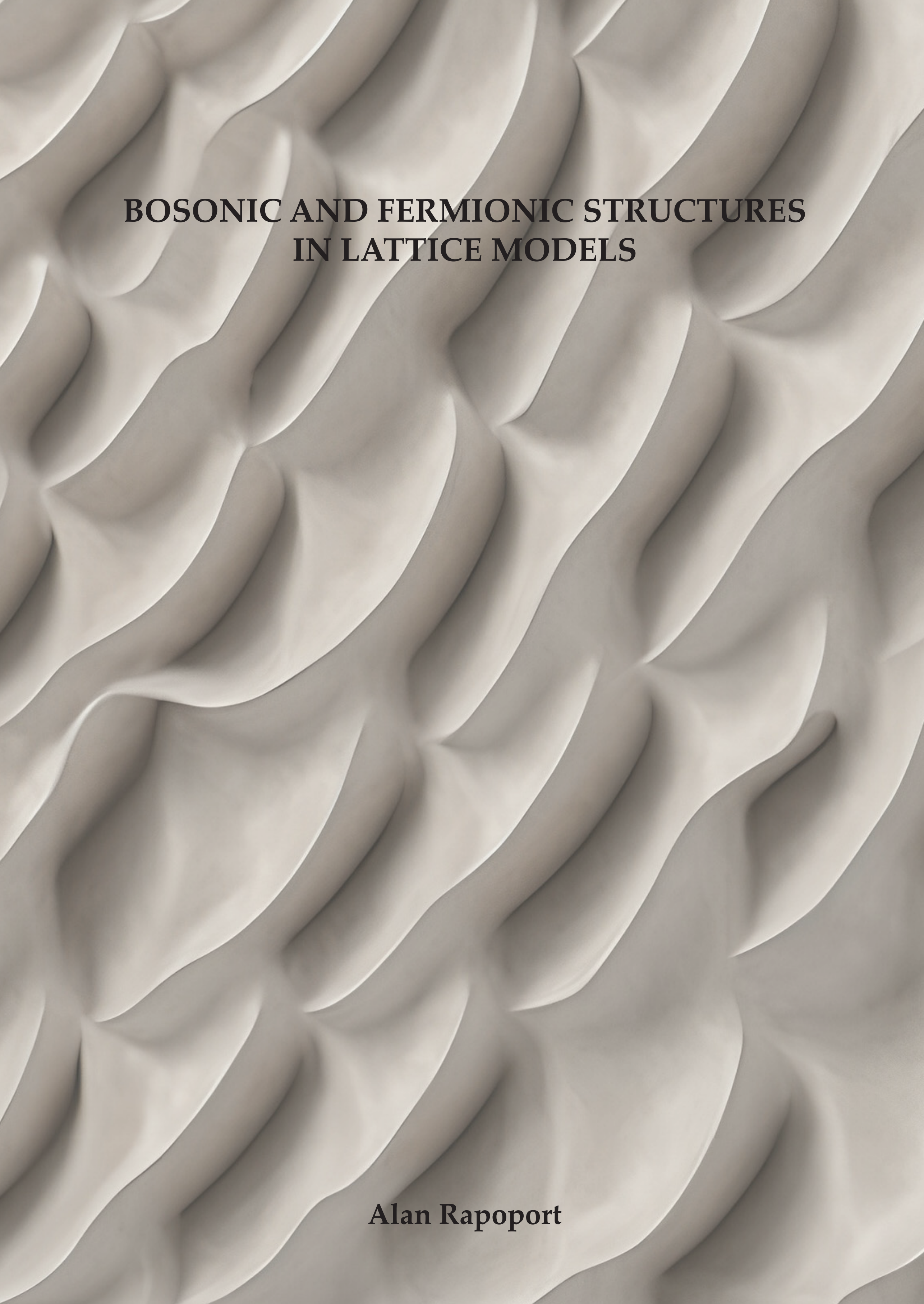

BOSONIC AND FERMIONIC STRUCTURES IN LATTICE MODELS

Alan Rapoport

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BOSONIC AND FERMIONIC STRUCTURES IN LATTICE MODELS

BOSONISCHE EN FERMIONISCHE STRUCTUREN IN ROOSTERMODELLEN

(met een samenvatting in het Nederlands)

Proefschrift

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Para los que están
y los que ya no.

Contents

Summary

E pur si muove ("And yet it moves").

—Galileo Galilei (probably)

In this thesis we study the behavior of gradients squared of Gaussian fields on different graphs and their relationship with certain lattice models. In particular, we study both commutative and anti-commutative squared Gaussian fields, and used them to calculate correlation functions of lattice models such as the *Abelian sandpile model (ASM)* and *uniform spanning tree (UST)*.

The first model to be studied is the gradient squared of the *bosonic* version of the *discrete Gaussian free field (dGFF)* on a subset of $\mathbb{Z}^d$. The name *bosonic* comes from the physics literature and is reserved for commutative fields, which in general tend to produce particles that attract each other. We prove that, as a distribution, this field converges to white noise in the thermodynamic limit. We also calculate its joint moments explicitly, which unveil a quasi permanental structure. We observe in particular a similarity with the so-called *height-one field* of the ASM. This is a lattice model that operates as a cellular automaton, in which every vertex on a finite grid has a value corresponding to the slope of a pile. This slope gradually increases as "grains of sand" are randomly added to the grid. When the slope surpasses a predetermined threshold the site collapses, redistributing sand to adjacent sites and increasing their slopes. When depositing grains of sand onto the grid in a random manner, each deposition might be inconsequential or trigger a chain reaction, impacting multiple sites. Once the system attains stationarity, we can define the height-one field as the indicator function of each site having a slope of size 1. In Dürre [32] the author studies its joint cumulants in the limit, which turn out to be uncannily similar to the ones we obtained for our Gaussian field squared, albeit with an important sign of difference, having a quasi determinantal structure, instead of permanental. This similarity then begs the question: *What modification of this field produces the same moments structure as the height-one field of the ASM?*

Here is where the *Grassmannian* or *fermionic* variables come into play, terms which are often used in physics to refer to anticommutative fields, which give rise to repelling particles. If we replace the Gaussian variables at each vertex of the graph (whose correlations are given by those of the dGFF) by fermionic Gaussian variables, we obtain the exact same moments expression as Dürre [32]. What is more, our calculation method allows us to generalize the proof to square lattices in any dimension, and to the triangular and hexagonal lattices in two dimensions, hinting towards a potential universality property. In fact, it has long

been conjectured (Ruelle [87]) that the ASM in the limit should correspond to a logarithmic conformal field theory, an example of which is the field theory whose action is given by the gradient squared of a free fermion. We believe that our joint moments correspondence hints in that direction.

This equality poses a new question: *Why?* What does the fermionic GFF have to do with the ASM, so that such a particular function like the squared norm of its gradient yields the exact same moments as the height-one field? On the one hand it is well-known that the height-one field configurations can be put in one-to-one correspondence with some realizations of the UST. On the other, it is also known that there is an explicit connection between UST configurations and fermionic variables. In particular, the probabilities of some edges belonging to a realization of the UST model can be calculated as determinants of some specific matrices, which themselves can be expressed as expectations of products of fermionic variables (or gradients thereof).

This allows us to study more observables of the UST, even if they do not correspond to height-one field realizations. Leveraging on the proofs of the previous results, we extend those techniques in order to find exact closed-form expressions of the probability mass functions of the degree field of the UST in the aforementioned lattices. To the author's knowledge, this is the first time such expressions are given in the literature.

Samenvatting

E pur si muove ("En toch beweegt het").

—Galileo Galilei (probably)

In dit proefschrift bestuderen we het gedrag van kwadraatgradiënten van Gaussische velden op verschillende grafen en hun relatie met bepaalde roostermodellen. In het bijzonder bestuderen we zowel commutatieve als anti-commutatieve kwadratische Gaussische velden, en gebruiken we deze om correlatiefuncties te berekenen van roostermodellen zoals het *Abelian sandpile model (ASM)* en de *uniform spanning tree (UST)*.

Het eerste model dat bestudeerd wordt is het kwadraat van de gradiënt van de *bosonische* versie van het *discrete Gaussian free field (dGFF)* op een subset van $\mathbb{Z}^d$. De naam *bosonic* komt uit de natuurkundeliteratuur en is gereserveerd voor commutatieve velden, die over het algemeen de neiging hebben deeltjes te produceren die elkaar aantrekken. We bewijzen dat dit veld in de thermodynamische limiet als distributie convergeert naar witte ruis. We berekenen ook expliciet de gezamenlijke momenten, waardoor een quasi permanente structuur wordt onthuld. We zien vooral een gelijkenis met het zogenaamde *height-one field* van het ASM. Dit is een roostermodel dat werkt als een cellulaire automaat, waarbij elke knoop op een eindige rooster een waarde heeft die overeenkomt met de helling van een hoop. Deze helling neemt geleidelijk toe naarmate er willekeurig "zandkorrels" aan de rooster worden toegevoegd. Wanneer de helling een vooraf bepaalde drempel overschrijdt, stort het punt in, waardoor het zand wordt herverdeeld naar aangrenzende locaties en de hellingen ervan toenemen. Wanneer zandkorrels op willekeurige wijze op het rooster worden aangebracht, kan elke afzetting onbeduidend zijn of een kettingreactie veroorzaken, die gevolgen heeft voor meerdere locaties. Zodra het systeem stationariteit heeft bereikt, definiëren we het *height-one field* als de indicatorfunctie van elke locatie met een helling ter grootte van 1. In Dürre [32] bestudeert de auteur de gezamenlijke cumulanten in de limiet, die geheimzinnig veel lijken op de waarden die we hebben verkregen voor ons kwadraat van het Gaussische veld, zij het met een belangrijk teken van verschil, met een quasi determinantstructuur, in plaats van permanent. Deze gelijkenis roept vervolgens de vraag op: *Welke wijziging van dit veld produceert dezelfde momentenstructuur als het* height-one field *van de ASM?*

Hier komen de *Grassmann* of *fermionische* variabelen in het spel, termen die in de natuurkunde vaak worden gebruikt om te verwijzen naar anti-commutatieve velden die aanleiding geven tot afstotende deeltjes. Als we de Gaussische variabelen bij elk hoekpunt van de graaf (waarvan de correlaties worden gegeven door die van het dGFF) vervangen door fermionische Gaussische variabelen, krijgen we exact dezelfde momentenuitdrukking als Dürre [32]. Bovendien stelt onze berekeningsmethode ons in staat om het bewijs te generaliseren naar vierkante roosters in elke dimensie, en naar de driehoekige en hexagonale roosters in twee dimensies, waardoor een potentiële universaliteitseigenschap wordt bereikt. Er wordt eigenlijk al lang vermoed (Ruelle [87]) dat de ASM in de limiet zou moeten overeenkomen met een logaritmische conforme veldentheorie, waarvan een voorbeeld de veldentheorie is waarvan de actie wordt gegeven door het kwadraatgradiënten van een vrije fermion. Wij geloven dat onze gezamenlijke momentencorrespondentie in die richting wijst.

Deze gelijkheid roept een nieuwe vraag op: *Waarom?* Wat heeft de fermionische GFF te maken met de ASM, zodat een dergelijke specifieke functie zoals de kwadrat van de norm van zijn gradiënt, exact dezelfde momenten oplevert als het *height-one field*? Enerzijds is het algemeen bekend dat het *height-one field* configuraties één op één in overeenstemming kunnen worden gebracht met enkele realisaties van de UST. Aan de andere kant is het ook bekend dat er een expliciet verband bestaat tussen UST-configuraties en fermionische variabelen. De kansen dat sommige zijden tot een realisatie van het UST-model behoren kunnen in het bijzonder berekend worden als determinanten van enkele specifieke matrices. Deze kunnen dan weer uitgedrukt worden als verwachtingen van producten van fermionische variabelen (of gradiënten daarvan).

Dit stelt ons in staat meer observabelen van de UST te bestuderen, zelfs als deze niet overeenkomen met de realisaties van het *height-one field*. Gebruikmakend van de bewijzen van de eerdere resultaten, breiden we deze technieken uit om exacte uitdrukkingen in gesloten vorm te vinden van de kansfuncties van het gradenveld van de UST in de bovengenoemde roosters. Voor zover de auteur weet is dit de eerste keer dat dergelijke uitdrukkingen in de literatuur worden gebruikt.

1

Introduction and preliminaries

The first principle is that you must not fool yourself –
and you are the easiest person to fool.

—-Richard P. Feynman

The person you are the most afraid to contradict is yourself.

—-Nassim N. Taleb, *The Bed of Procrustes*

In this chapter we provide an introduction to the thesis, giving the motivation behind and introducing the most important results. We then establish the common notation that we will use throughout the rest of the chapters, and review the main mathematical structures that we will need, such as the Gaussian free field, the Abelian sandpile model, the uniform spanning tree, and the fermionic counterpart of the discrete Gaussian free field.

