Distinguishing causal time from Minkowski time and a model for the black hole quantum eigenstates

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August 7, 2002

THU-97/36 gr-qc/9711053

Abstract

A discussion is presented of the principle of black hole complementarity. It is argued that this principle could be viewed as a breakdown of general relativity, or alternatively, as the introduction of a time variable with multiple 'sheets' or 'branches'.

A consequence of the theory is that the stress-energy tensor as viewed by an outside observer is not simply the Lorentz-transform of the tensor viewed by an ingoing observer. This can serve as a justification of a new model for the black hole atmosphere, recently re-introduced. It is discussed how such a model may lead to a dynamical description of the black hole quantum states.

1 Introduction

The gravitational force, as dictated by General Relativity, has a built-in instability. If a sufficiently large quantity of matter is concentrated in a sufficiently small volume, collapse is inevitable, and a black hole results. It is characterized by a space-time in which an event horizon occurs, as is sketched in Fig. 1a. The horizon is depicted as a transparant surface; it is a lightlike surface defined in such a way that any material object inside it, is unable to escape from this region as long as its velocity force vector stays within its local lightcone.

From the point of view of classical physics, one can predict precisely what will happen, and there seems to be no serious clash with astronomical observations. However, when one wishes to study the consequences of quantum field theory in these circumstances, new difficulties arise that are intensively being studied. It was discovered by Hawking [1] that quantum field theory will force the black hole to emit particles, with the intensity of a thermal heat source at temperature

$$T_H = \frac{\hbar c^3}{8\pi G M}\,, \tag{1.1}$$

where M is the mass of the black hole. Details of the derivation of this effect can be found in refs. [1, 2, 3, 4]

The fact that the radiation emitted, as described by (1.1), is *thermal*, opens up the possibility to approach this phenomenon from a thermodynamical point of view. [5]. Taking M to be the energy (using units in which $\hbar = c = 1$), and $T = T_H$ the temperature, one readily derives the *entropy* S:

$$T\mathrm{d}S = \mathrm{d}M\,; \qquad \mathrm{d}S = 8\pi G M \mathrm{d}M\,; \qquad S = 4\pi G M^2 + C\,, \tag{1.2}$$

where C is an unknown integration constant, to be referred to as the "entropy normalization constant". It is important to note that the expression obtained for the entropy S, apart from the integration constant, is always equal to $\frac{1}{4}A/G$, where A is the *area* of the horizon, a finding that will be very much at the center of our discussions. A physical interpretation of this entropy is that *information* appears to be distributed over the 2-dimensional area of the event horizon in such a way that one bit of information occupies an area exactly amounting to

$$4G\ln 2 \;=\; 0.724\times 10^{-65}\,\mathrm{cm}^2\,. \tag{1.3}$$

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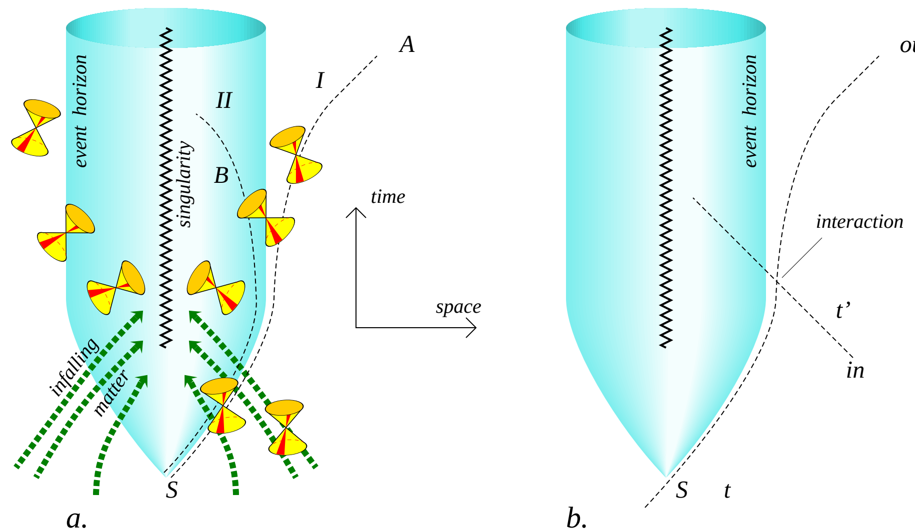

Fig. 1. Space-time sketch of gravitational collapse.

It is tempting to assume that a black hole can exist in e^S distinct quantum states, and, apart from the fact that this number can take huge values when the hole is large, these quantum states each can be compared with ordinary elementary particles. Furthermore, one would expect that the entire process of formation and subsequent desintegration of a black hole can be described by a scattering matrix, as in quantum scattering theory.

In the next section, however, it will be explained that this simple assumption appears to lead to conflicts; a genuine paradox is encountered. It is called the 'quantum information problem', and it is believed to have a deep and fundamental significance for the theory of quantum gravity.

The most promising avenue towards a resolution of the paradox is what is called the 'S-matrix Anstz', as explained in Sect. 3, but this does lead to the new and peculiar feauture of 'complementarity' (Sect. 4). In Sect. 5 we introduce a multivalued time parameter.

The second half of this paper is a description of a model based on these ideas. The outside observer here treats the Hawking particles he/she observes as genuine sources for the Einstein equations, in spite of the fact that ingoing observers would disagree. Although an unphysical singularity emerges at the origin ($r = 0$), the model allows for a detailed description of the black hole quantum states. It does appear to require an exotic equation of state for matter in the Planck regime.

