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A CHIRAL ALTERNATIVE TO THE VIERBEIN FIELD IN GENERAL RELATIVITY

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An alternative to the usual vierbein field in a (3 + 1)-dimensional (euclidean) space-time is proposed such that the internal index takes only three values and the external is a double: $e^a{}_{\mu\nu} = -e^a{}_{\nu\mu}$. In flat space-time this field reduces to the self-dual generalized Levi-Civita symbol $\eta^a{}_{\mu\nu}$. Like the vierbein field, our field determines the metric field $g_{\mu\nu}$ uniquely. It can be viewed upon as the "cube root" of the metric field. In euclidean space the internal symmetry group is SL(3). In Minkowski space, in a sense to be explained, the internal symmetry group is SU(3).

The Einstein–Hilbert action takes an elegant form in terms of this new field.

1. Introduction

For various reasons it is sometimes useful to introduce the "square root" of the metric tensor $g_{\mu\nu}$ in General Relativity, called the vierbein field $e^a{}_\mu$,

$$g_{\mu\nu} = e^a{}_\mu e^a{}_\nu . \tag{1.1}$$

Here the Greek indices μ and ν transform as usual vector indices, whereas the Latin indices a are "internal" indices.

Eq. (1.1) does not determine $e^a{}_\mu$ completely since $g_{\mu\nu}$, being symmetric in μ and ν, has 10 independent components whereas $e^a{}_\mu$ has 16. The remaining 6 degrees of freedom are in the internal O(3, 1) symmetry in the a index,

$$(e^a{}_\mu)' = S^{ab} e^b{}_\mu , \qquad S^{ab} S^{cb} = S^{ba} S^{bc} = \delta^{ac} . \tag{1.2}$$

Since this is a local symmetry one often also introduces a connection field $A^{ab}{}_\mu$ simply by demanding the covariant derivative of the e-field to vanish,

$$D_\mu e^a{}_\nu = \partial_\mu e^a{}_\nu - \Gamma^\lambda{}_{\mu\nu} e^a{}_\lambda + A^{ab}{}_\mu e^b{}_\nu = 0 . \tag{1.3}$$

Note that the Christoffel field $\Gamma^\lambda{}_{\mu\nu}$ has $10 \times 4 = 40$ degrees of freedom and $A^{ab}{}_\mu$

has $6\times4=24$ degrees of freedom. Eq. (1.3) has $4\times4\times4=64$ components. It is not difficult to deduce unique expressions for Γ and A from (1.3).

The Riemann curvature $R^{\alpha}{}_{\beta\mu\nu}$ can be expressed either in terms of the Γ-field or in terms of the A-field. This is because from (1.3) it also follows that

$$[D_\mu D_\nu]e^a{}_\beta = -R^\alpha{}_{\beta\mu\nu}e^a{}_\alpha + F^{ab}{}_{\mu\nu}e^b{}_\beta = 0, \tag{1.4}$$

where

$$F^{ab}{}_{\mu\nu} = \partial_\mu A^{ab}{}_\nu - \partial_\nu A^{ab}{}_\mu + [A_\mu, A_\nu]^{ab}, \tag{1.5}$$

so that, moving indices up and down with $e^a{}_\mu$,

$$R^{ab}{}_{\mu\nu} = F^{ab}{}_{\mu\nu}. \tag{1.6}$$

It is possible now to cast Einstein's equations in a Lagrange form containing only the e- and A-fields,

$$\mathscr{L} = \sqrt{g}\,R = \det(e)\,F^{ab}{}_{\mu\nu}e^\mu{}_a e^\nu{}_b, \tag{1.7}$$

where $e^\mu{}_a$ (Greek and Latin indices reversed) is the inverse of $e^a{}_\mu$.

