CAN WE MAKE SENSE OUT OF "QUANTUM CHROMODYNAMICS"?

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1. INTRODUCTION

"Quantum Chromodynamics" is a pure gauge theory of fermions and vector bosons that is assumed to describe the observed strong interactions. To get an accurate theory it is mandatory to go beyond the usual perturbation expansion. Not only must we explore the mathematics of solving the field equations non-perturbatively; it is more important and more urgent first to find a decent formualtion of these equations themselves, in such a way that it can be shown that the solution is uniquely determined by these equations. We understand how to renormalize the theory to any finite order in the perturbation expansion, but it is expected[1-5] that this expansion will diverge badly, for any value of the coupling constant. Thus the expansion itself does not yet define the theory. But the renormalization procedure is not known to work beyond the perturbation expansion. A clear example of the possible consequences of such an unsatisfactory situation was the recent suprising demonstration[6-10] that all non-Abelian gauge theories have parameters θ, in the form of an angle, that describe certain symmetry breaking phenomena in the theory, but never show up within the usual perturbation expansion because they occur in the combination

$$\exp\left[i\vartheta - \frac{8\pi^2}{g^2}\right]$$

where g is the gauge coupling constant. This discovery marks an increase in our understanding of non-perturbative field theory but this understanding is not yet complete and, in principle, more of such surprises could await us.

The situation can be compared with the infinity problems in the older theories of weak interactions. Those problems were solved by the gauge theories for which an acceptable and unique regularization and renormalization scheme was found. For our present strong interaction theory, again a "regularization scheme" must be found, this time for "regularizing" the infinities encountered in summing the perturbation expansion.

An interesting attempt to give a non-perturbative formulation of the (renormalized) theory is the introduction of a space-time lattice in various ways[11-13]. But, here also, a proof of uniqueness could not be given (will the continuum-limit yield one and only one theory?) and of course the θ phenomenon mentioned before was not observed in the lattice scheme. So, the lattice theories are still a long way off from answering our fundamental questions.

It is more important for us to make use as much as possible of the important pieces of information contained in the coupling constant expansion. Because of asymptotic freedom[14-16] this expansion tells us precisely what happens at asymptotically large external momenta and it would be a waste to throw that information away.

Which tools could we use to extend our definitions? One is a study of the theory at complex values of the coupling constant. That this is possible in some particular cases is explained in Section 4. We may find that Green's functions must become singular at certain points in the complex g^2 plane and stay regular at others.

Suppose we would discover that there are only singularities on the real axis for $-\infty \leq g^2 < -a$ (unfortunately the true situation is much less favourable). Then we could make a mapping:

$$g^2 \rightarrow u = 1 + 2g^{-2} a [1 - \sqrt{1 + a^{-1} g^2}]$$

with the inverse

$$g^2 = 4au(1-u)^{-2}$$

The whole complex plane is now mapped onto the interior of the unit circle, with the singularities at the edge. If we rewrite the perturbation expansion in terms of u instead of g then we have convergence everywhere inside the circle*), which implies convergence for all g^2 not too close to the negative real axis. A definite improvement. Unfortunately, the structure for complex g^2 we find in Section 4 is too complicated for this method to work: the origin turns out to be an essential singularity. Still, I shall show how this knowledge of the complex structure may be used to give a very slight improvement in the perturbation expansion (Section 9).

The other tool to be used is the Borel resummation procedure, to be explained in Section 5. A new set of Green's functions" is considered whose perturbation expansion terms are defined to be the previous ones divided by (n-1)! where n is the order of the expansion terms. The singularities of these new functions can be found and the analytic continuation procedure as sketched above can be applied to obtain better convergence. There seem to emerge two types of singularities: one due to the instantons in the theory, the other due to renormalization phenomena. The latter are slightly controversial, they are the only ones that should occur in

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*) We make use of a well-known theorem for analytic functions that says that the rate of convergence of an expansion around the origin is dictated by the singularity closest to the origin.

quantum electrodynamics, giving that theory an n! type divergence. The first few terms for the electron g-2 do not seem to diverge the way suggested by this singularity. Formally we can improve the perturbation expansion for QED but in practice the "improvement" seems to be bad.

Our work is not finished. What remains to be done is to prove that all singularities in the Borel variable have been found and to find a prescription how to deal with those singularities that are on the positive real axis. Finally, it must be shown that the integrals that link the Borel Green's functions with the original Green's functions make sense and a good theory is found (see Section 9).

The problem of quark confinement can probably be related to certain singularities on the positive real axis, because these singularities arise from infra-red divergences (Section 8).

2. DEFINITION OF THE COUPLING CONSTANT AND MASS PARAMETERS IN TERMS OF LARGE MOMENTUM LIMITS

This section contains the mathematical definitions of the parameters in the theory, so that the statements in the other sections can be made rigorous and free of unnecessary assumptions. It could be skipped at first reading.

The dimensional renormalization scheme is a convenient way of defining a perturbation series of off-mass shell Green's functions with some coupling constant $g_D(\mu)$ as an expansion parameter. The subscript D stands for dimensionally renormalized[17-20]. Each term of the perturbation series is finite. There is an arbitrariness in the choice of the subtraction point μ (which has the dimension of a mass). The theory is invariant under a simultaneous change in g_D and μ provided that

$$\frac{\mu\, dg_D^2}{d\mu} = \beta(g_D^2) = -\beta_1 g_D^4 + \beta_2 g_D^6 + \beta_3^D g_D^8 + \cdots \tag{2.1}$$

Here we show explicitly the minus sign for the first coefficient. This minus sign is a unique property of non-Abelian gauge fields and is responsible for "asymptotic freedom" (at increasing μ we get decreasing g^2, see Refs. 14-16).

The coefficients β_1 and β_2 are known[21-23]. Since we shall try to go beyond perturbation expansion we must be aware of two facts: First, the perturbation expansion is expected to diverge for all g^2 and is, therefore, at this stage, meaningless as soon as we substitute some finite value for g^2. Second, the dimensional procedure has only been defined in terms of the perturbation expansion (Feynman diagrams). Consequently, g_D^2 may not have any meaning at all as a finite number. The correct interpretation of these series is that they are asymptotic series valid for infinitesimal g only, or, equivalently, valid only for asymptotically large moment: $p^2 = O(\mu^2) \to \infty$. Thus, g_D^2 may not be so good to use as a variable for a study of analytic structures at finite complex values.

We shall now introduce another parameter g_R^2, that may just as well be used instead of g_D^2. It is defined by the following requirements:

When $\mu \to \infty$, then

$$g_R^2(\mu) = g_D^2(\mu) + O(g_D^6(\mu)) \tag{2.2a}$$

$$\frac{\mu\, d}{d\mu} g_R^2(\mu) \equiv -\beta_1 g_R^4 + \beta_2 g_R^6 \tag{2.2b}$$

The series in (2.2b) must stop after the second term. In perturbation theory these requirements have a unique solution for g_R^2. For instance, we get that the rest term in (2.2a) is

$$\frac{\beta_3^D}{\beta_1} g_D^6 + \ldots \qquad (2.3)$$

In the dimensional renormalization scheme also the mass parameter was cut-off-dependent (of course, one cannot define such a thing as a physical quark mass parameter, which would have been cut-off independent):

$$\frac{\mu d}{d\mu} m_D^2(\mu) = m_D^2(\mu)[-1 + \alpha_1 g_D^2 + \ldots] \qquad (2.4)$$

Again we define $m_R^2(\mu)$ by

$$m_R^2(\mu) = m_D^2(\mu)(1 + O(g^2(\mu)) \text{ for } \mu \to \infty \qquad (2.5a)$$

and

$$\frac{\mu d}{d\mu} m_R^2(\mu) = m_R^2(\mu)[-1 + \alpha_1 g_R^2] \qquad (2.5b)$$

where the series in (2.5b) stops after the α_1 term.

In QCD the parameters α_1, β_1, β_2 are known[21-23])

$$\begin{aligned} \beta_1 &= (8\pi^2)^{-1}(11 - 2N_f/3) \\ \beta_2 &= (8\pi^2)^{-2}(19N_f/3 - 51) \\ \alpha_1 &= -(2\pi^2)^{-1} \end{aligned} \qquad (2.6)$$

We emphasize that the new parameters g_R and m_R are better than the previous ones, because for any theory for which the perturbation expansion is indeed an asymptotic expansion, they are completely finite and non-trivial. Of course, they still depend on the subtraction point μ. At infinite μ they coincide with other definitions; at finite μ they are finite because we can solve Eqs. (2.2b) and (2.5b):

$$\frac{1}{\beta_1 g_R^2(\mu)} + \frac{\beta_2}{\beta_1^2}\log\left(\frac{\beta_1}{g_R^2(\mu)} - \beta_2\right) = \log(\mu/\mu_0) \tag{2.7a}$$

$$\frac{m_R^2(\mu)}{m_0^2} = \left(\frac{\beta_1}{g_R^2(\mu)} - \beta_2\right)^{\alpha_1/\beta_1} \tag{2.7b}$$

Here μ_0 and m_0 are integration constants. They are invariant under the renormalization group. Thus, μ_0 is a true parameter that fixes the gluon couplings, and has the dimensions of a mass. For each quark in the system we have a mass parameter m_0. As must be clear from the derivations, μ_0 and m_0 actually tell us how the theory behaves at asymptotically large energies and momenta.