2 The quantum information problem

Imagine the black hole formation process, assuming that the collapsing object started in a pure state. The wave packet of a particle moving along the trajectory labled A in Fig. 1a, enters the black hole region near the pint S (where the horizon opens up). There, it corresponds to a pure state $|\psi\rangle_{\rm in} = |\Omega\rangle$. During its evolution, this wave packet will split into two parts, one leaving the hole near the point A in region I, and one entering the hole in region II. This state thus evolves into a superposition of product states:

$$|\psi\rangle_{\rm in} \;=\; \prod_{\substack{\text{modes}\\ \omega}} \sqrt{1-e^{-8\pi M\omega}}\,\sum_n e^{-4\pi M\omega n}|n\rangle_I|n\rangle_{II} \;=\; \sum_i |\psi_i\rangle_I|\psi_i\rangle_{II}\,, \tag{2.1}$$

the last line being short-hand notation. n stands for the number of particles at a given frequency ω.

In terms of observable operators $\mathcal{O}$ that act only upon the visible states $|\psi_i\rangle_I$, one finds that this state is a *quantum mechanically mixed* state:

$$\langle\mathcal{O}\rangle \;=\; \sum_{ij}\langle\psi_i|_I\,\mathcal{O}\varrho_{ij}|\psi_j\rangle_I\;;\qquad \varrho_{ij} \;=\; {}_{II}\langle\psi_i|\psi_j\rangle_{II} \;=\; \mathrm{Tr}\varrho\mathcal{O}\,. \tag{2.2}$$

Here, the matrix ϱ is a quantum mechanical density matrix, and, since in Eq. (2.1), the quantity ωn is the energy of the state $|n\rangle$, one easily derives from (2.1) that this density matrix is a thermal one, corresponding to the Hawking temperature (1.1).

Now, imagine an amount of matter in a quantum mechanically *pure* state, undergoing gravitational collapse. The state $|\psi\rangle_{\mathrm{in}}$ can be taken to be pure as well. Then, if the states $|\psi\rangle_I$ are taken to be the only ones accessible to the outside observer (after some time has elapsed), one would have to conclude that the final state is a mixed state [6]. Did an evolution take place where a pure state evolved into a mixed state? This would be against the rules of conventional quantum mechanics. Alternatively, should we keep the states $|\psi\rangle_{II}$ in our Hilbert space forever? In that case, one deduces that the dimensionality of Hilbert space would quickly grow much larger than suggested by the rather low value (1.2) of the total entropy – the level density would become a continuum, instead of describing a denumerable, discrete set. Because of the infinite number of possible states, one would argue that such black holes would obey Boltzmann statistics instead of Bose-Einstein or Fermi-Dirac statistics, which is the law for ordinary matter. There would be no such thing as a scattering matrix.

From a physical point of view, one expects that the density matrix is merely approximately thermal, but that true thermality only occurs at the infinite size limit. The *S-matrix Ansatz* [7, 4] asserts that Hawking radiation is actually governed by an S matrix. The entropy (1.2) would indeed refer to the total number of possible states, and the *quantum information* of all particles that enter into the hole during its entire lifetime, does not disappear, but reemerges with the Hawking radiation. More precisely, if we compare two initial states that are orthogonal to each other (which already is the case if only one of its ingoing particles has its spin reversed), then the two corresponding pure out-states will be orthogonal as well, according to the laws of unitary evolution.

Consider Fig. 1b. Imagine a particle with spin up, entering the hole at late time t'. If we flip its spin, the outgoing particles, seen at all points A, should change into a new quantum state, orthogonal to the previous one. The paradoxical aspect of this, however, is that these outgoing particles *all* originate at the point S where the horizon first opened up. The change should therefore already be visible at that point, but S is located at time t much earlier than t'. Thus, the S-matrix Ansatz appears to violate causality.

In Fig. 1b, one can also see how causality can be safeguarded. We need to consider the *interactions* that take place where the ingoing particles meet the outgoing ones. In a linear theory, *i.e.* in the absence of interactions, there would be no causality. But the interactions cannot be ignored. Most of all, the *gravitational* interactions between in- and outgoing material is seen to diverge with the time difference $t' - t$.

Another way to see the difficulty for linearized theories, is to construct the equations for quantum fields in the Schwarzschild geometry near the horizon. After collapse, this space-time is described by the metric

$$\mathrm{d}s^2 = -\Big(1-\frac{2M}{r}\Big)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-2M/r} + r^2\mathrm{d}\Omega^2\,, \tag{2.3}$$

where Newton's constant G was normalized to one, and

$$\mathrm{d}\Omega^2 \equiv \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\,. \tag{2.4}$$

Working out the field equations, one finds it to be convenient to substitute

$$r - 2M = e^{\sigma}\ , \qquad -\infty < \sigma < \infty\,. \tag{2.5}$$

A free scalar field at angular momentum ℓ then obeys

$$\left[r\Big(\frac{\partial}{\partial t}\Big)^2 - \frac{1}{r^2}\frac{\partial}{\partial\sigma}r\frac{\partial}{\partial\sigma} + e^{\sigma}\left(\frac{\ell(\ell+1)}{r^2} + m^2\right)\right]\Phi = 0\,. \tag{2.6}$$

At $\sigma \to -\infty$, this produces plane waves moving in and out, but ingoing waves do not scatter back in any finite amount of time; hence a strictly continuous spectrum of states results, which is at odds with the finiteness of the entropy. Of course, gravitational interactions diverge as $\sigma \to \infty$.

If a *brick wall* would be introduced [8], with which we mean a reflecting boundary condition at some small but finite value of σ, a discrete spectrum could be enforced. We do notice that such a brick wall would correspond to a reflecting envelope at a small but finite distance away from the horizon in Fig. 1b. Clearly, it would be enough te reestablish an S-matrix but it is clear that such a boundary condition would violate general coordinate invariance. Or could it be that the divergent gravitational interactions mimick a brick wall?

Why is the information problem considered to be so important? First of all, it appears that what is at stake here is the extent of *predictability* of the fundamental dynamical processes at the Planck length. Some theories of the Universe [9] suggest that in many – extremely distant – regions of the

Universe, different laws of physics apply; for instance, the unifying gauge symetry may condense into gauge subgroups different from $SU(3) \times SU(2) \times U(1)$, which we are used to in the Standard Model. Is there indeed nothing more than the anthropic principle that determines the symmetry breaking and the associated constants? If so, this will mean that we will never be able to compute these effects from first principles more refined than this anthropic principle.

