NUCLEAR PHYSICS B
PROCEEDINGS
SUPPLEMENTS

Nuclear Physics B (Proc. Suppl.) 30 (1993) 200–203
North-Holland

COSMOLOGY IN 2+1 DIMENSIONS

G. 't Hooft[a]

[a] Institute for Theoretical Physics
University of Utrecht, P.O.Box 80 006
3508 TA Utrecht, the Netherlands

Summary

Classical (unquantized) gravitating point particles in 2+1 dimensions may seem to be a rather trivial system. There are no gravitational waves and hence no gravitons; space-time is locally flat, and the particles do not even attract each other when they are not moving. And even if they move, the system is exactly solvable [1]. Its physical interest could be that it describes the gravitational interactions of "cosmic strings" if these are all oriented parallel to each other. But of much more interest is the fact that it could serve as a prototype of a completely solvable classical cosmological model. One can play the game of creating and destroying toy universes, and attempts to find a quantum version of this system may be extremely instructive.

There are various claims that 2+1 dimensional gravity "can be quantized" or "is a renormalizable theory" [2], but these refer to the infinite number of degrees of freedom $g_{\mu\nu}(\mathbf{x},t)$ in stead of the finite number of coordinates and momenta of the particles. These particles are the "topological defects" of the theory. A particle at rest produces a conical singularity in the locally flat 2-space. If a particle moves one easily derives the structure of the surrounding space-time by performing a Lorentz transformation on a particle at rest. To follow directly the movements of the coordinates of such particles is a highly relevant and surprisingly complex exercise.

The first surprise was noted by J.R. Gott [3]. He discovered that if two particles are given a sufficiently large relative momentum the surrounding space-time apparently admits the presence of a closed timelike curve. The present author reacted upon this by stressing that such a phenomenon should not occur in a more precisely defined 2+1 dimensional cosmology, and searched for the cause of this apparent paradox. It was first established by various groups [4] that what is now called a "Gott pair" cannot exist in an open universe with physically acceptable boundary conditions; the combined gravitational fields are so powerful that the universe will be closed. However, Carroll, Farhi and Guth [5] suggested that a Gott pair can occur in a closed universe with physically acceptable initial conditions, namely when two heavy particles at $t = 0$ simultaneously decay, after which the decay products may meet each other head-on.

How can this observation be reconciled with causality requirements of this model? The resolution to this problem was our second surprise. The author [6] discovered that this system is much more complex than one would anticipate, and that if one tries to describe that piece of space-time surrounding such a Gott pair that would host a closed time-like curve, one actually finds that it is not devoid of other particles. Worse even, the entire universe ceases to exist before the closed timelike curve had a chance to form. There is a "final crunch" and this crunch replaces the closed timelike curve. Thus, if one considers an entire cosmology, nature itself prevents the occurrence of any closed timelike curves and the associated threat of violation of causality.

How could I have been so sure that there "had to" be a cure? The answer lies in the fact that one can formulate the construction of solutions of Einstein's equations by using Cauchy surfaces. A Cauchy surface is a spacelike surface that divides the entire universe into two pieces: the past and the future. By giving all necessary data on a Cauchy surface one can construct a new Cauchy

0920-5632/93/$06.00

surface in the near future, and by continuing this way one should be able to cover the entire universe with Caucy surfaces. But by definition a timelike curve can cross a given Cauchy surface only once, and hence it can never close into itself. So, in order to figure out what happened here I constructed Caucy surfaces in the neighborhood of a Gott pair.

Since space-time is locally flat I chose my Cauchy surfaces to consist of patches of flat surfaces, "polygons" glued together. Within every polygon the equation for the surface, $t = t_1$, defines a preferred Lorentz frame. At the seam between two polygons, the two corresponding Lorentz frames define a Lorentz transformation, needed if one wants to go from one polygon into the other. If we furthermore wish that in these preferred frames time always runs equally fast, one has to choose the seam between two polygons such that its velocity, as seen in both frames, is equal in magnitude but opposite in direction. A *particle* must be located at a corner point of a polygon, and the wedge it cuts out of flat space-time corresponds to the two adjacent edges of this single polygon, which have to be glued together. If one crosses this wedge one also has to perform a Lorentz transformation, but reenters into the same polygon.