The value of μ_0 for QCD is presumably of the order of the mass, and m_0 will be a few MeV for the up and down quarks, 100 MeV or so for the strange quark, etc. For simplicity, we will often drop the terms containing β_2, and the quark masses will be put equal to zero.

3. THE RENORMALIZATION GROUP EQUATION

Now let us consider the Green's functions of the theory. For definiteness, take only the two-point functions (dimensionally renormalized)

$$G^D(K^2,\mu,g_R^2) = a_0(K^2) + g_R^2\, a_1^R(K^2,\mu) + \dots \qquad (3.1)$$

They satisfy a renormalization group equation[24]:

$$\left[\frac{\mu\partial}{\partial\mu} + \beta^R(g^2)\frac{\partial}{\partial g_R^2} + \gamma(g_R^2)\right] G^D(K^2,\mu,g_R^2) = 0 \qquad (3.2)$$

Here β^R is the truncated, finite β function for the constant g_R^2 as it occurs in Eq. (2.2b), but $\gamma(g_R^2)$ is still an infinite series. We wish to do something about that also. Let us first make clear how to interpret Eq. (3.2). Consider the μ versus g^2 plane. Suppose we choose a special curve in that plane, where g_R^2 depends on μ such that

$$\frac{\mu d}{d\mu} g_R^2 = \beta^R(g_R^2) \qquad (3.3)$$

then it follows from (3.2) that

$$\frac{\mu d}{d\mu} G^D(K^2,\mu,g_R^2(\mu)) = -\gamma(g_R^2(\mu))G^D \qquad (3.4)$$

or

$$\frac{d}{dg_R^2(\mu)}\log G^D = -\frac{\gamma(g_R^2)}{\beta(g_R^2)} = \frac{z_0}{g_R^4} + \frac{z_1}{g_R^2} + z_2 + \dots \qquad (3.5)$$

That implies that, if we stay on one of the curves (3.3), then (3.2) reduces to (3.5) which can easily be integrated. But the integration constant will still depend on k^2 and on the curve chosen, that is, on the constant μ_0, which we get in solving (3.3), see Eq. (2.7a). Thus we get

$$G^D = Z(g_R^2)\,G(K^2, \mu_0) \tag{3.6}$$

with

$$Z(g_R^2) = \exp -\left(\frac{z_0}{g_R^2} + z_1 \log g_R^2 + z_2 \quad g_R^2 + \cdots\right) \tag{3.7}$$

and for dimensional reasons, $G(k^2, \mu_0)$ can only depend on the ratio k^2/μ_0^2.

For our purposes it is now important to observe the following. The coefficients z_0 and z_1 are clearly very important as $g^2 \to 0$, but the terms $z_2 g_R^2 + \ldots$ in (3.7) can simply be absorbed in a re-definition of the coefficients $a_1, \ldots$ in (3.1). That way we get new, improved functions G^R that can be written as

$$\begin{aligned} G^R(K^2, \mu, g_R^2) &= a_0(K^2) + g_R^2\, a_1^R(K^2, \mu) + \ldots \\ &= (g_R^2)^{-z_1} \exp\left(\frac{-z_0}{g_R^2}\right) G(K^2/\mu_0^2) \end{aligned} \tag{3.8}$$

In a typical example where we study the time ordered product of two operators

$$\bar\psi(0)\psi(0) \quad \text{and} \quad \bar\psi(x)\psi(x)$$

corresponding to the σ channel, we have

$$\gamma = 2 - \pi^{-2} g_R^2 + \ldots$$
$$z_0 = 2/\beta_1 \tag{3.9}$$
$$z_1 = -\pi^{-2}\beta_1^{-1} + 2\beta_2\beta_1^{-2}$$

4. ANALYTIC STRUCTURE FOR COMPLEX g^2

In Eq. (3.8) we can write (neglecting for simplicity the β_2 terms)

$$G(K^2/\mu_0^2) = \tilde{G}\left(\frac{1}{g^2} + \frac{1}{2}\beta_1 \log K^2/\mu^2\right) \tag{4.1}$$

This is a function of one single parameter

$$x = \frac{1}{g^2} + \frac{1}{2}\beta_1 \log K^2/\mu^2 \tag{4.2}$$

Complex x corresponds to either complex k^2, real g^2 or complex g^2, real k^2. Now, on physical grounds, we know what we should expect at real g^2, complex k^2 (Fig. 1). The singularities are at k^2 real and negative (i.e. Minkowskian). That is when $x = \text{real} + (\beta_1/2)\cdot(2n + 1)\pi i$, n integer. Choosing now k^2 real and positive we find the same singularities at

$$\frac{1}{g^2} = \text{real} + \frac{\beta_1}{2}(2n+1)\pi i \tag{4.3}$$

They are sketched in Fig. 2.

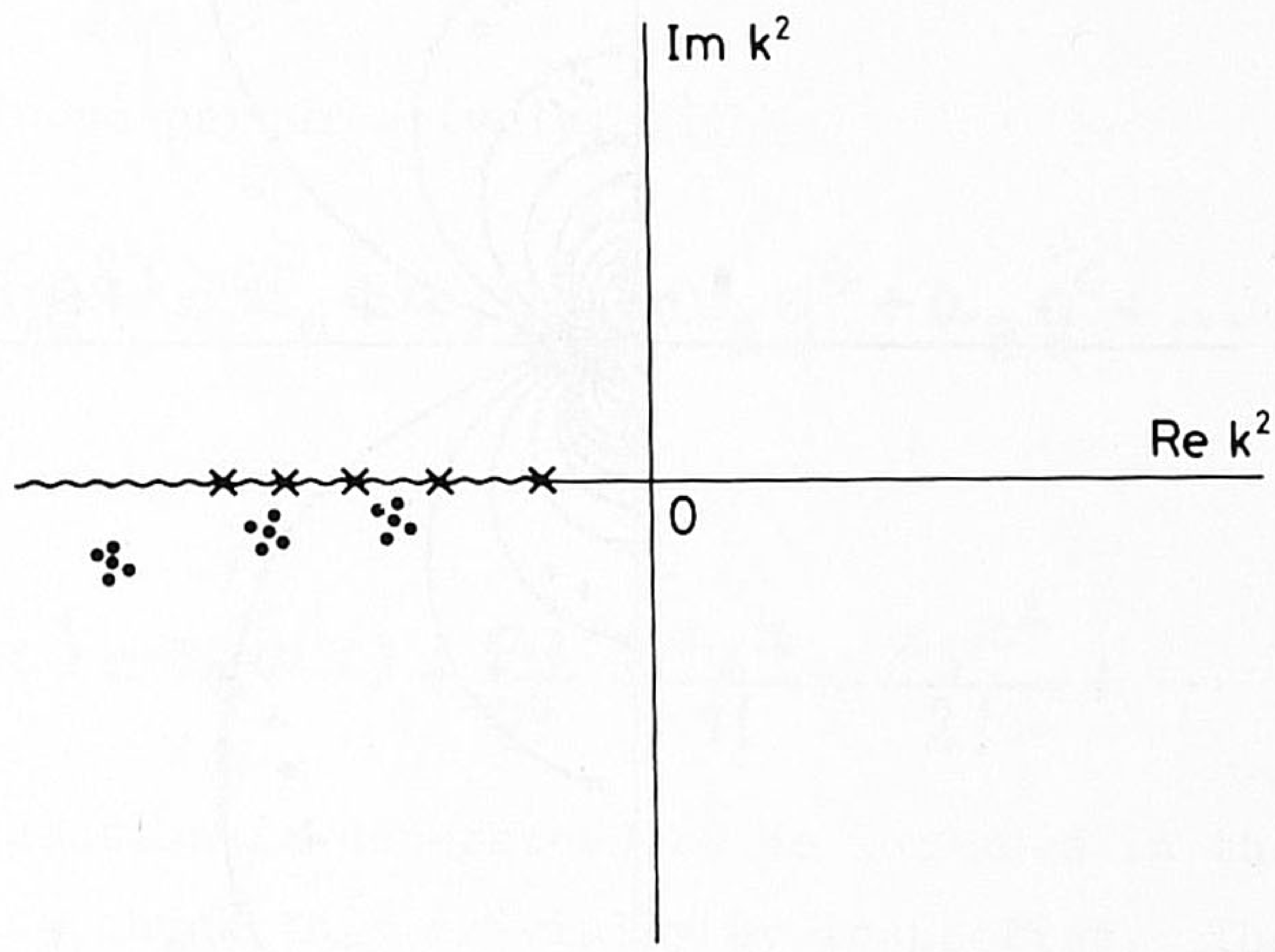

Fig. 1 Expected analytic structure of G^R for complex k^2. The wavy line is a cut (in Baryonic channels this cut starts away from the origin, in mesonic channels, since we have put $m_f = 0$, the cut starts at zero because the pion is massless). The dotted crosses are singularities to be expected in the second Riemann sheet (resonances).