Of even more importance, however, appears to be the fact that *assuming* the preservation of quantum information leads to interesting new insights in the forces of nature. Conservation of quantum information is likely to demand a new kind of conspiracy, and the resolution of the paradox may well lead to important new physics. Indeed, we see examples of important new developments in physics in the past, that sprouted from attempts to resolve paradoxes.

3 The S-matrix Ansatz

At first sight, it seems to be a necessary consequence of the imput theories, being quantum mechanics and general relativity, that the outgoing state is related to the ingoing state through a well-defined linear mapping, called the S-matrix. Yet, as was argued in the previous section, this appears to be not true if the laws are taken at face value and the calculation is performed. In this section, we *postulate* [7] that indeed there is an S-matrix. When this postulate is taken together with the known laws of physics, new features can be derived.

We begin by first concentrating on a small region of the black hole horizon, ignoring the local curvature of space-time. One then gets "Rindler space", pictured in Fig. 2a. The frame used to draw the picture is the one in terms of which the metric stays regular naer the horizon. In terms of the original coordinates r and t, the Schwarzschild metric (2.3) is singular at $r = 2M$. A few lines $t =$ constant and $r =$ constant, are indicated. A translation in time, $t \to t +$ const., corresponds to a Lorentz transformation in terms of regular coordinates.

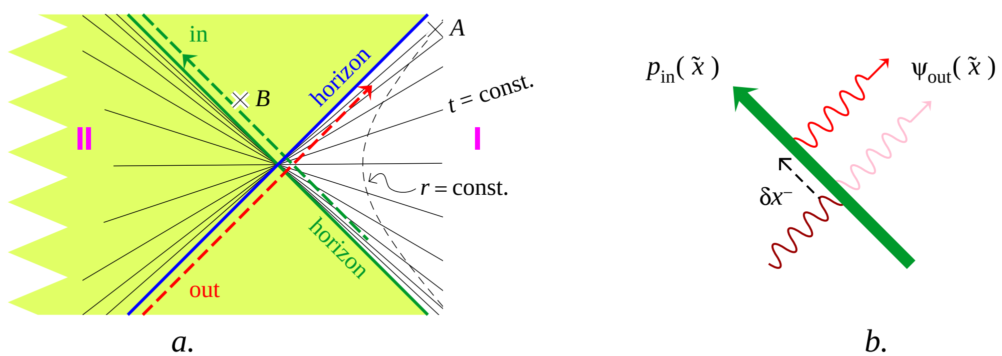

Fig. 2. a) Rindler space. The unshaded region is the visible part of the universe. b) The gravitational shift of an outgoing particle due to the gravity of an ingoing one.

According to the S-matrix Ansatz, it is to be expected that the ingoing particles (dotted line from lower right to upper left) *affect* the outgoing particle state (dotted line from lower left to upper right). This interaction must occur at the intersection point (which is actually a two-dimensional plane since we still have the angular variables θ and ϕ. The interactions due to the forces of the Standard Model are weak, and it is hard (but not impossible) to take these interactions into account. But the gravitational interactions are much more profound. Since time translations for the distant observer correspond to Lorentz transformations for a local observer, the late outgoing particles cross the trajectories of early ingoing ones at very large relative velocities. It is here where the gravitational interactions become important – growing as they do with energy.

The gravitational field due to a very fast particle, with small rest mass, can be computed by standard methods of general relativity. One finds that the field of an ingoing particle affects the outgoing ones by giving them a *shift* [10]. This is pictured in Fig. 2b.

The shift δx^- in the coordinate x^- for the outgoing particle depends on the transverse separation $\delta\tilde{x}$ between the two particles:

$$\delta x^-(\tilde{x}) = \int \mathrm{d}^2\tilde{x}' f(\tilde{x} - \tilde{x}')\, p_{\rm in}(\tilde{x}')\,, \tag{3.1}$$

where the momentum p_{in} is taken as a distribution over the transverse (angular) coordinates $\tilde{x} = (\theta, \phi)$. The function f has a logarithmic singularity where $\tilde{x} - \tilde{x}' \to 0$. Its form is dictated by Einstein's field equations.

Our general philosophy, exhibited in much more detail in Ref. [4], is as follows. Consider a black hole, formed by the collapse of a large amount of matter. It will produce outgoing particles in a set of very wide wave packets $\psi_{\mathrm{out}}(\tilde{x}')$. Now, consider an infinitesimal *change* in the quantum states of the ingoing particles. The distribution $\delta p_{\mathrm{in}}(\tilde{x})$ will undergo a tiny change, $\delta p_{\mathrm{in}}(\tilde{x})$. This causes a corresponding shift in the outgoing states, described by a quantum mechanical shift operator. Thus, indeed, we find that the outgoing state depends on the ingoing one. The most delicate part of the argument is then the requirement that this dependence be described by a unitary scattering matrix operator. A new version of Fock space must be introduced, and what is found shows a startling resemblance to string theory amplitudes – though it is not exactly string theory that we find.[1]

4 Black hole complementarity

Although, in principle, the theory outlined above should reveal the nature of the quantum states on the black hole horizon, the paradox mentioned earlier is not entirely resolved. The pivotal question is, how does an ingoing observer experience and identify these quantum states, and how can we reconcile the finiteness of their density with Lorentz invariance? Should Lorentz transformations not lead to strictly infinite numbers of states? And, secondly, how is the apparent clash with causality diverted?