1.1. INTRODUCTION

1.1.1. SQUARING GAUSSIANS

Gaussian fields have been extensively studied over the decades, being the discrete Gaussian field one of its most important examples. On the other hand, fields that arise from summing independent squared Gaussian variables have also been studied (see for example McCullagh and Møller [74]). In this thesis we study a particular case where independence does not hold. More specifically we will take the norm squared of the gradient of a dGFF, which gives a predefined correlation structure between each direction of differentiation. We will sometimes refer to the underlying Gaussian variables in this model as bosonic, signifying the fact that they take values in the real numbers, which in particular form a commutative algebra. This apparently unnecessary specification will become clear later on in Chapter 3, where we turn to the study of its fermionic counterpart and the consequences thereof.

Let us the begin by discussing the main ingredient of our construction. The Gaussian free field is one of the most prominent models for random surfaces. It appears as a scaling limit of observables in many interacting particle systems (see, for example, Jerison, Levine and Sheffield [59], Kenyon [64], Sheffield [90], and Wilson [98]). Its discrete counterpart, the discrete Gaussian free field (dGFF), is also very well-known among random interface models on graphs. Given a simple (that is, without loops or multiple edges) graph Λ, a (random) interface model is defined as a collection of (random) real heights $\Gamma = (\Gamma(x))_{x\in\Lambda}$, measuring the vertical distance between the interface and the set of points of Λ (Funaki [38] and Velenik [96]). The discrete Gaussian free field has attracted a lot of attention due to its links to random walks, cover times of graphs, and conformally invariant processes (see Barlow and Slade [6], Ding, Lee and Peres [30], Glimm and Jaffe [41], Schramm and Sheffield [89], and Sheffield [90], among others). In Chapter 2 we consider the dGFF on the square lattice, that is, we will focus on $\Lambda \subseteq \mathbb{Z}^d$, in which case the probability measure of the dGFF is a Gibbs measure with formal Hamiltonian given by

$$H(\Gamma) = \frac{1}{2d} \sum_{x,y:\, \|x-y\|=1} V\big(\Gamma(x) - \Gamma(y)\big), \tag{1.1}$$

where $V(\varphi) = \varphi^2/2$. We will always work with 0 boundary conditions, which means that we will set $\Gamma(x)$ to be zero almost surely outside Λ. For general potentials $V(\cdot)$ the Hamiltonian (1.1) defines a broad class of gradient interfaces which have been widely studied in terms of decay of correlations and scaling limits (Biskup and Spohn [12], Cotar, Deuschel and Müller [25], and Nadaf and Spencer [80]), among others.

The gradient Gaussian free field $\nabla\Gamma$ is defined as the gradient of the dGFF Γ along the edges of the square lattice. This field is a centered Gaussian process whose correlation structure can be written in terms of $M(\cdot,\cdot)$, the transfer current (or transfer impedance) matrix (Kassel and Wu [61]). Namely, if we consider the gradient $\nabla_i\Gamma(\cdot) := \Gamma(\cdot + e_i) - \Gamma(\cdot)$ in the i-th coordinate direction of $\mathbb{R}^d$, we have, for $x, y \in \mathbb{Z}^d$, $1 \le i, j \le d$, that

$$\begin{aligned}\mathbb{E}\left[\nabla_i \Gamma(x) \nabla_j \Gamma(y)\right] &= G_\Lambda(x,y) - G_\Lambda(x+e_i,y) - G_\Lambda(x,y+e_j) + G_\Lambda(x+e_i,y+e_j) \\ &= M(e,f)\end{aligned}$$

where $e = (x, x+e_i)$ and $f = (y, y+e_j)$ are directed edges of the grid and $G_\Lambda(\cdot,\cdot)$ is the discrete harmonic Green's function on Λ with 0-boundary conditions outside Λ. Here $M(e,f)$ describes a current flow between e and f.

The main object we will study in Chapter 2 is the following. Take U to be a connected, bounded subset of $\mathbb{R}^d$ with smooth boundary. Consider the recentered squared norm of the gradient dGFF, formally denoted by

$$\Phi_\varepsilon(\cdot) = :\|\nabla\Gamma\|^2:(\cdot) = \sum_{i=1}^{d} :\big(\Gamma(\cdot+e_i) - \Gamma(\cdot)\big)^2:$$

on the discretized domain $U_\varepsilon = U/\varepsilon \cap \mathbb{Z}^d$, $\varepsilon > 0$, $d \ge 2$, with Γ a 0-boundary dGFF on U_ε. The colon $:(\cdot):$ denotes the Wick centering of the random variables. From now on we will simply call Φ_ε the *gradient squared of the dGFF*. Let us remark that we do not consider $d = 1$ here since in one dimension the gradient of the dGFF is a collection of independent and identically distributed Gaussian variables.

k-point correlation functions Our first main result in Chapter 2 determines the k-point correlation functions for the field Φ_ε on the discretized domain U_ε and in the scaling limit as $\varepsilon \to 0$. We defer the precise statement to Theorem 2.6 in Section 2.2, which we will now expose in a more informal way. Let $\varepsilon > 0$ and $k \in \mathbb{N}$ and let the points $x^{(1)}, \ldots, x^{(k)}$ in $U \subset \mathbb{R}^d$, $d \ge 2$, be given. Define $x_\varepsilon^{(j)}$ to be a discrete approximation of $x^{(j)}$ in U_ε, for $j = 1, \ldots, k$. Let $\Pi([k])$ be the set of partitions of k objects and $S^0_{\text{cycl}}(B)$ be the set of cyclic permutations of a set B without fixed points. Finally, let $\mathcal{E}$ be the set of coordinate vectors of $\mathbb{R}^d$. Then the k-point correlation function at fixed "level" ε is equal to

$$\mathbb{E}\left[\prod_{j=1}^{k} \Phi_\varepsilon\big(x_\varepsilon^{(j)}\big)\right] = \sum_{\pi \in \Pi([k])} \prod_{B \in \pi} 2^{|B|-1} \sum_{\sigma \in S^0_{\text{cycl}}(B)} \sum_{\eta: B \to \mathcal{E}} \prod_{j \in B} \nabla^{(1)}_{\eta(j)} \nabla^{(2)}_{\eta(\sigma(j))} G_{U_\varepsilon}\big(x_\varepsilon^{(j)}, x_\varepsilon^{(\sigma(j))}\big). \tag{1.2}$$

Moreover if $x^{(i)} \neq x^{(j)}$ for all $i \neq j$, the scaling limit of the above expression is

$$\lim_{\varepsilon \to 0} \varepsilon^{-dk} \mathbb{E}\left[\prod_{j=1}^{k} \Phi_\varepsilon\big(x_\varepsilon^{(j)}\big)\right] = \sum_{\pi \in \Pi([k])} \prod_{B \in \pi} 2^{|B|-1} \sum_{\sigma \in S^0_{\text{cycl}}(B)} \sum_{\eta: B \to \mathcal{E}} \prod_{j \in B} \partial^{(1)}_{\eta(j)} \partial^{(2)}_{\eta(\sigma(j))} g_U\big(x^{(j)}, x^{(\sigma(j))}\big), \tag{1.3}$$

where $g_U(\cdot,\cdot)$ is the continuum Dirichlet harmonic Green's function on U. As a corollary (Corollary 2.9) we also determine the corresponding cumulants on U_ε and in the scaling limit.

Let us discuss some interesting observations in the sequel. The k-point correlation function of (1.3) has similarities to the k-point correlation that arises in permanental processes (see Eisenbaum and Kaspi [34], Hough *et al.* [48], and Last and Penrose [66] for relevant literature). In fact, in $d = 1$ one can show that the gradient squared is *exactly* a permanental process with kernel given by the diagonal matrix whose non-zero entries are the double derivatives of g_U (McCullagh and Møller [74, Thm. 1]). In higher dimensions, however, we cannot identify a permanental process arising from the scaling limit, since the directions of derivations of the dGFF at each point are not independent. Nevertheless, the 2-point correlation functions of Φ_ε are positive (see Equation (2.26) in Section 2.5), which is consistent with attractiveness of permanental processes (Last and Penrose [66, p. 139]), and the overall structure resembles closely that of permanental processes marginals.

Scaling limit The second main result of Chapter 2 is the scaling limit of the field towards white noise in some appropriate local Besov-Hölder space. As we show in Theorem 2.11, Section 2.2, as $\varepsilon \to 0$ the gradient squared of the discrete Gaussian free field Φ_ε converges as a random distribution to spatial white noise W:

$$\frac{\varepsilon^{-d/2}}{\sqrt{\chi}}\Phi_\varepsilon \xrightarrow{d} W, \tag{1.4}$$

for some explicit constant $0 < \chi < \infty$. The result is sharp in the sense that we obtain convergence in the smallest Hölder space where white noise lives. The constant χ, defined explicitly in (2.8), is the analogue of the susceptibility for the Ising model, in that it is a sum of all the covariances between the origin and any other lattice point. We will prove that this constant is finite and the field Φ_ε has a Gaussian limit. Note that Newman [81] proves the same result for translation-invariant fields with finite susceptibility satisfying the FKG inequality. In our case we do not have translation invariance since we work on a domain, so we are not able to apply directly this criterion. From a broader perspective there are several other results in the literature that obtain white noise in the limit due to an algebraic decay of the correlations. See for example Bauerschmidt, Brydges and Slade [7].

Note that our field can be understood in a wider class of models having correlations which depend on the transfer current matrix $M(\cdot,\cdot)$. An interesting point mentioned in Kassel and Wu [61] is that pattern fields of determinantal processes closely connected to the spanning tree measure and $M(\cdot,\cdot)$ (for example the spanning unicycle, the Abelian sandpile model (Dürre [32]) and the dimer model (Boutillier [13])) have a universal Gaussian limit when viewed as random distributions. Correlations of those pattern fields can be expressed in terms of transfer current matrices which decay sufficiently fast and assure the central limit-type behavior which we also obtain.

Let us comment finally on the differences between expressions (1.3) and (1.4). The scaling factors are different, and this reflects two viewpoints one can have on Φ_ε: The one of (1.3) is that of correlation functionals in a bosonic (as opposed to fermionic, as we will later see) Fock space (see Section 2.3), while in (1.4) we are looking at it as a Gaussian distributional field (compare also Theorems 2 and 3 in Dürre [32]). This is compatible, as there are examples of trivial correlation functionals which are non-zero as random distributions (Kang and Makarov [60]).

The novelty of the chapter lies in the fact that we construct the gradient squared of the Gaussian free field on a grid, determine its k-point correlation function and scaling limits. We determine tightness in optimal Besov-Hölder spaces (optimal in the sense that we cannot achieve a better regularity for the scaling limit to hold). Furthermore, we show the "dual" behavior in the scaling limit of the gradient squared of the dGFF as a Fock space field and as a random distribution. As mentioned before, we recognize a similarity to permanental processes, and it is worthwhile noticing that for general point processes there is a Fock space structure (see e.g. Last and Penrose [66, Sec. 18]).

Abelian sandpile model The Abelian sandpile model (ASM) was introduced by Bak, Tang and Wiesenfeld [5] as a prototype of a model displaying self-organized criticality. This refers to the property of a model to drive itself into a critical state, characterized by power-law behavior of certain observables such as the avalanche size, without fine-tuning any external parameter.

The ASM on a finite simple graph is defined as follows. Assign to each vertex of the graph an integer number (height configuration), modeling the amount of "grains of sand" associated with it. This random dynamical system runs in two steps: In the first step, a vertex of the graph is picked uniformly at random and a grain of sand is added to it, increasing its height by one. In the second step, the vertices that are unstable (that is, those which bear strictly more grains than the graph degree) topple, sending out one grain to each neighbor. This procedure is repeated until all the vertices are stable again. Grains sent outside the graph to so-called "sinks" are lost.

The Abelian property yields that the order of topplings does not matter, and the system will eventually reach a stable configuration thanks to dissipation at the boundary. This Markov chain has a known unique stationary measure: the uniform measure on the set of recurrent configurations. Thanks to the well-known *burning bijection* introduced by Majumdar and Dhar [73], which relates recurrent configurations for the ASM with spanning trees, one can determine the stationary measure and several quantities of the model explicitly.

What makes the model critical is the occurrence of long-range correlations at all scales, resulting from possible avalanches of topplings. Although the ASM is very simple, it is challenging to treat mathematically due to its non-locality and many questions still remain open.

In $d = 2$, the limiting k-joint cumulants of first order κ of our field are interestingly connected to the cumulants of the height-one field $\left(h_\varepsilon\left(x_\varepsilon^{(i)}\right) : x_\varepsilon^{(i)} \in U_\varepsilon\right)$ of the ASM (Dürre [32, Thm. 2]). Theorem 2.6 will imply that for every set of $\ell \geq 2$ pairwise

1

distinct points in $d = 2$ one has

$$-2 \lim_{\varepsilon \to 0} \varepsilon^{-2\ell} \kappa \left(\frac{C}{4} \Phi_\varepsilon (x_\varepsilon^{(1)}), \ldots, \frac{C}{4} \Phi_\varepsilon (x_\varepsilon^{(\ell)}) \right) = \lim_{\varepsilon \to 0} \varepsilon^{-2\ell} \kappa \left(h_\varepsilon (x_\varepsilon^{(1)}), \ldots, h_\varepsilon (x_\varepsilon^{(\ell)}) \right) \quad (1.5)$$

with

$$C = \frac{2}{\pi} - \frac{4}{\pi^2} = \pi \, \mathbb{E}\left[h_0(0) \right]. \quad (1.6)$$

See Dürre [33, Thm. 6].

