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Causality in (2+1)-dimensional gravity

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Abstract. A method is presented to characterize fully the evolution of an arbitrary set of spinless particles in (unquantized) (2+1)-dimensional gravity theory. The method produces a complete series of time ordered Cauchy surfaces, which are being triangulated. By construction, closed timelike curves never arise, even if the initial conditions contain a Gott pair. In particular our construction shows that the configuration proposed by Carrol *et al*, in which a Gott pair is formed in a closed universe, nevertheless does not admit closed timelike curves; this universe has a finite lifetime, ending in a 'big crunch'.

1. Introduction

In a recent flurry of papers several authors [1–3] showed that certain solutions of the equations of motion for particles gravitating in 2+1 dimensions [4, 5] admit the presence of 'closed timelike curves' (CTC). The key construction was first proposed by Gott [1] who considered two particles each with mass m and rapidity ξ approaching each other, such that

$$\cosh\xi \sin\pi m > 1. \tag{1.1}$$

Here, rapidity ξ is defined such that $\tanh\xi$ is the velocity v, and the mass m is normalized such that for a particle at rest there is a deficiency angle $\alpha = 2\pi m$. If one were allowed to extend the locally flat spacetime surrounding these particles as far as one wished (i.e. there are no obstructions caused by other particles) then, as Gott showed, a 'space traveller' can circle around these particles in such a way that his journey ends at the same spacetime point as where he started: there is a CTC.

We were quick to point out [6] that this 'Gott pair' of particles is surrounded by a boundary condition that has CTC also at infinity, so if one were to formulate constraining conditions on the boundary, requiring the absence of CTC there, then such a Gott pair could never be formed.

But this seems to be not the complete answer to the question how one can avoid CTC occurring, because it was shown by Carrol, Fahri and Guth [5] that in a *closed* spacetime (which we will refer to as the CFG universe), having no further boundary conditions at all, a pair of particles obeying Gott's condition (1.1) can emerge, in particular if one allows a heavy particle with mass M to decay into two lighter ones. Two such decay products (from two different decaying particles) can accidentally realize a close encounter, and then it seems that the spontaneous creation of a CTC in the immediate neighbourhood of these two particles is inevitable, in particular if all other particles stay far away.

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In this paper we show that although particles obeying (1.1) can indeed occur in a closed or open space, one can still formulate a causal theory for such situations. In an open space one can still give a boundary condition such that there is time ordering at the boundary, and in a closed space one can describe time-ordered Cauchy surfaces. CTC then of course do not form.

But then how can this be reconciled with the Gott/CFG construction? We will show that if the evolution of the particles is followed in a time-ordered manner then this system has a finite lifetime. The 2-volume of this universe decreases monotonically with time until a big crunch ends it all. If one attempts to make the coordinate transformation necessary to produce Gott's CTC one sees that the big crunch forms an obstruction here. Thus, there are no CTC. This paper is to explain how this mechanism takes place.

2. Triangulation of locally flat spacetime

To describe what happens we want to make optimal use of the fact that in between the particles spacetime is flat. The ideal method is 'triangulation', which implies that we consider Cauchy surfaces that are built from entirely flat triangular 2-simplexes, glued together. In practice however it is easier to take polygons rather than triangles, because in the generic case the vertex points will not connect more than three simplexes together. The general picture of such a Cauchy surface is sketched in figure 1, although in practice often the polygons will look somewhat more complex.

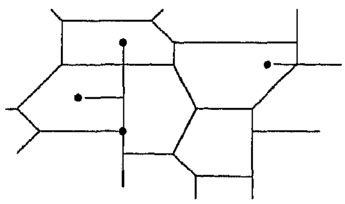

Figure 1. Cauchy surface built from polygons. The heavy dots are particles.

All particles in our theory must be at vertex points, but vertex points where no particles sit are of course also allowed.

If all particles are at rest then the situation is easy to visualize. At all vertices we can take the angles to add up to exactly 2π, unless there is a particle present, and then the angles add up to $2\pi - 2\pi m$. If the total mass exceeds 1 then this Cauchy surface closes, so that there must be further particles at the 'antipodes'. The total mass then automatically adds up to 2.

But if the particles move the situation is more complex. The polygons become deformed as a function of time, and in general the boundary between one polygon and another will involve a Lorentz transformation. It is now very instructive to formulate the rules that the moving polygons have to obey, if we insist that at the seams between all pairs of polygons spacetime remains locally flat. Most importantly, since we are constructing Cauchy surfaces we will never allow for time shifts at the seams.