The latter question can be phrased more precisely. Consider Rindler space as it is sketched in Fig. 2a. Consider operator-valued fields in the Heisenberg picture.[2] Thus, consider the point B in Fig. 2a, and fields $\Phi(x)$ with the space-time point x in the immediate neighbourhood of B. Its operators can only act on the states described by the early ingoing particles, but all visible outgoing particle wave packets are well outside the lightcone defined by B, and so one might conclude that $\Phi(x)$ should commute with *all* operators $\Phi(y)$ with y in the direct neighborhood of the point A, where an observer detects Hawking particles. If, however, the Hawking particles, all of which can be detected near the point A, would form a complete representation of Hilbert space, no operator acting on the ingoing particles could commute with all Hawking operators at A.

According to the S-matrix Ansatz, the operators at A will therefore *not* all commute with the operators at B. The reason why the light cone argument breaks down is the excessive relative energies of the particles in consideration; due to these energies, the space-time metric is distorted, and the light cone will not always stay in position.

Again consider an operator field acting near the point B. What is really meant by this field operator is that it acts in a certain way on the ingoing states, and this action can be derived by extrapolating the fields of all ingoing particles well into the forbidden (shaded) region of Fig. 2a. In making this extrapolation, it is tacitly assumed that the penetration into this region is unproblematic, since space and time are free from excessive curvature near the horizon.

Now, however, consider an observer performing a measurement near the point A. This observer detects Hawking particles. If he waits long enough, his time coordinate t is so large that the particles he detects all carry enormous amounts of energy with respect to the Lorentz frame of B. The action of A on any state in Hilbert space, in general will produce a state in which B does not see a near vacuum at the origin, but one or more extremely energetic particles there. These particles carry devastating gravitational fields. The ingoing particles are, in fact, deflected by these fields. The *tacit assumption* of the previous paragraph, that the operator in B could be defined as if the ingoing particles could reach this point undisturbed, has become untenable. This is why the commutators between fields at A and fields at B cease to vanish.

In fact, if observers are allowed to act on our states at sufficiently late points A, any attempt to define the action of fields at B will become meaningless. We have to *choose*: either we allow the operators defined at A, or the operators defined at B, but not both sets. Black hole complementarity [11] is the principle that a *complete* description of Hilbert space can be obtained either in the frame of the ingoing observer, allowing him to do measurements at points B, or by allowing all measurements on the emitted Hawking particles. This is an inevitable consequence of the S-matrix Ansatz. It also delivers a picture that appears to be more compatible with time reversal invariance than the classical picture of black holes losing information.

[1] The *string coupling constant* is found to be purely imaginary.
[2] The points A and B in Fig. 2a more or less correspond to A and B in Fig. 1a.

5 The forked time paramter

The linear mapping in the previous section, relating A and B, may be quite complicated. This is in contrast to the more usual general coordinate transformations, for which the transformation rules are direct and transparent. Indeed, the mapping will probably be as difficult to elaborate as the time evolution itself. This may be tantamount to saying that the general coordinate transformation itself is not able to provide us with the information we want about the evolution of these states, or in other words, the general coordinate transformation fails. The transformation works if we want to handle the states $|\psi\rangle_I$ and $|\psi\rangle_{II}$, but not for the states $|\psi\rangle_{\mathrm{in}}$ or $|\psi\rangle_{\mathrm{h}}$. Although the detailed description of the states seen by an observer going into the hole, does contain information concerning the phenomena seen by the late outside observer, this information is thoroughly encrypted.

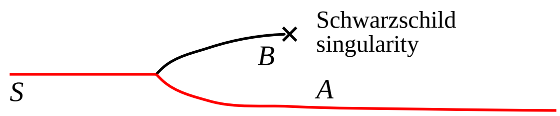

Fig. 3. The multi-valued time parameter.

In Fig. 3, this is further illustrated. The collapse is observed at the point S. The ingoing observer follows the path B towards the Schwarzschild singularity. The outside observer continues along the path A. After the collapse, Hilbert space is *not* described by the products of the states along path A and the ones along path B, but it is spanned either by the states at A alone, or by the states at B alone. The Hamiltonian H is defined along A and along B, and it dictates the evolution everywhere along the curve. Thus, we see that time has become a manifold more complicated than just a single line.

An interesting implication of this view is that the *stress-energy-momentum tensor* $T_{\mu\nu}(x)$ as seen by B, will be defined differently from the one seen by A. Energy, after all, is related to the time translation operator, but translation in the B direction is now no longer equivalent to translation in the A direction. Therefore, it has become illegal to relate $T_{\mu\nu}$ as seen by B to that of A through the usual coordinate transformations. Now, usually, it is assumed that B will detect a vacuum state ($T_{\mu\nu} \approx 0$). We are not allowed to infer from this that A will see a cancellation between the stress-energy-momentum of the Hawking particles against a large (and negative) contribution from the Casimir effects in the Schwarzschild metric, as was reported in earlier literature [12, 13].

This gave us the inspiration to perform the calculation sketched in the next sections. The stress-energy-momentum tensor as experienced by an observer of Hawking radiation is *not* considered to be the Lorentz transform of an empty space stress-energy-momentum tensor. We allow it to be the full tensor of genuine matter. This simply means that the subtraction of the vacuum contribution is not done as in Ref. [12]. The conformal anomaly obtained by [12] is here regarded as an anomaly in the general coordinate transformation itself. In some sense, one may state that general coordinate invariance is violated. The violation arises from the fact that Hilbert space, as viewed by an ingoing observer, does not contain the same states as the one handled by the outside observer. The cut-offs are performed differently.