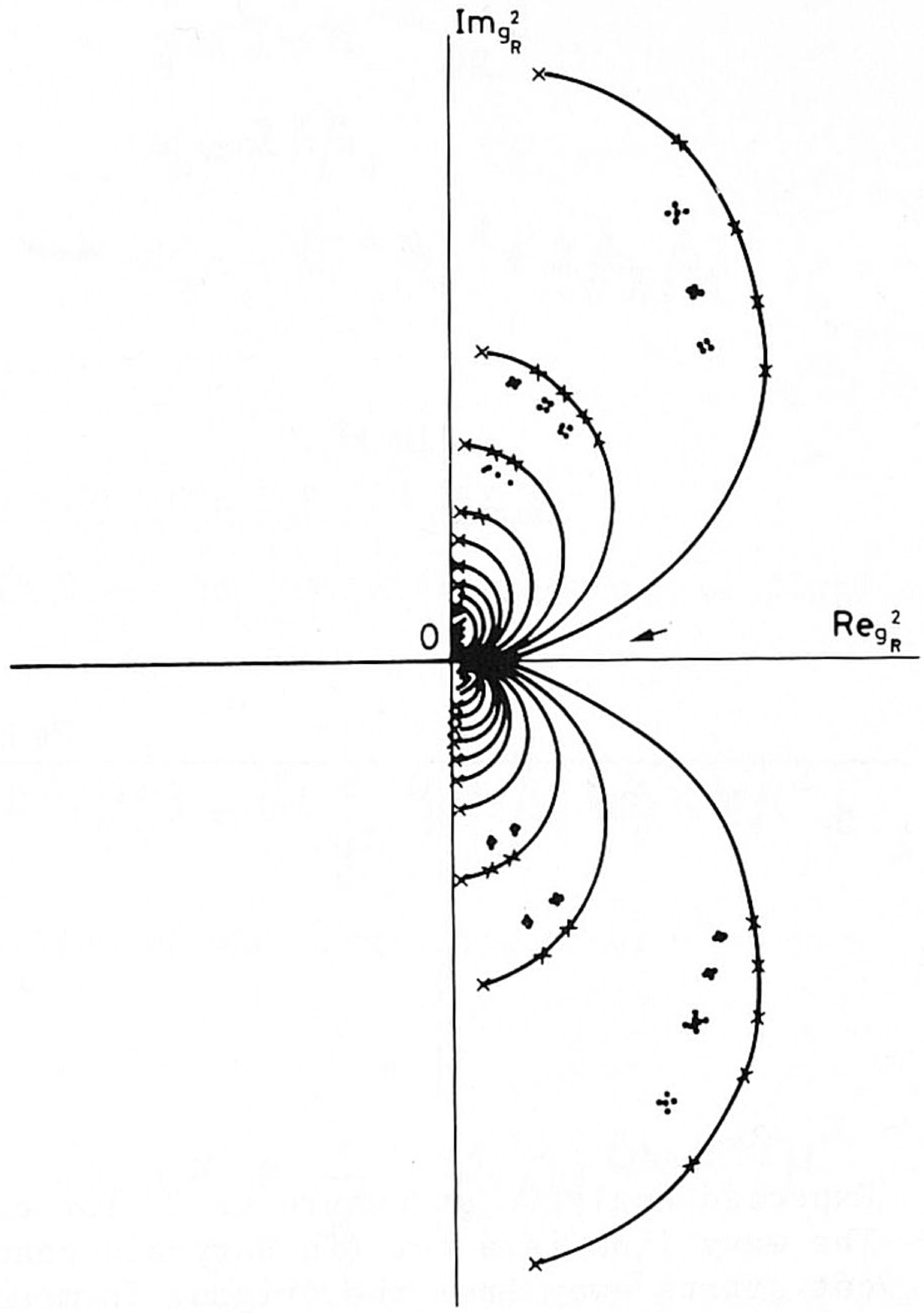

Fig. 2 Resulting analytic structure for complex g_R^2. The single cut of Fig. 1 now reproduces many times on semi circles. These semi-circles are only slightly distorted due to the β_2 term in Eq. (2.7a). The arrow shows the region where perturbation expansion is done. The cut on the left is due to the Z-factor in (3.8).

The conclusion of this section is obvious: we find such a bad accumulation of singularities at the origin that the analytic continuation procedure given in the introduction will never work. We must look for a more powerful technique.

5. BOREL RESUMMATION

We now assume that our Green's functions can be written as a Laplace transform of a special type

$$G^R(g^2) = \int_0^\infty F(z)\, e^{-z/g^2} dz \tag{5.1}$$

$F(z)$ can be found perturbatively: If

$$G^R(g^2) = a_0 + a_1 g^2 + a_2 g^4 + a_3 g^6 + \dots \tag{5.2}$$

then

$$F(z) = a_0 \delta(z) + \frac{a_1}{0!} + \frac{a_2 z}{1!} + \frac{a_3 z^2}{2!} + \dots \tag{5.3}$$

where the δ function is understood to be included in the integral (5.1). One can check this trivially by inspection. The importance of this is that the series (5.3) converges much faster than (5.2). Contrary to (5.2) it may very well have a finite radius of convergence (this is at present believed to be the case for all renormalizable field theories). If $F(z)$ can now be analytically continued to all real positive z, and if, for some g^2, the integral (5.1) converges, then the series (5.2) is called Borel summable. We will call $F(z)$ the Borel function corresponding to the Green's function $G^R(g^2)$.

First, we wish to find out where we can expect singularities in $F(z)$. Let us illustrate an interesting feature in the case of an over-simplistic "field theory", namely field theory at one space-time point. Remember that field theoretical amplitudes can be written as functional integrals, with a certain number of integration variables at each space-time point[10]. If we have one space-time point and one field, then there is just one integration to do

$$G = g \int dA \, \exp\left[-\frac{1}{2}A^2 - \frac{1}{g^2}V(gA)\right] \tag{5.4}$$

where $V(x) = V_3x^3 + V_4x^4 + \ldots$ and the factor g in front is just for convenience. g^2 comes out this way as the usual perturbation parameter. Note the minus signs in the integrand. This anticipates that we shall always consider Field theory in Euclidean space-time. Let us rescale the fields, and the action,

$$\begin{aligned} A' &= gA \\ -\frac{1}{2}A^2 - \frac{1}{g^2}V(gA) &= -\frac{1}{g^2}S'(A') \\ S'(A') &= \frac{1}{2}(A')^2 + V(A') \end{aligned} \tag{5.5}$$

Our integral becomes

$$G = \int dA' \, \exp\left[-\frac{1}{g^2}S'(A')\right] \tag{5.6}$$

Comparing this with the Borel expression (5.1), we immediately find $F(z)$:

$$F(z) = \int dA' \, \delta(z - S'(A')) \tag{5.7}$$

Thus at given z we must find all solutions of $S'(A) = z$, which we can call $A_i(z)$. The result of the integral is

$$\sum_i \left[\frac{\partial S'(A)}{\partial A}\right]^{-1}_{A=A_i(z)} \tag{5.8}$$

This outcome reveals the singularities in the z plane: we must find the solutions of

$$\frac{\partial S'(A)}{\partial A} = 0 \tag{5.9}$$

to be recognized as the classical field equation of this system. At a solution $\bar{A}$ of (5.9) we have

$$\begin{aligned} &\bar{z} = S'(\bar{A}) \\ &\left[\frac{\partial S'}{\partial A}\right]^{-1} \to \left[2\,\frac{\partial^2 S'}{\partial A^2}\,(z-\bar{z})\right]^{-1/2} \\ &\text{as } z \to \bar{z} \end{aligned} \tag{5.10}$$

and so we find a square root branch point at $z = \bar{z}$.

