BOREL SUMMABILITY OF A FOUR-DIMENSIONAL FIELD THEORY

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Borel summability of any renormalizable euclidean field theory which is planar, asymptotically free, massive, and whose coupling constant is sufficiently small can be proven. The simplest example is massive λ Tr ϕ^4 in the $N \to \infty$ limit, with the "wrong" sign of λ. An outline of the proof is given. Our methods also yield further information on the analyticity of the Borel function for massless theories.

There are two principal reasons for trying to construct absolutely convergent calculational schemes for quantum field theories in four space–time dimensions. One is, of course, the hope of enabling ourselves to compute spectra and S matrix elements reliably in a strong-interaction model such as QCD. Or to show that certain approximative schemes [1] do not deviate too far from the truth.

The other reason is that we do not know yet whether theories such as QCD can be constructed rigorously at all. If not, then our picture of the physical world at distance scales much smaller than the hadronic distance scale might have to be revised significantly [2].

Here we wish to report progress made for the case of planar field theories. These are field theories in which the field variables are $N \times N$ matrices, in the limit $N \to \infty$, λN fixed, where $\lambda(=g^2)$ is the coupling constant. We limit ourselves to asymptotically free field theories, though theories with a demonstrable ultraviolet fixed point [3] can perhaps also be treated in our way. For technical reasons we also introduce a mass for all particles. This is to ensure that Green functions in euclidean space never deviate too far from their perturbative values. Sending this mass to zero is not yet possible, as we will see.

We will not make use of unitarity, so any set of renormalizable Feynman rules, possibly with wrong metric or wrong signs for coupling constants, will do. Thus there are many examples. The simplest is a massive scalar planar field theory with the "wrong" sign for λ. We do not know of a planar asymptotically free Higgs theory but that might also exist.

The bulk of our argument is rather technical and will appear elsewhere. In ref. [2] we outlined how the euclidean Green functions are constructed. It is the exact treatment of the irreducible Green functions at euclidean exceptional momenta that makes our proofs so lengthy that we still had to postpone their publication, but we found how to handle those. We find that one can define exact running coupling constant(s) $\lambda(\mu)$ that satisfy equations of the form

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \lambda(\mu) = \beta(\mu, \{\lambda(\mu')\}) , \tag{1}$$

where β is a function of μ and a functional of $\lambda(\mu')$. It can be put in such a form that

$$\beta(\mu, \{\lambda(\mu')\}) = -\beta_0 \lambda^2(\mu) - \beta_1 \lambda^3(\mu) + \lambda^4(\mu) \rho(\mu, \{\lambda(\mu')\}) , \tag{2}$$

where

$$|\rho| \leqslant C , \tag{3}$$

if

$$|\lambda(\mu')| \leqslant \lambda^{\text{crit}} \tag{4}$$

at all μ', for some $C < \infty$ and $\lambda^{\text{crit}} > 0$. Basically, ρ is just a convergent expansion [4] in $\lambda(\mu')$. Eqs. (1)–(4) are our main result, from which we shall now derive Borel summability. For simplicity we restrict ourselves

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to one coupling constant only.

For massive theories we can restrict ourselves to

$$\mu' \geqslant m \,. \tag{5}$$

Of course eq. (2) with ρ replaced by zero can be solved exactly. Its integration constant defines the free parameter of the theory. In fact the mass m is defined similarly as an integration constant. Now we choose these parameters such that ineq. (4) is satisfied with a large enough margin. Then it is easy to show that by successively substituting this $\lambda(\mu')$ into the right-hand side of eq. (1) and integrating it one finds a series of functions $\lambda(\mu)$ that converge to a unique solution (this would not have been the case if the β_1 term had been absorbed into ρ). This solution defines our theory. From now on, for simplicity, we put $\beta_1 = 0$. Adding the effects due to β_1 only affects the lengths but not the nature of the following formulas. Crudely, our solution has the form

$$\lambda(\mu) = 1/(\beta_0 \log \mu + C) \,. \tag{6}$$

We write this as

$$\lambda(\mu) = 1/[\beta_0 \log(\mu/\mu_0) + \lambda^{-1}(\mu_0)] \,. \tag{7}$$

If we take this to be exact at infinite μ, then this is a good definition of $\lambda(\mu_0)$. Let us put

$$\lambda(\mu_0 = m) = \lambda \,. \tag{8}$$

Now remember that if ineq. (4) is satisfied then (7) is close to the exact solution. In euclidean space, $\beta_0 \log(\mu/\mu_0)$ is real and positive. So, in particular for complex λ it is easy to satisfy ineq. (4). It holds as soon as either

$$\mathrm{Re}\,\lambda \geqslant 0 \,, \qquad |\lambda| \leqslant \lambda^{\mathrm{crit}} \,, \tag{9}$$

or

$$|\mathrm{Im}(\lambda^{-1})| \geqslant (\lambda^{\mathrm{crit}})^{-1} \,. \tag{10}$$