One observes that the combination of det(e) with the inverse of the e-field in (1.7) invites one to simplify things there, using

$$\det(e)\,(e^\mu{}_a e^\nu{}_b - e^\nu{}_a e^\mu{}_b) = \tfrac{1}{2}\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{abcd}e^c{}_\alpha e^d{}_\beta, \tag{1.8}$$

so that

$$\mathscr{L} = \tfrac{1}{2}F^{ab}{}_{\mu\nu}\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{abcd}e^c{}_\alpha e^d{}_\beta. \tag{1.9}$$

One easily sees that varying (1.9) with respect to the A-field gives

$$D_\mu e^a{}_\nu - D_\nu e^a{}_\mu = 0, \tag{1.10}$$

which is sufficient to determine the A-field in terms of $\partial_\mu e^a{}_\nu$ (cf. eq. (1.3)). This implies that the F may be identified with the R, so that the A-field may be eliminated, after which variation with respect to the e-field now gives the same as varying $\sqrt{g}\,R$ with respect to $g_{\mu\nu}$: the Einstein equations.

If we use the first-order formalism where $A^{ab}{}_\mu$ and $e^a{}_\mu$ are varied independently then this is a beautifully simple polynomial lagrangian.

If we take the (2 + 1)-dimensional theory instead of the (3 + 1)-dimensional one, the ε's in eqs. (1.8) and (1.9) have one index less, so that the kinetic term in eq. (1.9) is quadratic. This allows one to quantize the theory around the values $e^a{}_\mu \simeq 0$ instead of the usual choice $e^a{}_\mu \simeq \delta^a{}_\mu$. Witten [1] proposed to treat the resulting topological theory as a "renormalizable Chern–Simons theory".

It is not clear whether his proposal is any improvement in the (2 + 1)-dimensional theory since before this trick was applied the theory was trivial: in 2 + 1 dimensions in the absence of matter space-time is completely flat [2].

In 3 + 1 dimensions one cannot perturb around the values $e^a{}_\mu \simeq 0$ because the lowest terms in $\mathscr{L}$ are cubic rather than quadratic in the fields. But could we not get something more interesting if we introduce a *two-Lorentz-index* dynamic field instead of $e^a{}_\mu$? It would certainly be spectacular if Quantum Gravity could be turned into a renormalizable "topological" theory this way.

Of course such a miracle should not be expected. In this paper we only derive equations that are mathematically equivalent with Einstein's equations, whereas it would definitely be necessary to introduce much more "new physics" to solve Quantum Gravity. Nevertheless, the two-Lorentz-index field variable that we will define now is interesting, and may inspire one towards new ideas that do involve new physics. Here is how it goes.

2. The new field variable

Consider first flat space-time. Here one may define an invariant tensor $\eta^a{}_{\mu\nu}$ (a = 1, 2 or 3) in the following way [3]:

$$\eta^a{}_{\mu\nu} = -\eta^a{}_{\nu\mu} = \varepsilon_{a\mu\nu} + \delta_{a\mu}\delta_{\nu 4} - \delta_{a\nu}\delta_{\mu 4}\,, \tag{2.1}$$

where ε is the three-dimensional Levi-Civita symbol (vanishing if one of the indices is given the value 4). It is self-dual,

$$\eta^a{}_{\mu\nu} = \tfrac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\eta^a{}_{\alpha\beta}\,. \tag{2.2}$$

We also define an anti-self-dual tensor,

$$\bar\eta^a{}_{\mu\nu} = -\bar\eta^a{}_{\nu\mu} = \varepsilon_{a\mu\nu} - \delta_{a\mu}\delta_{\nu 4} + \delta_{a\nu}\delta_{\mu 4}\,, \tag{2.3}$$

$$\bar\eta^a{}_{\mu\nu} = -\tfrac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\bar\eta^a{}_{\alpha\beta}\,. \tag{2.4}$$

In euclidean space-time these tensors are real. It is clarifying to view them as mappings of the antisymmetric (6) representations of SO(4) onto the $3_L \otimes 1_R$ and the $1_L \otimes 3_R$ representations of $SO(3)_L \otimes SO(3)_R$.

In Minkowski space $\eta^a{}_{\mu\nu}$ is complex. One then has

$$\bar\eta^a{}_{\mu\nu} = (\eta^a{}_{\mu\nu})^*\,. \tag{2.5}$$

More about Minkowski space in sect. 6. For the time being we will stick to euclidean space.