We point out that the apparently intricate structure of Equations (1.2)–(1.3) and of Dürre's Theorem 2 can be unfolded as soon as one recognizes therein the structure of a Fock space. We will discuss this point in more detail in Subsection 2.3, where in particular in Corollary 2.17 we will derive a Fock space representation of the k-point function for the height-one field.

Much like in the height-one field of the ASM, in Proposition 2.10 we show that in $d = 2$ the k-point correlation functions are conformally covariant (compare Dürre [32, Thm. 1], Kassel and Wu [61, Thm. 2]). This hints at Theorems 2 and 3 of Kassel and Wu [61], in which the authors prove that for finite weighted graphs the rescaled correlations of the spanning tree model and minimal subconfigurations of the Abelian sandpile have a universal and conformally covariant limit.

1.1.2. TURNING BOSONS INTO FERMIONS

So far we have been working with the traditional dGFF, which in particular satisfies the property of commutativity $\Gamma(a)\Gamma(b) = \Gamma(b)\Gamma(a)$, $a, b \in \Lambda$. In the physics literature this is called the *bosonic free field*, where the word "bosonic" comes from the fact that commutative fields are referred to as bosons. In Chapter 3 we will instead work with the anticommutative counterpart, called *fermionic GFF*, which satisfies $\Gamma(a)\Gamma(b) = -\Gamma(b)\Gamma(a)$. In particular this will allow us to obtain the same moments expressions as in the height-one field.

Lattice models and (log)-conformal field theories. Lattice models from statistical mechanics have been successfully used to describe macroscopic properties of interacting systems and model critical phenomena by specifying their random microscopic interaction. One of the major breakthroughs in theoretical physics was the development of conformal field theory (CFT) (see Belavin, Polyakov and Zamolodchikov [9] and Di Francesco, Mathieu and Sénéchal [29] for general references). This theory is based on the postulate that many critical lattice models of two-dimensional statistical mechanics have conformally invariant scaling limits.

To be able to understand the CFT structure emerging from scaling limits of such lattice models, one often resorts to (possibly non-commutative) algebras instead of probability measures. Those algebras allow us to describe quantities of interest, referred to as "observables" rather than random variables. We then measure these observables via suitable operators, referred to as "states", which are analogous to expectations in probability theory. The construction of such algebras and states is highly non-trivial. The construction, characterization, and understanding of these algebras and operators are common challenges in the study of CFTs.

Solvable CFTs can be studied in terms of representations of the Virasoro algebra, which is a complex Lie algebra. It allows us to identify continuum theories and universality classes corresponding to particular lattice models. It can lead to exact formulae for scaling limits of correlations, partition functions, and critical exponents.

Prominent examples of predictions obtained by the CFT approach are, among others, conformal invariant scaling limit and critical exponents of the Ising model (Belavin, Polyakov and Zamolodchikov [9, 10]), crossing probabilities in percolation (Cardy [19]), and cluster growth in diffusion limited aggregation (Davidovich and Procaccia [28]).

Some of the drawbacks of using CFT methods to understand critical lattice models are that often they are non-rigorous and that they use functions of local operators, making them less appropriate for analyzing global quantities. In fact, it is a known long-standing open question whether there is a direct link connecting CFTs and lattice models (Itoyama and Thacker [51]). There has been progress in recent years to rigorously establish the predictions obtained from the CFT approach regarding scaling limits of lattice models, leading to important mathematical contributions. Yet, the full picture is still far from well-understood.

Let us mention a few examples of important mathematical contributions. In Chelkak, Hongler and Izyurov [21] and Hongler and Smirnov [47] the authors related correlation functions of the Ising model and the relevant correlations of the associated CFT. In Camia, Garban and Newman [15, 16] the authors identified the scaling limit of the magnetization field at the critical/near-critical point. A Virasoro representation of the Gaussian free field as the simplest example of a Euclidean field theory was proved in Kang and Makarov [60]. Furthermore, the concept of *fermionic observables* in the context of discrete complex analysis, put forward by Kenyon and Smirnov, led to proving conformal invariance of the height function of the dimer model (Kenyon [63]), critical percolation (Smirnov [91]), Ising model (Smirnov [92]), or very recently the construction of conformal field theory at lattice level (Hongler, Kytölä and Viklund [46]). In this last article, the authors give a rigorous link between CFT local fields and lattice local fields for the Gaussian free field and the Ising model.

Several years after the introduction of CFTs, Gurarie [43] observed logarithmic singularities in correlation functions of certain CFTs. Typically, those logarithmic CFTs (logCFTs) describe the critical behavior of lattice models at second order phase transitions. The Virasoro representation in this case involves pairs of fields, a primary field and its logarithmic partner. LogCFTs are much less classified and understood, contrary to ordinary CFTs, both from the theoretical physics point of view and even less from the mathematical point of view. The reasons behind this fact are that computations are significantly harder due to the non-local features of the theory, and that it is not known what a generic logCFT looks like. The simplest, but still highly non-trivial, logCFT which is understood from the theoretical physics point of view is the *free symplectic fermion theory* with central charge $c = -2$ (see also Gaberdiel and Kausch [40], Gurarie [43], and Kausch [62]).

Models which display logarithmic singularities in the correlation functions and are conjectured to belong to this class include percolation (Cardy [20]), self-avoiding random walks (Duplantier and Saleur [31]), random spanning forests (Ivashkevich [52], Liu, Peltola and Wu [69]), the (double) dimer model (Adame-Carillo [2] and Izmailian *et al.* [53]), and the Abelian sandpile model (Piroux and Ruelle [84]). We refer to the overview article Hogervorst, Paulos and Vichi [45] for further references.

Abelian sandpile model and logCFTs. In a series of papers from the last two decades (see Jeng [57], Jeng, Piroux and Ruelle [58], Mahieu and Ruelle [71], Moghimi-Araghi, Rajabpour and Rouhani [79], Piroux and Ruelle [84], and Ruelle [87] and the review in Ruelle [88] with references therein) physicists have made significant contributions from the theoretical physics viewpoint to understand how and which logCFT emerges from a stochastic lattice model like the ASM, and attempted to identify it. In particular, they have computed the height probabilities of the ASM on different lattices (e.g. Euclidean lattices, triangular and hexagonal lattices), 2-, 3- and 4-point height correlation functions, studied different bulk and boundary observables, and identified logarithmic pairs of several fields.

Typically, an ansatz from the theoretical physics point of view to validate (or discard) the continuum theory is to take an educated guess for a field theory Φ and test it, in the sense that, if a lattice observable in one point x converges in the scaling limit towards $\Phi(x)$, then the corresponding correlation functions must also converge to the equivalent field theoretic correlators. The more identities are tested positively, the higher the conviction that the proposed theory is the correct one.

More precisely, consider the ASM on a rescaled lattice with mesh ε such that the points $x_i^\varepsilon \to x_i$ as $\varepsilon \to 0$, $i = 1, \ldots, n$ for general $n \in \mathbb{N}$. Formally, we expect that in the continuum scaling limit, the field Φ satisfies

$$\lim_{\varepsilon \to 0} \varepsilon^{-\sum_i D_i} \left\langle O^\varepsilon(x_1^\varepsilon) \cdots O^\varepsilon(x_n^\varepsilon) \right\rangle_{\text{lattice}} = \left\langle \Phi(x_1) \cdots \Phi(x_n) \right\rangle_{\text{field theory}}, \tag{1.7}$$

where the D_i's are related to the scale dimension of the field, and O^ε is a local observable of the ASM on the lattice.

The first educated guess for the ASM was to consider the free symplectic fermion theory mentioned above. Its Lagrangian is given by (see Ruelle [88, Eq. (27)])

$$S = \frac{1}{\pi} \int dz \, d\bar{z} \, \partial_z \theta \, \partial_{\bar{z}} \tilde{\theta}, \tag{1.8}$$

where $\theta, \tilde{\theta}$ are a pair of free, massless, Grassmannian scalar fields. In fact, the authors in Jeng, Piroux and Ruelle [58] (see also Brankov, Ivashkevich and Priezzhev [14] and Mahieu and Ruelle [71]) showed that the bulk dissipation field and the height-one field of the ASM can be realized as a logarithmic pair of a symplectic fermion theory. The height-one field can be identified as (Ruelle [88, Eq. (108)])

$$\Phi_\theta = -C \left(\partial_z \theta \, \partial_{\bar{z}} \tilde{\theta} + \partial_{\bar{z}} \theta \, \partial_z \tilde{\theta} \right), \tag{1.9}$$

where the constant C can be computed explicitly and is equal to the probability of the ASM to have height 1 on a generic site on $\mathbb{Z}^2$. In Ruelle [88, Sec. 8] it is argued that higher height fields are not described by the symplectic fermion theory, discarding the ansatz that this theory is the right logCFT to describe the ASM in the scaling limit. As we will see later, the height-one lattice field can be associated to a local observable of the uniform spanning tree. This ceases to be the case for higher heights, as the observables become non-local, which strongly suggests a qualitative different field theory.

Discrete fermionic fields. Let V be a finite set of vertices on some graph $\mathcal{G}$. We will consider two lattice fields, namely the height-one field of the ASM $(h(v))_{v\in V}$, which is the indicator that at a site v there is only one grain of sand, and the field $(\mathcal{X}_v)_{v\in V}$, given by the (normalized) degree of a site in the uniform spanning tree (UST).

For a set of generators $\{\psi_v, \overline{\psi}_v\}_v$ of a suitable (real) Grassmann algebra, we will define the lattice fields **X** and **Y** which are products over discrete gradients of the Grassmannian variables defined by

$$X_v = \frac{1}{\deg_{\mathcal{G}}(v)} \sum_{i=1}^{\deg_{\mathcal{G}}(v)} \nabla_{e_i}\psi(v)\nabla_{e_i}\overline{\psi}(v) \tag{1.10}$$

and

$$Y_v = \prod_{i=1}^{\deg_{\mathcal{G}}(v)} \left(1 - \nabla_{e_i}\psi(v)\nabla_{e_i}\overline{\psi}(v)\right). \tag{1.11}$$

We will evaluate those fields according to an operator $\langle\cdot\rangle$, referred to as the *fermionic Gaussian free field* (fGFF) state on the lattice. This operator can be seen as the natural counterpart to the expectation of the usual Gaussian free field on the lattice, but whose "spins" take values in a Grassmannian algebra rather than on $\mathbb{R}$. The fGFF is a type of Gaussian integral over Grassmannian variables, and it is a known tool to treat Matrix-Tree-type theorems (see for instance Abdesselam [1], Bauerschmidt *et al.* [8], and Caracciolo, Sokal and Sportiello [17]).

In the following, we will highlight the most relevant results in a qualitative way and discuss their implications. We defer the precise formulations of the results to Section 3.1. We will work on the Euclidean lattice $\mathbb{Z}^d$ and triangular lattice **T**. Let us call such a lattice **L**.

The first results in Chapter 3 (Theorems 3.2 and 3.24) are a representation of the n-point function of the height-one field of the ASM in terms of functions of Grassmannian variables. For $V \subset$ **L**, the height-one field can be represented in terms of fermionic variables as

$$\mathbb{E}\left(\prod_{v\in V} h(v)\right) = \left\langle \prod_{v\in V} X_v Y_v \right\rangle,$$

where $\langle\cdot\rangle$ is the fGFF state.

The second relevant result is the scaling limit of the joint cumulants of first order and determination of the constant C appearing in (1.9) in terms of permutations of double gradients of the continuum harmonic Green's function. This is stated in Corollary 3.10, which follows from the more general scaling limit for the field **XY** stated in Theorem 3.7 for $\mathbb{Z}^d$ and in Theorem 3.25 item 2 for the triangular lattice. Let $U \subset \mathbb{R}^d$, $V \subseteq U$ be a set of points, and U, V satisfy "nice" properties. Consider the renormalized graph $\mathcal{G}_\varepsilon := U/\varepsilon \cap L$. There exists an explicit constant C_L such that the joint cumulants κ of the height-one field scale as

$$\lim_{\varepsilon \to 0} \varepsilon^{-d|V|} \kappa\big(h^\varepsilon(v) : v \in V\big) = -C_L^{|V|} \sum_\sigma \sum_\eta \prod_v \partial^{(1)}_{\eta(v)} \partial^{(2)}_{\eta(\sigma(v))} g_U\left(v, \sigma(v)\right), \quad (1.12)$$

where h^ε is a suitable embedding of h in $\mathcal{G}_\varepsilon$ and g_U is the harmonic Green's function on U with Dirichlet boundary conditions, σ's are certain permutations of V and η's are directions of derivations.