Because of the higher-order corrections we must choose λ^{crit} in (9) and (10) somewhat smaller than $\underline{\lambda^{\mathrm{crit}}}$ in (4).

Consequently our euclidean Green functions are analytic in the entire region (9)–(10). From this it follows that if we write

$$G = \int_0^\infty \mathrm{d}z\, F(z) \mathrm{e}^{-z/\lambda} \,, \tag{11}$$

then

$$|F(z)| < A \mathrm{e}^{|z|/\lambda^{\mathrm{crit}}} \tag{12}$$

as soon as

$$\mathrm{Re}\, z \geqslant 0 \,.$$

From arguments of ref. [5] one can deduce that there is a number $r > 0$ such that $F(z)$ is analytic for $|z| < r$. Actually, here we can do more. Let us consider the change δG in our Green functions obtained if β in (1) is replaced by zero for $\mu > \mu_1$ for some μ_1, and unchanged below μ_1. Then essentially in (7) μ/μ_0 remains below μ_1/μ_0. In that case a third region of analyticity is added to (9) and (10):

$$\mathrm{Re}\,\lambda^{-1} < -\beta_0 \log(\mu_1/m) - (\lambda^{\mathrm{crit}})^{-1} \,, \tag{13}$$

so that the Borel transform is entirely analytic. But the error in $\lambda(\mu')$ causes an error in β also for $\mu < \mu_1$. Since β is given by convergent diagrams only, one can argue that this error goes like

$$(\mu/\mu_1)^{2-\epsilon} \tag{14}$$

for arbitrarily small ϵ. Thus, the error δG is limited by

$$|\delta G| \leqslant A(m/\mu_1)^{2-\epsilon} \,. \tag{15}$$

For any λ we choose μ_1 such that (13) is saturated. Then

$$|\delta G| \leqslant A \exp\left(\frac{2-\epsilon}{\beta_0} \mathrm{Re}\,\lambda^{-1}\right) \tag{16}$$

for some A. Thus G and $G + \delta G$ have the same Borel transform at

$$-(2-\epsilon)/\beta_0 \leqslant \mathrm{Re}\, z \leqslant 0 \,. \tag{17}$$

Since ϵ can be made arbitrarily small this implies that the Borel transform of G is analytic in $-2/\beta_0 < \mathrm{Re}\, z < +\infty$. Now at $z = -2/\beta_0$ we expect a singularity called "renormalon" [6,7]. We imagine that perhaps subtractions can be made to replace (14) by higher powers, so that the branch points at $-2n/\beta_0$ can be demonstrated, using operator product expansions. This we have not yet worked out.

If the mass is sent to zero then the analyticity region (10) remains. Now we replace β by zero both at $\mu > \mu_1$ and $\mu < \mu_2$, for $\mu_2 \ll \mu_1$. Using similar arguments we find analyticity of the Borel transform in

$$-2/\beta_0 < \mathrm{Re}\, z < 2/\beta_0 \,,$$

and perhaps the branch points (infrared renormalons) at $+2n/\beta_0$ can be found. So for massless planar QCD it is likely that the entire branch point structure of the

Borel transform can be proven. But although the physics of these infrared renormalons is clear [8,9], we do not know whether the Borel integral can be made sufficiently convergent after suitable subtractions. This is why we are unable to Borel sum the massless theory. A correction must be made to ref. [2]: it is highly dubious whether the proposed "block spin method" for considering the infrared behaviour is applicable at all, since we are working with infinite component field theories!

The author of ref. [1] expresses the philosophy that a sort of floating coupling constant should be used as an expansion parameter. Basically our work is just a realization of that idea in mathematically more precise terms (we take the entire irreducible four-point function as "expansion parameter"). But the price we pay for mathematical precision is further restrictions on the theory such as planarity (infinite-color limit) and massiveness.

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