observe a rather curious similarity between eq. (14) and the well-known Veneziano amplitudes from string theories. One explanation for this is that, as in string theories, we are dealing with a two-dimensional world; the shock waves are three-dimensional null surfaces, but the null direction (the vanishing eigenvector of the induced metric) does not appear in the equations for the displacement vector. The displacement vector in eqs. (3) and (4) only depends on $\tilde{x}$. But one may also imagine that there is a deeper connection between string theories and this kind of scattering [3].

How exact is the amplitude (14)? Exchange of other massless particles would alter it. For instance, it is easily seen that electric charges $e^{(1)}$, $e^{(2)}$, just cause a shift:

$$Gs \to Gs + e^{(1)} e^{(2)}/4\pi \quad (16)$$

in eqs. (14) and (15). In many respects, eq. (14) can be seen to be an "eikonal approximation" [4]. However, we claim that other quantum field theoretic effects, for instance those due to exchange of massive particles, will be swamped by eq. (14) at sufficiently large Gs. This is not only because of the obvious divergence at $t \to 0$. Take a particle (1) with a definite impact parameter b with respect to particle (2). If b is large we only have effects from the graviton (and other massless particles). But if b is small, we have the divergence of the logarithm in eqs. (3) and (4): particle (1) is being shifted along in the direction of $p_\mu^{(2)}$. Whatever it does, these effects will only be seen at much later times by any observers in $R^4_{(+)}$.

Finally one may ask what happens if Gs becomes much larger than one. In that case it seems more appropriate to consider both particles (1) and (2) and their gravitational fields in the CM frame. The computation of the general relativistic effects when both shock waves collide is complicated, however [5]. In general a spacelike singularity is expected. If the impact parameter is less than the Schwarzschild radius corresponding to the CM energy then one obviously expects a black hole to form, and classical gravitational waves will be emitted (in particle terms these are just coherent many-graviton modes). The corresponding computations are technically difficult, but should follow completely from the well- known laws of general relativity, although we hasten to add to this that questions of a proper formulation of the initial conditions and the actual existence of solutions are far from settled mathematically [6]. In any case, non-trivial quantum field theoretical phenomena are well hidden behind the horizon.

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