6 A model

For studying the complete set of quantum states of a black hole, it is not sufficient to limit oneself to just the single case where ingoing observers see only empty space, the so-called Hartle-Hawking state. The modes to be studied in this paper are stationary in time. In some sense, they are unphysical. This is because a black hole in equilibrium with an external heat bath cannot be stable; it has a negative heat capacity, and it is not difficult to deduce that thermal oscillations therefore diverge. Thus, the states that will be discussed in this paper, being stationary in time, are good candidates for the quantum microstates, but do not represent the Hartle-Hawking state. This way we claim to be able to justify the calculations carried out in this paper: we simply take the stress-energy-momentum tensor generated by Hawking radiation as if the Hawking particles represented an ‘atmosphere’. This atmosphere is taken as the source for the gravitational field equations. We ignore its instability, which is relevant only for much larger time scales. Thermal, non-interacting, massless particles obey the equation of state,

$$p = {}^1\!/_3\varrho\,, \tag{6.1}$$

where p is the pressure and ϱ the energy density. If we have N massless (bosonic) particle types, at temperature $T = 1/\beta$, we have

$$\varrho = 3p = \frac{\pi^2}{30}\frac{N}{\beta^4} \,. \tag{6.2}$$

In this paper, we regard Eq. (6.1) or (6.2) as describing matter near the black hole, and at a later stage we will substitute the true Hawking temperature for β^{-1}. Taking Hawking radiation as a description of the boundary condition at some large distance from the horizon, we continue the solution of Einstein's equations combined with the equation of state as far inwards as we can. What is found is that Hawking radiation produces radical departures from the pure Schwarzschild metric near the horizon. In fact, the horizon will be removed entirely, but eventually a singularity is reached at the origin ($r = 0$). This should have been expected; our system is unphysical in the sense that, when M is sufficiently large, the Shandrasekhar limit is violated, so that a solution that is regular everywhere, including the origin, cannot exist.

We must mention that the differential equations for this system, the so-called Tolman-Oppenheimer-Volkoff equations [14], have been studied before by Zurek and Page [15], and although they use slightly different variables, many of their conclusions were identical to the ones rederived by the author [16].

The negative-mass singularity is the only way in which this approach departs (radically) from earlier Einstein-matter calculations [17]. But for the remainder of our considerations this singularity is harmless. It being repulsive, all particles will keep a safe distance from this singularity. This then, enables us to compute quantum states in the metric obtained. Thus, we compute the entropy due to the scalar fields. We find that this entropy is finite, so that the Hawking 'blanket' itself apparently acts as a soft alternative to the 'brick wall' introduced in Ref. [8]. Furthermore, we find the total entropy of all particles to be independent of N, but to slightly overshoot Hawking's value. Presumably, the equation of state (6.1) was too much of a simplification.

It so happens that the metric can also be calculated if Eq. (6.1) is replaced by

$$p = \kappa\varrho \,, \tag{6.3}$$

where κ is a coefficient ranging anywhere between 0 and 1. Moreover, the total entropy can also be calculated in this case. One may decide to adjust the value of κ such that the entropy calculation matches precisely, but this could be premature, because here one cannot ignore the quantum corrections to Einstein's equations, since we are operating in the Plankian regime. It may nevertheless be of interest to note that a complete match is achieved if κ is set equal to 1, which is the case that will be elaborated further in the Appendix. It is a rather singular and unphysical case. We do conclude that, with interactions taken into account, it may well be possible to obtain a self-consistent approach towards microcanonical quantization of black holes, using ordinary Hartree-Fock methods in standard gravity theories. We do stress that the price paid was a (mild) singularity at the origin. A more thorough analysis of the exact role played by this singularity in a more comprehensive theory of quantum gravity is still to be performed.

7 The equations for $\kappa = \frac{1}{3}$

We only consider spherically symmetric, stationary metrics in $3 + 1$ dimensions, of the form

$$\mathrm{d}s^2 = -A(r)\mathrm{d}t^2 + B(r)\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \,, \tag{7.1}$$

having as a material source a perfect fluid with pressure $p(r)$ and energy density $\varrho(r)$. The Einstein equations read[3]

$$1 - \partial_r\left(\frac{r}{B}\right) = 8\pi r^2\varrho \,, \qquad \frac{\partial_r(AB)}{AB^2 r} = 8\pi(\varrho + p) \,, \tag{7.2}$$

where ∂_r stands for $\partial/\partial r$. The relativistic Euler equations for a viscosity-free fluid can be seen here to amount to [18]

$$\partial_r p = -(\varrho + p)\partial_r \log\sqrt{A(r)} \,, \tag{7.3}$$

and the relation between p and ϱ is governed by an equation of state. We will consider the case

$$p = \kappa\varrho \,, \tag{7.4}$$

[3] Here, units are chosen such that $G = 1$.

where κ is a fixed coefficient. Massless, non-interacting particles in thermal equilibrium have $\kappa = \frac{1}{3}$. The calculations described here can be performed for any choice of κ between 0 and 1, but for brevity we concentrate on the special choice $\kappa = \frac{1}{3}$. For the general case we refer to Ref. [15, 16]. In particular, if $\kappa = 1$, complications arise [16]. Our liquid will be viscosity-free and free of vortices, so that Eq. (7.3) can be integrated to yield

$$8\pi\varrho A^2 = C\,, \tag{7.5}$$

where C is a constant, later to be called $3\lambda^2/(2M)^2$. After inserting this equation into Eqs (7.2) –(7.4) , the latter can be cast in a Lagrange form, which will not be further discussed here. What is needed is to observe the scaling behavior as a function of r. Scale-independent variables are X and Y, defined by

$$X = C^{-1/2}A\,r^{-1}\,, \qquad \text{and} \qquad Y = B\,. \tag{7.6}$$

This turns the equations into:

$$\frac{r\partial_r X}{X} = \frac{Y}{3X^2} + Y - 2\,, \qquad \frac{r\partial_r Y}{Y} = 1 - Y + \frac{Y}{X^2}\,. \tag{7.7}$$

Eliminating r, yields a first order, non-linear differential equation relating X and Y.

The result of a numerical analysis of this equation is presented in Fig. 4. For large r, all solutions spiral towards the point Ω. For small r, only one curve allows B to approach the value one, so that, according to Eq. (7.2) , the density ϱ stays finite, and the metric (7.1) stays locally flat. This is the only regular solution. Further away from the origin, this solution requires ϱ always to be large.