We consider each polygon in its own rest frame. In general the sides all move in this rest frame, and their lengths change. Now in principle we could allow that on each polygon time runs at different speeds, that is, have g_{00} different for different polygons (as long as time runs forward), but we will impose the simplifying extra requirement that time runs equally fast at each polygon, which corresponds to choosing $g_{00}=1$ everywhere. In that case the evolution is uniquely determined, and furthermore the rules at the seams become very simple.

(1) The lengths of matching edges of two adjacent polygons, as measured in the rest frame of each, are equal. This is not a completely trivial statement because the matching goes with a Lorentz transformation.

(2) The velocities with which the matching edges of two adjacent polygons move (always in a direction orthogonal to the orientation of the edge) in the rest frame of each, are the same, but the signs may differ. In general the signs are such that if the edge of one polygon recedes, the matching edge in the other one recedes also, because in the other case the matching becomes trivial.

(3) The identification of points at two matching edges is such that a point moving in an orthogonal direction on one edge, remains identified with a point moving orthogonally on the matching edge, as seen in the corresponding frames.

(4) The vertices between three polygons move in such a way that rules 1, 2 and 3 remain valid. When new vertices are created (one vertex may split into several), special attention should be paid to whether they represent flat space or particles. Often this comes out all right because of energy-momentum conservation at such spacetime points.

The proof that polygons describing locally flat space can only move according to the above rules is not difficult. One must insist that $g_{\mu\nu}$ stays continuous. Thus, if we take $dt=0$ and dx in the direction of the seam, then $ds^2=dx^2$ must be the same for two adjacent polygons. This proves requirement (1).

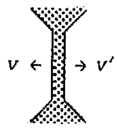

Figure 2. Matching edges of two adjacent polygons.

Now take a line segment (dx, dt) in 3-space on the edge of one polygon. Take the edge to be in the direction dy, so that $dx=v\,dt$, and dy is arbitrary. We have

$$ds^2=(v^2-1)\,dt^2+dy^2 \tag{2.1}$$

and this must be the same for the two adjacent polygons. If the corresponding point on the other polygon had

$$dx'=v'\,dt \qquad dy'=dy+\kappa\,dt \tag{2.2}$$

then

$$ds'^2=(v'^2+\kappa^2-1)\,dt^2+dy^2+2\kappa\,dy\,dt. \tag{2.3}$$

This must coincide with (2.1). Therefore, $\kappa=0$, which proves requirement (3), and $v'=\pm v$, which proves requirement (2).

Requirement (4) has to be checked explicitly when vertices split. At first sight these rules may seem to be trivial but they are far from that, as we will see, in particular because the matching goes with Lorentz transformations at every seam. Indeed, these rules will allow us to construct complete spacetimes with relatively little effort. Since time runs forward everywhere and always, this method allows us to resolve the questions concerning CTC.

The connected edges of two polygons I and II form 2-surfaces in spacetime determined by the equation

$$t_{\mathrm{I}} = t_{\mathrm{II}}. \tag{2.4}$$

The location of this surface in the frame of polygon I completely determines the Lorentz boost of polygon II (apart from an irrelevant spacelike rotation of the frame of polygon II). The resulting equations are given in the appendix.

The most elementary way in which our Cauchy surface can evolve further is pictured in figure 3. At a certain moment t_0 a boundary line between polygons II and III disappears and a new boundary between I and IV forms. As we explained, the angles of the edges of each polygon and the velocities of the edges completely fix the velocities of each frame with respect to the others. Therefore the angles and velocities at the new boundary between I and IV are all completely determined, if there are no particles at the vertices (if there are particles the evolution is as described in the next section). For the technical details we refer to the appendix.

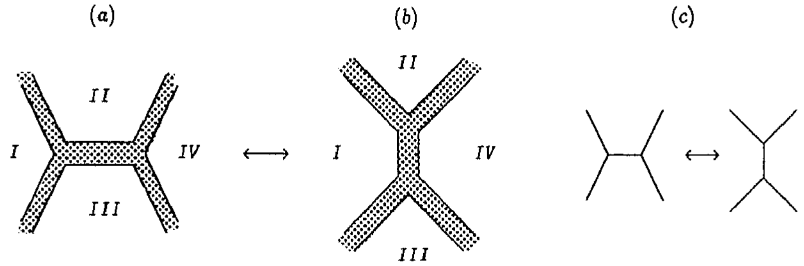

Figure 3. (*a*, *b*) Formation of a new edge in flat space. (*c*) Diagrammatic notation for this event.

3. Evolution of polygons near moving particles

A spinless particle at rest can easily be described in a triangulated spacetime. All we need is a single polygon. The particle is at one of its vertices and the two edges that join at this vertex are glued together. They must have equal length and the receding velocity v at both sides vanishes. The spacetime has its familiar conical structure (figure 4(*a*)).