The third result concerns the scaling limit of the cumulants of the renormalized degree field of the uniform spanning tree T. The precise statement can be found in Corollary 3.5 and Theorem 3.25. We define

$$\mathcal{X}_v := \frac{\deg_T(v)}{\deg_{\mathcal{G}}(v)}.$$

Let V, U and $\mathcal{G}_\varepsilon$ as above. There exists an explicit constant $c_L < 0$ such that the joint cumulants κ of the normalized degree field scale to

$$\begin{aligned}\lim_{\varepsilon \to 0} \varepsilon^{-d|V|} \kappa\big(\mathcal{X}^\varepsilon_v : v \in V\big) &= \lim_{\varepsilon \to 0} \varepsilon^{-d|V|} \kappa\big(X_v : v \in V\big) \\ &= -c_L^{|V|} \sum_\sigma \sum_\eta \prod_v \partial^{(1)}_{\eta(v)} \partial^{(2)}_{\eta(\sigma(v))} g_U\left(v, \sigma(v)\right), \quad (1.13)\end{aligned}$$

for a suitable embedding $\mathcal{X}^\varepsilon$ of the field $\mathcal{X}$ in $\mathcal{G}_\varepsilon$.

Let us discuss the implications of our results. We prove in Theorem 3.3 and Corollary 3.4 that the scaling limit of the cumulants of the height-one field in $\mathbb{Z}^2$ (see Equation 1.12) match those of the field $-C_2\,\mathbf{X}$, where $C_2 > 0$ is equal to C_L in $\mathbb{Z}^2$. The field **X** can be interpreted as the *lattice realization of a free symplectic fermion* and is responsible for the structure of the field. Furthermore, we deduce that the auxiliary field **Y** will act as a multiplicative constant and can be thought of as a *lattice correction term*. The field $-\mathbf{X}$ is an ideal candidate to validate the claim that the height-one field is represented as a free symplectic fermion theory. Note that the constant from (1.9) and our C_L match as well. Remark that we do not determine the limiting field which bears those specific cumulants. See also the next section on open problems and further discussions.

Expression 1.13 suggests that the degree field of the UST is described by the free symplectic fermion theory **X** as well, which hints towards a positive answer to the question posed by Liu, Peltola and Wu [69] that the UST can be described by a logCFT. Furthermore, the same theorem implies that the symplectic theory of the height-one field in the ASM and degree field are not the same, yet very similar.

Another interesting consequence of expressions 1.12 and 1.13 is the potential universality for both fields. Universality was conjectured already in Hu and Lin [49] and Poncelet and Ruelle [85], where the authors proved that the critical exponents of avalanche size probabilities of the ASM are the same for several lattices as well.

Finally, we will prove in Theorem 3.11 that the fields **XY** and $-\mathbf{X}$ viewed as random distributions converge to a non-trivial limit (not white noise) on test functions with disjoint supports, using the same scaling as in expressions 1.12 and 1.13. Note however that one would obtain white noise if we relax the assumption that the test functions have disjoint support and adjust the scaling to $\varepsilon^{d|V|/2}$ instead of $\varepsilon^{d|V|}$, analogously to Theorem 2.11 (see also Kassel and Wu [61, Thm. 5]).

To the best of the author's knowledge, there are several novel aspects in Chapter 3. Firstly, we give a rigorous representation of the height-one field in the ASM and degree field of the UST in terms of Grassmannian variables at the lattice level in $\mathbb{Z}^d$ and $\mathbf{T}$. This suggests a certain lattice realization of a free symplectic fermion theory. A similar concept appears in the article Moghimi-Araghi, Rajabpour and Rouhani [78], where the authors use Grassmannian Gaussians with a different "covariance" to derive a formal treatment of the limiting theory for the height-one field. Secondly, we prove scaling limits and certain universality of the mentioned fields. This is the first rigorous proof showing convergence of correlations to the analogous continuum correlators for a fermionic system in d dimensions and on the triangular lattice. The proof relies on a careful analysis of the structure of the cumulants, identifying which terms survive the scaling limit and which cancel out. Our analysis is substantially different from the one used in Dürre [32], which is written out only in $\mathbb{Z}^2$ and not generalizable to different graphs in an obvious manner. Thirdly, our proof technique to analyze cumulants of fermionic fields is very general and robust, and permits to determine the lattice constants C_L and c_L explicitly. Those can be used as multiplicative constants in the definition of the continuum field. We find that the constant C_L for the ASM is explicitly related to the height-one probability on the underlying lattice.

1.1.3. EXPANDING ON UNIFORM SPANNING TREES

Recapitulating, in Chapter 3 we study the joint moments of the height-one field of the Abelian sandpile model, by means of a construction of a local field with fermionic variables on a graph. This was achieved given the fact that the height-one field of the ASM at stationarity can be put into correspondence with certain realizations of the uniform spanning tree (Majumdar and Dhar [72], Járai [55], Dürre [32]). By doing so, we also managed to obtain closed-form expressions of the joint moments of the degree field of the UST. In Chapter 4 we build up on those techniques to obtain, among other results, a closed-form expression for the probability mass function of the UST.

Our first observation is a general recipe to calculate probabilities of given edges to be or not to be in the UST in terms of fermionic variables, which is the result given in Proposition 4.2 in Section 4.1. Namely, for any finite simple graph $\mathcal{G} = (\Lambda, E)$ and directed edges $\{f_i\}_i, \{g_j\}_j$ with tail points $\{v_i\}_i, \{w_j\}_j$ respectively,

$$\mathbb{P}\left(\{f_i\}_i \in \mathrm{UST}, \{g_j\}_j \notin \mathrm{UST}\right) \\ = \left\langle \prod_i \nabla_{f_i}\overline{\psi}(v_i)\nabla_{f_i}\psi(v_i) \prod_j \left(1 - \nabla_{g_j}\overline{\psi}(w_j)\nabla_{g_j}\psi(w_j)\right) \right\rangle.$$

On the other hand, there is a well-known connection between these UST probabilities and determinants of the *transfer-current matrix* M, which was originally studied to model electric networks. We gave a brief introduction on page 2, but we expand on it here. If $\mathcal{G}$ is considered as a network where each edge represents a conductance equal to 1, for any two edges e and f the value of $M(e,f)$ is the current measured through f when a battery imposes a unit current through e. These values can also be related to local times of a random walk on $\mathcal{G}$, so M can also be expressed in terms of gradients of the Green's function of the graph in question (see e.g. Kassel and Wu [61]). With this ingredient, the aforementioned fermionic expected values can be written in terms of determinants of M.

Afterwards, for $v \in \Lambda$ we can define the fields

$$X_v^{(k_v)} := \sum_{\mathcal{E} \subseteq E_v : |\mathcal{E}| = k_v} \prod_{e \in \mathcal{E}} \nabla_e \overline{\psi}(v) \nabla_e \psi(v)$$

for $k_v \in \{1, \ldots, \deg_{\mathcal{G}}(v)\}$, being $\deg_{\mathcal{G}}(v)$ the degree of v on the graph $\mathcal{G}$, E_v the edges incident to v (cf. (1.10)), and

$$Y_v := \prod_{e \in E_v} \left(1 - \nabla_e \overline{\psi}(v) \nabla_e \psi(v)\right).$$

(cf. (1.11)). With these fields we obtain the joint probability mass functions of the degree field D_v of the UST as

$$\mathbb{P}\left(D_v = k_v, v \in V\right) = \left\langle \prod_{v \in V} X_v^{(k_v)} Y_v \right\rangle,$$

establishing a clear connection between the fermionic formalism and the UST. This is the result of Theorem 4.6. We highlight that fermionic variables have already been used to study problems of random trees, as in Caracciolo, Sokal and Sportiello [17] and Bauerschmidt *et al.* [8].

By means of the transfer-current matrix, this result can be further expanded to yield an explicit expression of the joint moments of the fields $\left(X_v^{(k_v)} Y_v\right)_v$ in terms of the Green's function of the graph $\mathcal{G}$, given in Theorem 4.8. To the best of the author knowledge, in the literature there is no full general expression for the exact distribution of the degree field of a UST on a general graph. This can be applied, for example, to calculate the probability of a vertex of the UST on the complete graph K_n to have degree k, for any $n \geq 1$. Taking $n \to \infty$, we show that the degree variable behaves as a Poisson variable plus 1, a result which was already known

(Aldous [3] and Pemantle [83]), but it comes in a more straight-forward manner with our approach since we have an explicit expression of the probability mass function of the degree variable for any $n \geq 1$ at any given point.

Finally, if we take a bounded subset $U \subset \mathbb{R}^d$ and restrict ourselves to a finite subset of regular lattices $\mathbf{L}$ like $\mathbb{Z}^d$ or the triangular or hexagonal lattices in $d = 2$ by taking the intersection $U/\varepsilon \cap \mathbf{L}$ with $\varepsilon > 0$, we can obtain a limiting expression for the joint cumulants of the variables $\left(X_v^{(k_v)} Y_v\right)_v$ when we take the limit of the whole infinite lattice, as

$$\tilde{\kappa}(v_1, \ldots, v_n) := \lim_{\varepsilon \to 0} \varepsilon^{-dn} \kappa\left(\left(X_v^{k_v}\right)^{\varepsilon} Y_v^{\varepsilon} : v \in V\right)$$
$$= -\left[\prod_{v \in V} C_{\mathbf{L}}^{(k_v)}\right] \sum_{\sigma} \sum_{\eta} \prod_{v \in V} \partial_{\eta(v)}^{(1)} \partial_{\eta(\sigma(v))}^{(2)} g_U\left(v, \sigma(v)\right), \tag{1.14}$$

where σ are cyclic permutations on V and η are the directions of derivation on $\mathbb{R}^d$. The constants $C_{\mathbf{L}}^{(k_v)}$ are explicitly calculated in terms of the Green's function values of $\mathbf{L}$. We observe that the expression for the limiting cumulants are the same for all lattices up to a constant, hinting towards a potential universality property of the system. Unlike in Chapter 3, the proof of this now more general limiting result is unified for all the lattices considered, which makes the necessary conditions of the lattice more clear for our proof to work. The reader will also observe that expression (1.14) has exactly the same functional form as that of the height-one field of the ASM (see Dürre [32] and Chapter 3 of this thesis), albeit with a different constant in front, meaning that the limiting joint cumulants expressions are affected by the values of $(k_v)_v$ only through $C_{\mathbf{L}}^{(k_v)}$, but otherwise remain the same.

1.2. NOTATION AND PRELIMINARIES

In this section we will fix most notation introduce preliminary concepts used throughout the thesis.

Functions and (Euclidean) sets For the rest of the thesis we will work in dimension $d \geq 2$. We will write $|A|$ for the cardinality of a set A. For $n \in \mathbb{N}$, let $[n]$ denote the set $[n] := \{1, \ldots, n\}$ and $[\![-n, n]\!] := \{-n, \ldots, -1, 0, 1, \ldots, n\}$. We will use $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ to denote the $\ell^2(\mathbb{Z}^d)$ inner product and norm, respectively. By an abuse of notation we will use the same symbols for the inner product and norm in $L^2(\mathbb{R}^d)$.

Let $g_1, g_2 : \mathbb{R}^d \to \mathbb{R}^d$ be two functions. We will use $g_1(x) \lesssim g_2(x)$ to indicate that there exists a constant $C > 0$ such that $|g_1(x)| \leq C|g_2(x)|$, where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$. If we want to emphasize the dependence of C on some parameter (for example U, ε) we will write $\lesssim_U$, $\lesssim_\varepsilon$ and so on. We use the Landau symbol $g_1 = \mathcal{O}(g_2)$ if there exist $x_0 \in \mathbb{R}^d$ and $C > 0$ such that $|g_1(x)| \leq C|g_2(x)|$ for all $x \geq x_0$. Similarly $g_1 = o(g_2)$ means that $\lim_{x \to 0} g_1(x)/g_2(x) = 0$.

Throughout the thesis $\mathbf{L}$ will denote a lattice. In particular, we will consider what we will call the hypercubic lattice $\mathbb{Z}^d$, the two dimensional triangular lattice $\mathbf{T}$, and we will also make some remarks on the two dimensional hexagonal lattice $\mathbf{H}$ (precise reminders of the definitions of these will be given in due course). Since these lattices are regular, we write $\deg_{\mathbf{L}}$ for the degree of any vertex, which is $2d$ for $\mathbb{Z}^d$, 6 for $\mathbf{T}$, and 3 for $\mathbf{H}$.

We will denote an oriented edge f on the lattice $\mathbf{L}$ as the ordered pair (f^-, f^+), being f^- the tail and f^+ the tip of the edge. Denote $\{e_i\}_{i\in[\deg_{\mathbf{L}}]}$ the set of edges with tail in the origin. The e_i's define a natural orientation of edges which we will tacitly choose whenever we need oriented edges (for example when defining the matrix M in Definition 1.9). The opposite vectors will be written as $e_{\deg_{\mathbf{L}}+i} := -e_i, i = 1, \ldots, \deg_{\mathbf{L}}$, whenever the vector exists. Furthermore

$$\tilde{e}_i := (0, \ldots, 0, \underbrace{1}_{\text{i-th position}}, 0, \ldots, 0), \quad i = 1, \ldots, d$$

denotes the d standard coordinate vectors of $\mathbb{R}^d$.