Consider now multidimensional integrals of the same type. The integral (5.7) then corresponds to an integral over the contours in $\vec{A}$ space defined by the equation $S'(\vec{A}) = z$. You get singularities only at those values of z that are equal to the total action S' of a solution of the equation

$$\frac{\partial S'}{\partial \vec{A}} = 0 \tag{5.11}$$

because those are the contours that shrink to a point (in the case of a local extremum) or have crossing points (saddle points). Again, Eq. (5.11) is nothing but the classical Lagrange equation for the fields $\vec{A}$. Conclusion: to find singularities in F we have to search for finite solutions of the classical field equations in Euclidean space-time. Their rescaled action S' corresponds to singularity points $\bar{z}$ in the z place for the function F. In general, these singularities are branch points. In our actual four-dimensional field theories that are supposed to describe strong (or weak and electromagnetic) interactions; such solutions indeed occur, and are called "instantons", because they are more or less instantaneous and local in the Euclidean sense[25,26,6,10]. Their action (in the case of QCD) is $S' = 8\pi^2 n$, where n counts the "winding number"[8-10] and so we may expect singularities in the complex z plane at $z = 8\pi^2 n$.

6. UNIVERSALITY OF THE BOREL SINGULARITIES

The student might wonder whether the conclusions of the previous sections were not jumped to a little too easily. The connected Green's functions in field theories are not just multi(infinite) dimensional integrals but rather the ratio of such integrals with some source insertion and an integral for the vacuum, and then often differentiated with respect to those source insertions. Do all these additional manipulations not alter or replace these singularities and/or create new ones? Do different Green's functions perhaps not have their own singular points?

Let us for a moment forget the renormalization infinities, to which we devote a special section. Then the answer to these equations is reassuring. Multiplications, divisions, exponentiations can be carried out, after which we shall always find the singularities back in the same place as they were before, possibly with a different power behaviour. To understand this general property of Borel transforms, let us formualte some simple properties.

Let

$$G_i(g^2) = \int_0^\infty F_i(z)\, e^{-z/g^2}\, dz \tag{6.1}$$

Then, if

$$G_3(g^2) = G_1(g^2)\, G_2(g^2)$$

then

$$F_3(z) = \int_0^z F_1(z_1)\, F_2(z - z_1)\, dz_1 \tag{6.2}$$

And let

$$G_2(g^2) = [G_1(g^2)]^{-1}, \quad G_1(0) = 1$$

then

$$F_2(z) = 2\delta(z) - F_1(z) - \int_\varepsilon^{z-\varepsilon} F_2(z')\, F_1(z - z')\, dz' \tag{6.3}$$

Here the ε symbols are there just to tell us to leave the δ symbols out of the integration. It is easy to show that if ($\overset{6}{5}$.3) is solved iteratively, then the series converges for all z, as long as F_1 stays finite between ε and z.

Now note that we may choose the contours (0,z) so that they avoid singularities. Only if z is a singularity of either F_1 or F_2, or both, then $F_3(z)$ in Eq. (6.2) will be singular. Also $F_2(z)$ in Eq. (6.3) is only singular if $F_1(z)$ is singular. Note, however, that if a singularity lies between 0 and z then the contour can be chosen in two (or more) ways, and, in general, we expect the outcome to depend on that. Thus if we start with pure pole singularities, they will propagate as branch points in the other Borel functions.

In quantum field theories, the Green's functions are related through many Schwinger-Dyson equations and Ward-Slavnov-Taylor identities. Since singularities survive the multiplications and divisions in these equations without displacement, they must occur in all Borel-Green's functions at the same universal values of z (unless miraculous cancellations occur; I think one can safely exclude that possibility). These singularities will, in general, be of the branch-point type.

In particular, those singularities that we obtained through the solutions of the classical equations, will stay at the same position for all Green's functions.

7. SINGULARITIES IN F(z) DUE TO INSTANTONS

Let us consider these classical equations in Euclidean space-time for the various field theories. First take $\lambda\phi^4$ theory, for simplicity, without mass term. Rescaling the fields and action the usual way

$$\begin{aligned} \varphi' &= \sqrt{\lambda}\,\varphi \\ S &= -\frac{1}{\lambda} S' \end{aligned} \tag{7.1}$$

we have

$$S' = \int d^4x \left[\frac{1}{2} (\partial\varphi')^2 + \frac{1}{4!} (\varphi')^4 \right] \tag{7.2}$$

It turns out[25] that a purely imaginary solution exists for the equations $\delta S/\delta\phi = 0$ (here, δ indicates derivative in the Euler sense), namely

$$\varphi = \frac{\rho i \sqrt{48}}{x^2 + \rho^2} \tag{7.3}$$

Here ρ is an arbitrary scale parameter (after all, our classical action is scale invariant).

In spite of this solution being purely imaginary, it is important to us because it indicates a singularity in F(z) away from the positive real axis. The corresponding value for S' is

$$S' = -16\pi^2 \tag{7.4}$$

So the singularity occurs at $z = -16\pi^2$, indeed away from the positive real axis. Such singularities are relatively harmless, since F is only needed for positive z. We may invoke the analytic continuation procedure sketched in the Introduction to improve convergence for the series in z.

Now, let us turn our attention to Quantum Chromodynamics. Here we have a real solution in Euclidean space:

$$A_\mu^{a'} = g A_\mu^a = \frac{2\eta_{a\mu\nu} x^\nu}{x^2 + \rho^2} \tag{7.5}$$

where $\eta_{a\mu\nu}$ are certain real coefficients[6,27] and ρ is again a free scale parameter. One finds for the action

$$S = -\frac{1}{g^2} S', \quad S' = 8\pi^2 \tag{7.6}$$

Thus $z = 8\pi^2$ is a singularity on the positive real axis. In fact, we can also have n instantons far apart from each other, so we also expect singularities*) at $z = 8\pi^2 n$. Now, Green's functions are obtained from F(z) by integrating from zero to infinity, over the positive real axis. Do the singularities on the real axis give unsurmountable problems? I think not, although the correct prescription will be complicated. A clue is the following. The single-instanton contribution to the amplitudes has been computed directly

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*) A more precise analysis suggests that only those multi-instantons with zero total winding number (that is, as many instantons as anti-instantons) will give rise to ordinary singularities that limit the radius of convergence of F(z). The others give discontinuities rather than singularities.

in the small coupling constant limit. A typical result goes like[27-30])

$$\Delta G(g^2) \rightsquigarrow C g^{-12} e^{-8\pi^2/g^2}(1+O(g^2)) \qquad (7.7)$$

That is already a Green's function, the one we would like to obtain after integrating

$$\Delta G(g^2) = \int_0^\infty \Delta F(z)\, e^{-z/g^2}\, dz \qquad (7.8)$$

A function F(z) that yields (7.7) exists

$$F(z) \to C\left(\frac{d}{dz}\right)^6 \delta(z-8\pi^2) \qquad (7.9)$$

Indeed, a "singularity" at $z = 8\pi^2$. We see that, since all Green's functions will show the same exponential in their g dependence, the universality theorem of the previous section is obeyed. What is important is that by first computing (7.7) one can short-circuit the problem of defining an integration over such singular points. Thus the instanton singularities at the right-hand side on the real axis will not destroy our hopes of obtaining a convergent theory. The reason is that the physics of the instanton is understood. The situaiton is less clear for the other type of singularities that we discuss in the next section.

8. OTHER SINGULARITIES IN F

In principle, the instanton-singularities in F can also be understood within the context of ordinary perturbation expansion, by a statistical treatment of Feynman diagrams[31]). We do not show the derivation here, but the following argument has been given. In the previous section, we have never bothered about the renormalization procedure that is supposed to make all diagrams finite. Suppose we had a strictly finite theory, with bounded propagators, bounded integrals and all that. Individual diagrams in such a theory are then bounded by a pure power law as a function of their order n. The only way that factors n! can arise is because there are n! diagrams at n^{th} order and they may not cancel each other very well. This is how in the statistical treatment the instanton singularity occurs. But in realistic four-dimensional renormalizable field theories, the power law for individual Feynman diagrams no longer holds. A simple example is quantum-electrodynamics. We consider the diagrams of the type shown in Fig. 3.