When a particle moves there are various possibilities, but we will always take the simplest one. Again we take one single polygon, and the particle sits at one of its vertices. The two edges there are glued together, as before, but now they both move with velocity v, and because of requirement (2) these must be equal at both sides. It is easy to see that the geometry implies that, in the frame of this polygon, the particle moves in the direction of the bisector at that point (figure 4(*b*)).

Indeed, this is the only way to describe the spacetime surrounding a moving particle such that the identification at the seam does not involve a time shift. Our prescription will *never* allow for time shifts when two polygons are glued together because then we are no longer dealing with Cauchy surfaces.

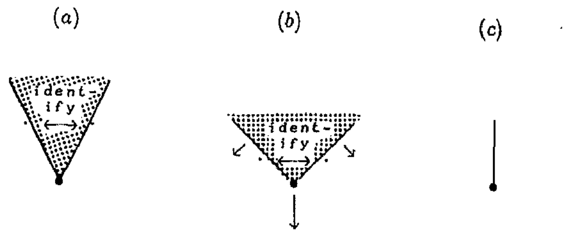

Figure 4. (*a*) Particle at rest. The shaded region is excised from flat space and the two edges are glued together. The two dots are points to be identified. (*b*) Particle moving downwards (arrow). We get the Lorentz contraction of picture (*a*), therefore the angle widens. Since the two dots in (*a*) are at the same height, they still will be at the same height in (*b*), and after the Lorentz transformation there will be no relative time shift. (*c*) Diagrammatic notation for a particle.

The generic situation is then that a particle sits on a vertex at the end of the two identified edges, and in figure 1 three particles are drawn like that. The fourth is in an exceptional position. We see in figure 4 how the moving particle can be obtained from a static one by means of a Lorentz transformation. Only if the excised region is chosen symmetrically with respect to the particle's trajectory does the identification of points across the excised region go without any time shift. We may either choose this excised region to trail behind the particle or to lie ahead of it. In this paper we will let them trail behind.

If ξ is the particle's rapidity, we see that due to the Lorentz contraction the deficiency angle β of the moving particle is related to the one of the static particle $(2\pi m)$ by

$$\tan\tfrac{1}{2}\beta = \tan(\pi m)\cosh\xi. \tag{3.1}$$

The orthogonal velocity of the polygon's edges is given by

$$v = \tanh\eta = \sin\tfrac{1}{2}\beta\,\tanh\xi. \tag{3.2}$$

As long as a particle stays in one polygon its geometry is given as in figure 5. But of course it may happen that a particle will reach the boundary of a polygon and try to enter into the neighbouring polygon. The crossover is described in figure 5(*b*). This figure is obtained as follows. Before the particle P crosses the seam it has a known velocity and a known deficiency angle. In figure 5(*a*) the points A and A′ are identified. The line AA′ must always be orthogonal to the trajectory of P. Now the vertical seam is assumed to move with velocity $v \equiv \tanh\eta$ in both polygons I and II, but in opposite directions. Therefore when the particle enters polygon II its velocity is boosted by a

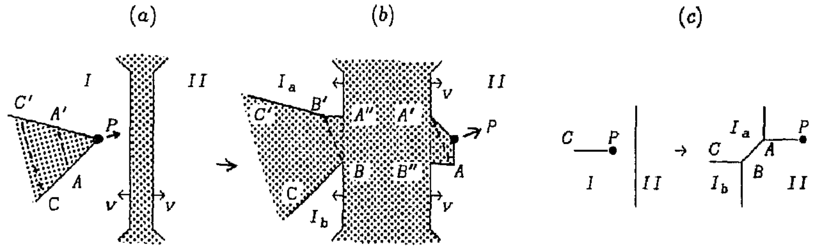

Figure 5. (*a*), (*b*) Particle traversing a seam. Polygon I is split into two polygons I_a and I_b. Shaded regions are the excluded ones. (*c*) Diagrammatic notation for this transition.

Lorentz boost 2η, in the horizontal direction. So we know its new velocity in II. The sign of the vertical component in I and II remains the same. As soon as it is in polygon II the particle P creates two edges PA and PA′ again at equal angles with its new trajectory, of which A′ is on the old edge, and identified with A″ on the edge of I. Similarly there is a line BB′ at the left of which the geometry was as before, and B is on the old seam, to be identified with B″ on the edge of II. Now in general there is no reason why B″, A and P should form a straight line, and in general they will not. Neither will C′, B′ and A″ form a straight line. But one will discover that B″A and B′A″ come out to have equal length. This is because we *know* that at the seam AB there is no curvature and hence our requirements (1)–(4) must be valid.

It is not difficult to convince oneself that all other such crossings will be uniquely defined. Thus any Cauchy surface that was once 'triangulated' with polygons will evolve in a unique manner into new Cauchy surfaces, also triangulated by polygons. Every time a particle crosses a boundary the new Cauchy surface may appear to obtain more edges than the previous ones, but we found that in practice other transitions occur that tend to keep the number of edges limited.

