Class. Quantum Grav. 10 (1993) S79–S91. Printed in the UK

# Classical $N$-particle cosmology in 2 + 1 dimensions

G 't Hooft

Institute for Theoretical Physics, University of Utrecht, PO Box 80 006, NL 3508 TA Utrecht, The Netherlands

**Abstract.** In 2 + 1 dimensional cosmology particles are topological defects in a universe that is nearly everywhere flat. We use a time dependent triangulation procedure of space to formulate the laws of evolution and to construct phase space of this system. A theorem is derived from which it follows that many configurations may have a big bang in their past or a big crunch in their future. The dimensionality of phase space for $N$ particles is $4N - 11 + 12g$, where $g$ is the genus of 2-space. This suggests to consider $2N - 6 + 6g$ pairs of canonical variables and one time variable. The quantum version of this model is speculated about.

## 1. Introduction

In 2 + 1 dimensional General Relativity [1,2] Einstein's equations for the vacuum,

$$R_{\mu\nu} = 0 \tag{1.1}$$

imply also that the entire Riemann tensor $R^{\alpha}_{\beta\mu\nu}$ vanishes. Hence the vacuum is flat. A particle at rest, at the position $\boldsymbol{x} = \boldsymbol{a}$, produces a curvature proportional to $\delta^2(\boldsymbol{x} - \boldsymbol{a})$. From that the global structure of space-time surrounding the particle can easily be seen to be a cone, and it is described by excising a wedge out of the plane (see figure 1), after which one identifies points at each side of the wedge (for instance the two arrows in figure 1 are identified). The angle of the wedge, the conical deficiency angle $\beta$, is proportional to the mass $M$ of the particle:

$$\beta = 2\pi GM \tag{1.2}$$

where $G$ is Newton's constant, which we will choose to be one from now on.

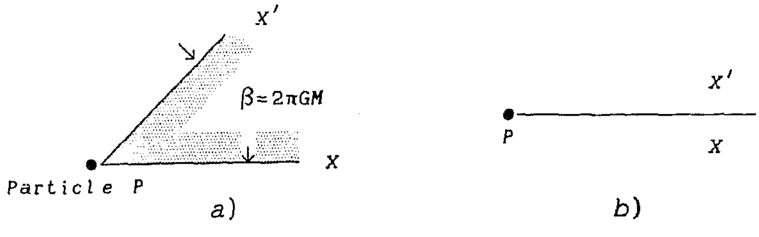


**Figure 1.** ($a$) Excised wedge near a particle $P$ (shaded). ($b$) Diagrammatic notation.

0264-9381/93/0S0079+13$07.50 

If we wish to use Cartesian coordinates to describe locations in the neighbourhood of this particle we have to attach strings to each of these locations, and then the coordinates depend on how a string attached as indicated in figure 1 can also be described by the coordinates of $\boldsymbol{x}'$ with string as drawn, provided that

$$\boldsymbol{x}' = \boldsymbol{a} + \Omega(\boldsymbol{x} - \boldsymbol{a}) \tag{1.3}$$

where $\boldsymbol{a}$ is the location of $P$ and $\Omega$ is the rotation

$$\Omega = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \qquad \beta = 2\pi M. \tag{1.4}$$

To describe a *moving* particle, one has to Lorentz transform this space-time. The relationship between the two coordinate frames then becomes

$$\begin{aligned} \begin{pmatrix} \boldsymbol{x}' \\ t' \end{pmatrix} &= L\left\{ \begin{pmatrix} \Omega & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{x} - \boldsymbol{a} \\ t \end{pmatrix} + \begin{pmatrix} \boldsymbol{a} \\ 0 \end{pmatrix} \right\} L^{-1} \\ &= A \begin{pmatrix} \boldsymbol{x} \\ t \end{pmatrix} + \begin{pmatrix} \boldsymbol{b} \\ b^0 \end{pmatrix} \end{aligned} \tag{1.5}$$

where $L$ is the Lorentz transformation that gives the particle the velocity $\boldsymbol{v}$ from its rest frame. $A$ is a new Lorentz transformation and $\boldsymbol{b}$ and $b^o$ are some shift vectors. Note that now the particle, its mass and its velocity, are determined by just giving an element of the Poincaré group. However not all elements of the Poincaré group specify a particle because the particle's space-time trajectory is given by the points satisfying

$$(\boldsymbol{x}', t') = (\boldsymbol{x}, t) \tag{1.6}$$

an equation that for generic elements of the Poincaré group has no solutions, as we will see.

Consider now two particles, moving with respect to each other. Following a path around both we find that points $x$ and $x'$ are identified by the product of two transformations of the type (1.5) (see figure 2).