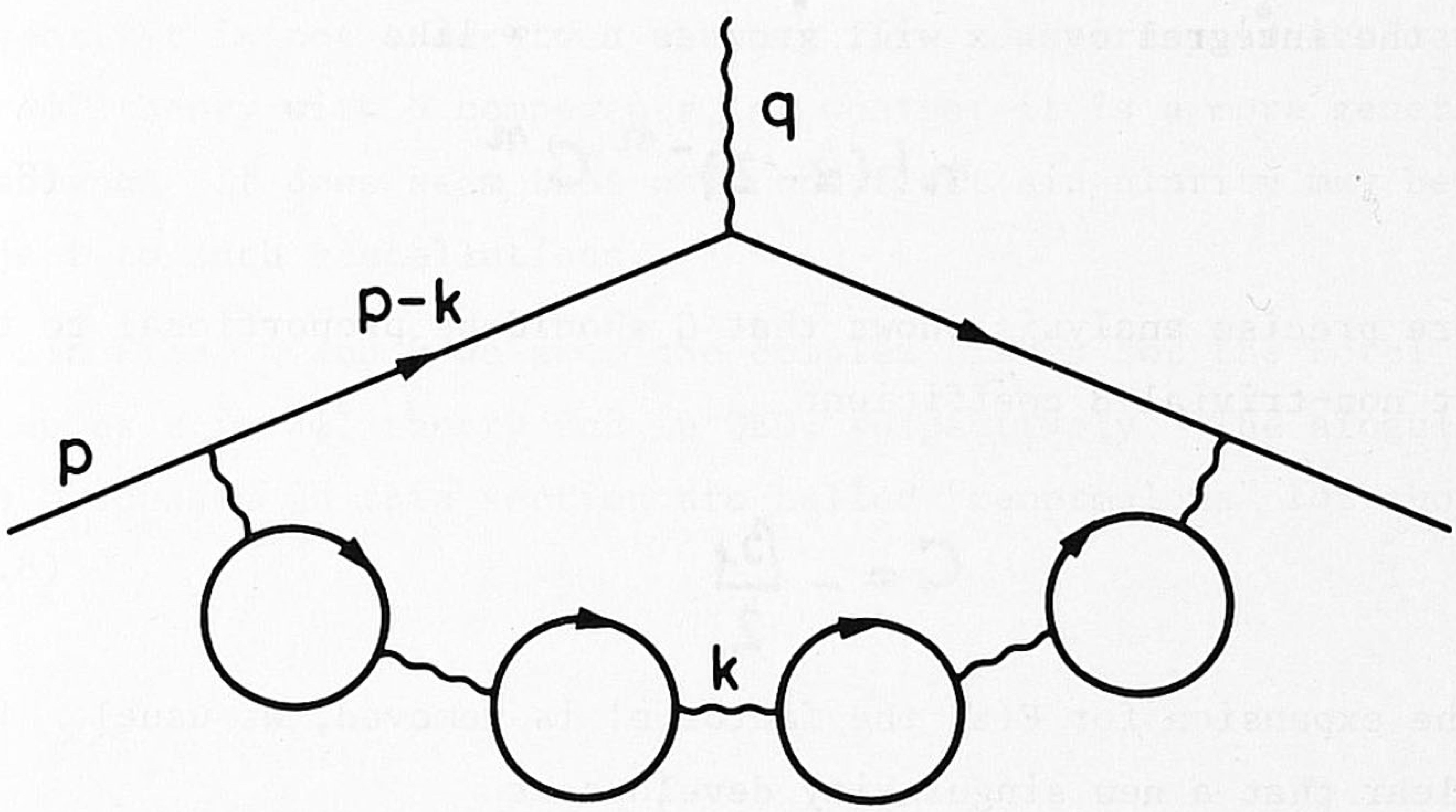

Fig. 3 Fourth member of a subclass of diagrams discussed in this section.

It is the class of diagrams with n electron bubbles in a row, which in itself closes again a loop. It is well known that each electron bubble separately behaves for large k^2 as

$$C K^2 \log K^2 \tag{8.1}$$

and each propagator as $(k^2)^{-1}$. Thus, for large k^2 the integrand in the k variable behaves as

$$\frac{d^4K}{(K^2)^\alpha}(\log K^2)^n C^n \tag{8.2}$$

where α is some fixed power. After having made the necessary subtractions to make the integral converge, and in order to obtain physically relevant quantities, such as a magnetic moment, the leading coefficient α becomes 3 or larger. Let us replace $\log k^2$ by a new variable x, then (8.2) becomes proportional to

$$dx \cdot x^n e^{-(\alpha-2)x} C^n \tag{8.3}$$

Thus the integral over x will grow as $n \to \infty$ like

$$n!\,(\alpha-2)^{-n} C^n \tag{8.4}$$

A more precise analysis shows that C should be proportional to the first non-trivial β coefficient

$$C = -\frac{\beta_1}{2} \tag{8.5}$$

In the expansion for F(z) the factor n! is removed, as usual. It is clear that a new singularity develops at

$$\bar{z} = (\alpha-2)\left(-\frac{2}{\beta_1}\right) \tag{8.6}$$

It seems to be a universal phenomenon for all field theories, and not related to any instanton solution. Our definition for β_1 was positive for asymptotically free theories and negative otherwise. So, the singularity is at negative real z and therefore harmless if our theory is asymptotically free, but for non-asymptotically free theories such as QED and $\lambda\phi^4$, we have singularities on the positive real axis. Since a detailed understanding of the ultraviolet behaviour of non-asymptotically free theories is lacking, there may exist no cure for these singularities then. This is in contrast with the instanton singularities.

An important observation has been made by G. Parisi[32]. The ultraviolet behaviour of $\lambda\phi^4$ and QED are well understood in the limit $N \to \infty$, where N is the number of field components. A systematic study of the singular point (8.6) is then possible. Parisi found in $\lambda\phi^4$ theory a conspiracy between diagrams such that the first singularity at $\alpha = 3$ cancels. In total the integrals do behave as (8.2) but with $\alpha > 3$, after all necessary subtractions. At present, it is not understood whether this conspiracy is accidental for $\lambda\phi^4$ theory with N components, or whether it is a more general phenomenon. It does seem that only the first singularity may be subject to such cancellations.

In Figs. 4 and 5 we show the complex planes for the Borel variables z in $\lambda\phi^4$ theory and in QED, respectively. The singularities discussed in this section are called "renormalons" for short.

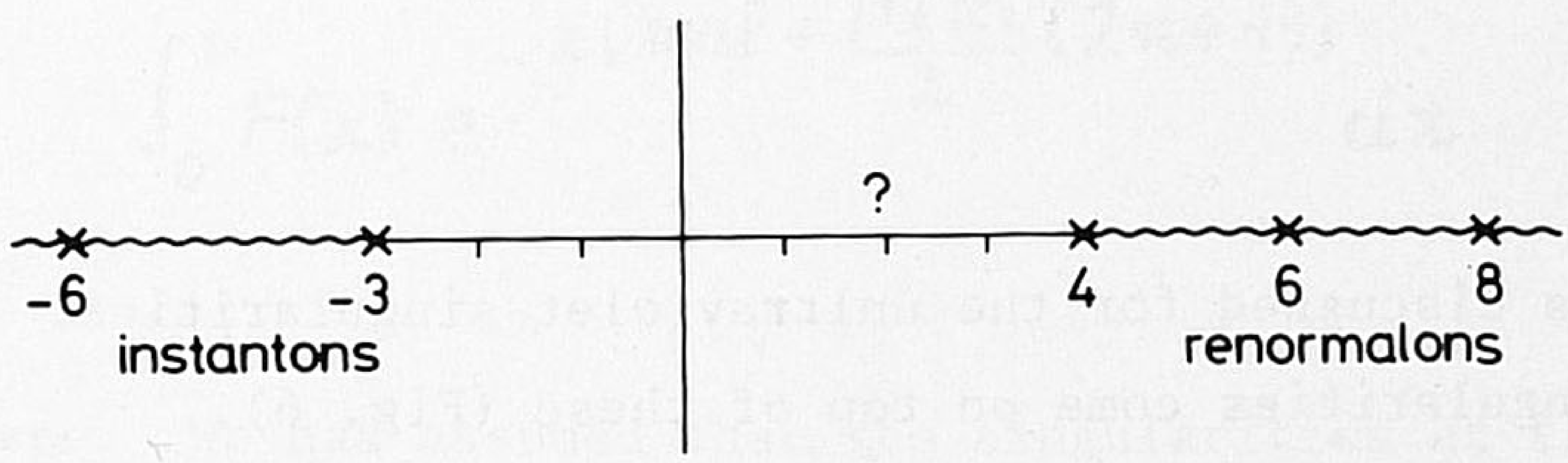

Fig. 4 Singularities in the Borel z variable for $\lambda\phi^4$. The units are $16\pi^2/3$. The question mark denotes the singularity that may be cancelled according to Parisi's mechanism.