There is a small danger in this procedure. In principle one could obtain particle-like singularities at the new node points A and B. Are those points really flat? Indeed they are, provided that we gave particle P the correct mass and velocity to begin with. In that case energy and momentum conservation prevents the emergence of any particles at A and B. So we obey requirement (4).

A transition that will also often occur is that a particle crosses a seam sideways/backwards. This transition is pictured in figure 6. Here the number of edges remains the same. It occurs if the sides XA and A″Y move faster than the particle. Note that the points A, A′ and A″ are physically identified, a notation that we will be using frequently.

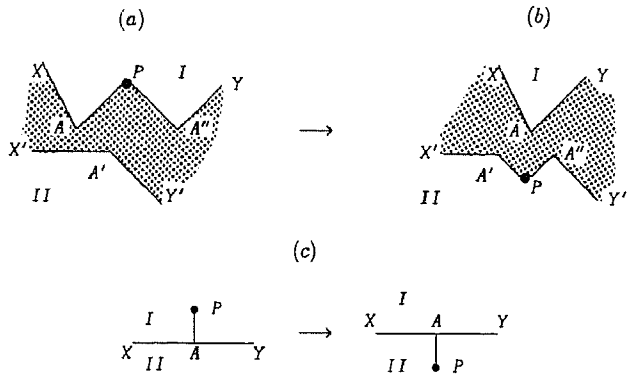

Figure 6. Particle P crosses a seam backwards. In polygon I the edges AP and PA″ disappear and in II new edges A′P and PA″ form. (*c*) Diagrammatic notation.

We made the simplifying requirement that at all polygons time runs equally fast. As stated earlier, one might relax on this requirement, but then the rules are much more complicated. For our purposes it suffices to have time run equally fast everywhere.

In the appendix we give some useful relations that enable one to work out the angles and velocities of new boundaries such as the new edges AB in figure 5 and AP in figure 6(*b*), as they develop during the evolution.

4. The CFG universe

Clearly then, *by construction*, any spacetime described by our triangulated Cauchy surfaces, can never admit any CTC. So how can this be reconciled with the findings of Gott, Guth and others?

In an open universe, Gott's system satisfies bad boundary conditions [6], so that in his Lorentz frame Cauchy surfaces cannot be constructed properly because they cannot be matched with the boundary. However, we *can* consider two particles approaching each other head on with velocities satisfying Gott's inequality (1.1) as an initial condition, and as Cutler showed [2], we then may still *start* with good Cauchy surfaces. Furthermore, the CFG universe is closed. It has no boundary. Its initial conditions are entirely regular. In all these cases we should be able to apply our method. How could the closed timelike curves that these authors found suddenly have disappeared? Let us see what we get. To make our point, the closed CFG universe is more suitable because it is finite, so that our polygons have a finite size.

CFG start with two heavy *static* particles, each having a mass $M_1 = M_2 = M$, which we take to be equal for simplicity. A third particle, X, also static, which we will call the 'antipode', closes the universe. It has mass $2-2M$ (if $2M<1$ we would need more than one antipode particle, but that is not the interesting case). The two particles M_1, M_2 in their (common) rest frame both decay simultaneously into two particles: $M_1 \rightarrow P+R$, $M_2 \rightarrow Q+S$, where P, Q, R and S each have equal masses m and rapidity ξ. It is easy to see that if β is defined as in equation (3.1) then

$$2\pi M = 2\beta \tag{4.1}$$

so that both the parent particle and the pairs of decay products shortly after the decay can be described with a single polygon such that they give the same contribution to the total deficiency angle. The geometry shortly after the decay is pictured in figure 7.

By comparing (3.1) with (1.1) we see that, if the masses m are taken to be fairly small then the Gott condition (1.1) can be satisfied if $\tan \frac{1}{2}\beta$ is somewhat larger than one, so that M has to be somewhat larger than $\frac{1}{2}$, which is why we said that the universe has to be closed and one antipode particle suffices. The decay products can of course go in any direction they want, as long as their centre of mass remains static. We take the case that, accidentally, one decay product P of M_1 makes a close encounter with