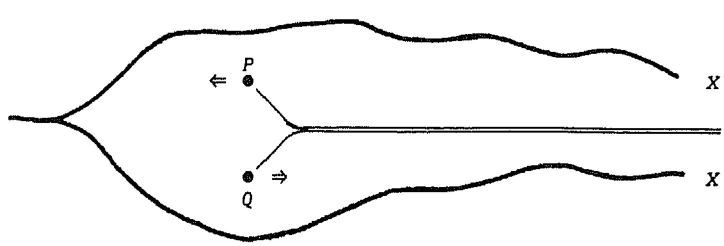


**Figure 2.** Two moving particles, $P$ and $Q$.

The lines in figure 2 indicate the separation between the different coordinate frames used. The relation between the different coordinate frames for the point $x$, indicated as $x$ and $x'$, is obtained by the product of the transformations (1.5) for the two particles $P$ and $Q$. Let us now consider the *center of mass frame*. This is the coordinate frame that is chosen such that

$$\begin{pmatrix} \boldsymbol{x}' \\ t' \end{pmatrix} = A_{cm} \begin{pmatrix} \boldsymbol{x} \\ t \end{pmatrix} + \begin{pmatrix} \boldsymbol{b} \\ b^o \end{pmatrix}_{cm} = \begin{pmatrix} \Omega(\boldsymbol{x} - \boldsymbol{a} \\ t \end{pmatrix} + \begin{pmatrix} \boldsymbol{a} \\ \delta t \end{pmatrix} . \qquad (1.7)$$

In general one can indeed find a Lorentz frame such that $A_{cm}$ has only non-trivial elements in the first 2×2 block, as indicated, so that it is a pure rotation. It is natural to define the corresponding rotation angle to be the center of mass energy. Also one can find a vector $\boldsymbol{a}$ such that it acts as the origin of this rotation so that it can be interpreted as the location of the particle. But the new thing is the additional *time shift* $\delta t$ as we encircle the two particles. It is not hard to verify that $\delta t$ can be identified as being the *total angular momentum* [2]. The angles $\beta$ over which one rotates are additive, and so are the time shifts $\delta t$. Thus one recovers the conservation laws of energy and angular momentum.

A complication is now that one can no longer define the trajectory of the center of mass as being the set of points invariant under (1.7). If the time shift is non-zero there are no invariant points. One has to go to the center of mass frame to locate the center of mass.

It is not difficult to derive the center of mass energy. Since the trace of a Lorentz matrix is Lorentz invariant one finds the cosine of the deficiency angle for the center of mass by taking the trace of the product of the Lorentz matrices corresponding to the two individual particles. The outcome of that (simple) calculation is [2]:

$$\cos \pi m_{cm} = \cos \pi m_1 \ \cos \pi m_2 - \sin \pi m_1 \ \sin \pi m_2 \ \cosh \gamma \qquad (1.8)$$

where $m_{1,2}$ are the masses of the particles 1 and 2; $m_{cm}$ is the center of mass energy; $\gamma$ is the Lorentz boost parameter connecting the rest frame of 1 with that of 2. Note that in this expression we have the trigonometric functions of half the deficiency angles in stead of the angles themselves. This is because it is more convenient to work with the $SL(2, R)$ representation of the Lorentz group than it is with the $SO(2,1)$ representation.

We see that there only is a center of mass if the condition

$$\cosh \gamma \leqslant \frac{\cos \pi m_1 \ \cos \pi m_2 + 1}{\sin \pi m_1 \ \sin \pi m_2} \qquad (1.9)$$

is satisfied. If this condition is *not* met $P$ and $Q$ are said to be a Gott pair [3]. The identification of the Cartesian coordinate frames connected by a loop around the particles is a Lorentz boost rather than a rotation. Such a pair can therefore always be surrounded by a closed timelike curve (CTC). If no other particles occur it cannot be avoided that the universe is surrounded by an unphysical boundary condition [4, 5], even though CTC will *not* occur if we go sufficiently far to the past or to the future [6].

An open universe has an acceptable boundary if the total energy (corresponding to the sum of all deficiency angles in one coordinate frame divided by $2\pi$) is less than 1. A universe closes if its energy is exactly 2. It then has the topology of a sphere ($S_2$, $g = 0$). But a torus ($g = 1$) or even 2-dimensional spaces with higher genus are also possible. The total energy vanishes for $g = 1$ and is negative in higher genus spaces. This is not in contradiction with positivity of the rest masses of the particles because when they move there can be negative energy in the surrounding gravitational fields as we shall see.