One has the identities

$$\eta^{p}{}_{\alpha\beta}\eta^{p}{}_{\mu\nu} = \delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\mu} + \varepsilon_{\alpha\beta\mu\nu}\,, \tag{2.6}$$

$$\eta^{a}{}_{\mu\alpha}\eta^{b}{}_{\nu\alpha} = \delta_{ab}\delta_{\mu\nu} + \varepsilon^{abc}\eta^{c}{}_{\mu\nu}\,, \tag{2.7}$$

and a bunch of others.

We could now propose to introduce a field $e^{a}{}_{\mu\nu}$ in curved space-time that takes the values $\eta^{a}{}_{\mu\nu}$ in a locally flat coordinate frame, so that in a general coordinate frame

$$e^{a}{}_{\mu\kappa}e^{a}{}_{\nu\lambda}g^{\kappa\lambda} \overset{?}{=} 3g_{\mu\nu}\,, \tag{2.8}$$

but then there would be a difficulty. This field would have $3\times 6 = 18$ components and the internal symmetry group O(3) would be 3-dimensional. Since $g_{\mu\nu}$ has only 10 independent components this leaves 5 field components too many; these 5 then cannot be rotated away by gauge rotations, and should then correspond to physically observable fields. There are no obvious candidates for such fields in Nature (they would correspond to a $5_{\mathrm{L}}\otimes 1_{\mathrm{R}}$ representation of $\mathrm{SO(3)_L}\otimes\mathrm{SO(3)_R}$, i.e. transform like the self-dual part of the Weyl curvature tensor).

We need a larger internal symmetry group with 8 generators. A natural candidate (in euclidean space) is SL(3). Indeed it is possible to replace eq. (2.8) by an SL(3) invariant expression,

$$\varepsilon_{abc}e^{a}{}_{\mu\nu}e^{b}{}_{\kappa\lambda}e^{c}{}_{\rho\sigma}\varepsilon^{\mu\nu\kappa\rho} = 24\sqrt{g}\,g_{\lambda\sigma}\,, \tag{2.9}$$

where the factor $\sqrt{g}$ is inserted in order to have the left- and right-hand side transform the same way under general coordinate transformations (see however sect. 5, where we choose to work with other fields that transform anomalously). Eq. (2.9) holds for the η-tensors in flat space, and it is invariant under any transformations of the form

$$e^{a}{}_{\mu\nu} \Rightarrow S^{a}{}_{b}e^{b}{}_{\mu\nu}\,, \tag{2.10}$$

if $\det S = 1$.

As was done for the vierbein field, we introduce an SL(3) connection field $A^{a}{}_{b\mu}(x)$ by demanding

$$D_{\mu}e^{a}{}_{\alpha\beta} = \partial_{\mu}e^{a}{}_{\alpha\beta} - \Gamma^{\lambda}{}_{\mu\alpha}e^{a}{}_{\lambda\beta} - \Gamma^{\lambda}{}_{\mu\beta}e^{a}{}_{\alpha\lambda} + A^{a}{}_{b\mu}e^{b}{}_{\alpha\beta} = 0\,. \tag{2.11}$$

We have

$$A^{a}{}_{a\mu} = 0\,, \tag{2.12}$$

but no further constraints such as antisymmetry for the indices of A, therefore eq. (2.11) with its $4\times3\times6=72$ components may determine the 40 Christoffel fields Γ and the $8\times4=32$ A-fields unambiguously.

To be precise, we must assume that $g_{\mu\nu}$, as defined by eq. (2.9) from the e-fields, has an inverse $g^{\mu\nu}$, and furthermore that the bilinear

$$K^{ab}=\tfrac{1}{8}\varepsilon^{\alpha\beta\mu\nu}e^{a}{}_{\alpha\beta}e^{b}{}_{\mu\nu} \tag{2.13}$$

has an inverse K_{ab}. Multiplying eq. (2.11) with $\varepsilon^{\alpha\beta\sigma\tau}e^{c}{}_{\sigma\tau}$ gives an equation for $A^{a}{}_{b\mu}K^{bc}$. One then proves from eqs. (2.9) and (2.11) that $D_\mu g_{\alpha\beta}=0$, from which $\Gamma^{\lambda}{}_{\mu\nu}$ can be determined uniquely.