The collection of all e_i, $i \in \{1, \ldots, \deg_{\mathbf{L}}\}$, will be called E_o, where o is the origin. By abuse of notation but convenient for the thesis, if $f = (f^-, f^- + e_i)$ for some $i \in [\deg_{\mathbf{L}}]$, we denote by $-f$ the edge $(f^-, f^- - e_i)$ whenever it exists; that is, the reflection of f over f^-.

Let $A \subseteq \mathbb{R}^d$ denote a countable set. For every $v \in A$, denote by E_v the set $E_o + v$, and let $E(A) = \bigcup_{v\in A} E_v$.

Let $U \subseteq \mathbb{R}^d$ and $e \in E_o$. For a function $f : U \to \mathbb{R}^d$ differentiable at x we define $\partial_e f(x)$ as the directional derivative of f at x in the direction corresponding to e, that is

$$\partial_e g(x) = \lim_{t\to 0^+} \frac{g(x + te) - g(x)}{t}.$$

Likewise, when we consider a function in two variables $g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, we write then $\partial_e^{(j)} f(\cdot, \cdot)$ to denote the directional derivative in the j-th entry, $j = 1, 2$. When $e = e_i$ for some i, we might abuse notation by writing $\partial_i^{(j)} f$ instead of $\partial_{e_i}^{(j)} f$.

For $A, B \subseteq \mathbb{R}^d$ let $\operatorname{dist}(A, B) := \inf_{(x,y)\in A\times B} \|x - y\|$.

Let $U \subset \mathbb{R}^d$ be a non-empty bounded connected open set with $\mathcal{C}^1$ boundary. Denote by $(U_\varepsilon, E_\varepsilon)$ the graph with vertex set $U_\varepsilon := U/\varepsilon \cap \mathbf{L}$ and edge set E_ε defined as the bonds induced by the lattice $\mathbf{L}$ on U_ε. Since we will use approximations via grid points, we need to introduce, for any $t \in \mathbb{R}^d$, its floor function as

$$\lfloor t \rfloor := \text{the unique } z \in \mathbb{Z}^d \text{ such that } t \in z + [0, 1)^d.$$

Graphs and Green's function As we use the notation (u, v) for a directed edge we will use $\{u, v\}$ for the corresponding undirected edge.

Recall that a graph is said to be *simple* if it has no loops and no multiple edges. For a finite and simple (unless stated otherwise) graph $\mathcal{G} = (\Lambda, E)$ we denote the degree of a vertex v as $\deg_{\mathcal{G}}(v) := |\{u \in \Lambda : u \sim v\}|$, where $u \sim v$ means that u and v are nearest neighbors.

Definition 1.1 (Discrete derivatives)**.** For a function $g : \mathbf{L} \to \mathbb{R}$ its discrete derivative $\nabla_{e_i} f$ in the direction $i = 1, \dots, \deg_{\mathbf{L}}$ is defined as

$$\nabla_{e_i} g(u) \coloneqq g(u + e_i) - g(u), \quad u \in \mathbf{L}.$$

When no ambiguities arise, we will write $\nabla_i^{(1)}$ for $\nabla_{e_i}^{(1)}$, and analogously for the second argument. Analogously, for a function $g : \mathbf{L} \times \mathbf{L} \to \mathbb{R}$ we use the notation $\nabla_{e_i}^{(1)} \nabla_{e_j}^{(2)} g$ to denote the double discrete derivative defined as

$$\nabla_{e_i}^{(1)} \nabla_{e_j}^{(2)} g(u, v) \coloneqq g(u + e_i, v + e_j) - g(u + e_i, v) - g(u, v + e_j) + g(u, v),$$

for $u, v \in \mathbf{L}$, $i, j = 1, \dots, \deg_{\mathbf{L}}$.

Remark 1.2. Throughout this thesis we will work with directed edges to encode discrete derivatives in observables of interest. However, whenever we are referring to graphs, the Laplacian operator, and probabilistic models on graphs (for example sandpiles or spanning trees), we will always think of $\mathbf{L}$ as an undirected graph. In fact, one can show that all of the fields $\mathbf{X}$ and $\mathbf{Y}$ (defined in Chapters 3 and 4) remain the same if one changes the direction of any/all edges.

Definition 1.3 (Discrete Laplacian on a graph)**.** We define the (unnormalized) *discrete Laplacian* on $\mathbb{Z}^d$ as

$$\Delta(u, v) \coloneqq \begin{cases} -|\{w \in \mathbf{L} : w \sim u\}| & \text{if } u = v, \\ 1 & \text{if } u \sim v, \\ 0 & \text{otherwise.} \end{cases}$$

where $u, v \in \mathbf{L}$ and $u \sim v$ denotes that u and v are nearest neighbors. For any function $g : \mathbf{L} \to \mathcal{A}$, where $\mathcal{A}$ is an algebra over $\mathbb{R}$, we define

$$\Delta g(u) \coloneqq \sum_{v \in \mathbf{L}} \Delta(u, v) g(v) = \sum_{v \sim u} (g(v) - g(u)).$$

Note that we define the function taking values in an algebra because we will apply the Laplacian both on real-valued functions and functions defined on Grassmannian algebras, which will be introduced in Section 1.6.

We also introduce $\Delta_\Lambda \coloneqq (\Delta(u, v))_{u, v \in \Lambda}$, the restriction of Δ to Λ. Notice that for any lattice function f we have that for all $u \in \Lambda$,

$$\Delta_\Lambda g(u) = \sum_{v \in \Lambda} \Delta(u, v) g(v) = \Delta g_\Lambda(u), \tag{1.15}$$

where g_Λ is the lattice function given by $g_\Lambda(u) \coloneqq g(u) \mathbb{1}_{u \in \Lambda}$.

The interior boundary of a set Λ will be defined by

$$\partial^{\text{in}} \Lambda \coloneqq \{u \in \Lambda : \exists v \in \mathbf{L} \setminus \Lambda : u \sim v\},$$

1

and the outer boundary by

$$\partial^{ex}\Lambda := \{u \in \mathbf{L} \setminus \Lambda : \exists v \in \Lambda : u \sim v\}.$$

We also consider the interior of Λ, given by $\Lambda^{in} := \Lambda \setminus \partial^{in}\Lambda$. The notation ∂U will also be used to denote the boundary of a set $U \subseteq \mathbb{R}^d$.

Definition 1.4 (Discrete Green's function)**.** Let $u \in \Lambda$ be fixed. The Green's function $G_\Lambda(u,\cdot)$ with Dirichlet boundary conditions is defined as the solution of

$$\begin{cases} -\Delta_\Lambda G_\Lambda(u,v) = \delta_u(v) & \text{if } v \in \Lambda, \\ G_\Lambda(u,v) = 0 & \text{if } v \in \partial^{ex}\Lambda, \end{cases}$$

where Δ_Λ is defined in (1.15).

Definition 1.5 (Infinite volume Green's function, [68, Sec. 1.5–1.6])**.** With a slight abuse of notation we denote by $G_0(\cdot,\cdot)$ two objects in different dimensions:

- $d \geq 3$: G_0 is the solution of

$$\begin{cases} -\Delta G_0(u,\cdot) = \delta_u(\cdot) \\ \lim_{\|v\|\to\infty} G_0(u,v) = 0 \end{cases}, \quad u \in \mathbf{L} \tag{1.16}$$

  ($\mathbf{L} = \mathbb{Z}^d$ in this case).

- $d = 2$: G_0 is given by

$$G_0(u,v) = \frac{1}{\deg_{\mathbf{L}}(u-v)} a(u-v), \quad u,v \in \mathbf{L}$$

  (here $\mathbf{L}$ can be $\mathbb{Z}^2$, $\mathbf{T}$, or $\mathbf{H}$), where $a(\cdot)$ is the potential kernel defined as

$$a(u) = \sum_{n=0}^{\infty} \left[\mathbb{P}_o(S_n = o) - \mathbb{P}_o(S_n = u)\right], \quad u \in \mathbf{L},$$

  and $\{S_n\}_{n\geq 0}$ is a random walk on the plane starting at the origin and $\mathbb{P}_o$ its probability measure.

Remark 1.6. The reason for the discrepancy in the definition of G_0 for $d = 2$ is that the solution of Equation (1.16) for $d = 2$ diverges for every $u, v \in \mathbf{L}$. The potential kernel deals with this by renormalizing the expression by subtracting a cancelling factor. Since we will only work with discrete differences of G_0, this subtraction could even be applied to the definition for $d \geq 3$ and the results would hold the same.

Points in $\partial^{ex}\Lambda$ will be later identified with g, which we call *the ghost vertex*, to define a graph with wired boundary conditions. Define $\Lambda^g := \Lambda \cup \{g\}$, and consider another Laplacian given by

$$\Delta^g(u,v) := \begin{cases} \Delta_\Lambda(u,v) & u,v \in \Lambda, \\ |\{w \in \partial^{ex}\Lambda : u \sim w\}| & u \in \partial^{in}\Lambda, v = g, \\ |\{w \in \partial^{ex}\Lambda : v \sim w\}| & u = g, v \in \partial^{in}\Lambda, \\ -|\partial^{in}\Lambda| & u = v = g, \\ 0 & \text{otherwise.} \end{cases}$$

As said, this is equivalent to looking at the graph given by $\Lambda \cup \partial^{ex}\Lambda$ and identifying all elements of $\partial^{ex}\Lambda$ as the ghost.

Definition 1.7 (Continuum Green's function)**.** The continuum Green's function g_U on $\overline{U} \subset \mathbb{R}^d$ is the solution (in the sense of distributions) of

$$\begin{cases} \Delta g_U(\cdot, y) = -\delta_y(\cdot) & \text{on } U, \\ g_U(\cdot, y) = 0 & \text{on } \partial U \end{cases} \tag{1.17}$$

for $y \in U$, where Δ denotes the continuum Laplacian and $\overline{U}$ is the closure of U.

For an exhaustive treatment on Green's functions we refer to Evans [36] and Lawler and Limic [67] and Spitzer [93].

Permutations For any finite set A we define $\Pi(A)$ as the set of all partitions of A. Let $\mathrm{Perm}(A)$ denote the set of all possible permutations of the set A (that is, bijections of A onto itself). When $A = [k]$ for some $k \in \mathbb{N}$, we might also refer to its set of permutations as S_k. If we restrict S_k to those permutations without fixed points, we denote them as S_k^0. Call $S_{cycl}(A)$ the set of the full cyclic permutations of A, possibly with fixed points. More explicitly, any $\sigma : A \to A$ bijective is in $S_{cycl}(A)$ if $\sigma(A') \neq A'$ for any subset $A' \subsetneq A$ with $|A'| > 1$. When this condition is relaxed to all A' with $|A'| > 0$ we obtain the set of all cyclic permutations without fixed points which is called $S_{cycl}^0(A)$.

Cumulants We now give a brief recap of the definition of cumulants and joint cumulants for random variables. Let $n \in \mathbb{N}$ and $\mathbf{X} = (X_i)_{i=1}^n$ be a vector of real-valued random variables, each of which has all finite moments.

Definition 1.8 (Joint cumulants of random vector)**.** The cumulant generating function $K(\mathbf{t})$ of $\mathbf{X}$ for $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{R}^n$ is defined as

$$K(\mathbf{t}) := \log\left(\mathbb{E}\left[e^{\mathbf{t}\cdot\mathbf{X}}\right]\right) = \sum_{\mathbf{m}\in\mathbb{N}^n} \kappa_{\mathbf{m}}(\mathbf{X}) \prod_{j=1}^n \frac{t_j^{m_j}}{m_j!},$$

where $\mathbf{t} \cdot \mathbf{X}$ denotes the scalar product in $\mathbb{R}^n$, $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{N}^n$ is a multi-index with n components, and

$$\kappa_{\mathbf{m}}(\mathbf{X}) = \frac{\partial^{|m|}}{\partial t_1^{m_1} \cdots \partial t_n^{m_n}} K(\mathbf{t})\Big|_{t_1=\ldots=t_n=0},$$

being $|m| = m_1 + \cdots + m_n$. The joint cumulant of the components of $\mathbf{X}$ can be defined as a Taylor coefficient of $K(t_1, \dots, t_n)$ for $\mathbf{m} = (1, \dots, 1)$; in other words

$$\kappa(X_1, \dots, X_n) = \frac{\partial^n}{\partial t_1 \cdots \partial t_n} K(\mathbf{t})\Big|_{t_1=\ldots=t_n=0}.$$

In particular, for any $A \subseteq [n]$, the joint cumulant $\kappa(X_i : i \in A)$ of $\mathbf{X}$ can be computed as

$$\kappa(X_i : i \in A) = \sum_{\pi \in \Pi(A)} (|\pi| - 1)!(-1)^{|\pi|-1} \prod_{B \in \pi} \mathbb{E}\left[\prod_{i \in B} X_i\right],$$

with $|\pi|$ the cardinality of π.