S M Carroll *et al* [5] proposed to consider a universe where the following happens. At $t = 0$ both $M_1$ and $M_2$ decay simultaneously each into a pair of light particles with masses:

$$M_1 \to m_1\ m_2 \qquad M_2 \to m_3\ m_4. \tag{1.10}$$

One easily finds that the two particles $m_1$ and $m_2$ created by the decay of $M_1$ move away from each other at some angle $\pi(1 - M_1)$ (see figure 3). With respect to each other they can never violate condition (1.9) because their center of mass energy is $M_1$. But $m_1$ and $m_3$ can meet each other head-on, and thus their relative boost parameter $\gamma$ can exceed the value (1.9). Thus they may form a Gott pair. It turns out that this may only happen if

$$\mathrm{tg}^2 \tfrac{1}{2}\pi M_1 > \mathrm{tg}^2 \pi m + 1 \tag{1.11}$$

so that we must have $\frac{1}{2} < M_1 < 1$. This implies that the universe must be closed. So we add an 'antipode particle' $X$ with mass $M_X = 2 - 2M_1$. The problem raised by these authors was the question whether closed timelike curves can then also arise in this universe.

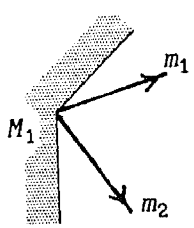


**Figure 3.** $M_1 \to m_1\ m_2$.

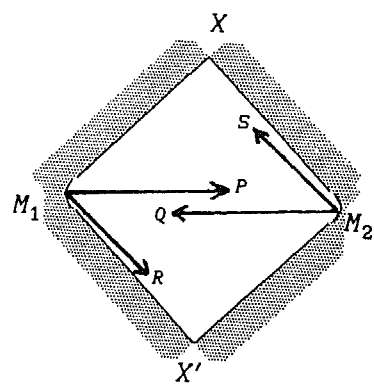


**Figure 4.** The CFG university at $t = 0$.

## 2. Triangulation and Cauchy surfaces

We now propose an approach to 2+1 dimensional gravity/cosmology that starts with construction of Cauchy cross sections of space-time. These are purely spacelike surfaces that are time-ordered, and together span the entire universe. We make optimal use of the fact that in between the particles space-time is flat. The ideal method is 'triangulation', which implies that we consider Cauchy surfaces that are built from entirely flat triangular 2-simplexes, glued together [7]. In practice however it is easier to take polygons rather than triangles, because in the generic case the vertex points will not connect more than three simplexes together. The general picture of such a Cauchy surface is sketched in figure 5, although in practice often the polygons will look somewhat more complex. Indeed, ultimately the most economic description will be using just one polygon, with rather elaborate prescriptions as to how the edges are to be sewn together.

All particles in our theory must be at vertex points, but vertex points where no particles sit are of course also allowed. In general we will put a particle at a 1-vertex; at such a point the adjacent sides of the corresponding polygon are glued together leaving the appropriate deficiency angle.

If all particles are at rest with respect to each other then the situation is easy to visualize. At all vertices we can take the angles to add up to exactly $2\pi$, unless there is a particle present, and then the angles add up to $2\pi - 2\pi m$. If the total mass exceeds 1, then the Cauchy surface closes, so that there must be further particles at the 'antipodes'. The total mass then automatically adds up to 2.

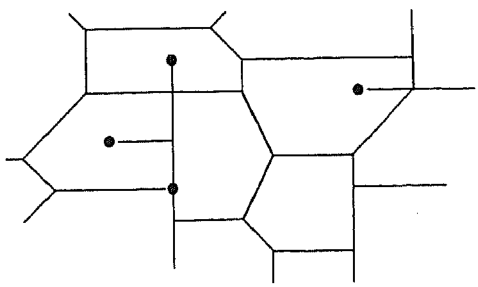

**Figure 5.** Cauchy surface built from polygons. The heavy dots are particles.

But if the particles move the situation is more complex. The polygons become deformed as a function of time, and in general the boundary between one polygon and another will involve a Lorentz transformation. It is now very instructive to formulate the rules that the moving polygons have to obey, if we insist that at the seams between all pairs of polygons space-time remains locally flat.

We consider in each polygon (which is actually a 3-simplex in space-time) a Cartesian coordinate frame. In general the sides all move in this frame, and their lengths change. Now in principle we could allow that on each polygon time runs at different speeds (as long as time runs forward), but we will impose the simplifying extra requirement that time runs equally fast at each polygon. In that case the evolution is uniquely determined, and furthermore the rules at the seams [7] become very simple:

1) The lengths of matching edges of two adjacent polygons, as measured in the coordinate frame of each, are equal. This is not a completely trivial statement because the matching goes with a Lorentz transformation.