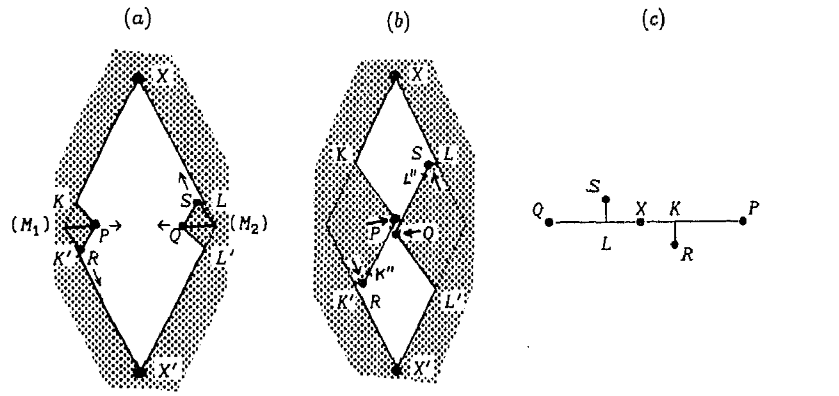

Figure 7. (*a*) The CFG universe shortly after the decays. (*b*) At the time of close encounter. (*c*) Diagram, to be obtained by pulling the polygon over a 2-sphere, and gluing the edges together.

a decay product Q of M_2. They miss each other by a small but non-vanishing impact parameter b. The geometry is given by figure 7(a). The particle R moves close to the boundary KX, which is identified with the line K'X', and similarly the particle S.

Just after the close encounter the geometry is as pictured in figure 7(b). The question is now how they proceed. The answer given by Gott would be the following. We rotate the excised regions near the particles such that they remain turned away from each other. One gets the geometry of figure 8.

Near P and Q the cusps are chosen in a direction orthogonal to their trajectories. This is of course against the rules of section 2, and therefore there are now time shifts. In the rest frame of P the points A and A' would be identified without time shift, but in the moving frame there is a time shift, and because of that the points A and A' move far apart in our reference frame, as drawn (our reference frame can be seen to be the one of the antipode particle X). The points B and B' are identified with the opposite time shift because the velocity of Q is opposite to that of P. Under Gott's condition (1.1) these time shifts can become so large that a CTC is possible (the trajectory ABB'A'A in figure 8). Note that, incidentally, the angle APA' here is the same as the angle β, equation (3.1). There are right angles in figure 8.

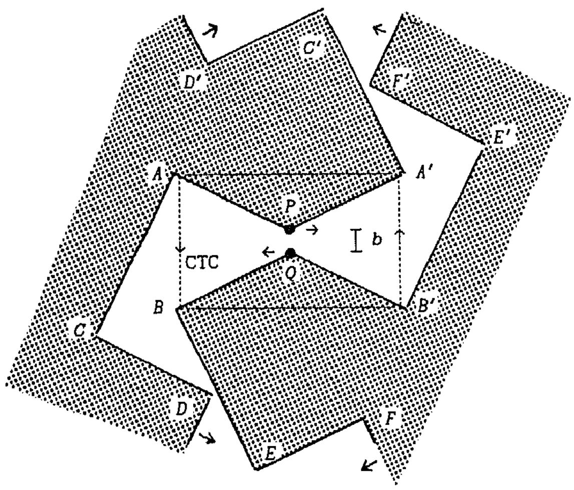

Figure 8. Rotation of the forbidden regions. The path ABB'A'A is a closed timelike curve. Only the unshaded region in the centre is supposed to be the physical space.

Further away from the encounter region we may turn the boundaries back so that they match with the original ones. Thus we draw the lines PACD to be matched with PA'C'D' and QBEF to be matched with QB'E'F'.

One might suggest that this rotation should be allowed, in particular if the impact parameter b is small, so that figure 8 may be seen as a very large magnification of the encounter in the centre of figure 7(b).

5. Cauchy surfaces in the CFG universe

But this is not the case, as we shall see. Let us now treat the evolution of figure 7 strictly abiding by the rules. When P and Q reach (simultaneously) the edges of the polygon the situation is as described in figure 9(*a*). We see that the original polygon splits in two. A transition takes place that is a limiting case of figure 5 (more precisely, two transitions, one for P and one for Q). The edge PQ′ of polygon I now matches with edge P′Q of polygon II. If the original boundaries were moving with rapidity η, according to equation (3.2), the rapidity of PQ′ and P′Q will be 2η, so from now on these edges will move faster. In figure 9(*b*), a snapshot of a few moments later, these edges, now called B″A′ and BA″, have therefore run ahead of the other boundaries parallel to them.

After crossing the boundaries the particles P and Q both have gained a larger velocity. Their rapidity, ξ_2, can be calculated to be given by

$$\cosh \xi_2 = \cosh \xi (1 + 4 \sin^2(\pi m) \sinh^2 \xi) \tag{5.1}$$

and the motion is directed as indicated in figure 9(*b*). Clearly the velocities increased. One might wonder how this can be reconciled with energy conservation, but we know that the total energy is topologically fixed to be 2. Just because the two particles are in different polygons and therefore in different Lorentz frames one cannot add the energies to see if energy is conserved. One can also say that in our coordinate frame there is energy in the gravitational field.

