QUANTUM FIELD THEORY AND GRAVITATION

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Abstract

The formulation of a generally covariant quantum field theory is described. The scaling limit of physical states and its relation to the metric of space-time is discussed.

One of the long standing challenges concerning the conceptual structure in theoretical physics is the question as to how a synthesis of the ideas of general relativity and quantum theory will look. Between 1945 and 1975 there has been (sociologically and perhaps even ideologically) a rather clean separation between the scientific communities of these two areas. It is a rather recent phenomenon that elementary particle theorists have become actively interested in gravitation theory and conversely, scientists working on differential geometry and general relativity, in elementary particle physics. The success of local gauge theories forced differential geometry on particle physicists. Hawking's suggestion of particle emission from a black hole, the realization that a local formulation of supersymmetry needs a supergravity theory and the projection that the "grand unification" of the weak and strong interactions leads to a mass scale which is not far from the Planck mass (which is defined by the gravitational constant) have contributed to a general feeling that the time is ripe to think about the synthesis of gravitation and quantum field theory.

My talk here will not be a review of the immense amount of work which could be subsumed under the given title. Rather I shall concentrate on some aspect of the problem, in particular the formulation of a generally covariant quantum field theory and the rôle of a scaling limit of states in relation to the space-time geometry.

Let me first recall a few corner stones of the classical theory of gravitation.

i) The effect of a gravitational field on matter.

In Newton's theory there is a gravitational field strength $\Gamma(x,t)$ due to which a test particle (mass point) experiences an acceleration

$$\frac{d^2\vec{x}}{dt^2} = \vec{\Gamma}(\vec{x},t) \tag{1}$$

The surprising fact is, that the mass of the test particles does not appear in the equation of motion. This universality leads to the principle of equivalence between inertial and gravitational mass.

In Einstein's theory this universality of the orbits of all (uncharged) particles is used to define "straight lines" in the 4-dimensional space time continuum. In a symmetric notation we denote coordinates of a space-time point x by $x^{\mu} = (t,x)$; $\mu = 0, 1, 2, 3$. To define straight lines in a general manifold we need the concept of "parallel transport", described by the "affine connection coefficients" $\Gamma^{\nu}_{\mu\rho}$ (x) and Newton's equation (1) is replace by

$$\frac{d^2x^{\nu}}{ds^2} = - \Gamma^{\nu}_{\mu\rho}(x) \frac{dx^{\mu}}{ds} \frac{dx^{\rho}}{ds} \tag{2}$$

where s is a suitably chosen parameter on the orbit. Γ is interpreted as an intrinsic geometric property of the 4-dimensional space-time continuum. We can choose an arbitrary coordinate system and know how Γ transforms from the one to the other viz:

$$\Gamma'^{\lambda}_{\mu\nu}(x') = \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \Gamma^{\rho}_{\tau\sigma}(x) + \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial^2 x^{\rho}}{\partial x'^{\mu}\partial x'^{\nu}} . \tag{3}$$

I have written down this well known formula to remind us of the fact that if Γ vanishes at some point in one coordinate system is does not vanish in another due to the inhomogeneous second term. We will encounter a corresponding transformation formula later. The intrinsic nature of Γ (and related quantities) is usually expressed as the principle of general covariance. The laws do not give any preference to any particular coordinate system. From Γ we can obtain the Riemann curvature tensor and the Ricci Tensor $R_{\mu\nu}$.

ii) The generation of a gravitational field by matter.

In Newton's theory a matter density distribution generates a gravitational field

$$\vec{\Gamma}(\vec{x},t) = - g \int \rho(\vec{x}',t) \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} d^3x' \tag{4}$$

In Einstein's theory it is assumed that space-time is a pseudo-Riemannian manifold equipped with a metric form $g_{\mu\nu}(x)$ which defines both a causal structure and a length scale. This metric form determines the affine connection Γ and allows to define the curvature scalar $R = g^{\mu\nu} R_{\mu\nu}$. The equation (4) of Newton's theory is then replaced by the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \gamma T_{\mu\nu} \tag{5}$$

where $T_{\mu\nu}$ is the energy momentum tensor of the matter distribution.