2) The velocities with which the matching edges of two adjacent polygons move (always in a direction orthogonal to the orientation of the edge) in the coordinate frame of each, are the same, but the signs may differ. In general the signs are such that if the edge of one polygon recedes, the matching edge in the other one recedes also, because in the other case the matching becomes trivial.

3) The identification of points at two matching edges is such that a point moving in an orthogonal direction on one edge, remains identified with a point moving orthogonally on the matching edge, as seen in the corresponding frames.

4) The vertices between three polygons move in such way that rules 1, 2 and 3 remain valid. When new vertices are created (one vertex may split into several), special attention should be paid to whether they represent flat space or particles. Often this comes out all right automatically because of energy-momentum conservation at such space-time points.

The proof that polygons describing locally flat space can only move according to the above rules is not difficult. We will not here repeat the arguments already given in [7].

It is important to note that when a particle moves relatively to the coordinate frame chosen for the polygon that it is in, its velocity vector is only allowed to be in the direction of the bisector of the cusp, see figure 6.

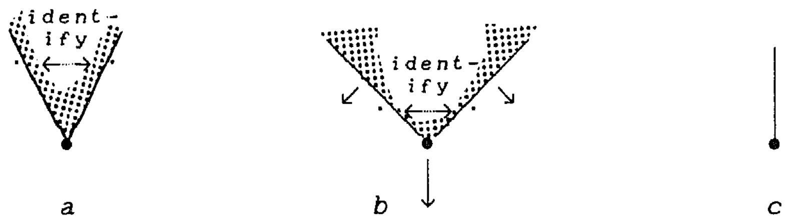


**Figure 6.** (*a*) Particle at rest. The shaded region is excised from flat space and the two edges are glued together. The two dots are points to be identified. (*b*) Particle moving downwards (arrow). We get the Lorentz contraction of picture (*a*), therefore the angle widens. Since the two dots in (*a*) are at the same height, they still will be at the same height in (*b*), and after the Lorentz transformation there will be no relative time shift. (*c*) Diagrammatic notation for a particle.

If $\xi$ is the particle's rapidity, we see that a moving particle is Lorentz contracted by a factor $\cosh\xi$. Therefore it gets a widened deficiency angle $\beta$:

$$\mathrm{tg}\,\tfrac{1}{2}\beta = \mathrm{tg}(\pi m)\cosh\xi. \tag{2.1}$$

The boost parameter $\eta$ describing the velocity $\mathrm{th}\eta$ with which the adjacent edges of the polygon widen or retract, is given by

$$\mathrm{th}\eta = \sin\tfrac{1}{2}\beta\ \mathrm{th}\xi. \tag{2.2}$$

Later it will turn out to be useful to eliminate $\xi$ or $\beta$ out of these expressions. One then obtains

$$\cosh\eta\ \cos\tfrac{1}{2}\beta = \cos\pi m \tag{2.3a}$$

$$\sinh\xi\ \sin\pi m = \sinh\eta. \tag{2.3b}$$

When three polygons meet at a vertex, see figure 7(*a*), and if there is no particle present precisely at this vertex (in the generic case there if none), then we can use the fact that space-time is locally flat here (note that although we talk about three different polygons, they could equally well be different regions of just one polygon, meeting at this particular vertex). The three coordinate frames can be seen as three points in a hyperbolic space (figure 7(*b*)). Since the sides of two adjacent polygons recede or widen with the same boost factor $\eta$ in each coordinate frame, the distance between these two frames is the boost given by $2\eta$. Hence in hyperbolic space the lengths of the sides of the triangle formed by the frames I, II and III are $2\eta_i$. The angles $A_i$ in this triangle correspond to the angles $\alpha_i$ of the three polygons at the vertex, as follows:

$$A_i = \pi - \alpha_i . \tag{2.4}$$

These observations make it easy to understand the relations at each vertex as they were reported in ref. @: writing

$$\sin\ \alpha_i = s_i \qquad \cos\ \alpha_i = c_i \qquad \sinh\ 2\eta_i = \sigma_i \qquad \cosh\ 2\eta_i = \gamma_i . \tag{2.5}$$

It was found [7] that:

$$s_1 : s_2 : s_3 = \sigma_1 : \sigma_2 : \sigma_3; \tag{2.6}$$

$$\gamma_2 s_3 + s_1 c_2 + c_1 s_2 \gamma_3 = 0; \tag{2.7}$$

$$c_1 = c_2 c_3 - \gamma_1 s_2 s_3; \tag{2.8}$$

$$\gamma_1 = \gamma_2\gamma_3 + \sigma_2\sigma_3 c_1; \tag{2.9}$$

$$\cot\alpha_2 = -\cot\alpha_1\ \cosh 2\eta_3 - \coth 2\eta_2\ \sinh 2\eta_3 / \sin\alpha_1 \tag{2.10}$$

and all cyclic permutations. These are nothing but the trigonometric properties of the triangle of figure 7(*b*) in hyperbolic space.