In figure 9(*b*) we see two new segments, QA, to be glued against QA′, and PB, to be glued onto PB′. Their opening angles are widened because of the increased rapidities of P and Q.

But this is not the final state. It is inevitable that P and Q hit the other boundaries again. We get transitions of the type of figure 3 and figure 6. The velocities increase again, but the general topological shape of the polygons returns to its previous shape, figure 9(*b*). And even that is not the end. As long as the antipode particle X stays out of the way, P and Q will never cease to cross the polygon boundaries, interchanging their positions in I and II, in a spiralling motion. This spiralling motion cannot end because the intersection point x of the lines CA and DB″ in figure 9(*b*) moves with velocity

$$v_\perp = \tanh \xi \tan(\tfrac{1}{2}\beta) = \tan(\pi m) \sinh \xi \tag{5.2}$$

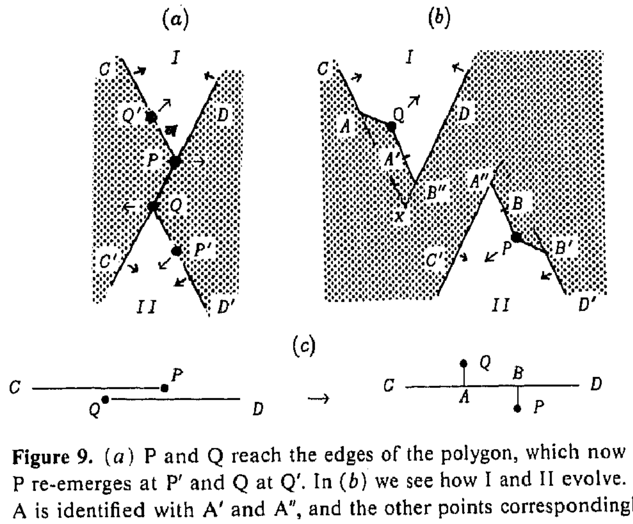

Figure 9. (*a*) P and Q reach the edges of the polygon, which now splits in two, I and II. P re-emerges at P′ and Q at Q′. In (*b*) we see how I and II evolve. As in previous figures, A is identified with A′ and A″, and the other points correspondingly.

where we substituted equation (3.1). Now if we assume Gott's inequality (1.1) to hold, or

$$\cosh^2 \xi > 1/\sin^2(\pi m) \tag{5.3}$$

then we must have

$$\sinh^2 \xi > 1/\sin^2(\pi m) - 1 = 1/\tan^2(\pi m) \tag{5.4}$$

and so precisely in that case this point moves faster than the speed of light. So there is no situation possible such that P and Q can outrun the boundaries. There is an eternal cycle as indicated diagrammatically in figure 10.

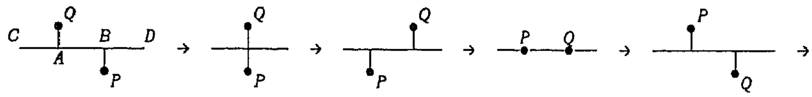

Figure 10. One evolution cycle of a Gott pair, diagrammatic notation.

Each time the boundaries are crossed the particle velocities are boosted in the direction of the antipode particle X. We also see that the surface area of the two polygons rapidly decreases. The author was able to follow the development of this situation numerically. The sizes of the edges reach essentially the same values after every cycle, and the duration of each cycle rapidly goes to a constant. It all ends when the particles P and Q meet the other particles, R, S and the antipode particle X. By that time they have a tremendous velocity. The total 2-volume of space has decreased to zero. Thus we have a 'big crunch'. We briefly discuss this crunch in section 7.

6. What happened to the CTC?

Now back to the Gott/CFG construction. Suppose we try to make the coordinate transformation towards figure 8. Its smallest CTC circles around the pair of particles at some distance, proportional to the impact parameter b. The points B and A′ of the CTC are at some distance in the *future* of the Gott pair. Since this CTC obviously does not exist in our legal description of this universe we must conclude that these points are simply too far in the future. The big crunch has already occurred. This is a completely acceptable conclusion. One might object that by tuning the impact parameter b towards a very small value one can bring B and A′ arbitrarily close to P and Q, which are pictured at an instant well before the crunch. However one must realize that figure 8 is in a Lorentz frame close to the rest frames of both particles. Now the smaller the impact parameter b, the faster these particles will be boosted to very near light velocity (much closer than in figure 8). Due to time dilation the big crunch will then be encountered by these particles extremely quickly, if they are followed in their own rest frames. Regardless the impact parameter, the big crunch, and the X, R and S particles were illegally not represented in figure 8. So we conclude that no contradictions arise. There is no CTC but a final catastrophe in the CFG universe.