Let us remark that, by some straightforward combinatorics, it follows from the previous definition that

$$\mathbb{E}\left[\prod_{i \in A} X_i\right] = \sum_{\pi \in \Pi(A)} \prod_{B \in \pi} \kappa(X_i : i \in B). \tag{1.18}$$

If $A = \{i, j\}$, $i, j \in [n]$, then the joint cumulant $\kappa(X_i, X_j)$ is the covariance between X_i and X_j. We stress that, for a real-valued random variable X, one has the equality

$$\kappa(\underbrace{X, \dots, X}_{n \text{ times}}) = \kappa_n(X), \quad n \in \mathbb{N},$$

which we call the n-*th cumulant of* X.

1.3. UNIFORM SPANNING TREES

Let $\mathcal{G} = (\Lambda, E)$ be any finite connected graph. We define a *spanning tree* T of $\mathcal{G}$ as any connected subgraph of $\mathcal{G}$ containing every vertex of Λ and having no loops (see Figure 1.1).

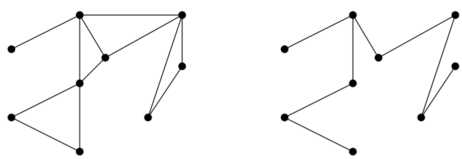

Figure 1.1 – *Example of a graph (left) and possible spanning tree (right).*

Let us denote by $\mathbb{T}$ the set of all possible spanning trees of $\mathcal{G}$. We use $\mathbf{P}$ to denote the uniform measure on all possible such trees. That is, for $T \in \mathbb{T}$ we have

$$\mathbf{P}(T) = \frac{1}{|\mathbb{T}|}.$$

We call $\mathbf{P}$ the *uniform spanning tree (UST)* measure. It can be shown that

$$\mathbf{P}(T) = \frac{1}{\det(-\Delta_\Lambda)},$$

being Δ_Λ the discrete Laplacian defined in Definition 1.3. This is called the *matrix-tree theorem* (see e.g. Lyons and Peres [70, Ch. 4] for a proof).

One of the fundamental properties of the UST is that of *negative associations*, which says that

$$\mathbf{P}(f \in T \,|\, g \in T) \le \mathbf{P}(f \in T)$$

for $f, g \in E$, $f \neq g$, and T a realization of the UST (see Grimmett [42]). Another important characteristic is the *spatial Markov property*. As in Hutchcroft and Nachmias [50, Subsec. 2.2.1], let A and B be subsets of E, and write $(\mathcal{G}\backslash B)/A$ for the graph formed by removing from $\mathcal{G}$ the edges in B, and contracting edges of A. Let us assume that $\mathbf{P}(A \subseteq T, B \cap T = \emptyset) > 0$. Then, for any cylinder event $\mathcal{A} \subseteq \{0,1\}^E$ we have

$$\mathbf{P}_{\mathcal{G}}(T \in \mathcal{A} \,|\, A \subseteq T, B \cap T = \emptyset) = \mathbf{P}_{(\mathcal{G}\backslash B)/A}(T \cup A \in \mathcal{A}),$$

where the subindex in $\mathbf{P}$ emphasizes the graph in which the measure takes place. We will see a similar property for the Gaussian free field.

Finally, we need the notion of the *transfer-current matrix*, a key ingredient in many expressions we obtain in our theorems.

Definition 1.9 (Transfer-current matrix)**.** We define the *transfer-current matrix* M_Λ as

$$M_\Lambda(f,g) := \nabla^{(1)}_{\eta^*(f)} \nabla^{(2)}_{\eta^*(g)} G_\Lambda(f^-, g^-), \quad f, g \in E(\Lambda), \tag{1.19}$$

where $\eta^*(f) \in E_o$ is the coordinate direction induced by $f \in E(\Lambda)$ on f^-, in the sense that $\eta^*(f) = e_i$ if $f = (f^-, f^- + e_i)$ for some $i \in [\deg_L]$. When there is no room for confusion, we might sometimes write f (resp. g) instead of $\eta^*(f)$ (resp. $\eta^*(g)$) to lighten notation. Hereafter, to simplify notation we will omit the dependence of M_Λ on Λ and simply write M.

Remark 1.10. As stated in Lyons and Peres [70], there is another definition of M in terms of electrical networks, as follows: Let $\mathcal{G}$ represent an electric network with impedance 1 on each edge. Defining $\phi_f(x)$ as the voltage at vertex $x \in \Lambda$ when a battery of 1 volt is connected between vertices g^- and g^+ by removing the resistance on g and setting the voltage at f^- to 0, $M(f,g)$ is given by

$$M(f,g) = \phi_f(g^+) - \phi_f(g^-).$$

An important property of this matrix, upon which we will rely, is the following theorem (see again Lyons and Peres [70] for a proof).

1

Theorem 1.11 (Transfer-current theorem)**.** *For any set of* k *distinct edges* $f_1, \dots, f_k \in E$, $k \in \mathbb{N}$, *it follows that*

$$\mathbf{P}(f_1, \dots, f_k \in T) = \det\left(M(f_i, f_j)\right)_{1 \le i,j \le k}.$$

1.4. ABELIAN SANDPILE MODEL

Motivation The *Abelian sandpile model (ASM)* was the first dynamical stochastic model to exhibit *self-organized criticality*; that is, it presents power law decay of correlations without the need of fine tuning any parameter, something which caught the interest of physicists, especially those working on statistical mechanics. It is a system with very simple dynamics and yet with interesting and unexpected behaviors, treatable from a mathematical point of view, although many results are still either not exact, rigorous, or complete.

Mathematical formulation Let $\Lambda \subseteq \mathbf{L}$ be finite. Call a height configuration a map $\rho : \Lambda \to \mathbb{N}$. We can think of this as a sandpile of particles or grains of sand in the sites Λ, being $\rho(v)$ the amount of them at the site v. We identify all vertices of $\mathbf{L} \setminus \Lambda$ into the ghost vertex g, which will play the role of a sink, in the sense that every particle that falls in g is lost.

The *Abelian sandpile model (ASM)* with associated toppling matrix $(\Delta(u,v))_{u,v \in \Lambda^g}$ is a discrete-time Markov chain on $S_\Lambda = \prod_{v \in \Lambda}\{1, \dots, \deg_{\mathbf{L}}\}$. Given $\rho \in S_\Lambda$ the Markov chain evolves as follows: Choose uniformly at random a site $w \in \Lambda$ and increase the height by one. For a site w which is unstable, that is, such that $\rho(w) > \deg_{\mathbf{L}}$, we decrease the height at w by $\deg_{\mathbf{L}}$ and increase the height at each nearest neighbor of w by one. At the ghost vertex g, particles leave the system. All unstable sites are toppled until we obtain a stable configuration. It is proved that the order in which the topplings take place does not matter, and hence the word "Abelian" (see e.g. Meester, Redig and Znameski [75] for a proof).

The unique stationary measure of the ASM is the uniform measure on all recurrent configurations $\mathcal{R}_\Lambda$. Furthermore, we know that $|\mathcal{R}_\Lambda| = \det(-\Delta_\Lambda)$. In fact, there is a bijection (the "burning algorithm" or "burning bijection" by Majumdar and Dhar [73]) between spanning trees of Λ^g and recurrent configurations of $\mathcal{R}_\Lambda$. Here lies the importance of the uniform spanning tree model, introduced in Section 1.3. Let us see this in more detail.

The burning algorithm goes as follows: Begin with all sites unburned, except for the sink. At time $t \ge 0$, a site is burnable if its height is greater than the number of unburned neighbors. In the subsequent time we burn any one of the burnable sites, if any, and all other sites that become burnable after this one. The Abelian property ensures that the order in which we burn sites does not matter towards the final configuration. It can be shown (see e.g. Járai [55] and Meester, Redig and Znameski [75]) that every site will eventually be burned if and only if the configuration is recurrent. Moreover, this algorithm gives a unique spanning tree. To construct it, at each step add an edge between the site that gets burned in that moment and its neighbors that become burnable. It can happen that more than one edge in this step goes to the same burnable site x. In that case, take any ordering of the neighbors of x, such as $(e_1, \dots, e_{\deg_L})$. Then set u_x as the number of unburned neighbors of x and take $p_x = \rho(x) - u_x$. Now choose the p_x-th edge from the ordered list of potential edges connecting to the site x in this step. Repeat this until all sites are reached. Finally, join the sink with each burnable site at $t = 1$, obtaining a spanning tree rooted at the sink; that is, with wired boundary. See Figure 1.2 for an example. In that case we start by the sites with height 4 at the boundary, which are the only burnable ones, and in the following step we connect them with those that become burnable after burning those sites with height 4. Repeat the process until every site has bee reached. At the end we see that the site in the upper-left corner can be joined with the boundary either by above or by the left. In this case $\rho(x) = 4$ and $u_x = 2$, so $p_x = 4 - 2 = 2$. We then take an arbitrary ordering of its edges, like $(e_1, e_2, -e_1, -e_2)$ and extract $(e_2, -e_1)$, which are the potential edges connecting to this x. We take the one in the position p_x, that is, the second, which in this case is $-e_1$, the left edge.

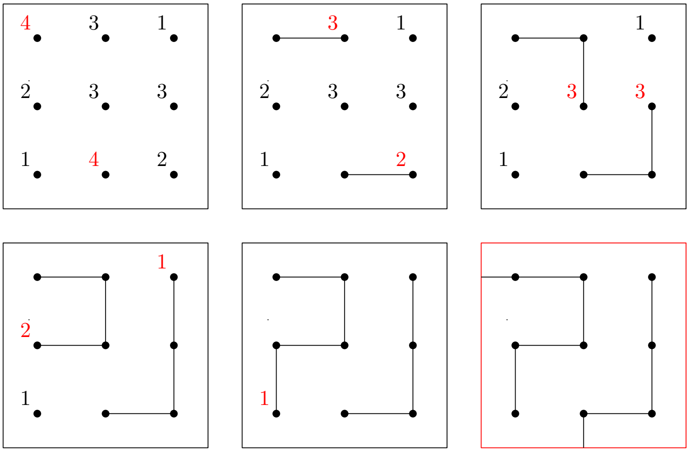

Figure 1.2 – *Example of the algorithm to construct the spanning tree associated to a recurrent sandpile configuration. In red are the burnable sites at a given time, and the boundary at the end once every site has been reached. The order goes from left to right and from top to bottom.*

Remark 1.12. It is immediate to see that any site with height 1 will correspond to a leaf in its corresponding spanning tree, but the converse is not true. See Figure 1.3.

Height-one field We now introduce the main observable of the Abelian sandpile model with which we will work throughout the thesis.

Definition 1.13 (Height-one field)**.** For $z \in \Lambda$, let $h_\Lambda(z) = \mathbb{1}_{\{\rho(z)=1\}}$ be the indicator function of having height 1 at the site $z \in \Lambda$. We call h the *height-one field* of the ASM.

Let us now give an important result of h that will be of use later on.

4	3	1	2
3	3	2	3
1	3	2	4
2	3	4	2

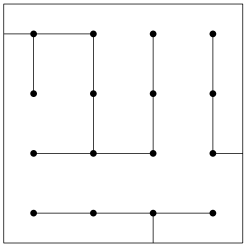

Figure 1.3 – *Example of a recurrent configuration and its associated spanning tree with wired boundary. Observe that every site with height* 1 *corresponds to a leaf of the tree, but the converse is not true.*

Definition 1.14 (Good set)**.** We call $A \subseteq \Lambda$ a *good set* if it does not contain any nearest neighbors, and for every site $v \in \Lambda \setminus A$ there exists a path P of nearest-neighbor sites in Λ^g so that P and A are disjoint.

Lemma 1.15 (Dürre [33, Lem. 24])**.** *Let $V \subseteq \Lambda$. The expected value $\mathbb{E}\left[\prod_{v \in V} h_\Lambda(v)\right]$, which is the probability of having height one on V under the stationary measure for the ASM on Λ, is non-zero if and only if V is a good set.*

1.5. DISCRETE GAUSSIAN FREE FIELD

The *discrete Gaussian Free Field (dGFF)* is, loosely speaking, a generalization of the one-dimensional Gaussian random walk to d dimensions. Although we will only deal with the discrete version in this thesis, we should highlight that the continuum counterpart also exists, although its definition and proof of well-posedness are significantly more tedious in $d \geq 2$. For a more complete exposition we refer the reader to Werner and Powell [97], Sheffield [90], or Berestycki [11].

There are two possible approaches to define the dGFF on a subset $D \subset \mathbf{L}$ with edges set E, as follows.

Definition 1.16 (dGFF via density function)**.** The discrete Gaussian Free Field on Λ with Dirichlet boundary conditions on $\partial^{ex}\Lambda$ is the centered Gaussian vector $(\Gamma(x))_{x\in\Lambda}$ such that its density function at the configuration $(\gamma(x))_{x\in\Lambda}$, is (a multiple of)

$$\exp\left(-\frac{1}{2\deg_{\mathbf{L}}}\sum_{e\in E}|\nabla\gamma(e)|^2\right),$$

where $\gamma(x) \in \mathbb{R}$ if $x \in \Lambda$, and $\gamma(x) = 0$ if $x \in \partial^{ex}\Lambda$.