I have separated the description of gravitation into the parts i) and ii) because the first, in which the principle of general covariance and the intrinsic-geometric rôle of gravitation appear are more readily incorporated in a quantum theory than the second where the gravitational constant enters. This constant γ adds to $\hbar$ and c one more constant of presumably universal significance. In units in which $\hbar = c = 1$ i.e. where times and lengths are measured in cm and energies and masses in cm^{-1}, γ takes the value 10^{-66} cm². Thus, there is now a natural length unit

$$\lambda_0 = \left(\frac{\hbar\gamma}{c^3}\right)^{\frac{1}{2}} \approx 10^{-33}\ \mathrm{cm} \qquad \text{"Planck length"}$$

and a natural mass unit

$$M_0 = \frac{\hbar}{\lambda_0 c} \approx (\hbar c \gamma^{-1})^{\frac{1}{2}} = 10^{19}\ M_{Proton} \qquad \text{"Planck mass"}$$

M_0 is the mass for which the Compton wave length and the Schwarzschild radius become equal. One expects that the physics at lengths below λ_0 will become radically different. In some sence M_0 must act as a natural cut-off, λ_0 as a fundamental length reminiscent of the expectations 40 years ago (when, however, the fundamental length was assumed to be of the order of 10^{-13} cm i.e. larger than λ_0 by a factor 10^{20}). We cannot hope to get any direct experimental evidence about this regime of lengths below λ_0. However there are important implications: the theory should be finite (no renormalization necessary), absolute values of masses should be calculable in terms of M_0. It is tempting to speculate whether such a theory will eventually abandon the space-time continuum in favour of a quantum geometry of elementary events in which irreversibility appears on a fundamental level (rahter than due to subjective ignorance as in statistical mechanics). I shall not pursue this line here but focus attention on the more modest question: if we accept a 4-dimensional space-time continnum and the standard quantum theoretical concepts, how far can we get towards a satisfactory inclusion of gravity; in particular how can we incorporate general covariance?

Let me sketch quickly the frame in which the symbiosis of special relativity and quantum physics may be described. Here the space-time continuum is Minkowski space on which we have a physically distinguished set of coordinate systems (the globally inertial ones). We have the symmetry group of Poincaré transformations P. A group element $(a,\Lambda) \in P$ consists of a translation 4-vector a and a homogeneous Lorentz transformation Λ. In the quantum theory we may take a separable Hilbert space H, assuming that its rays correspond to pure physical states. We represent the action of a Poincaré transformation on the physical states by a unitary operator $U(a,\Lambda)$ acting on H. The U(g) with $g \in P$ form a unitary representation of P (up to a phase factor).

The relation of the quantum theoretic observables to space-time is established in a simple way. To each (open) region $\mathcal{O}$ of space-time we have an algebra $\mathfrak{A}(\mathcal{O})$ of operators acting on H. It is the algebra generated by all observables in $\mathcal{O}$. It is at this point not relevant whether we consider only bounded observables, in which case the algebras $\mathfrak{A}(\mathcal{O})$ may be taken as C^*-algebras or W^*-algebras, or whether we take them as Wightman-Borchers-algebras generated by smeared out field operators whose test functions have support in $\mathcal{O}$. All that is needed is that they are *-algebras over the complex numbers (i.e. addition and multiplication of elements is defined as well as multiplication of an element with a complex number and the adjoint operation $A \to A^*$). For finer analysis, of course, a topology in these algebras has to be used. For many purposes it is preferably to consider the algebras as abstract algebras (not yet represented by operators on a Hilbert space) and P as the abstract Poincaré group. Then a physical state ω is a positive linear form on the algebra $\mathfrak{A} = \cup \mathfrak{A}(\mathcal{O})$ (resp. on the covariance algebra generated by $\mathfrak{A}$ and P) and, given one such state, we can reconstruct a Hilbert space and a representation of $\mathfrak{A}$ (resp. $\mathfrak{A}$ and P) by operators acting on H (GNS-construction). Altogether we have a net of local algebras $\mathcal{O} \to \mathfrak{A}(\mathcal{O})$; the Poincaré group acts on $\mathfrak{A} = \cup \mathfrak{A}(\mathcal{O})$ by automorphisms in an obvious manner and, in the representations generated from physical states the Poincaré automorphisms are implemented by unitary operators $U(a,\Lambda)$. I shall abstain from writing down the obvious structural relations and mention only three which have relevance to the subsequent discussion:

1) Causality.

   a) If $\mathcal{O}_1$, $\mathcal{O}_2$ lie space-like to each other the elements of $\mathfrak{A}(\mathcal{O}_1)$ commute with those of $\mathfrak{A}(\mathcal{O}_2)$.

   b) If $\mathcal{O}_2$ lies in the causal shadow of $\mathcal{O}_1$, then $\mathfrak{A}(\mathcal{O}_2) \subset \mathfrak{A}(\mathcal{O}_1)$.

2) Stability.

   The infinitesimal generators P_μ of the (represented) translation subgroup U(a) are interpreted as energy-momentum operators of the system. Their simultaneous spectrum is assumed to be contained in the closed forward light cone. Usually one assumes also that there is a distinguished state ω_0, the vacuum, which is Poincaré invariant and hence a ground state for the energy.

3) In a Hilbert space representation we may always pass from the algebras $\mathfrak{A}(\mathcal{O})$ to the associated von Neumann algebras $R(\mathcal{O})$. We usually assume that in the representation generated (via GNS) from the vacuum state, all the $R(\mathcal{O})$ are factors i.e. they have trivial center.

What changes if we go over from Minkowski space to a more general manifold? The first step is the consideration of a pseudo-Riemannian manifold. The metric structure is then fixed and describable by a classical metric field $g_{\mu\nu}$. We want to describe quantum fields living on such a non flat space-time manifold. In that case the concept of the local net of algebras, the causality principles 1) and the assumption 3) can be kept without change. We loose the Poincaré group as a symmetry and thereby the distinguished vacuum state and this means that the stability condition 2) has to be formulated in a different manner. We have made a proposal how this should be done in a recent paper [1]. The basic idea was that by a scaling limit one can define a net of algebras associated with the tangent space at a point of the manifold and that all allowed physical states coalesce to one distinguished state on each tangent space algebra. Under some continuity assumptions this state is invariant under translations in tangent space and therefore the stability requirement can be formulated there. In [1] some of the arguments were somewhat handwaving and some proofs lacked rigour. This has been remedied by K. Fredenhagen [2] and it turned out that the same technique can be applied to a more general situation in which a priori no given Riemannian structure of the space-time manifold is assumed. This is the subject of a joint work by K. Fredenhagen and myself [3] which is still in progress and of which I want to report here some results.

To formulate the theory in a generally covariant way we adopt the point of view which had been advocated for many years by my fondly remembered friend, Hans Ekstein [4]. Ekstein pointed out that one should distinguish between observation procedures and observables. A "procedure" means typically that certain instruments are placed in a certain space-time region. There may be quite different procedures (associated with different space-time regions) which are equivalent in the sense that they yield identical expectation values in all physical states. An "observable" is then an equivalence class of procedures. Two procedures A_1, A_2 are equivalent if $\omega(A_1-A_2) = 0$ for all physically allowed states ω. This implies that $A_1 - A_2$ generates a proper 2-sided ideal in $\mathfrak{A}$. We have then, for each region $\mathcal{O}$ an ideal $J(\mathcal{O})$ generated by the equivalence relation of procedures. The customary "observable algebras" are the quotients

$$\mathfrak{A}_{obs}(\mathcal{O}) = \mathfrak{A}(\mathcal{O}) / \mathcal{J}(\mathcal{O})$$

In the following we shall take $\mathfrak{A}$ as a "flexible algebra" whose elements correspond to procedures. The physical laws may be characterized either by specifying the local ideals $J(\mathcal{O})$ directly (equations of motion, commutation relations ...) or, alternatively, by characterizing the class of all physically relizable states on $\mathfrak{A}$. In the latter case the relevant ideals are formed by the elements of $\mathfrak{A}$ on which all physical states vanish. We shall pursue here this second alternative.