At points where three polygons meet one has the Lorentz boosts η_i , $i = 1,2,3,\dots$, and the three angles α_i , of the three polygons. Suppose at this point space-time is flat (there is no particle directly on this point, which should be the generic situation). By multiplying the corresponding Lorentz boosts one then finds a couple of equations that the η_i and α_i should satisfy. Let η_i be the boost parameters for the edges, and $2\eta_i$ the boosts from the k^{th} polygon to the j^{th} , $i,j,k = 1,2,3$. Define

$$s_i = \sin\alpha_i\,, \quad c_i = \cos\alpha_i\,, \tag{1}$$

$$\sigma_i = \sinh(2\eta_i)\,, \quad \gamma_i = \cosh(2\eta_i)\,, \tag{2}$$

then we have

$$s_1 : s_2 : s_3 = \sigma_1 : \sigma_2 : \sigma_3\,, \tag{3}$$

$$\gamma_2 s_3 + s_1 c_2 + c_1 s_2 \gamma_3 = 0\,, \tag{4}$$

$$c_1 = c_2 c_3 - \gamma_1 s_2 s_3\,, \tag{5}$$

$$\gamma_1 = \gamma_2\gamma_3 + \sigma_2\sigma_3 c_1\,, \tag{6}$$

$$\begin{aligned} \cot\alpha_2 &= -\cot\alpha_1\cosh(2\eta_3) \\ &- \coth(2\eta_2)\sinh(2\eta_3)/\sin\alpha_1\,, \end{aligned} \tag{7}$$

and all cyclic permutations[1].

Thus, the requirement that time runs equally fast on each polygon essentially fixes the velocity with which the edges of each polygon contract or recede, once the shapes have been given. Now when we follow these movements as a function of our Cauchy time variable, we find that occasionally an edge may shrink to zero. The two matching polygons will then cease to have one edge in common and evolve further disconnectedly, whereas two other polygons that were not connected before now grow a common edge. All angles of this new edge, as well as its velocity as seen in both frames, are completely determined by the above equations.

But this is not the only way the configuration of polygons will change as a function of time. The polygons will in general not be convex. So it may happen that one corner of a polygon hits one of the edges at the opposite side. The polygon then splits in two, and new boundary lines will grow at the point where the opposite edge was hit. Furthermore, a polygon may shrink as a whole. Its final shape will always be a triangle or a kite (if it contains a particle), and then the polygon may disappear altogether. All in all, these transitions turn this model into a complex, discrete, dynamical system. The shapes of the polygons after any transition is always completely fixed by the equations.

Upon studying these equations more closely one powerful theorem was discovered:

- *Suppose that on the Cauchy surface at one given time $t = t_0$ all edges of all polygons are contracting* (i.e. all Lorentz boosts η_i have the same sign such that the polygons all decrease in size) , *then this situation will persevere at all future times.*

Indeed, the boosts will tend to increase. The universe will shrink faster and faster, and nothing can divert a disaster: the final crunch. And while

[1] One may recognize in these equations the relations between the sides and angles of a triangle on a hyperbolic sphere.

this happens the particles will continue to fly past each other at ever decreasing distances (or scattering parameters). The particles in this universe can be compared with a ping-pong ball between a bat and a table, while the bat is pressed rapidly against the table. We could establish that just before the big crunch an infinite series of polygon transitions takes place, not unlike the infinite number of bounces of the ping-pong ball before it is pressed hard against the table. There seems to be no way to extend the space-time to periods after the crunch.