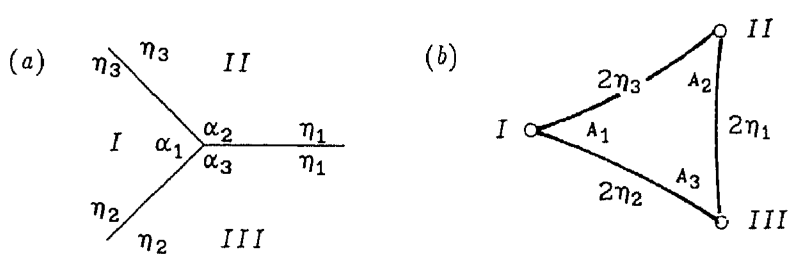


**Figure 7.**

It will be important to observe the ranges of values the angles can take: not more than one of the three angles $\alpha_i$ is allowed to exceed $\pi$. It is not hard to convince oneself that if there would be two angles $\alpha_i > \pi$ then there would be points in space-time that occur at two different spots on our Cauchy surface, which is not allowed. Now suppose one only knew the three boost parameters $\eta_i$ then from (2.9) the three angles can be determined, except for an ambiguity $\alpha \Leftrightarrow 2\pi - \alpha$. But because the relative signs of $\sin\alpha_i$ are fixed by (2.6), and no more than one $\alpha$ is allowed to exceed $\pi$, this ambiguity can be resolved completely. *So the angles are completely determined by boosts* $\eta_i$.

## 3. Evolution

The polygons of the previous section must be seen as fixed time cross sections of 3-simplexes. At any given time $t$ they together form a Cauchy surface. We now may ask how such a Cauchy surface evolves. There are several kinds of mutations that can take place at a given moment. First, an edge connecting two polygons (or different regions of a single polygon) may shrink to zero and be replaced by another edge, see figure 8($a$). It is also possible that a particle hits an edge of the polygon it is in. It then crosses over into the next polygon (figure 8($b$)). When the edge a particle is on shrinks to zero it hops into the adjacent polygon (figure 8($c$). Finally, if one of the angles of a polygon is greater than $\pi$ then that vertex may hit another edge and cause a 'vacuum cross-over' (see figure 8($d$)).

In all these cases the angles and boost parameters of the newly opened edges are all fixed by the trigonometric equations, in particular (2.9) and (2.10). These fix all properties of the edge #1, if the boosts $\eta_2$, $\eta_3$ and the relative angle $\alpha_1$ of the other edges are known. It is easy to understood why all these properties are fixed: if the boosts and the angles of the transformations between frames I and II, frames II and IV, frames II and III, and frames III and IV are well known, then of course the transformation leading from I to IV (see figure 8($a$)) is determined by that. And so are all other new edges in figure 8($a$)–($d$).