We saw in section 5 that as soon as two particles meet each other with Gott's inequality their behaviour in our Cauchy planes becomes such that they begin a spiralling motion, increasing their speed closer to that of light at every turn, until the final crunch ends it all. Closely before that there might be interactions with the antipode particle(s) but that will hardly slow them down. This mechanism will always successfully suppress any CTC.

Until close before the big crunch, our description of the orbiting particles is in the rest frame of particle X. One may ask whether an observer at X will really see the particles orbiting. This is not so, because the orbiting particles are approaching X faster than the speed of light. The first signals from the orbiting particles that reach X will *not* be sent in a straight line within the polygon we constructed because that is impossible. Instead, the signals travel entirely in (the analytic extension of) the last frame. The observer at X will always see the other particles. We suspect that the trajectory of the light rays involved in this observation will be very close to the apparent CTC. The first real CTC in figure 8 is screened off by the other particles and the big crunch.

Consider the first CLC (closed *light*like curve) that opens up in figure 8. It is approximately, but not precisely the dotted line there (the first CLC, which contrary to a claim in [2] is also a lightlike geodesic, is actually a bit tilted because it occurs a little before P and Q are directly opposed to each other). Then in a very large universe this CLC is the natural boundary where the big crunch occurs.

The fact that no CTC is allowed in (2+1)-dimensional gravity was already stated in [5] without proof. This insight had been based on considerations of Cauchy surfaces. In the present paper we made the argument explicit. It is clear that the statement only holds if one uses appropriate boundary conditions. For a closed universe this means that one should take a possible final crunch into account; in an open universe one should not allow for CTC at the boundary. A safe way to consider open spaces is to view them as limiting cases of infinitely large closed spaces. This will imply automatically that certain regions of spacetime are screened off by a big crunch.

7. Why a crunch and not a 'bounce'?

One could have thought that the CFG universe may not end as violently as in a big crunch. It would be natural to suspect that, because the particles R, S and X are encountered just before the end, this catastrophe might be averted in the last minute. Maybe the velocities are turned around and the particles move outward again: a big bounce rather than a big crunch. Alas, the inhabitants of this universe have no such luck. This can be derived from the formulae in the appendix. Our argument goes as follows.

Numerical calculations, which can be checked analytically, show that at the first stage in the CFG universe the situation can be described in terms of a single polygon. All angles α_i at its edges are between 0° and 180°, so they have

$$\sin \alpha_i > 0 \tag{7.1}$$

except those edges where a particle moves inwards. Secondly, very soon all edges move *inwards* so that at all edges L the quantities σ_L (see appendix) have the same sign (or are zero). One now uses the equations (A.1)–(A.6) to show that this situation cannot possibly alter. Any of the transitions, figures 3, 5, 6, 9 and 10 will keep this situation as it was. Therefore the 2-volume of this universe will decrease monotonically with time, and only a finite amount of time is needed to make it shrink to zero. The big crunch cannot be averted, even if we take all other particles into account.

Another danger is the 'Achilles and the Tortoise' effect: it could be that transitions follow each other at geometrically decreasing time intervals, ending in a 'coordinate singularity' rather than in a physical singularity. Our numerical calculations however did not indicate that this ever happens. It is also difficult to conceive how the various Lorentz frames could give rise to such a behaviour unless there were a real physical singularity at that spot. In any case the Gott configuration does *not* lead to any Achilles and the Tortoise effect.

Acknowledgment

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Appendix

The evolution of the system can be formulated by giving the sizes and shapes of the polygon(s) as a function of time, and also the velocities at the edges. In the generic case we can take all particles to be 'inside' polygons, such that at the location of a particle just two edges of one polygon join. In addition then we have vertices where, in the generic case, three polygons touch each other, but no particle. Given the initial configuration the evolution of this is unique. Changes can take place either when a particle hits the edge of the polygon it is in, or when the length of an edge shrinks to zero. In the first case we use figure 5, and in the second case figures 3 and 6.

In the case of figure 3 we have (locally) entirely flat space. This makes the algebra there easy. In general one has four different Lorentz frames, I, II, III and IV. The boundaries are simply the planes determined by the equations $t_1 = t_2$, $t_1 = t_3$, etc. In a computer program one would make a list of all angles and all velocities, and then one needs a prescription to determine the new angles and velocities of the new edge. This we do by giving some relations that hold as soon as three polygons meet at a vertex in flat space (there is no particle assumed to be at the vertex). In figure A.1 we indicate the angles α_1, α_2 and α_3, and the rapidities of the edges η_1, η_2 and η_3 (note that for symmetry reasons the rapidity of the edge between I and II is called η_3; the signs may now be defined such that with positive rapidities the polygons all grow, so that in section 7 most signs are negative).