Of course the time reverse of the big crunch is the big bang. If at one instant all edges of all polygons widen then our theorem tells us that there must be a big bang in the past. Our theorem gives a sufficient condition for the crunch to occur, but it is by no means a necessary one. Indeed, one can construct universes with both a big bang in the past and a big crunch in the future. The converse, a universe with neither a bang nor a crunch, seems not to be possible, but we have not examined this possibility closely yet.

There are even more surprises. One is the extreme complexity of the model; it can behave in a near chaotic way, in particular near the crunch. Another is that as the big crunch approaches the particles move with exorbitant energies; the Lorentz γ factors at which they are boosted explode as $\exp\{1/(t-t_0)^\kappa\}$, if t_0 is the moment of the crunch. Here κ is some power. It may also come as a surprise that the topology of 2-space needs not be that of $S(2)$; it may be a torus or even a Riemann space with higher winding numbers (as long as time runs in a fixed direction). At first sight namely this seems to be at odds with energy conservation: the total Pontryagin number of the universe, 4π for the sphere, 0 for the torus, and negative for higher topological spaces, represents the total energy. If all particles carry a positive mass and do not attract each other when at rest, one would easily have thought that the total Pontryagin number would have to be positive, so that only the sphere is allowed. But this is not at all true. One reads off from the equations that if the polygons are moving with respect to each other the angles α_i are such that they can easily build up negative curvature. Apparently the gravitational fields of moving particles often harbors negative energy, just as in the four-dimensional case.

Yet another surprise is the total number of physical degrees of freedom of the N-particle system in a universe with winding number g. When one counts one finds for this number D at given time t:

$$D = 4N + 12g - 11\,, \tag{8}$$

hence an odd number! Usually one has a matching number of coordinates and momenta. One can understand the situation by realizing that time t is really a variable that has no physical meaning, just like the choice of the origin in 2-space. One should remove this unobservable parameter, and then one gets

$$D' = 4N + 12g - 12\,. \tag{9}$$

Further work on this topic is under way. A simple simulation program on a computer showed that there are a few more possible polygon transitions and a minor complication (there could be self-overlapping polygons!). These are relatively unimportant and do not affect any of our results. A description of this work is forthcoming.

Our construction of these 2+1 dimensional cosmologies should shed a new light on attempts to formulate their quantum version [7] (if such a version exists at all...). The fact that time must be removed as an observable suggests that it may be hard to find a Hamilton formulation. Secondly, one has to take into account the very special asymptotic states at the beginning and the end of the universe. A thorough redefinition of what one would like to call an S-matrix would obviously be required.

REFERENCES

1 A. Staruszkiewicz, *Acta Phys. Pol.* **24** (1963) 734; J.R. Gott and M. Alpert, *Gen. Rel. Grav.* **16** (1984) 243; S. Giddings, J. Abbot and K. Kuchar, *Gen. Rel. Grav.***16** (1984) 751; S. Deser, G. 't Hooft and R. Jackiw, *Ann. Phys.* **152** (1984) 220.

2 E. Witten, *Nucl. Phys.* **B 311** (1988) 46; S. Carlip, *Nucl. Phys.* **B 324** (1989) 106, and in:

"Physics, Geometry and Topology" *NATO ASI Series B, Physics*, Vol 238, H.C. Lee ed., Plenum Press, p. 541.

3 J.R. Gott, *Phys. Rev. Lett.* **66** (1991) 1126; A. Ori, *Phys. Rev.* D **44** (1991) R2214; C. Cutler, *Phys. Rev. D* **45** (1992) 487.

4 S. Deser, G. 't Hooft and R. Jackiw, *Phys. Rev. Lett.* **68** (1992) 267; D.N. Kabat, *MIT preprint CTP 2034*, 1991; H. Waelbroeck, *UNAM Mexico preprint*, 1991.

5 S.M. Carroll, E. Farhi and A. Guth, *Phys. Rev. Lett.* **68** 267 (1992).

6 G. 't Hooft, *Class. Quantum Grav.* **9** (1992) 1335.

7 G. 't Hooft, *Comm. Math. Phys.* **117** (1988) 685.