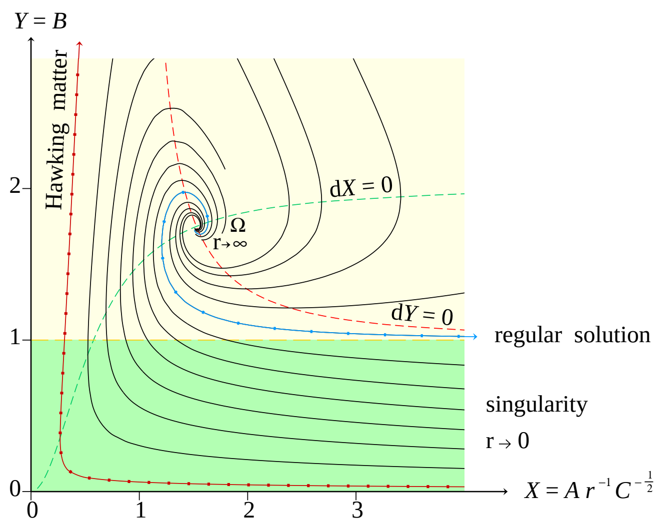

Fig. 4. Solutions to Eqs. (2.9) – (2.10).
Solutions entering the shaded region must have a singularity at $r = 0$.

We are interested in a very different class of solutions, the ones which, far from the origin, approach a black hole surrounded by a very tenuous cloud of matter, Hawking radiation. This is the case where $Y \approx 1$ and $X \gg 1$. Observe that, for very large r, all solutions will eventually spiral into the point Ω:

$$\Omega = \left(X = \sqrt{7C/3}\,,\ Y = \tfrac{7}{4} \right)\,. \tag{7.8}$$

Physically, this means that, since the universe is actually filled with radiation, the curvature at large distances becomes substantial. For the study of Hawking radiation, this large distance effect is immaterial and will henceforth be ignored.

Thus, our boundary condition far from the origin will be chosen to be

$$A(r) = 1 - \frac{2M}{r}\,, \qquad B(r) = \left(1 - \frac{2M}{r}\right)^{-1}\,, \qquad 8\pi\varrho(r) = \frac{3\lambda^2}{(2MA)^2}\,, \tag{7.9}$$

in the region

$$\lambda \ll \frac{r}{2M} - 1 \ll \frac{1}{\lambda}\,. \tag{7.10}$$

Here, the constant C was replaced by $3\lambda^2/(2M^2)$, since ϱ has dimension (mass) $\times$ (length)$^{-3}$ = (mass)$^{-2}$, and the factor 3 is for later convenience. λ will be dimensionless:

$$\lambda^2 = \frac{4\pi^3}{45}\frac{N}{\beta^4}(2M)^2\,, \tag{7.11}$$

where β is the inverse temperature. In Planck units, Hawking radiation has

$$\beta = 8\pi G M = 8\pi M \qquad \to \qquad 2M\lambda = \frac{1}{24}\sqrt{\frac{N}{5\pi}}\,, \tag{7.12}$$

and hence for large black holes, λ is very small. The approximation (7.10) holds over a huge domain.

When r approaches the horizon, the effects of ϱ are nevertheless felt, and the solution becomes more complicated. Actally, A never goes to zero, so there is no horizon at all. At small r, A diverges as a constant$/r$, which means that there is a negative-mass singularity.

Following the line of the solution of interest in Figure 4, one observes that, for small enough λ, the solution can be found analytically. It will be elaborated in the next section.

8 The solution for small λ

Eqs. (7.7) become slightly easier if we substitute X and Y by P and Q:

$$P = 3X^2/Y\ ; \qquad Q = 1/Y\ ; \qquad \text{hence} \qquad X = 3^{-\frac12}P^{\frac12}Q^{-\frac12}\,. \tag{8.1}$$

Defining $L = \log r$, the equations become

$$\frac{\partial P}{\partial L} = \frac{3P}{Q} - 5P - 1\ ; \qquad \frac{\partial Q}{\partial L} = -\frac{3Q}{P} - Q + 1\,. \tag{8.2}$$

In all limiting cases of interest to us, the r.h.s. of these equations will simplify sufficiently to make them exactly soluble.

In order to integrate these equations down towards $r \to 0$, we have to glue together different regions, where different approximations are used. All in all, we cover the line $0 < r \ll \frac{2M}{\lambda}$ with six overlapping regions, as depicted in Figure 5 (in the $\kappa = 1$ case, these will be 8 regions).

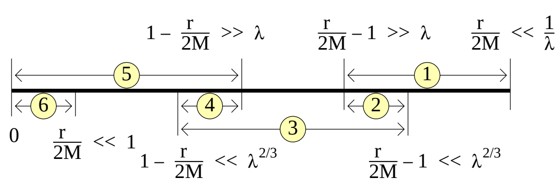

Fig. 5. Covering the line $0 < r \ll 1/\lambda$ with regions (1) – (6).

The region of Eq. (7.10) is region (1). The effects of ϱ on the metric are negligible, and so A and B still obey Eqs. (7.9). These expressions agree with Eqs. (8.2) if we take $P \gg Q$.

In region (3), a different approximation gives the solution:

$$\begin{aligned} r &= 2M\,e^{\lambda\left(\omega - 1/\omega\right)} \quad ; & A &= \lambda\omega\ ; \\ 8\pi\varrho &= \frac{3}{\omega^2(2M)^2} \quad ; & B &= \frac{\omega^3}{\lambda(1+\omega^2)^2}\,, \end{aligned} \tag{8.3}$$

holding as long as

$$\left|\frac{r}{2M} - 1\right| \ll \lambda^{\frac{2}{3}}\,. \tag{8.4}$$

This overlaps with region (5), where integrating the equations yields

$$\frac{r}{2M} = (1+5P)^{-\frac{1}{5}} \quad ; \qquad A = \frac{\lambda^2}{P}\left(\frac{2M}{r}\right)^6 \; ;$$
$$B = \frac{\lambda^2}{P^3}\left(\frac{2M}{r}\right)^{14} \quad ; \qquad 8\pi\varrho = \frac{3P^2}{\lambda^2(2M)^2}\left(\frac{r}{2M}\right)^{12} . \tag{8.5}$$