The situation for QCD is more complex. Not only do we have the renormalons at points on the negative real axis but also there are such singularities on the positive real axis. They are due to the infra-red divergence of the theory. The mechanism is otherwise the

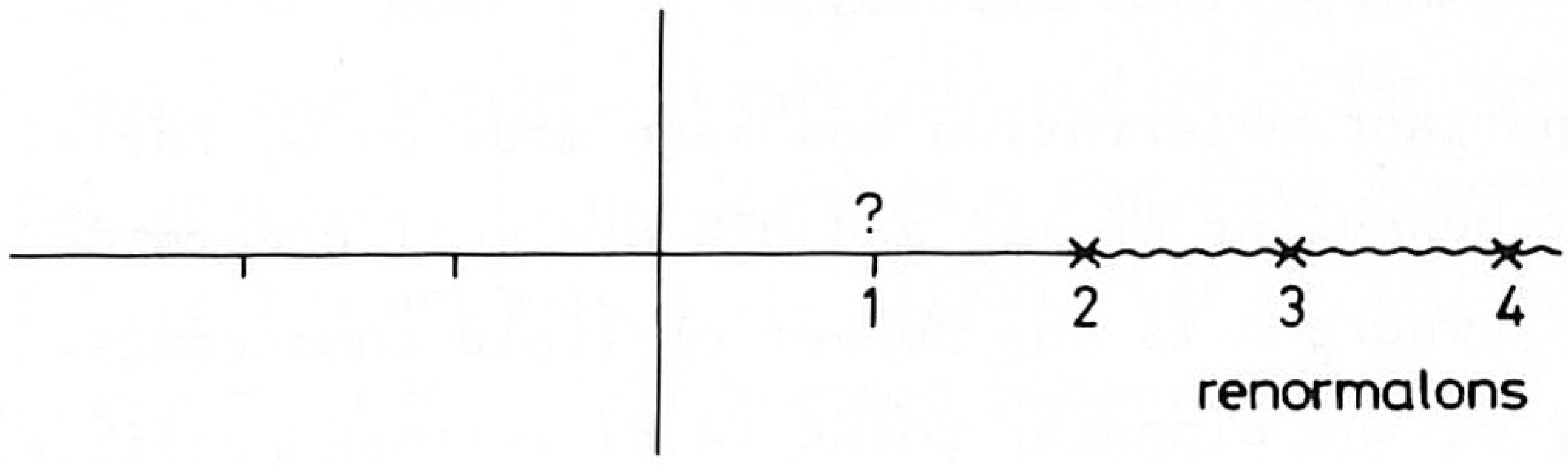

Fig. 5 Singularities for QED. Here the units are 3π, if α is the original expansion parameter.

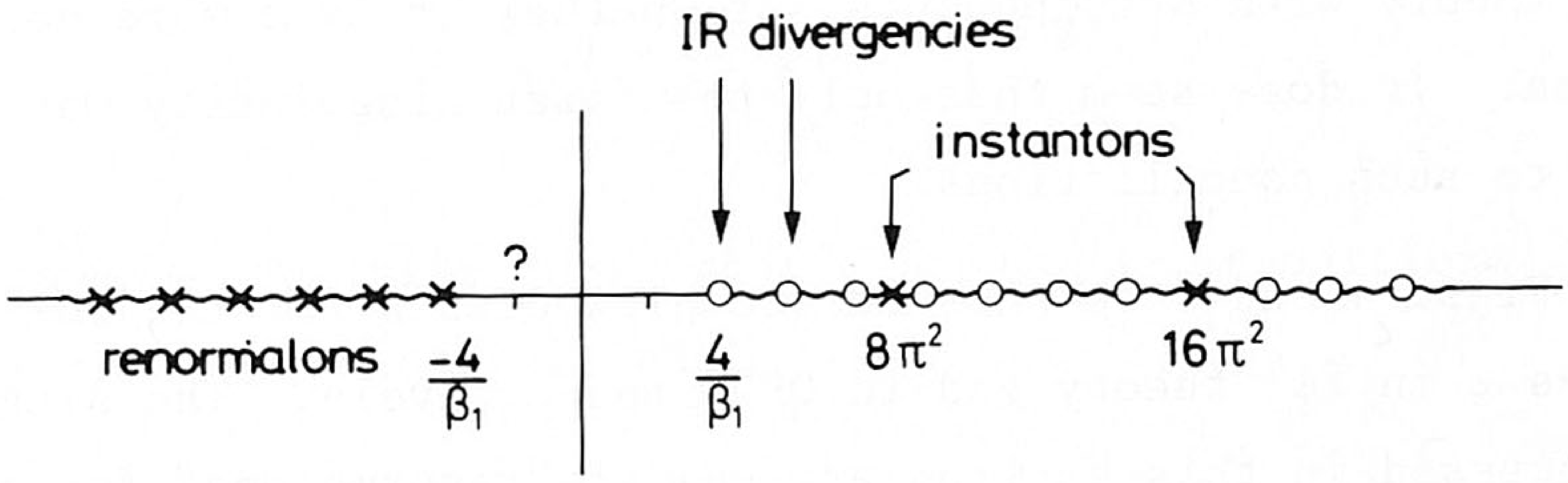

Fig. 6 Borel z plane for QCD. The circles denote IR divergences that might vanish or become unimportant in colour-free channels.

the same as discussed for the unltraviolet singularities. The instanton singularities come on top of these (Fig. 6).

An interesting speculation is that these infra-red singularities are only surmountable in colourless channels, but the integration over these singularities becomes impossible in single quark- or gluon- channels. It is likely that these singularities are related to the quark confinement mechanism.

9. SECOND BOREL PROCEDURE

Many features of the singularities in the complex z plane of the Borel functions F(z) are still uncertain and ill-understood. But from the foregoing we derive some hopes that it will be possible to obtain F(z) for $0 \leq z < \infty$ for asymptotically free theories, such as QCD. The only thing to be investigated then is how the integral in

$$G(g^2) = \int_0^\infty F(z)\, e^{-z/g^2}\, dz \tag{9.1}$$

behaves at ∞. Does the integral converge? The answer to this is almost certainly: no. Consider massless QCD and its singularities in complex g_R^2 plane as derived in Section 3. According to Eq. (3.3) there are singularities when

$$1/g^2 = \text{real} + \frac{\beta_1}{2}(2n+1)\pi i \tag{9.2}$$

where the real number may be arbitrarily large. Substituting that in Eq. (9.1) we find that

$$\int_0^\infty F(z)\, e^{-z\left(\text{real} + \frac{\beta_1 \pi i}{2}(2n+1)\right)}\, dz \tag{9.3}$$

must diverge. We had assumed that the singularities at finite z did not give rise to divergences. So F(z) must diverge at large z worse than any exponential of z. Note that (9.3) contains an oscillating term. It is likely then, that at $z \to \infty$, F(z) does not only grow very fast, but also oscillates with periods $4/\beta_1$ or fractions thereof.

Can we cure this disease? We have no further clue at hand which could provide us with any limit on the large -z behaviour of F. But there is a way to express the unknown Green's functions in terms of a more convergent integral than (9.3). Let us treat the divergent integral (9.1) on the same footing as the divergent perturbation expansions which we had before. We consider a new, better converging integral

$$W(s) = \int_0^\infty \frac{F(z)\,dz\, s^{\beta_1 z/2}}{2\Gamma(\beta_1 z + 2)} \tag{9.4}$$

We may hope that this has a finite region of convergence, from which we can analytically continue. Note the analogy between (9.1) and (9.4) on the one hand, and (5.2) and (5.3) on the other. The integral relation between W and G, analogous to (5.1), is

$$G(g^2) = \int_0^\infty ds\, W(s) \exp[-\sqrt{s}\, e^{1/\beta_1 g^2} + 2/\beta_1 g^2] \tag{9.5}$$

Now, remembering that instead of varying g^2 we could vary k^2, replacing

$$2/\beta_1 g^2 \to \log K^2 \tag{9.6}$$

So that, ignoring the Z factor that distinguishes G from G^R (see Eq. (3.8)), one gets

$$G(K^2) \to K^2 \int_0^\infty ds\, W(s)\, e^{-\sqrt{s} K^2} \tag{9.7}$$

Now we can easily prove that, if our theory makes any sense at all, there may be no singularities in W(s) on the positive real axis, and the integral (9.7) must converge rapidly. Thus, <u>if</u> the integral (9.4) makes sense, then our problems are solved. The proof goes as follows:

$G(k^2)$ satisfies a dispersion relation: it is determined by its imaginary part. We have, at $k^2 = -a^2$, $a > 0$,

$$G(K^2 = -a^2 + i\varepsilon) - G(K^2 = -a^2 - i\varepsilon) = -2\pi i \rho(a) \tag{9.8}$$

where ρ is usually a positive spectral function. Substituting (9.8) into (9.7) we get

$$\begin{aligned} \frac{-2\pi i \rho(a^2)}{a^2} &= \int_0^\infty 2\omega \, d\omega \, W(\omega^2)[e^{-i\omega a} - e^{i\omega a}] \\ &= -2\int_{-\infty}^{\infty} e^{i\omega a} \, \omega \, W(\omega^2) \, d\omega \end{aligned} \tag{9.9}$$

Thus, $\rho(a)$ and $W(\omega^2)$ are each other's Fourier transform. The inverse of (9.9) is

$$W(s) = \frac{1}{\sqrt{s}} \int_0^\infty \frac{\rho(a^2)}{a^2} \sin a\sqrt{s} \, da \tag{9.10}$$

A possible singularity at $a^2 = 0$ is an artefact of our simplifications and can be removed. It is important to observe that (9.10) severely limits the growth of $W(s)$ at large s so that (9.7) is always convergent.