3. The Riemann curvature

Following the analogon of the vierbein field closely we deduce from eq. (2.11)

$$\left[D_\mu,D_\nu\right]e^{a}{}_{\alpha\beta}=F^{a}{}_{b\mu\nu}e^{b}{}_{\alpha\beta}-R^{\lambda}{}_{\alpha\mu\nu}e^{a}{}_{\lambda\beta}-R^{\lambda}{}_{\beta\mu\nu}e^{a}{}_{\alpha\lambda}=0\,, \tag{3.1}$$

where

$$F^{a}{}_{b\mu\nu}=\partial_\mu A^{a}{}_{b\nu}-\partial_\nu A^{a}{}_{b\mu}+\left[A_\mu,A_\nu\right]^{a}{}_{b}\,. \tag{3.2}$$

Now in any point x we can use a tangent coordinate frame where $e^{a}{}_{\mu\nu}=\eta^{a}{}_{\mu\nu}$; in that coordinate frame at the point x,

$$K^{ab}=\delta^{ab}\,, \tag{3.3}$$

and after a little algebra (multiplying eq. (3.1) with $\eta^{c}{}_{\alpha\beta}$),

$$F^{a}{}_{c\mu\nu}=\tfrac{1}{2}\varepsilon_{acd}\eta^{d}{}_{\lambda\alpha}R_{\lambda\alpha\mu\nu}\,, \tag{3.4}$$

from which it follows that

(i) the curvature F of the A-field only represents the *self-dual part* of the Riemann tensor, and

(ii) the curvature F is *antisymmetric* in its internal indices a and c, which means that the curvature is all within an SO(3) subgroup of the internal symmetry group SL(3).

This second point can be understood by realizing that there is a symmetric tensor K^{ab}, eq. (2.13), which satisfies

$$D_\mu K^{ab}=0\,, \tag{3.5}$$

because the covariant derivative of $e^{a}{}_{\mu\nu}$ vanishes (eq. (2.11)). Thus, if we fix the internal gauge freedom SL(3) into an SO(3) subgroup by requiring

$$K^{ab} = \delta^{ab}\,, \tag{3.6}$$

then in this gauge the field $A^a{}_{b\mu}$ is antisymmetric in its indices a and b.

In the tangent coordinate frame we have also, using eq. (2.6),

$$\tfrac{1}{2}F^a{}_{c\mu\nu}\varepsilon_{acd}\eta^d{}_{\mu\beta} = R_{\nu\beta}\,, \qquad \tfrac{1}{2}F^a{}_{c\mu\nu}\varepsilon_{acd}\eta^d{}_{\mu\nu} = R\,, \qquad (3.7),(3.8)$$

so the Ricci tensor can be found from F.

Note: eq. (3.6) contains 6 equations. They not only reduce SL(3) (which has 8 generators) to SO(3) (which has 3 generators), but also fix the conformal factor in the general coordinate transformations. Under the latter K^{ab} transform as $\sqrt{g}$, as we can read off from the defining equation (2.13). We get

$$\det K = g^{3/2}\,. \tag{3.9}$$

Since eq. (3.9) holds trivially in the tangent frame it holds in all coordinate frames and all SL(3) gauges.

4. The Einstein action

Eq. (2.9) gives us $g_{\mu\nu}$ and $\sqrt{g}$ in terms of $e^a{}_{\mu\nu}$, and eq. (3.8) gives us R. The Einstein action is

$$S = \int \mathrm{d}^4x\, \mathscr{L}(x)\,, \tag{4.1}$$

with the gauge-invariant expression for $\mathscr{L}$ being

$$\mathscr{L} = \tfrac{1}{4}F^a{}_{c\mu\nu}\hat{K}^{cb}\varepsilon_{abd}e^d{}_{\alpha\beta}\varepsilon^{\mu\nu\alpha\beta}\,, \tag{4.2}$$

where

$$\hat{K}^{cb} = K^{cb}(\det K)^{-1/3}\,.$$

The determinant was necessary to give $\mathscr{L}$ the correct weight under conformal transformations.