Finally, in region (6) we have $\frac{r}{2M} \ll 1$, and our solution simplifies into

$$P = \frac{1}{5}\left(\frac{2M}{r}\right)^5 \quad ; \qquad A = 5\lambda^2\,\frac{2M}{r} \quad ; \qquad B = 125\lambda^2\,\frac{r}{2M} . \tag{8.6}$$

The distance (in Planck units) between the point $r = 2M$ (more or less in the middle of region (3)) and the origin at $r = 0$ follows from Eqs. (3.8):

$$\int_0^{2M}\sqrt{B(r)}\mathrm{d}r \;=\; \int_0^1 2M\sqrt{\lambda}\frac{\mathrm{d}\omega}{\sqrt{\omega}} \;=\; 4M\sqrt{\lambda} \;=\; \sqrt{\frac{M}{3}}\left(\frac{N}{5\pi}\right)^{\frac{1}{4}} . \tag{8.7}$$

In contrast, region (5) is small (in spite of the fact that r runs from 0 to nearly $2M$), its geodesic length being proportional to an even smaller power of the black hole mass M. With $y = (2M/r)^5 = 5P$, the metric here can be written as

$$\mathrm{d}s^2 \;=\; -\frac{5\lambda^2 y^{\frac{6}{5}}}{y-1}\mathrm{d}t^2 \;+\; 5(2M\lambda)^2\frac{y^{\frac{2}{5}}}{(y-1)^3}\mathrm{d}y^2 \;+\; y^{-\frac{2}{5}}\mathrm{d}\tilde{x}^2 . \tag{8.8}$$

Substituting (7.12), and rescaling the time t, shows that this metric will be universal for all black holes. It is the metric of a horizon, as seen by a 'Rindler observer'.

The gravitational potential $A(r)$ takes an absolute minimum inside this region (5):

$$\frac{\partial A}{\partial r} = 0 \quad \to \quad y = 6 \quad , \quad \frac{r}{2M} = \frac{1}{\sqrt[5]{6}} , \tag{8.9}$$

and here, the matter density takes the extreme value

$$\varrho^{\mathrm{extr}} = \frac{3(y-1)^2}{200\pi(2M\lambda)^2}y^{-\frac{12}{5}} = \frac{5\cdot 6^{3/5}}{N} . \tag{8.10}$$

Observe, that this is inversely proportional to the number of fields N contributing to the Hawking radiation.

9 The entropy

According to local observers, the entropy density s of matter at any κ value, is

$$s = \beta(r)(1+\kappa)\varrho . \tag{9.1}$$

Here, however, one has to substitute the locally observed temperature, which, due to redshift, is given by

$$\beta(r) = \sqrt{A(r)}\,\beta , \tag{9.2}$$

where β is the inverse temperature as experienced by the distant observer.

The entropy for general κ was also calculated by the author, but it also can be deduced from Ref. [15], who use the coefficient $\gamma = \kappa + 1$, or $n = 1 + \frac{1}{\kappa}$. The main contribution comes from region (5). As Eqs (8.5) were computed for the case $\kappa = 1/3$, we first give the entropy for that case:

$$\frac{S}{\Sigma} = \frac{2\pi^2 N}{45\beta^3}\frac{2M}{5\lambda^2}\int_1^\infty \frac{\mathrm{d}y}{y^2} , \tag{9.3}$$

where Σ is the area of the horizon. With the value (7.12) for λ, one obtains

$$S/\Sigma \;=\; 2/5 , \tag{9.4}$$

instead of Hawking's value $1/4$. Now, for the more general equation of state, Eq. (7.4), this calculation can be repeated (see the next section), with the result

$$\frac{S}{\Sigma}=\frac{\kappa+1}{7\kappa+1}\,, \tag{9.5}$$

and this equals the desired value $\frac{1}{4}$ if $\kappa=1$. It is difficult to imagine ordinary matter with such a high κ value. Free massless fields generate the entropy density

$$s=C\,N\,T^3\,, \tag{9.6}$$

where N is the number of non-interacting field species. C is a universal constant. It appears that, at very high temperature, this should be replaced by

$$s=C\,T^{1/\kappa}\,. \tag{9.7}$$

A striking feature of the result of Eq. (9.5) is the independence on N. But, if N is made to depend strongly on temperature, this does affect the equation of state. Now copare Eqs (9.6) and (9.7). If a strong kind of unification takes place at Planckian temperatures, such that, at those temperatures N suddenly decreases strongly, one could imagine an effective increase of κ beyond the canonical value $\frac{1}{3}$. It is more likely, however, that this argument is still far too naive, and that our approach must merely be seen as a rough approximation. An error of 60% is perhaps not so bad. On the other hand, it is tempting to speculate that the case $\kappa\to 1$ has physical significance. This is a highly peculiar case. The total entropy receives its main contribution from a very tiny region in space where the matter density reaches values diverging exponentially with the small parameter λ.