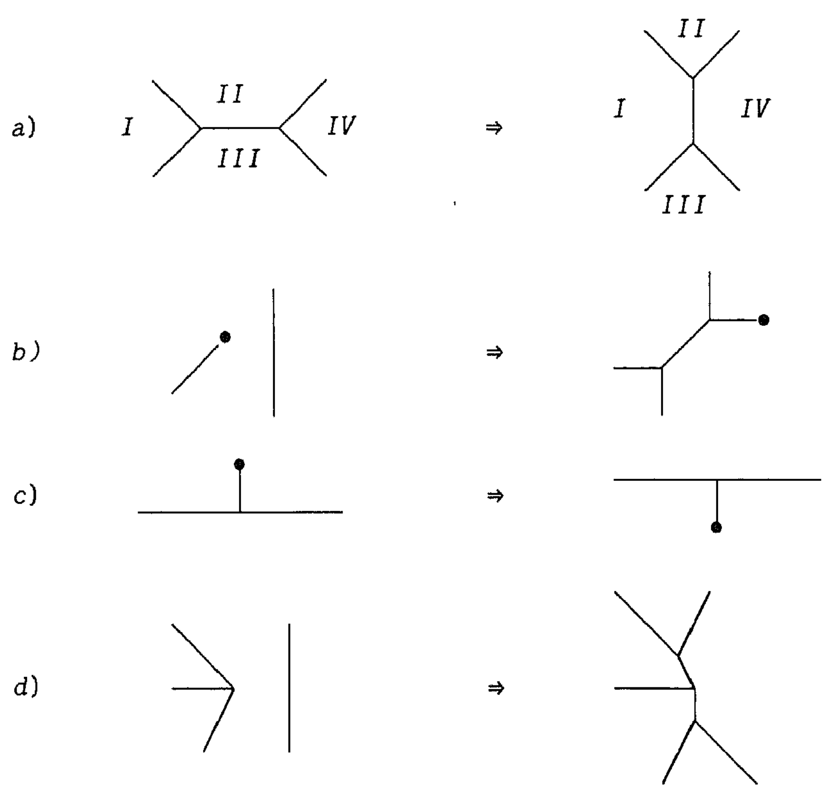


**Figure 8.** ($a$) Exchange; ($b$) cross-over; ($c$) hop; ($d$) vacuum cross-over.

In practice one needs no more than one single polygon, with identification rules at its edges. An example of a five-particle universe is pictured in figure 9(*a*). The identifications among the various edges are indicated by arrows.

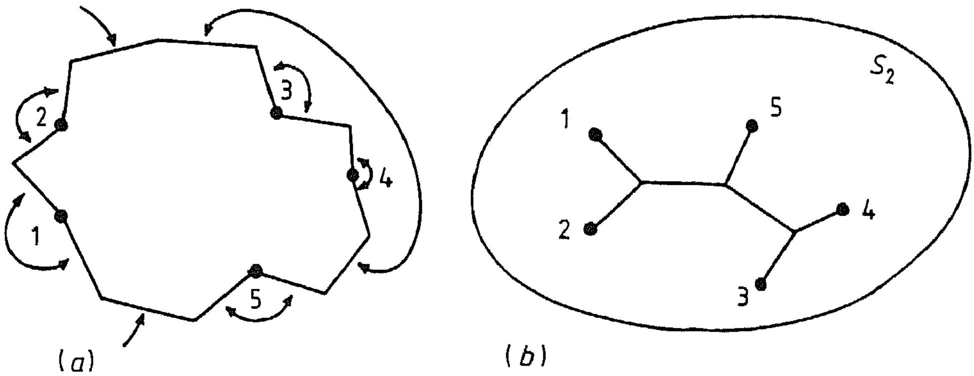


**Figure 9.** (*a*) Polygon representing a universe at given time $t$. (*b*) Diagrammatic notation of the same universe.

It is convenient to indicate the identification rules for the edges by drawing a diagram, obtained by drawing the polygon on an $S_2$ sphere. One can then glue the edges together. The seams obtained are shown in the diagram (figure 9(*b*)).

## 4. Crunch and bang theorem

An important property of 2+1 dimensional universes was indicated in [7]. It can actually be formulated in terms of a theorem:

*Theorem.* Let at one moment $t = t_1$ all edges of all polygons describing a closed or open universe be contracting, which means that all Lorentz boosts $\eta$ at all edges are negative. (We will refer to this condition as the *Crunch condition*), then this condition will remain satisfied at all times $t > t_1$.

Note that the crunch condition implies that at all vacuum vertices (the 3-vertices containing no particle, so that (2.6)–(2.10) apply), the angles $\alpha_i$ are convex: $\alpha_i < \pi$, because due to (2.6) all $\sin\alpha_i$ have the same sign and because of the remark at the end of section 2. Where the particles are, the polygons have angles $\alpha = 2\pi - \beta$, where $\beta$ is given by (2.1) or (2.3*a*). Thus $\alpha > \pi$ if $m < \frac{1}{2}$, and $\alpha < \pi$ if $m > \frac{1}{2}$. So where we have particles with mass $m < \frac{1}{2}$ the polygons are not convex. See figure 9(*a*). Evidently, we will not need figure 8(*d*).

The proof of the theorem can be given by checking all possible mutations, figure 8(*a*)–(*c*), one by one. Consider figure 8(*a*). The angles all start out being convex ($< \pi$). If $\eta$ at the newly opened edge were positive all four adjacent angles would have to be $> \pi$, which is clearly not allowed. We see right away that that is geometrically impossible. So this transition keeps the Crunch situation as it was; the newly opened edge has $\eta < 0$ just like the others.

Next consider the cross-over, figure 8(*b*). This transition is only possible for a particle with mass $< \frac{1}{2}$ (otherwise it could only be involved with the hop transition, figure 8(*c*)). Again the angles start out being all convex. First look at the lower vertex that appears. If the new $\eta$ that arises there would not be negative the two adjacent angles would come out larger than $\pi$, which is not allowed. We can then repeat the argument for the $\eta$ of the edge attached to the particle after it entered the other polygon. One could argue alternatively that the particle as it enters gets an additional boost in the direction of the new polygon it is entering. This way also one finds unambiguously that the $\eta$ of the particle (see equation (2.2)) must again be negative. Thus at this transition also the Crunch condition remains valid.