We can see that configurations with large gradients $|\nabla\gamma(e)|$ are penalized due to the large negative contribution in the exponent of the density function. In this way, the dGFF tries to "keep neighbors close". It can be shown that this definition is equivalent to:

Definition 1.17 (dGFF via covariance function)**.** The discrete Gaussian Free Field on Λ with Dirichlet boundary conditions is the centered Gaussian vector $(\Gamma(x))_{x\in\Lambda}$ whose covariance function is equal to the Green's function G_Λ; that is,

$$\mathbb{E}\left[\Gamma(x)\Gamma(y)\right] = G_\Lambda(x,y), \quad x,y \in \Lambda.$$

This definition accentuates the Gaussian structure of the field. In particular, the distribution at a given $x \in \Lambda$ is Gaussian with mean 0 and variance $G_\Lambda(x,x)$.

One very important characteristic of the dGFF is the so-called *Markov property*. In rough terms, this property says that if we condition the field on a subset of $D \subseteq \Lambda$, the remaining field is also a dGFF with different boundary conditions. More precisely, let Γ^D denote the dGFF conditioned on the values on D. Then

$$\Gamma^D = \Gamma_0^{\Lambda\setminus D} + h^D,$$

where $\Gamma_0^{\Lambda\setminus D}$ is a dGFF with Dirichlet boundary conditions on D and h^D is harmonic on D. Moreover, $\Gamma_0^{\Lambda\setminus D}$ is independent of h^D. See Werner and Powell [97] for more details.

1.6. A PRIMER ON GRASSMANN VARIABLES

In this section we will introduce notions and results about Grassmannian variables and integration. We refer to Abdesselam [1] and Meyer [76] for further reading.

Definition 1.18 (Abdesselam [1, Def. 1])**.** Let $M \in \mathbb{N}$ and $\xi_1, \ldots, \xi_M$ be a collection of letters. Let $\mathbb{R}[\xi_1, \ldots, \xi_M]$ be the quotient of the free non-commutative algebra $\mathbb{R}\langle\xi_1, \ldots, \xi_M\rangle$ by the two-sided ideal generated by the anticommutation relations

$$\xi_j \xi_j = -\xi_i \xi_j, \tag{1.20}$$

where $i, j \in [M]$. We will denote it by Ω^M and call it the Grassmann algebra in M variables. The ξ's will be referred to as *Grassmannian variables* or *generators*. Due to anticommutation these variables are called "fermionic" (as opposed to commutative or "bosonic" variables).

Notice that, due to the anticommutative property, we have that for any variable Pauli's exclusion principle holds (Abdesselam [1, Prop, 2]):

$$\xi_i^2 = 0, \quad i \in [M]. \tag{1.21}$$

An important property for elements of Ω^M is the following (see e.g. Caracciolo, Sokal and Sportiello [18, Prop. A.6]).

Proposition 1.19. *The Grassmann algebra Ω^M is a free $\mathbb{R}$-module with basis given by the 2^M monomials $\xi_I = \xi_{i_1} \cdots \xi_{i_p}$ where $I = \{i_1, \ldots, i_p\} \subseteq [M]$ with $i_1 < \cdots < i_p$. Each element $F \in \Omega^M$ can be written uniquely in the form*

$$F = \sum_{I\subseteq[M]} a_I \xi_I, \quad a_I \in \mathbb{R}.$$

Next we will define Grassmannian derivation and integration.

Definition 1.20 (Grassmannian derivation, Abdesselam [1, Eq. (8)])**.** Let $j \in [M]$. The derivative $\partial_{\xi_j} : \Omega^M \to \Omega^M$ is a map defined by the following action on the monomials $\xi_{i_1} \cdots \xi_{i_p}$. For $I = \{i_1, \dots, i_p\}$ one has

$$\partial_{\xi_j} \xi_I = \begin{cases} (-1)^{\alpha-1} \xi_{i_1} \cdots \xi_{i_{\alpha-1}} \xi_{i_{\alpha+1}} \cdots \xi_{i_p} & \text{if there is } 1 \le \alpha \le p \text{ such that } i_\alpha = j, \\ 0 & \text{if } j \notin I. \end{cases}$$

The following characterization of Grassmanian integration can be found in e.g. Swan [95, Eq. (2.2.7)]. Grassmann–Berezin integrals on fermionic spaces are completely determined by their values on Grassmann monomials as these form a basis for the space.

Definition 1.21 (Grassmannian–Berezin integration)**.** The Grassmann–Berezin integral on fermionic spaces is defined as

$$\int F d\xi := \partial_{\xi_M} \partial_{\xi_{M-1}} \cdots \partial_{\xi_2} \partial_{\xi_1} F, \quad F \in \Omega^M.$$

On the grounds of this definition, for the rest of the thesis Grassmannian–Berezin integrals will be denoted by $\left(\prod_{i=1}^M \partial_{\xi_i}\right) F$.

Definition 1.22. Let $f : \mathbb{R} \to \mathbb{R}$ be an analytic function given by $f(x) = \sum_{k=0}^\infty a_k x^k$, and let $F \in \Omega^M$ be an element of the Grassmann algebra. We define the composition of an analytic function with an element of the Grassmann algebra, $f(F)$, by

$$f(F) := \sum_{k=0}^\infty a_k F^k.$$

Notice that the series $f(F)$ is, in fact, a finite sum since the ξ's are nilpotent. An important special case arises when M is even, $M = 2m$, and the generators $\xi_1, \dots, \xi_M$ are divided into two groups $\psi_1, \dots, \psi_m$ and $\overline{\psi}_1, \dots, \overline{\psi}_m$, where we think of each ψ_i as paired with its corresponding $\overline{\psi}_i$. Since we have an even number of generators of the Grassmann algebra, we have that

$$\prod_{i=1}^m \partial_{\overline{\psi}_i} \partial_{\psi_i} = (-1)^{m(m-1)/2} \left(\prod_{i=1}^m \partial_{\overline{\psi}_i}\right) \left(\prod_{i=1}^m \partial_{\psi_i}\right),$$

and this can be identified as being a collection of "complex" fermionic variables in the language of Caracciolo, Sokal and Sportiello [18, Eq. (A.60)]. We stress that the notation $\overline{\psi}$ is only suggestive of complex conjugation and does not have anything to do with complex numbers. We will use bold to denote the collection of Grassmannian variables, for instance $\boldsymbol{\psi} = (\psi_i)_{i=1}^m$. In particular in the following $\boldsymbol{\psi}$ and $\overline{\boldsymbol{\psi}}$ will be treated as $m \times 1$ vectors.

The next result (Caracciolo, Sokal and Sportiello [18, Prop. A.14]) computes the integral of so-called Grassmannian Gaussians.

Proposition 1.23 (Gaussian integral for "complex" fermions)**.** *Let* A *be an* $m \times m$ *matrix with coefficients in* $\mathbb{R}$*. Then*

$$\left(\prod_{i=1}^{m} \partial_{\overline{\psi}_i} \partial_{\psi_i}\right) \exp\left(\langle \psi, A\overline{\psi} \rangle\right) = \det(A).$$

Another result (Caracciolo, Sokal and Sportiello [18, Thm. A.16]) is the analog of Wick's formula for Grassmannian Gaussians and also will be important to study properties of "transformed" normal variables in the fermionic context.

For a given matrix $A = (A_{i,j})_{i \in I_0, j \in J_0}$, and $I \subseteq I_0, J \subseteq J_0$, such that $|I| = |J|$, we write $\det(A)_{IJ}$ to denote the determinant of the submatrix $(A_{i,j})_{i \in I, j \in J}$. When $I = J$, we simply write $\det(A)_I$.

Theorem 1.24 (Wick's theorem for "complex" fermions)**.** *Let* A *be an* $m \times m$*,* B *an* $r \times m$ *and* C *an* $m \times r$ *matrix respectively with coefficients in* $\mathbb{R}$*. For any sequences of indices* $I = \{i_1, \ldots, i_r\}$ *and* $J = \{j_1, \ldots, j_r\}$ *in* $[m]$ *of the same length* r*, if the matrix* A *is invertible we have*

1. $$\left(\prod_{i=1}^{m} \partial_{\overline{\psi}_i} \partial_{\psi_i}\right) \prod_{\alpha=1}^{r} \psi_{i_\alpha} \overline{\psi}_{j_\alpha} \exp\left(\langle \psi, A\overline{\psi} \rangle\right) = \det(A) \det\left(A^{-T}\right)_{IJ},$$

2. $$\left(\prod_{i=1}^{m} \partial_{\overline{\psi}_i} \partial_{\psi_i}\right) \prod_{\alpha=1}^{r} (\psi^T C)_\alpha (B\overline{\psi})_\alpha \exp\left(\langle \psi, A\overline{\psi} \rangle\right) = \det(A) \det(BA^{-1}C).$$

If $|I| \neq |J|$*, the integral is* 0 *in both cases.*

1.6.1. THE FERMIONIC GAUSSIAN FREE FIELD

Unless stated otherwise, let $\Lambda \subseteq \mathbf{L}$ be finite and connected in the usual graph sense. We will also consider its wired version Λ^g as described on page 17.

Grassmannian algebra on Λ**.** We construct the (real) Grassmannian algebra $\Omega^{2\Lambda}$ resp. $\Omega^{2\Lambda^g}$ with generators $\{\psi_v, \overline{\psi}_v : v \in \Lambda\}$ resp. $\{\psi_v, \overline{\psi}_v : v \in \Lambda^g\}$. Note that $\Omega^{2\Lambda}$ is a subset of the algebra $\Omega^{2\Lambda^g}$.

In the following we will now define the fermionic Gaussian free field with pinned and Dirichlet boundary conditions. We use the notation $\langle f, g \rangle_{\Lambda^g} := \sum_{v \in \Lambda^g} f_v g_v$.

Definition 1.25 (Fermionic Gaussian free field)**.**

Pinned boundary conditions. The unnormalized fermionic Gaussian free field state on Λ^g pinned at g is the linear map $[\cdot]^{\mathbf{p}}_{\Lambda} : \Omega^{2\Lambda^g} \to \mathbb{R}$ defined, for $F \in \Omega^{2\Lambda^g}$, as

$$[F]^{\mathbf{p}}_{\Lambda} := \left(\prod_{v \in \Lambda^g} \partial_{\overline{\psi}_v} \partial_{\psi_v}\right) \psi_g \overline{\psi}_g \exp\left(\langle \psi, -\Delta^g \overline{\psi} \rangle_{\Lambda^g} + \langle \delta_g, \psi\overline{\psi} \rangle_{\Lambda^g}\right) F.$$

Dirichlet boundary conditions. The unnormalized fermionic Gaussian free field state with Dirichlet boundary conditions is the linear map $[\cdot]^{\mathbf{0}}_{\Lambda} : \Omega^{2\Lambda} \to \mathbb{R}$ defined, for $F \in \Omega^{2\Lambda}$, as

$$[F]^{\mathbf{0}}_{\Lambda} \coloneqq \left(\prod_{v \in \Lambda} \partial_{\overline{\psi}_v} \partial_{\psi_v} \right) \exp\left(\langle \psi, -\Delta_{\Lambda} \overline{\psi} \rangle \right) F.$$

We are borrowing here the terminology of "state" from statistical mechanics (compare Friedli and Velenik [37, Def. 3.17] since we do not associate a probability measure to the fermionic fGFFs. Note that we can define the normalized counterpart of the fGFF with pinned boundary conditions as

$$\langle F \rangle^{\mathbf{p}}_{\Lambda} \coloneqq \frac{1}{\mathcal{Z}^{\mathbf{p}}_{\Lambda}} [F]^{\mathbf{p}}_{\Lambda},$$

where $\mathcal{Z}^{\mathbf{p}}_{\Lambda} \coloneqq [1]^{\mathbf{p}}_{\Lambda}$. The normalization constant can be determined from Theorem 1.24 as

$$\mathcal{Z}^{\mathbf{p}}_{\Lambda} = \det(-\Delta_{\Lambda}).$$

We can also define the normalized expectation $\langle \cdot \rangle^{\mathbf{0}}_{\Lambda}$ in a similar fashion using Proposition 1.24 (noting that the normalization constant $\mathcal{Z}^{\mathbf{0}}_{\Lambda}$ also equals $\det(-\Delta_{\Lambda})$). To avoid cluttering notation, in our definitions we write Λ as subindex, even though $[\cdot]^{\mathbf{0}}_{\Lambda}$, $[\cdot]^{\mathbf{p}}_{\Lambda}$ and their normalized counterparts live on Λ^{g}.

We will also consider gradients of the generators in the following sense:

Definition 1.26 (Gradient of the generators)**.** For $i = 1, \ldots, \deg_{\mathcal{G}}(v)$, the gradient of the generators in the i-th direction is given by

$$\nabla_{e_i} \psi(v) = \psi_{v+e_i} - \psi_v, \quad \nabla_{e_i} \overline{\psi}(v) = \overline{\psi}_{v+e_i} - \overline{\psi}_v, \quad v \in \Lambda.$$

Remember that $e_{d+i} \coloneqq -e_i$ for $1 \le i \le d$, as introduced in Section 1.2.