The interpretation of $\mathfrak{A}$ as a flexible algebra of procedures allows us to consider the group of local diffeomorphisms of the manifold as acting by automorphisms on $\mathfrak{A}$. This makes the formulation manifestly covariant under general coordinate transformations. We shall not be concerned here with topological properties of the manifold at large but confine attention to a sufficiently small, contractible region for which a single chart is adequate. The diffeomorphisms mentioned shall be understood to transform this region into itself. We may then interpret such transformations either in the passive sense as changing the description (coordinatization) and leaving the physical situation unchanged or in the active sense, as keeping the reference frame unchanged but going over to a different physical situation (state).

Let us now illustrate this in the simplest case: a scalar quantum field on a 4-dimensional manifold which we do not a priori endow with a (physically distinguished) Riemann structure.We take as the flexible algebra of procedures the free Wightman-Borchers algebra, i.e. the tensor algebra of test functions on the part of the manifold considered. Its elements are formal linear combinations of monomials $f^{(n)}$ where $f^{(n)} = f^{(n)}(P_1, \ldots, P_n)$ is a smooth function on MxMx ... xM with support in $\mathcal{O}$ in each argument. In a chart φ which coordinatizes the point $P \in M$ by $x = \varphi(P) \in \mathbb{R}^4$, the monomial is described by the function

$$f_{\varphi}^{(n)}(x_1, \ldots, x_n) = f^{(n)}(\varphi^{-1}(x_1), \ldots, \varphi^{-1}(x_n))$$

Usually, using the field operators $\phi(x)$, the monomial $f^{(n)}$ is written as $\int f_{\varphi}^{(n)}(x_1, \ldots, x_n)\, \phi(x_1) \ldots \phi(x_n)\, d^4 x_1 \ldots d^4 x_n$ but we shall not use this notation because it introduces here unnecessary ambiguities. The algebraic product of $f^{(n)}$ with $g^{(m)}$ is just the tensor product

$$f^{(n)} \cdot g^{(m)} = h^{(n+m)} \; ; \quad h^{(n+m)}(P_1, \ldots P_{n+m}) = f^{(n)}(P_1, \ldots P_n) \cdot g^{(m)}(P_{n+1}, \ldots P_{n+m})$$

A state on the algebra is given by a hierarchy of distributions $\omega^{(n)} \in D'(\mathcal{O}\times\mathcal{O}\dots\times\mathcal{O})$. $\omega^{(n)}(f^{(n)})$ is the expectation value in ω of the monomial element $f^{(n)}$ of the algebra. In each chart we have, correspondingly, a set of distributions $\omega_\varphi^{(n)}$ on $\mathbb{R}^4 \times .. \mathbb{R}^4$ and, of course, for two charts

$$\omega_\varphi^{(n)}(f_\varphi^{(n)}) = \omega_\psi^{(n)}(f_\psi^{(n)})$$

$$f_\psi^{(n)}(y_1,..y_n) = f_\varphi^{(n)}(\chi^{-1}y_1,..\chi^{-1}y_n) \tag{6}$$

where $\chi = \psi\varphi^{-1}$ is the transition function between the charts.