Since the relativistic velocity of frame III relative to frame I can be obtained by adding relativistically the velocities I→II and II→III, one obtains a number of goniometric equations. We use the short-hand notation:

$$\sin\alpha_i = s_i \qquad \cos\alpha_i = c_i \qquad \sinh 2\eta_i = \sigma_i \qquad \cosh 2\eta_i = \gamma_i \qquad \text{(A.1)}$$

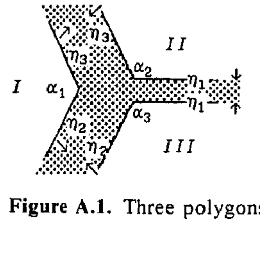

Figure A.1. Three polygons meet at a vertex in flat space.

then we find that

$$s_1 : s_2 : s_3 = \sigma_1 : \sigma_2 : \sigma_3 \tag{A.2}$$

$$\gamma_2 s_3 + s_1 c_2 + c_1 s_2 \gamma_3 = 0 \tag{A.3}$$

$$c_1 = c_2 c_3 - \gamma_1 s_2 s_3 \tag{A.4}$$

$$\gamma_1 = \gamma_2 \gamma_3 + \sigma_2 \sigma_3 c_1 \tag{A.5}$$

$$\cot \alpha_2 = -\cot \alpha_1 \cosh 2\eta_3 - \coth 2\eta_2 \sinh 2\eta_3 / \sin \alpha_1 \tag{A.6}$$

and all cyclic permutations.

Equation (A.4) tells us that if all angles add up to 360° then the rapidities have to be zero. The last two equations are particularly useful because they give all data needed for the edge between II and III if we know only the properties of polygon I. So we find easily practically all we want to know about the newly opened edges in figures 3, 5, 6, 9 and 10.

Note however that equations (A.5) and (A.6) only give the rapidity η_1 up to a sign, and the angles α_2 and α_3 up to a multiple of 180°. This corresponds to the fact that the *orientation* of this edge is not yet determined. One has to determine this by actually constructing the corresponding polygon, and by requiring that a newly formed edge must be a growing one; it cannot start by shrinking. The growth rate g_1 of an edge 1 gets two contributions, g_{A1} and g_{B1} from the two adjacent vertices. The growth g_{A1} at vertex A of edge labelled 1 is given by

$$g_{A1} = (v_1 \cos \alpha_3 + v_2)/\sin \alpha_3 = (v_1 \cos \alpha_2 + v_3)/\sin \alpha_2 \tag{A.7}$$

where the velocities are defined as

$$v_i = \tanh \eta_i = \sigma_i/(1+\gamma_i). \tag{A.8}$$

The second equality in equation (A.7) follows from the earlier equations.

Another useful identity is the growth rate g_P of an edge ending in a particle P with mass m, if we know the rapidity η at that edge. This turns out to be

$$g_P = \frac{\tanh \eta \cos \pi m}{\sqrt{\cosh^2 \eta - \cos^2 \pi m}}. \tag{A.9}$$

It should be clear that these rules completely fix the geometry of figures 3, 5, 6, 9 and 10. Equations (A.7) and (A.9) are needed to determine the shapes of the polygon(s) at some time after a transition, and for calculating the time intervals between events. We used those to observe that no Achilles effect takes place at a Gott pair (section 7).

Although we did not encounter it in our calculations, it should be mentioned that there could be a transition as pictured in figure A.2, where an angle $\alpha_1 > 180°$ of a

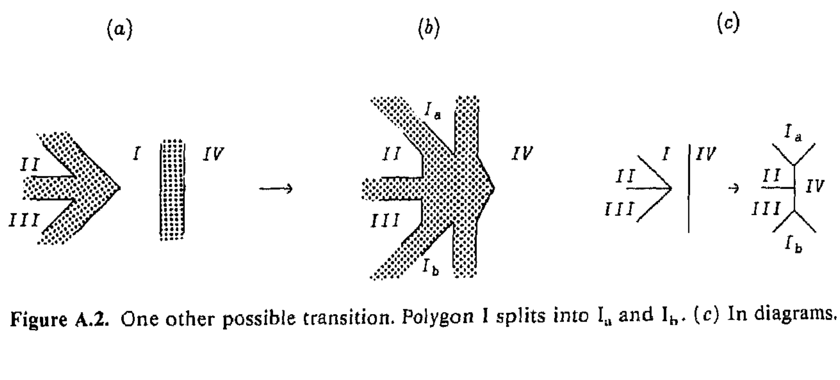

Figure A.2. One other possible transition. Polygon I splits into I_a and I_b. (*c*) In diagrams.

polygon meets one of its other sides. The computation of the newly opened edges goes again by using the identities (A.5) and (A.6).

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