Varying this with respect to $g_{\mu\nu}$, considering the other fields as functions of $g_{\mu\nu}$ and its derivatives, gives of course Einstein's equations. Alternatively, we could take $A^a{}_{b\mu}$ and $e^a{}_{\mu\nu}$ as the primary field variables. Keeping S stationary with respect to variations in A gives

$$\varepsilon_{\mu\nu\alpha\beta}D_\nu\left(\hat{K}^{cb}\varepsilon_{abd}e^d{}_{\alpha\beta}\right) = 0\,. \tag{4.3}$$

This gives an equation for A which in the tangent frame (as defined in sect. 3) reads

$$A_{\nu\ e}^{\ c}\varepsilon_{aed}\eta^{d}_{\ \mu\nu}-A_{\nu\ a}^{\ f}\varepsilon_{fcd}\eta^{d}_{\ \mu\nu}=-\partial_{\mu}(\hat{K}\varepsilon e)^{c}_{\ a}\equiv X_{\mu\ a}^{\ c}. \tag{4.4}$$

The author verified that eq. (4.4) can indeed be inverted to determine $A_{\nu\ b}^{\ a}$ uniquely from $X_{\mu\ b}^{\ a}$. An elegant way to do this is by introducing the operator

$$U^{ab}_{\ \ \mu\nu}=\varepsilon^{abc}\eta^{c}_{\ \mu\nu}, \tag{4.5}$$

and writing eq. (4.4) as

$$U^{A}A+U^{B}A=X, \tag{4.6}$$

where

$$(U^{A}A)_{\mu\ a}^{\ c}=U^{cb}_{\ \ \mu\nu}A_{\nu\ a}^{\ b},\qquad (U^{B}A)_{\mu\ a}^{\ c}=U^{ab}_{\ \ \mu\nu}A_{\nu\ b}^{\ c}. \tag{4.7}$$

Both X and A consist of the $SU(2)^{\mathrm{Left}}\otimes SU(2)^{\mathrm{Right}}$ representations $(2_{\mathrm{L}}\otimes 2_{\mathrm{R}})\otimes(3_{\mathrm{R}}\oplus 5_{\mathrm{R}})=2_{\mathrm{L}}\otimes(2_{\mathrm{R}}\oplus 4_{\mathrm{R}}\oplus 4_{\mathrm{R}}\oplus 6_{\mathrm{R}})$. The first three of these can be written as X_{μ} or $X_{\mu}^{\ a}$ and are obtained by contracting the original tensor using ε or η. To obtain A_{μ} and the two $A_{\mu}^{\ a}$-fields from the correspondingly contracted X-fields is easy. One uses

$$U^{2}=U+2. \tag{4.8}$$

Next, one deduces that, when acting on the $2_{\mathrm{L}}\otimes 6_{\mathrm{R}}$ representation,

$$U^{A}U^{B}=1. \tag{4.9}$$

On this representation one then uses both eqs. (4.8) and (4.9) to derive

$$U^{B}+U^{A}=-1. \tag{4.10}$$

Beware that U^{A} and U^{B} do not commute when acting on the other representations.

Now we can *define* the Christoffel symbols $\Gamma^{\lambda}_{\ \mu\nu}$ from eq. (2.11) (the fact that they follow uniquely from (2.11) is elementary). From sect. 3 it follows that indeed (4.1) is the Einstein action, and the only fields left to be varied independently are the e-fields. Since they determine the metric $g_{\mu\nu}$ via eq. (2.9), we conclude that (4.1) and (4.2) indeed generate Einstein's equations for the metric (2.9), *if the fields e and A are used as canonical variables*. F is the curvature defined by (3.1), and K is defined by (2.13).

In short: variation of S with respect to the A-fields gives us the A-fields in terms of e and its derivatives, after which variation with respect to the e-fields gives us Einstein's equations.