To get some handle on the characterization of the physically allowed states we consider their scaling limits at the points of the manifold. In a chart φ (coordinates x) we define the scaling to a point $\bar{x}$ by the 1-parameter group

$$\delta_\lambda x = \bar{x} + \lambda(x-\bar{x}) \tag{7}$$

contracting $\varphi(\mathcal{O})$ to the point $\bar{x}$ for $\lambda \to 0$. This corresponds to a 1-parameter subgroup of diffeomorphisms in M, contracting to $P = \varphi^{-1}(\bar{x})$ in a way which depends on the chosen chart φ. Correspondingly we consider the sequence of test functions

$$(f_\varphi)_{\bar{x},\lambda} = (\lambda^{-4}N(\lambda))^n f_\varphi(\delta_\lambda^{-1}x_1, .\ \delta_\lambda^{-1}x_n) \tag{8}$$

whose supports shrink to a point for $\lambda \to 0$. $N(\lambda)$ is a suitably chosen scaling factor. We say, that the state ω has a scaling limit of order $N(\lambda)$ if

$$\lim_{\lambda\to 0} \omega_\varphi((f_\varphi)_{\bar{x},\lambda}) \equiv \omega_{\bar{x}}^\varphi(f_\varphi) \tag{9}$$

is finite for all $f \in D'(\mathcal{O}^n)$ and is nonvanishing for some.

One may object that the assumption of the existence of such scaling limits disregards the existence of the Planck length and that therefore part ii) of the gravitation theory cannot be accomplished in an ultimately satisfactory manner once this assumption is made. This is probably true. Nevertheless it is an interesting first step, in which a synthesis of some principles of general relativity with quantum theory is achieved while the other part must be left on a semiclassical level.

One shows easily that the scaling factor $N(\lambda)$ is the same in every chart. It follows also [2] that $N(\lambda)$ is "almost" a power:

$$\lim_{\mu\to 0} \frac{N(\lambda\mu)}{N(\mu)} = \lambda^\alpha \tag{10}$$

We have split off the factor λ^{-4} in (8) so that α is positive if the distributions are singular (compared to the Lebesque measure). For each chart φ and each contraction point $\bar{x}$ the limit distributions (9) define a state on the tensor algebra of smooth functions on $\mathbb{R}^4$. The support restriction of the test functions in the manifold becomes wiped out because $\delta_\lambda^{-1}(\mathcal{O})$ tends to all of $\mathbb{R}^4$ in the limit $\lambda \to 0$. The dependence of $\omega_{\bar{x}}$ on the chart is found to be [2]

$$\omega_{\bar{y}}^{\psi}(F) = \omega_{\bar{x}}^{\varphi}(F \circ \overset{\circ}{\chi}{}^{-1}) \tag{11}$$

where $\overset{\circ}{\chi}$ is the affine (inhomogeneous linear) transformation, obtained from the transition function $\chi = \psi\varphi^{-1}$ between the charts by

$$(\overset{\circ}{\chi} x)^{\mu} = \bar{y} + \frac{\partial \chi^{\mu}}{\partial x^{\nu}}(x^{\nu} - \bar{x}^{\nu}) \; . \tag{12}$$

We can get, therefore, a chart independent definition of the scaling limits of a state ω at a point $P \in M$ by

$$\omega_P(f) = \omega_{\varphi(P)}^{\varphi}(f \circ \overset{\circ}{\varphi}{}^{-1}) \tag{13}$$

where f is now in the tensor algebra of smooth functions on the tangent space at P and

$$\overset{\circ}{\varphi}(z) = \varphi(P) + d\varphi(z) \tag{14}$$

maps the tangent vector $z \in T_P$ into $\mathbb{R}^4$.[1)] If we use in tangent space the frame which corresponds to the coordinatization φ then $\overset{\circ}{\varphi}$ is simply

$$\overset{\circ}{\varphi}(z) = \bar{x} + z \quad ; \qquad \overset{\circ}{\varphi}{}^{-1}(x) = x - \bar{x} \; . \tag{15}$$

The essential observation which leads to the linear (affine) transformation law (11) is the following. If we expand the transition function χ in the neighborhood of $x = \bar{x}$ in powers of $x - \bar{x}$:

$$(\chi x)^{\mu} = \bar{y}^{\mu} + a^{\mu}{}_{\nu}(x^{\nu} - \bar{x}^{\nu}) + \tfrac{1}{2} b^{\mu}{}_{\nu\rho}(x^{\nu} - \bar{x}^{\nu})(x^{\rho} - \bar{x}^{\rho}) + \dots \tag{16}$$