Conclusion: this section results in an improvement on perturbation theory. The physically relevant quantities can be expressed in terms of integrals of the type (9.4), which converge better than the original ones of type (9.1). It is not known whether this improvement is sufficient, i.e., whether (9.4) actually converges in some neighbourhood of the origin.

Even if the important open questions mentioned in these lectures cannot be answered we think that refinement of these techniques will lead to an improved treatment of strong coupling theories.

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DISCUSSION

CHAIRMAN: Prof. G. 't Hooft

Scientific Secretaries: E. Mottola, C. Tao

DISCUSSION No.1

- BAULIEU:

How do you obtain the circles in the complex g^2 plane?

- 't Hooft:

Originally, in the theory, k^2 and μ^2 are the only free parameters. The coupling constant g^2 is related to μ^2 through the renormalization group equation:

$$\mu \frac{d}{d\mu} g^2(\mu) = -\beta_1 g^4 + \cdots \quad \text{(higher order terms which are not important here)}$$

the solution of which is:

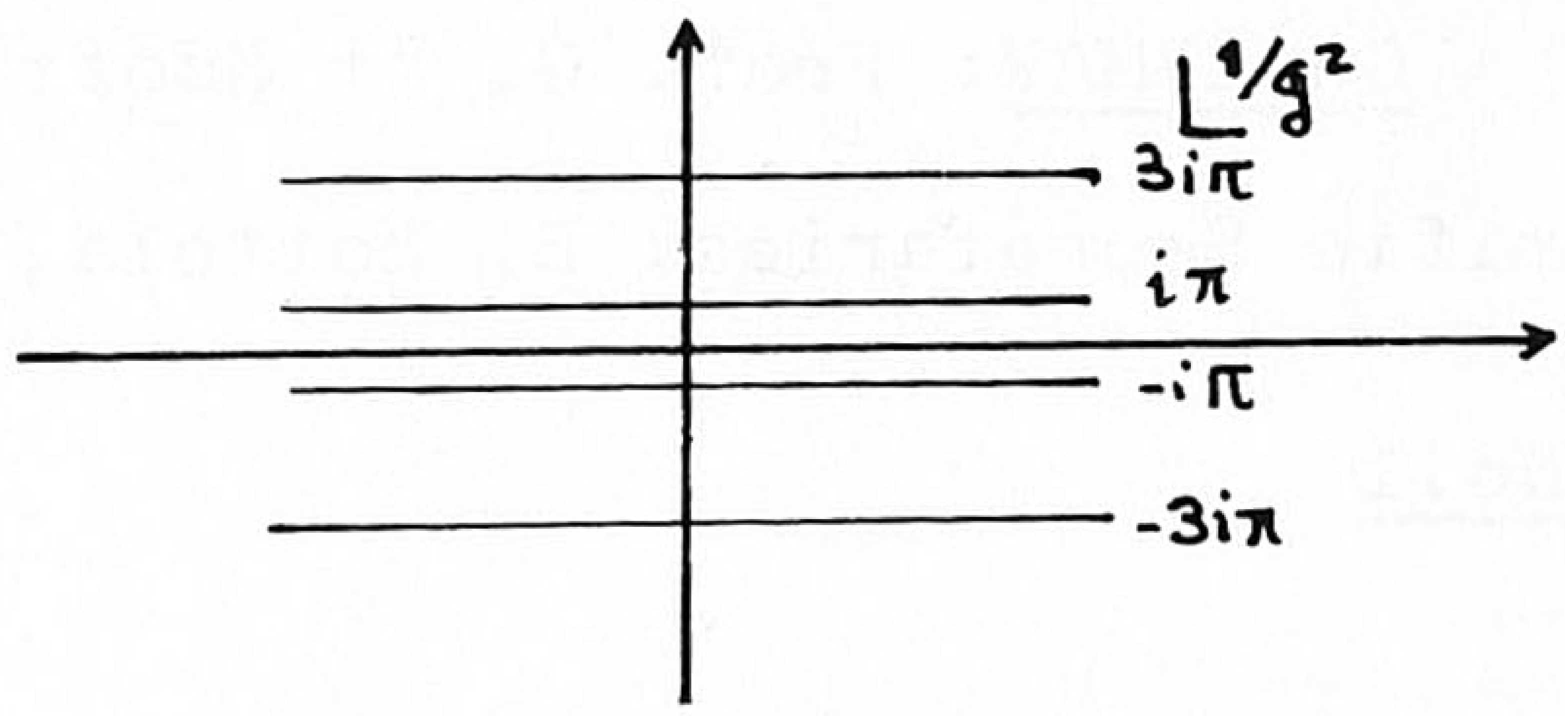
$$\frac{1}{g^2(\mu)} = \beta_1 \log\left(\frac{\mu}{\mu_0}\right)$$

where μ_0 is a free integration constant with dimension of a mass. Now the point is that I am free to choose the μ anyway I like so that I can just as well consider μ fixed and $g(\mu)$ the free parameter or the integration constant μ_0 as the free parameter.

Simple dimensional arguments show that any Green's function can depend only on $k^2/\mu_0^2 = x$, apart from some renormalization factors in front, which I'll leave out for simplicity.

Also from elementary physics, we know what the analytic structure in the complex k^2 plane should be: just poles and cuts on the negative real axis. So for $x = -a$, one such negative real singular point, we may as well consider k^2 real and μ_0^2 complex since there is only one parameter $x = k^2/\mu_0^2$. What does complex μ_0^2 imply about g^2?

If μ_0^2 is a real negative number, then $\log \mu_0^2 = (2n+1)i\pi +$ +(real) so that in the $1/g^2$ plane, the singularities must lie on lines.



Now, if we go to the g^2 plane, the result is just a set of circles by inversion about the origin.

- BUDNY:

You mentioned that instantons have the effect of introducing the physical parameter θ in QCD. Since we cannot get this from perturbation theory, there is the possibility of many such parameters. This would have the effect of preventing us from calculating things just as in non-renormalizable theories. Is there any reason to hope that there is a small number of such non-perturbative physical parameters?

- 't HOOFT:

It is possible that there is an infinity of those parameters.

- COLEMAN:

There are exact soluble models having such parameters and, in those cases, there is typically only one. There may, of course, be any number but there is no reason whatever to expect that there are more.

- WIGHTMAN:

Please describe the relation between your analysis of the singularities in the complex g^2 plane and N. Khuri's study of the trajectories in the complex g^2 plane.

- 't HOOFT:

I am afraid I am not familiar with that and I cannot answer.

- COLEMAN:

Khuri did not use Gerard's clever trick of defining the coupling constant in such a way that the Callan-Symanzik functions were exactly known. Therefore his analysis is bogged down quite a lot in technical assumptions about the analytic behaviour of the Callan-Symanzik functions. Gerard essentially shortcircuits about 95% of Khuri's work.

- PAFFUTI:

Is the Borel summation the "unique" way of regularization of the perturbation expansion?

- 't HOOFT:

This is exactly the question I am trying to investigate in these lectures. There may be two sources of trouble that make the method ambiguous, not unique. One is that while integrating over z, from 0 to ∞, one might encounter singularities on the positive real axis. And our problem is then to find a unique prescription of integrating over them. The second is that the integral might well diverge at ∞. Any ad hoc cut-off there, would of course make the method ambiguous. I will elaborate on this tomorrow.

- PREPARATA:

Could you please clarify the differences between QCD and QED, as far as the coupling constant singularities are concerned?

It seems to me that the difficulties you mention come from imposing that the theory exhibits a spectrum of the dual model type.

- COLEMAN:

You don't need to make any assumption about rising Regge trajectories or any confinement to get that circle of singularities. It suffices to say that you have normal thresholds on the real axis. For sufficiently large timelike k^2 or cuts on the left hand side, you certainly have a pion and that certainly means that you get 2π thresholds, 4π thresholds, 6π thresholds and ad infinitum. Those are branchpoints and not poles but that's irrelevant. When Gerard makes his transformations, they go on that circle tangent to the real axis and converge on the origin. So because of multiparticle states, resonances or not, you still have zero opening angle.

A cut is movable but the branchpoint is immovable, so you always have it. These tricks are also based on the fact we have an asympotically free theory with a negative leading term. In QED, I don't know, but maybe Gerard can speak for that.