5. More elegant fields

The lagrangian (4.2), when written in full, reads

$$\mathscr{L}=\tfrac{1}{32}\varepsilon^{\kappa\lambda\rho\sigma}e^{c}{}_{\kappa\lambda}e^{b}{}_{\rho\sigma}F^{a}{}_{c\mu\nu}\varepsilon_{abd}e^{d}{}_{\alpha\beta}\varepsilon^{\mu\nu\alpha\beta}(\det K)^{-1/3}, \tag{5.1}$$

but this can be simplified by redefining

$$e^{a}{}_{\mu\nu}(\det K)^{-1/9}=f^{a}{}_{\mu\nu}, \tag{5.2}$$

so that (5.1) becomes

$$\mathscr{L}=\tfrac{1}{32}\varepsilon^{\kappa\lambda\rho\sigma}f^{c}{}_{\kappa\lambda}f^{b}{}_{\rho\sigma}F^{a}{}_{c\mu\nu}\varepsilon_{abd}f^{d}{}_{\alpha\beta}\varepsilon^{\mu\nu\alpha\beta}. \tag{5.3}$$

The transformation (5.2) is invertible, since the quantity

$$W^{ab}=\tfrac{1}{8}\varepsilon^{\kappa\lambda\rho\sigma}f^{a}{}_{\kappa\lambda}f^{b}{}_{\rho\sigma}=K^{ab}(\det K)^{-2/9} \tag{5.4}$$

has the determinant

$$\det W=(\det K)^{1/3}=\sqrt{g}. \tag{5.5}$$

Transformation (5.2) implies that the fields f in our lagrangian (5.3) transform with an anomalous weight under conformal coordinate transformations. Eq. (5.3) generates Einstein's equations and as such is the analogue of eq. (1.9). The relationship between f and the metric g is

$$\varepsilon_{abc}f^{a}{}_{\mu\nu}f^{b}{}_{\kappa\lambda}f^{c}{}_{\rho\sigma}\varepsilon^{\mu\nu\kappa\rho}=24g_{\lambda\sigma}, \tag{5.6}$$

where the factor $\sqrt{g}$ cancelled out.

Thus, if the conventional vierbein field $e^{a}{}_{\mu}$ is called the "square root" of the metric tensor, our fields f may be dubbed "the cube root" of $g_{\mu\nu}$.

We did not achieve the optimistic goal mentioned in sect. 1, a bilinear lagrangian; the lagrangian (5.3) is quadrilinear in the fundamental fields, and therefore a perturbative treatment for small f, analogous to the small-e perturbation theory for (2 + 1)-dimensional gravity, is still not possible.

On the other hand however, we could impose a *gauge condition*,

$$W^{ab}=\tfrac{1}{8}\varepsilon^{\kappa\lambda\rho\sigma}f^{a}{}_{\kappa\lambda}f^{b}{}_{\rho\sigma}=\delta^{ab}. \tag{5.7}$$

In this gauge (which fixes SL(3) into SO(3) and fixes $\det g_{\mu\nu}=1$) the e-fields are identical to the f-fields. In this gauge the lagrangian is bilinear,

$$\mathscr{L}=\tfrac{1}{4}F^{a}{}_{b\mu\nu}\varepsilon_{abd}f^{d}{}_{\alpha\beta}\varepsilon^{\mu\nu\alpha\beta}, \tag{5.8}$$

however, the constraint (5.7) must be added by means of a Lagrange multiplier Z_{ab},

$$\mathscr{L}=\tfrac{1}{4}F^a{}_{b\mu\nu}\varepsilon_{abd}f^d{}_{\alpha\beta}\varepsilon^{\mu\nu\alpha\beta}+\tfrac{1}{2}\left(\tfrac{1}{8}\varepsilon^{\kappa\lambda\rho\sigma}f^a{}_{\kappa\lambda}f^b{}_{\rho\sigma}-\delta^{ab}\right)Z_{ab} \tag{5.9}$$

In a quantum version of the theory of course a gauge constraint and the Faddeev–Popov ghost field should be added.

If space-space and space-time components of the f- and A-fields are considered separately one gets more familiar looking vector fields, and our action then appears to be related to the ones found in ref. [4].

Our lagrangian now has an interesting bilinear kinetic term (with a trilinear interaction part due to the bilinear term in the curvature F. But of course the constraint term renders the theory non-renormalizable.

6. Minkowski metric

We call a metric $g_{\mu\nu}$ a Minkowski metric if exactly one of its eigenvalues is negative and the others are positive. The fact that in tangent space (the rectangular local coordinate frame) the fields e or f become the self-dual η-tensor, defined in (2.1), implies that in a space-time with Minkowski metric it should be taken to be complex. This is because the $i4$ components of η, in spite of having one time index, are real.