1) Notation: φ maps a neighborhood of $P \in M$ into $\mathbb{R}^4$ (chart). $d\varphi$ is the corresponding map of the tangent space at P on the tangent space of $\mathbb{R}^4$ at $\bar{x}$ which is naturally identified with $\mathbb{R}^4$.

and define

$$\chi_\lambda = \delta^{-1}_{\bar{y},\lambda} \circ \chi \circ \delta_{\bar{x},\lambda} \tag{17}$$

where $\delta_{\bar{x},\lambda}$ is the linear contraction (7) to the point $\bar{x}$, $\delta_{\bar{y},\lambda}$ the linear contraction to $\bar{y}$, then

$$(\chi_\lambda x)^\mu = \bar{y}^\mu + a^\mu{}_\nu (x^\nu - \bar{x}^\nu) + \lambda \, \tfrac{1}{2} \, b^\mu{}_{\nu\rho} (x^\nu - \bar{x}^\nu)(x^\rho - \bar{x}^\rho) + O(\lambda^2) \tag{18}$$

So χ_λ converges to $\mathring{\chi}$ for $\lambda \to 0$. On the other hand

$$\omega_\psi\left((f)_{\bar{y},\lambda}\right) = \left(\lambda^{-4} N(\lambda)\right)^n \omega_\varphi\left(f \circ \chi_\lambda \circ \delta^{-1}_{\bar{x},\lambda}\right) \to \omega^\varphi_{\bar{x}}(f \circ \mathring{\chi}).$$

There are two properties of the tangent space states which follow generally. The first is the transformation law under dilations in tangent space

$$\omega_P^{(n)}\left(f^{(n)} \circ \delta_\lambda\right) = \left(\lambda^{\alpha-4}\right)^n \omega_P^{(n)}\left(f^{(n)}\right) \tag{19}$$

$$\left(f^{(n)} \circ \delta_\lambda\right)(z_1, \ldots z_n) \equiv f^{(n)}(\lambda z_1, \ldots \lambda z_n). \tag{20}$$

This follows directly from the definition and (10). The second is the invariance under translations in tangent space

$$\omega_P(T_a f) = \omega_P(f) \tag{21}$$

$$(T_a f)(z_1, \ldots z_n) \equiv f(z_1 - a, \ldots z_n - a). \tag{22}$$

This follows if we assume that ω_P depends smoothly on P.

Let us calculate the right hand side of (13) in the chart φ using as a basis in the tangent spaces the one corresponding to the coordinate axes, so $\mathring{\varphi}$ is given by (15). Then

$$\omega^\varphi_{\bar{x}}\left(f^{(n)} \circ \mathring{\varphi}^{-1}\right) = \lim \left(\lambda^{-4} N(\lambda)\right)^n \omega \, f\left(\frac{x_1 - \bar{x}}{\lambda}, \ldots \frac{x_n - \bar{x}}{\lambda}\right) \tag{23}$$

If we are allowed to interchange differentiation with respect to $\vec{x}$ and the formation of the limit $\lambda \to 0$ then

$$\frac{\partial}{\partial \bar{x}^{\mu}} \omega^{\varphi}_{\bar{x}} \left(f^{(n)}_{0} \circ \varphi^{-1} \right) = - \lim \left(\lambda^{-4} N(\lambda) \right)^{n} \cdot \lambda^{-1} \omega_{\varphi} \left(P_{\mu} f \Big|_{z_i = \frac{x_i - \bar{x}}{\lambda}} \right) \tag{24}$$

where

$$P_{\mu} f = \sum_{i} \frac{\partial f}{\partial z_i^{\mu}} . \tag{25}$$

Since the right hand side involves one factor λ more in the denominator this can be finite only if the scaling limit of ω $(P_{\mu} f)$ with the original scaling factor $(N(\lambda)\lambda^{-4})^{n}$ vanishes. This is the infinitesimal form of (21), (22). [1)]