The case of the hop transition, figure 8(*c*), is again easy to check if the mass is $\geqslant \frac{1}{2}$. Then again it is the same geometrical argument that tells us that it is impossible for the two new angles at the vertex both to exceed $\pi$. If the mass is $< \frac{1}{2}$ this argument does not work. But careful analysis shows the theorem still to hold there. It is again geometry. The particle is entering a polygon at a convex point, where the two edges are contracting. The particle must outrun the edges and hence it must be receding also. It must have a negative $\eta$. We herewith checked all possible transitions. The crunch condition remains valid forever.

Of course the theorem can be time-inverted. Reversing the sign of all $\eta$ coefficients we see that if one has at $t = t_1$ a situation such that all polygons are expanding at all sides, then this expansion must have existed at all times $t < t_1$. This is why we call this theorem the 'Crunch and Bang theorem'. If the Crunch condition is satisfied the universe must end in a Big Crunch. If the converse or 'Bang condition' is satisfied there must be a Big Bang in the past.

The theorem applies to the CFG universe. At the moment that the two particles $M_i$ decay simultaneously into particles $m_i$ all polygons satisfy the Crunch condition. The theorem tells us that the universe will continue to shrink forever. The particles will all approach each other with smaller and smaller impact parameters until they crunch. Before the crunch occurs there cannot possibly be a closed timelike curve simply because we were dealing with Cauchy surfaces all along. The purported CTC only is obtained when analytic extensions of space-time are considered beyond the Big Crunch. But we showed that the crunch singularity is an essential one; all particles meet each other there. Analytic extensions beyond that point are illegal.

The question could be asked whether perhaps the occurrence of a crunch is frame-dependent. But since the particles approach each other with ever decreasing impact parameters, and since in practice we found that at each mutation the crunch accelerates, we found that this question must be answered in the negative. There is no timelike path for which the moment of the crunch, in terms of its eigen time, can be postponed.

## 5. Degrees of freedom

An important question arises in these constructions. We would like to know exactly how to characterize the 'physical' degrees of freedom of 2+1 dimensional cosmologies. This is particularly of importance if one wishes to 'quantize' such models. Several quantization schemes have been proposed. It would be illustrative if a scheme could be devised that starts off with the degrees of freedom of a universe containing a finite number of particles, replacing the Poisson brackets of this finite dimensional phase space by commutators, and it would be of importance to consider the large distance, low mass limit of such a model where it should approach an ordinary Fock space consisting of a small number of non-interacting scalar particles, which we certainly know how to quantize. This question calls for the most economic way to represent a 2+1 dimensional universe with $N$ particles at a given time.

Consider a universe of topology $S_2$, and containing $N$ particles. Take coordinates in the rest frame of one of the particles, say $M_0$. We now construct a Cauchy surface at time $t$, corresponding to a moment $\tau$ in the eigen time of $M_0$. We spread its borders further and further out until points would be included that can be connected by timelike curves. A prescription can be worked out corresponding to moving the borders of our polygon about until a situation is reached that the edges touch each other without any time shifts. A typical situation is then shown in figure 9. The diagrammatic description of figure 9(*b*) is most appropriate. The construction allowed us furthermore to postulate that one particle is at rest: $\eta_1 = 0$. The direction in which the cusp of this particle is rotated is then also immaterial.

The remaining $N - 1$ particles are connected by a tree graph having $L = 2(N-1) - 3$ lines and $V = (N-1) - 2$ vertices. All diagrams of this sort satisfy

$$2L = 3V + (N - 1). \tag{5.1}$$

There is one boost parameter at each line, and these fix all angles at the vertices via the identities (2.6)–(2.10). This gives us

$$L = 2N - 5 \tag{5.2}$$

dimensionless degrees of freedom. Now these also fix the cusp angles $\beta$ where the particles are, via equation (2.3*a*), and all other angles including the cusp of particle $M_0$ were determined. Now since the polygon must close into itself, all external angles, $\pi - a_i$, must add up to $2\pi$. This gives us one more constraint, and so we are left with

$$P = 2N - 6 \tag{5.3}$$

degrees of freedom that determine the *angles* between the *velocities* of the particles. They can be compared with the *momentum* variables of the particles involved.