10 Summary of the cases for $0<\kappa<1$, and the case $\kappa=1$

For general κ, Eqs. (7.5) and (7.6) are replaced by

$$8\pi\varrho=\frac{\lambda^2}{\kappa(2M)^2}A^{-\frac{1+\kappa}{2\kappa}}\quad,\qquad X=A\,r^{-\frac{4\kappa}{1+\kappa}}\,. \tag{10.1}$$

It is convenient to define

$$P=\frac{(2M)^2}{\lambda^2}\,\frac{X^{\frac{1+\kappa}{2\kappa}}}{Y}\;;\qquad Q\quad=\quad 1/Y\,. \tag{10.2}$$

The field equations (8.2) become

$$\frac{\mathrm{d}P}{\mathrm{d}L}=\frac{3\kappa+1}{2\kappa}\,\frac{P}{Q}-\frac{7\kappa+1}{2\kappa}P-\frac{1-\kappa}{2\kappa}\,,\qquad \frac{\mathrm{d}Q}{\mathrm{d}L}=1-Q-\frac{Q}{\kappa P}\,. \tag{10.3}$$

The solution of these equations can be obtained along the same lines as above [15]. The most important is the result in region 5:

$$\mathrm{d}s^2=-a\,(y-1)^{\frac{-2\kappa}{1-\kappa}}\,y^{\frac{4\kappa(1+3\kappa)}{(1-\kappa)(7\kappa+1)}}\,\mathrm{d}t^2+b\,(y-1)^{\frac{-2}{1-\kappa}}\,y^{\frac{4\kappa(1+5\kappa)}{(1-\kappa)(7\kappa+1)}}\,\mathrm{d}y^2+y^{\frac{-4\kappa}{7\kappa+1}}\,\mathrm{d}\tilde{x}^2 \tag{10.4}$$

where $\mathrm{d}\tilde{x}^2$ stands for $(2M)^2\mathrm{d}\Omega^2$, and

$$y=\left(\frac{2M}{r}\right)^{\frac{7\kappa+1}{2\kappa}}\;;\qquad a=\lambda^{\frac{4\kappa}{1-\kappa}}\left(\frac{1-\kappa}{7\kappa+1}\right)^{\frac{-2\kappa}{1-\kappa}}\;;\qquad b=\frac{4\kappa^2(7\kappa+1)^{\frac{2\kappa}{1-\kappa}}}{(1-\kappa)^{\frac{2}{1-\kappa}}}\,\lambda^{\frac{4\kappa}{1-\kappa}}(2M)^2\,, \tag{10.5}$$

and the entropy density turns out to be

$$s(r)=\frac{\lambda^2\beta}{8\pi\kappa(2M)^2}(1+\kappa)\,(A(r))^{-\frac{1}{2\kappa}}\,, \tag{10.6}$$

where $A(r)$ stands for the term in front of $\mathrm{d}t^2$ in Eq. (10.4).

The case $\kappa = 1$ is particularly delicate [16]. Here we just quote the result. We now find 8 regions, and of these, region 7 (which includes the origin $r = 0$) contains the most interesting physics. The λ dependence turns into an exponential one:

$$A = \lambda^{7/2}\left(\frac{2M}{r}\right) e^{\frac{1}{\lambda^2}\left[\left(\frac{r}{2M}\right)^4 - 1\right]}, \qquad B = \lambda^{-1/2}\left(\frac{r}{2M}\right) e^{\frac{1}{\lambda^2}\left[\left(\frac{r}{2M}\right)^4 - 1\right]}. \tag{10.7}$$

The entropy density is

$$s = \frac{\lambda^2\beta}{8\pi\kappa(2M)^2}(1+\kappa)A^{-\frac{1}{2\kappa}}, \tag{10.8}$$

so that

$$\frac{S}{\Sigma} = \int_0^{2M}\left(\frac{r}{2M}\right)^3 \frac{\mathrm{d}r}{2M} = \tfrac{1}{4}. \tag{10.9}$$

References

[1] S.W. Hawking and G.F.R. Ellis, "The Large Scale Structure of Space-time", Cambridge: Cambridge Univ. Press, 1973.

[2] S.W. Hawking, Comm. Math. Phys. **43** (1975) 199.

[3] R.M. Wald, Commun. Math. Phys. **45** (1975) 9; W.G. Unruh, Phys. Rev. **D14** (1976) 870;

[4] G. 't Hooft, J. Mod. Phys. **A11** (1996) 4623 (gr-qc/9607022)

[5] J.D. Bekenstein, Nuovo Cim. Lett. **4** (1972) 737; Phys. Rev. **D7** (1973) 2333; ibid. **D9** (1974) 3292; R.M. Wald, Phys. Rev. **D20** (1979) 1271

[6] S.W. Hawking, Phys. Rev. **D14** (1976) 2460; Commun. Math. Phys. **87** (1982) 395.

[7] G. 't Hooft, Phys. Scripta **T15** (1987) 143; Nucl. Phys. **B335** (1990) 138

[8] G. 't Hooft, Nucl. Phys. **B256** (1985) 727

[9] A. Linde, Phys. Lett. **B259** (1991) 38

[10] P.C. Aichelburg and R.U. Sexl, J. Gen. Rel. Grav. **2** (1971) 303; T. Dray and G. 't Hooft, Nucl. Phys.**B253** (1985) 173.

[11] L. Susskind, L. Thorlacius and J. Uglum, Phys. Rev. **D48** (1993) 3743 (hep-th 9306069); G. 't Hooft, "Dimensional Reduction in Quantum Gravity", Essay dedicated to Abdus Salam, Utrecht preprint THU-93/26; gr-qc/9310026

[12] P.C.W. Davies, S.A. Fulling and W.G. Unruh, Phys. Rev. **D13** (1976) 2720

[13] W.G. Unruh and R.M. Wald, Phys. Rev. **D29** (1984) 1047.

[14] R.C. Tolman, *Relativity, Thermodynamics, and Cosmology* (Clarendon, Oxford, 1934), p.p. 242-243; R. Oppenheimer and G. Volkoff, Phys. Rev. **55** (1939) 374.

[15] W.H. Zurek and D.N. Page, Phys. Rev. **D29** (1984) 628.

[16] G. 't Hooft, "The self-screening Hawking atmosphere", Proc. Strings97, Amsterdam, June 1997, to be publ. (gr-qc/9706058); see also: R.D. Sorkin, R.M. Wald and Z.Z. Jiu, Gen. Rel Grav. **13** (1981) 1127

[17] H. Bondi, Proc. R. Soc. London **A282** (1964) 303

[18] L.D. Landau and E.M. Lifshitz, *Fluid Mechanics*, Pergamon Press, 1959, p. 500.