From the general properties of ω_P mentioned and the assumption that $\alpha = 1$ (corresponding to the canonical dimension of a scalar field) it follows that the 1-point-distribution $\omega_P^{(1)} = 0$ and the 2-point-distribution $\omega_P^{(2)}$ is given by a (positive) measure on $\mathbb{R}^4$ ("momentum space", intrinsically the cotangent space at P) which is homogeneous of degree-2 with respect to dilations. If, in addition, we believe that there are no quantum observables at a point and that among the physical states there are "primary" ones i.e. states with generate (by the GNS-construction) a Hilbert space representation in which $R(\mathcal{O})$ is a factor, then all the states in one such primary folium (e.g. all vector states of such a representation) give the same scaling limit ω_P because $\bigcap_{\lambda} \mathcal{A}(\mathcal{O}_{\lambda})$ is the algebra of procedures at the point $\bar{x}$ and this should lie in the center of the algebra. We still have to incorporate the remnants of the causality and stability conditions of the Minkowski space theory. The simplest assumption is the ω_P arising from a primary folium are invariant under a subgroup of linear transformations in tangent space isomorphic to the Lorentz group and that they satisfy the spectrum condition with respect to the translation operators T_a in tangent space. In that case the transformation law of ω_P under dilations with $\alpha = 1$ fixes ω_P uniquely. It must be the vacuum state of a <u>free</u>, <u>massless</u> scalar theory [2)] in Minkowski space [1]. The state (or rather the primary folium of states) determines then not only a causal structure but a metric form in the tangent space at each point. The difference of perspective as compared to [1]

1) Actually for (21), (22) the differentiability of ω_P would not be necessary, only some uniform continuity. A precise proof has been given by Fredenhagen [2].

2) This does not mean that the theory itself must be free since $N(\lambda)$ can differ from λ for instance by logarithmic factors as it happens in "asymptotically free" theories.

is that we do not assume a priori that the manifold is equipped with a Riemann structure. Instead each primary folium of states determines a Riemann structure. The physical laws may allow different Riemann structures. They are separated by superselection rules as long as we assume the existence of scaling limits and of physical states which are primary on the algebras of arbitrary small regions.

To finish, let us compute the transformation law of the left hand side of (24) under coordinate change and its relation to an affine connection on M. Denoting this quantity by $(\omega^{\varphi}_{\bar{x}})_{,\mu}$ we have

$$(\omega^{\psi}_{\bar{y}})_{,\mu}(F) = (a^{-1})_{\mu}{}^{\nu}\left[\omega^{\varphi}_{\bar{x}}(F\circ A) - (a^{-1})_{\sigma}{}^{\rho}\, b^{\sigma}{}_{\nu\beta}\, \omega^{\varphi}_{\bar{x}}\left(\sum_j z^{\beta}_j \frac{\partial}{\partial z^{\rho}_j} f\circ A\right)\right]$$

where

$$(Az)^{\mu} = a^{\mu}{}_{\nu} z^{\nu} \; ; \qquad a^{\mu}{}_{\nu} = \frac{\partial y^{\mu}}{\partial x^{\nu}}(\bar{x}) \; ; \quad b^{\mu}{}_{\nu\rho} = \frac{\partial^2 y^{\mu}}{\partial x^{\nu}\partial x^{\rho}}(\bar{x}) .$$

If one assumes that to each point P there exists a coordinate system ψ, depending on P, in which $(\omega^{\psi}_{\psi(P)})_{,\mu} = 0$ (geodesic system at the point) then in another system

$$\nabla_{\mu}\, \omega^{\varphi}_{\varphi(P)} = 0$$

$$\nabla_{\mu} = \frac{\partial}{\partial x^{\mu}} - \Gamma^{\rho}_{\mu\beta}\, K_{\rho}{}^{\beta}$$

$$K_{\rho}{}^{\beta}\, \omega(f) = \omega\left(\sum_j z^{\beta}_j \frac{\partial}{\partial z^{\rho}_j} f\right)$$

and the Γ have the transformation properties of Christiphel symbols (3). In intrinsic language they define an affine connection which is determined by the primary folium of states considered.

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