Then the lengths of all lines are arbitrary, giving us again $2N - 5$ degrees of freedom with the dimension of a length. Now not only the velocity of particle $M_0$ was fixed to be zero, also its position with respect to the other particles is completely determined now by the geometrical exercise of closing the polygon. Therefore we find

$$Q = 2N - 5 \tag{5.4}$$

degrees of freedom with the dimension of a length. It may seem to be surprising that $Q$ is one bigger than $P$ and that it is odd. This surprise disappears if one realizes that *time t* was introduced completely arbitrarily. If we look at the total number of degrees of freedom for *cosmologies*, where time is an undetermined parameter running over a certain domain of real numbers, we find that cosmologies span a

$$P + Q - 1 = 4N - 12 \tag{5.5}$$

dimensional phase space.

In an earlier stage of this research the author thought that 2-spaces with non-vanishing genus should not be allowed, because Euler's theorem would limit the total energy to be zero or negative. Now the total energy is obtained by adding up all deficiency angles of all particles *and* the angular deficits at all vertices where polygons meet. Indeed, moving particles all contribute here by amounts $\beta_i$, all having the same sign. But the deficits at the vertices count in the other direction. If for instance the Crunch condition of the previous section is satisfied then it is easily seen that equation (2.8), with $\gamma > 1$, gives

$$\cos\alpha_1 < \cos(\alpha_2 + \alpha_3) \Rightarrow \sum \alpha_i > 2\pi. \tag{5.6}$$

Thus, the vacuum vertices in general have negative curvature and hence they tend to cancel and reverse the contributions of the particles. We may interpret this as a negative contribution to the energy by the gravitational field. Due to this phenomenon total curvature can easily be negative, so that we can have a torus ($g = 1$, total deficit angles zero) or even higher genus surfaces. In these spaces the particles cannot be at rest; they have to move and the situation is far from stationary.

Careful study of the torus gave us that here

$$P + Q = 4N + 1. \tag{5.7}$$

In the general case the number of degrees of freedom is

$$P + Q = 4N + 12g - 11. \tag{5.8}$$

The one extra positional degree of freedom is again the time parameter. In these less trivial space parameters with dimensions of angles and boosts on the one hand and those with dimensions of lengths and times are not so easy to distinguish anymore.

An open question seems to be how exactly to characterize the topological shape of the spaces (5.3)–(5.8). Our difficulty here is that the transitions of figures 8($a$)–($d$) are mappings of these spaces into themselves; we would like to have definitions of Poisson brackets that are continuous under these mappings. Only then Poisson brackets may be replaced by commutators if one wishes to quantize the theory.

It is natural to view the parameters counted by equation (5.3) as being the 'momenta', they would turn into ordinary momenta of particles in Fock space in the large $N$ limit, an ordinary non-interacting scalar particle model. The parameters counted by (5.4) are the coordinates plus a time variable. Considering the complicated topology of phase space it is however unclear to the present author whether and how a self consistent Fock space can be set up describing a cosmology with a fixed number of particles. An open question is also whether or not these particles, interacting only gravitationally can be pair created or destroyed. Classically (in the $\hbar \Rightarrow 0$ limit) as well as in the large $N$ limit (where we have ordinary Fock space of free scalar particles), this does not happen. Existing quantization schemes [8] suggest that topology change occurs but are inconclusive as to whether particle pair creation and annihilation take place.

## Acknowledgments

The author thanks R Jackiw, S Deser and A Guth for extensive correspondence leading to this work. He thanks H B Nielsen and G Gibbons for discussions concerning the last section.

## References

[1] Staruszkiewicz A 1963 *Acta Phys. Polon.* **24** 734
Gott J R and Alpert M 1984 *Gen. Rel. Grav.* **16** 243
Giddings S, Abbot J and Kuchar K 1984 *Gen. Rel. Grav.* **16** 751
[2] Deser S, Jackiw R and 't Hooft G 1984 *Ann. Phys.* **152** 220
[3] Gott J R 1991 *Phys. Rev. Lett.* **66** 1126
[4] Deser S, Jackiw R and 't Hooft G 1992 *Phys. Rev. Lett.* **68** 267
[5] Carroll S M, Farhi E and Guth A 1992 *Phys. Rev. Lett.* **68** 263 (and A Guth personal communication)
[6] Cutler C 1992 *Phys. Rev.* D **45** 487 (see also Ori A 1991 *Phys. Rev.* D **44** R2214)
[7] 't Hooft G 1992 *Class. Quantum Grav.* **9** 1335
[8] Witten E 1988 *Nucl. Phys.* B **311** 46
Carlip S 1989 *Nucl. Phys.* B **324** 106; 1990 *Physics, Geometry and Topology (NATO ASI Series B, Physics)* **238** ed H C Lee (New York: Plenum) p 541