Let us take the fields $f^a{}_{\mu\nu}$, which define a metric $g_{\mu\nu}$ according to eq. (5.6). We define $\hat{f}_a{}^{\mu\nu}$ by

$$W^{ba}\hat{f}_a{}^{\mu\nu}=\tfrac{1}{2}\varepsilon^{\mu\nu\alpha\beta}f^b{}_{\alpha\beta}\,, \tag{6.1}$$

where W is as defined in (5.4). This relation has a symmetry. If, analogously to eq. (5.4), we take $\hat{W}$ to be

$$\hat{W}_{ab}=\tfrac{1}{8}\varepsilon_{\mu\nu\alpha\beta}\hat{f}_a{}^{\mu\nu}\hat{f}_b{}^{\alpha\beta}\,, \tag{6.2}$$

then it is easy to derive that $\hat{W}$ is the inverse of W. If the hat operation (6.1) is always combined with the condition that upper and lower indices are interchanged, then we see that the hat operation is its own inverse.

We could now take as a *reality condition* in the case of a Minkowski metric

$$\hat{f}_a{}^{\mu\nu}=\left(f^a{}_{\mu\nu}\right)^*. \tag{6.3}$$

Comparing with eqs. (2.1)–(2.5) we see that the η's obey this,

$$\hat{\eta}_a{}^{\mu\nu}=\bar{\eta}^a{}_{\mu\nu}=\left(\eta^a{}_{\mu\nu}\right)^*, \tag{6.4}$$

where the first equation holds because lowering a time index gives a minus sign in flat Minkowski space. If W is real, condition (6.3) tends to make the fields f complex because $\varepsilon^{1230} = -i$. Observe however that ε also occurs in the definition of W, and that eq. (6.3) may *also* hold in euclidean space.

In some sense, the condition looks like a unitarity condition on f, because $\hat{f}$ scales like the inverse of f.

We should note that the condition (6.3) is not invariant under general coordinate transformations (it compares upper with lower Lorentz indices). This means that a general coordinate transformation must be accompanied by a shift of the functional integration contour in the complex plane. The important motive for requiring (6.3) is that it allows a Minkowskian metric structure of $g_{\mu\nu}$ as defined by (5.6).

Now we see that the transformations on the internal indices a, b, that leave eq. (6.3) invariant are the unitary ones. The ε-tensor in eq. (5.6) restricts us to the internal symmetry group SU(3).

A warning is in order. Since eq. (6.3) is not invariant under general coordinate transformations, we could just as well have replaced it by

$$f^a{}_{ij} \text{ is real}, \qquad f^a{}_{i0} \text{ is imaginary}. \tag{6.5}$$

This however is not even invariant under flat space Lorentz transformations, which is why we prefer (6.3). Condition (6.5) would leave SL(3) as the internal symmetry group.

Conversely, our reality condition (6.3) could also be imposed in euclidean space, yielding an internal SU(3) symmetry there. The condition that all f-components should be real (giving SL(3) symmetry) is preferred there only for its simplicity.

7. Conclusion

We did not obtain a renormalizable theory. It should be stressed that such a thing would never be possible along the lines followed. This is because renormalizability would require the linearized theory to be entirely topological in nature (because of the absence of a metric tensor), and such a theory cannot sustain any locally observable degree of freedom. Only if we would be able to cast all observables in quantum gravity in some coordinate-free language a successful result would be conceivable.

Now this was exactly what Ashtekar et al. [5] were trying to do. Self-duality plays an important role in these theories, as it does in our approach. Perhaps our cube root of the metric tensor may be helpful.

One might suggest that the constraint part of the lagrangian (5.9) has a dynamical origin, but then the above words of caution apply. Our main motivation for writing this paper was our curiosity for the use of a cube root as a fundamental degree of freedom, as well as the emergence of SU(3) as an internal symmetry group if the metric is Minkowskian.

Using this "chiral alternative to the vierbein" in a theory with fermions seems to be rather difficult, not only because products of odd numbers of gamma matrices are difficult to define, but also because spinorial representations of SO(3) cannot be extended to SL(3).

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