- 't HOOFT:

In QED, the same analysis would give circles that go the other way round, so that you would have an opening angle to the right.

However, there are difficulties in the procedure here due to non-asymptotic freedom and the presence of a mass term.

- TOWNSEND:

If the subtraction point can be chosen, so that $\beta = -\beta_1 g^4 + \beta_2 g^6$ exactly, and if β_1 and β_2 are subtraction independent, why is it that the whole theory, and zeroes of β in particular, cannot be determined from perturbation theory by calculation of β_1 and β_2 ?

- 't HOOFT:

What is wrong with your suggestion is that this mathematical trick only gives the positions of the singularities, which is only a very small part of all the information that you want. For instance, the residues of the poles are not given by those two coefficients.

- WEEKS:

Could you please state a working version of the Euclidicity postulate?

- 't HOOFT:

A vacuum expectation value of two operators at different times in Euclidian space is:

$$\langle A(t) \,|\, A'(0)\rangle = \sum_{E_i} \langle A(0)|E_i\rangle\langle E_i|A'(0)\rangle\, e^{-E_i t}$$

and in Minkowski space:

$$\langle A(t) \,|\, A'(0)\rangle = \sum_{E_i} \langle A(0)|E_i\rangle\langle E_i|A'(0)\rangle\, e^{-iE_i t} .$$

One should be the analytic continuation of the other.

- COLEMAN:

The statement that you analytically continue any time ordered product into Euclidian space, i.e., imaginary time, is just to say that all the energies are positive : because that means that all your sums get more and more convergent as an oscillating exponential becomes a damped exponential.

There is a precise mathematical theory, the Osterwalder-Schrader theory, that tells you, if you have a set of Green's functions in Euclidian space, what assumptions they must obey in order to be continued analytically back to Minkowski space, such that this analytical continuation obeys all Wightman's axioms, including positivity.

Now, what happens when you know the Euclidian Green's functions only approximately?

There, it depends on what you are computing. If you are computing energies, masses, or residues of poles, or matters of that kind, then you can show that the analytical continuation is stable. You get the values approximately, but you cannot introduce factors that are small in the Euclidian region and get large when you continue them in the Minkowski region. That can be demonstrated explicitly.

For quantities like S-matrix, I don't know.

- PAAR:

You showed that in QED an expression for

$$(g-2)/2 = \sum_n a_n \left(\frac{\alpha}{\pi}\right)^n$$

has about 137 decreasing terms which make the series look convergent and then ever increasing terms which make it diverge. Why should we believe that the intermediate result after 137 terms is the correct result?

- 't HOOFT:

Compare for instance the perturbation expansion for the integral $\int_0^\infty \exp(-z/\lambda) \frac{d\lambda}{1+z}$

Here, it is not difficult to show that the true value always lies in between the nth and the(n+1)th approximation, for all n. So the best result is obtained if you do the expansion until the nth correction is equal in magnitude to the(n+1)th. There is still an inaccuracy We assume that this strategy also makes sense in Quantum Field Theory.

- WIGHTMAN:

The usual theorems on Borel summability say that a function f, analytic in a sectorial neighborhood of the origin $-\pi/2-\varepsilon < \arg z < \pi/2+\varepsilon$, $0<|z|<R$, bounded in the closure of the vectorial neighborhood, and such that $f^{(n)}(0+)$ exist, n=0,1,2,3,... has a Borel summable Taylor series. The horn-like region which you have displayed is not of that type. How do you make it plausible that your series is Borel summable?

- 't HOOFT:

This is indeed an important problem and the answer is not obvious, and I will lecture on this tomorrow.

DISCUSSION No. 2 (Scientific Secretaries: B. Celmaster, E. Mottola, C. Tao.)

- CARTER:

Could you explain again where the universality of the singularity structure of the Borel function F(z) comes from? I understand that the instanton singularities are located by stationary values of the action and are therefore universal. In QED, I also understand that the renormalon poles are located by β_1 and hence are also universal. Are these observations the basis for "universality" or is the statement something very general?

- 't HOOFT:

There are different ways to observe this universality property. One is, as you say, to notice the fact that the instanton action is a single universal number and β_1 is the same for all Green's functions: But there is an independent and more general way to see this by considering the Dyson equations among the Green's functions. We have corresponding equations between the Borel transforms of these. You see then, that barring miraculous cancellations, if any Green's function has a singularity somewhere, the singularity will recur at the same spot in all other Green's functions. I must mention that we do not yet understand everything here. There might be other singularities besides the instantons and the "renormalons".

- LIPKIN:

What is the basic physical difference between QED and QCD? Is it the large coupling constant or the non-Abelian character of QCD that is most significant? In particular, why are quarks confined in QCD when electrons are not in QED?

- 't HOOFT:

The basic difference between QED and QCD arises from the non-Abelian character of QCD. However, the models one encounters in weak interaction theory, Weinberg-Salam for example, are also non-Abelian gauge theories. Yet the Weinberg-Salam model is not so very different from

QED. The reason is that the spontaneous symmetry breaking affects the non-Abelian gauge field properties of the theory in such a way that there is a closer resemblence to conventional Abelian QED. So, I would say that QCD and QED differ because QCD is not only non-Abelian but also an unbroken gauge field theory.

Concerning your question on confinement, opinions vary and the answer is still unknown. My personal feeling is that the essential infra-red divergence in "colored" amplitudes will make the quark or gluon poles disappear entirely in the completely solved theory and that this implies confinement. Such a thing can only occur in a theory with this particular behaviour under scale transformations.

- LIPKIN:

We know that the physics of QED is the same for $\alpha = 1/137$ as for $\alpha = 10^{-25}$. The hydrogen atom would have the same spectrum; only the scale changes. Is this also true for QCD? Would quarks be confined in QCD even if $g^2 = 10^{-25}$ or is there a phase transition?

- 't HOOFT:

We can perform scale transformations until the effective coupling constant is 10^{-25} (which is $10^{10^{25}}$ GeV). Quark confinement still occurs, but at that scale it occurs at $10^{10^{25}}$ units of length; at those energies, the dynamics are very much like the behavior at 10^{137} GeV. This is just the renormalization group statement of scaling, which is very similar to your remark on the hydrogen atom.

- LIPKIN:

How do you decide whether or not there is a phase transition?

- 't HOOFT:

The effective coupling constant of the theory is a smooth function of energy scale and all amplitudes exhibit a dependence on g^2 which shows no evidence of a phase transition singularity, except possibly at g=0.

Confinement formally takes place already at $g = 10^{-25}$ so that we do not expect any phase transition in QCD as long as the coupling constant is non-zero.

Some people argue that QED has a phase transition towards confinement at some large value of α but that happens only if you put QED on a lattice which explicitly introduces a length scale into the theory.

- WEILER:

Are there physical interpretations for the positions of the singularities of $(F(z))_{QCD}$? In particular, why are the instanton singularities only on the positive real axis, and why are the singularities at multiples of $8\pi^2$? I would have expected the positions to depend on the color group theoretic factors, as do the IR and UV singularities at $\pm\ 4/\beta_1$, etc.

- 't HOOFT:

F(z) is an integral transform of a Green's function of the theory so its singularities have an interpretation entirely different from the singularities in the original Green's function in the g^2 or k^2 plane.

- COLEMAN:

As I will show in my lecture tomorrow, $8\pi^2$ is a group theoretic factor. However, due to Bott's theorem, it is independent of the particular group.

- 't HOOFT:

Note that the relative positions of instanton and renormalon singularities do depend on the group as well as on the fermion representations, etc.

- BOHR:

My question concerns the singularities of the Borel function at infinity. Can you show more carefully how the convergence of $G(k^2)$ is obtained by introducing the function W(s)?

- 't HOOFT:

I do not really know a priori whether or not my definition of $W(s)$ in terms of the z-integration converges; in any case, its convergence properties can only be better than the integral for $G(k^2)$, not worse. However, if it does still diverge, the story stops here. On the other hand, the relation between $W(s)$ and $G(k^2)$ or $\rho(m^2)$ is a completely convergent, well-defined Fourier transformation. Thus, we obtain through this relation a convergent re-definition of $G(g^2)$.

- TOWNSEND:

What is the importance of the distinction between instanton poles of $F(z)$ which appear on the negative real axis as opposed to the positive real axis? Is there a connection with asymptotic freedom?

- 't HOOFT:

We have to consider the integral $\int_0^\infty F(z) \exp(-z/g^2) dz$. If there are poles of $F(z)$ on the positive real axis this causes the integral to be ill-defined. This is not the case if the poles are on the negative real axis. Whether the instanton poles are on the positive or negative real axis is almost certainly completely unrelated to asymptotic freedom, because there are theories which are not asymptotically free but which have instanton poles on the positive real axis.