Although the fermionic setting does not carry the notion of realization of random variables, we interpret the evaluation of the states over observables as expectations, so we can extend the notion of cumulants to the fermionic setting via the analogous expression for usual probability measures. Therefore, we define cumulants as follows.

Definition 1.27 (Cumulants of Grassmannian observables)**.** Let $V \subseteq \Lambda^{g}$. The joint cumulants $\kappa^{\bullet}_{\Lambda}(W_v : v \in V)$ of the Grassmannian observables $(W_v)_{v \in V}$ are defined as

$$\kappa^{\bullet}_{\Lambda}(W_v : v \in V) = \sum_{\pi \in \Pi(V)} (|\pi| - 1)!\, (-1)^{|\pi|-1} \prod_{B \in \pi} \left\langle \prod_{v \in B} W_v \right\rangle^{\bullet}_{\Lambda}, \tag{1.22}$$

where the bullet $\bullet$ indicates that we are considering states both under the pinned and the Dirichlet conditions.

As before one has

$$\left\langle \prod_{v \in V} W_v \right\rangle_\Lambda^\bullet = \sum_{\pi \in \Pi(V)} \prod_{B \in \pi} \kappa_\Lambda^\bullet (W_v : v \in B).$$

Example 1.28. *Let $v \in \Lambda$ and $F = \psi_v \overline{\psi}_v$. By Theorem 1.24 item 1 we have that*

$$\langle \psi_v \overline{\psi}_v \rangle_\Lambda^{\mathbf{0}} = \frac{1}{\det(-\Delta_\Lambda)} \det(-\Delta_\Lambda) G_\Lambda(v,v) = G_\Lambda(v,v).$$

The two-point function for $v, w \in \Lambda$ such that $v \neq w$ is equal to

$$\begin{aligned} \langle \psi_v \overline{\psi}_v \psi_w \overline{\psi}_w \rangle_\Lambda^{\mathbf{0}} &= \det \begin{pmatrix} G_\Lambda(v,v) & G_\Lambda(v,w) \\ G_\Lambda(w,v) & G_\Lambda(w,w) \end{pmatrix} \\ &= G_\Lambda(v,v) G_\Lambda(w,w) - G_\Lambda(v,w)^2. \end{aligned}$$

In particular, for $v \neq w$, $\kappa_\Lambda^{\mathbf{0}}(\psi_v \overline{\psi}_v, \psi_w \overline{\psi}_w) = -G_\Lambda(v,w)^2 < 0$; that is, we have negative "correlations", or more precisely, negative joint cumulants of second order.

2

Gradient Squared of the Discrete GFF

Ut est rerum omnium magister usus ("Experience is the best teacher").

—-Julius Caesar

What I learned on my own I still remember.

—-Nassim N. Taleb, *The Bed of Procrustes*

In *this chapter we study the properties of the centered (norm of the) gradient squared of the discrete Gaussian free field in* $U_\varepsilon = U/\varepsilon \cap \mathbb{Z}^d$, $U \subset \mathbb{R}^d$ *and* $d \geq 2$*. The covariance structure of the field is a function of the transfer current matrix and this relates the model to a class of systems (e.g. height-one field of the Abelian sandpile model or pattern fields in dimer models) that have a Gaussian limit due to the rapid decay of the transfer current. Indeed, we prove that the properly rescaled field converges to white noise in an appropriate local Besov-Hölder space. Moreover, under a different rescaling, we determine the* k*-point correlation function and joint cumulants on* U_ε *and in the continuum limit as* $\varepsilon \to 0$*. This result is related to the analogue limit for the height-one field of the Abelian sandpile (Dürre [32]), with the same conformally covariant property in* $d = 2$*.*

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2

THE structure of this chapter is as follows. In Section 2.1 we fix notation, introduce the fields that we study and provide the definition of the local Besov-Hölder spaces where convergence takes place. Section 2.2 is devoted to stating the main results in a precise manner. Section 2.3 analyzes the field from a Fock space point of view, and the subsequent Section 2.5 contains necessary preliminary results, a recapitulation of Feynman diagrams, and the proofs our main results.

Throughout this chapter we will work on the lattice $\mathbb{Z}^d$ for any $d \geq 2$. The set U_ε denotes $U_\varepsilon = U/\varepsilon \cap \mathbb{Z}^d$, with $U \subset \mathbb{R}^d$.

2.1. PRELIMINARIES

2.1.1. FUNCTIONS OF THE GAUSSIAN FREE FIELD AND WHITE NOISE

Remember the definition of the dGFF in the previous chapter (Section 1.5). Define, for an oriented edge $e = (e^-, e^+) \in E_\varepsilon$, the *gradient dGFF* $\nabla_e \Gamma$ as

$$\nabla_e \Gamma(e^-) := \Gamma(e^+) - \Gamma(e^-).$$

In the following, we will define the main object of interest of this chapter.

Definition 2.1 (Gradient squared of the dGFF)**.** The discrete stochastic field Φ_ε given by

$$\Phi_\varepsilon(x) := \sum_{i=1}^{d} :\left(\nabla_{e_i} \Gamma(x)\right)^2:, \quad x \in U_\varepsilon,$$

is called the gradient squared of the dGFF, where $: \cdot :$ denotes the Wick product; that is, $:X: = X - \mathbb{E}[X]$ for any random variable X.

The family of random fields $(\Phi_\varepsilon)_{\varepsilon>0}$ is a family of distributions, which is defined to act on a given test function $f \in \mathcal{C}_c^\infty(U)$ as

$$\langle \Phi_\varepsilon, f \rangle := \int_U \Phi_\varepsilon\left(\lfloor x/\varepsilon \rfloor\right) f(x)\, dx, \tag{2.1}$$

where we take $\Phi_\varepsilon\left(\lfloor x/\varepsilon \rfloor\right) = 0$ in case $\lfloor x/\varepsilon \rfloor \notin U_\varepsilon$, which can happen if ε is not small enough.

Definition 2.2 (Gaussian white noise)**.** The d-dimensional Gaussian white noise W is the centered Gaussian random distribution on $U \subset \mathbb{R}^d$ such that, for every $f, g \in L^2(U)$,

$$\mathbb{E}\left[\langle W, f \rangle \langle W, g \rangle\right] = \int_U f(x) g(x) dx.$$

In other words, $\langle W, f \rangle \sim \mathcal{N}\left(0, \|f\|^2_{L^2(U)}\right)$ for every $f \in L^2(U)$.

2.1.2. BESOV-HÖLDER SPACES

In this subsection we will define the functional space on which convergence will take place. We will use Furlan and Mourrat [39] as a main reference. Local Hölder and Besov spaces of negative regularity on general domains are natural functional spaces when considering scaling limits of certain random distributions or in the context of non-linear stochastic PDE's (see e.g. Furlan and Mourrat [39] and Hairer [44] especially when those objects are well-defined on a domain $U \subset \mathbb{R}^d$ but not necessarily on the full space $\mathbb{R}^d$). They are particularly suited for fields which show bad behaviour near the boundary ∂U.

2

We first need to introduce the concept of a *multiresolution analysis*. To fix concepts, for the moment let us work with $d = 1$; that is, in $\mathbb{R}$. A multiresolution analysis consist of a function $\phi \in L^2(\mathbb{R})$, sometimes called *father wavelet*, such that

$$\int \phi(x)\phi(x+k)dx = \delta_{k,0} \quad \forall k \in \mathbb{Z}$$

and for some structure constants $(a_k)_k$ it satisfies

$$\phi(x) = \sum_{k \in \mathbb{Z}} a_k \phi(2x - k). \tag{2.2}$$

It is immediate to see that the indicator function satisfies these two conditions. However, it is not even continuous. It turns out (Daubechies [27]) that there exists such compactely supported function ϕ for any desired regularity.

Now set $\phi^n_x(y) = 2^{n/2}\phi(2^n(y-x))$ and $\Lambda_n = \mathbb{Z}/2^n$. Calling $V_n \subset L^2(\mathbb{R})$ the subspace generated by $\{\phi^n_x : x \in \Lambda_n\}$, from (2.2) it follows that $V_n \subset V_{n+1}$ for all n. Moreover, the complement $V_n^\perp$ of V_n within V_{n+1} can be written in terms of ϕ itself. It turns out that one can always find constants $(b_k)_k$ such that $V_n^\perp$ is generated by $\{\psi^n_x : x \in \mathbb{Z}\}$, where

$$\psi(x) = \sum_{k \in \mathbb{Z}} b_k \phi(2x - k)$$

and $\psi^n_x(y) = 2^{n/2}\psi(2^n(y-x))$. The function ψ is sometimes called the *mother wavelet*. This way, what we obtain is that the functions ϕ^n_x provide a description of a function up to scales 2^{-n}, while the ψ^n_x "fills" the details at finer scales. We should emphasize that, in principle, one could do without the ψ^n_x and always work with the ϕ^n_x and an explicit expression of the constants b_k, but that complicates notation.

Now that we intuitively understand the role of these functions, let us go to $\mathbb{R}^d$. The only complication that arises now is thatwe will need $2^d - 1$ functions $(\phi^{(i)})_{1 \le i < 2^d}$ to extend the previous results, as it can be seen in Hairer [44]. Let us then define the setting, and see how this related to Besov spaces.

Let $(V_n)_{n \in \mathbb{Z}}$ be a dense subsequence of subspaces of $L^2(\mathbb{R}^d)$ with trivial intersection; that is, $\bigcap_{n \in \mathbb{Z}} V_n = \{0\}$. Denote by W_n the orthogonal complement of V_n in V_{n+1} for all $n \in \mathbb{Z}$. Furthermore, we assume the following properties. The function $f \in V_n$ if and only if $f(2^{-n} \cdot) \in V_0$. Let $(\phi(\cdot - k))_{k \in \mathbb{Z}^d}$ be an orthonormal basis of V_0 and $(\psi^{(i)}(\cdot - k))_{i < 2^d, k \in \mathbb{Z}^d}$ an orthonormal basis of W_0. Note that ϕ, $(\psi^{(i)})_{i < 2^d}$

both belong to $\mathcal{C}_c^r(\mathbb{R}^d)$ for some positive integer $r \in \mathbb{N}$; that is, they belong to the set of r times continuously differentiable functions on $\mathbb{R}^d$ with compact support. For more details see Daubechies [26] and Meyer and Salinger [77].

Define $\Lambda_n = \mathbb{Z}^d/2^n$ and

$$\phi_{n,x}(y) = 2^{dn/2}\phi\big(2^n(y-x)\big)$$

resp.

$$\psi_{n,x}^{(i)}(y) = 2^{dn/2}\psi^{(i)}\big(2^n(y-x)\big),$$

which makes $(\phi_{n,x})_{x\in\Lambda_n}$ an orthonormal basis of V_n resp. $(\psi_{n,x}^{(i)})_{x\in\Lambda_n, i<2^d, n\in\mathbb{Z}}$ an orthonormal basis of $L^2(\mathbb{R}^d)$. Every function $f \in L^2(\mathbb{R}^d)$ can be decomposed into

$$f = \mathcal{V}_k f + \sum_{n=k}^{\infty} \mathcal{W}_n f$$

for any fixed $k \in \mathbb{Z}$, where $\mathcal{V}_n$ resp. $\mathcal{W}_n$ are the orthogonal projections onto V_n resp. W_n defined as

$$\mathcal{V}_n f = \sum_{x\in\Lambda_n} \langle f, \phi_{n,x}\rangle \phi_{n,x}, \quad \mathcal{W}_n f = \sum_{i<2^d, x\in\Lambda_n} \langle f, \psi_{n,x}^{(i)}\rangle \psi_{n,x}^{(i)}.$$

Definition 2.3 (Besov spaces)**.** Let $\alpha \in \mathbb{R}$, $|\alpha| < r$, $p, q \in [1, \infty]$ and $U \subset \mathbb{R}^d$. The Besov space $\mathcal{B}_{p,q}^{\alpha}(U)$ is the completion of $\mathcal{C}_c^{\infty}(U)$ with respect to the norm

$$\|f\|_{\mathcal{B}_{p,q}^{\alpha}} := \|\mathcal{V}_0 f\|_{L^p} + \left\| \left(2^{\alpha n}\|\mathcal{W}_n f\|_{L^p}\right)_{n\in\mathbb{N}} \right\|_{\ell^q}.$$

The local Besov space $\mathcal{B}_{p,q}^{\alpha,\mathrm{loc}}(U)$ is the completion of $\mathcal{C}^{\infty}(U)$ with respect to the family of semi-norms

$$f \mapsto \|\widetilde{\chi} f\|_{\mathcal{B}_{p,q}^{\alpha}}$$

indexed by $\widetilde{\chi} \in \mathcal{C}_c^{\infty}(U)$.

We will use the following embedding property of Besov spaces in the tightness argument.

Lemma 2.4 (Furlan and Mourrat [39, Rmk. 2.12])**.** *For any* $1 \le p_1 \le p_2 \le \infty$, $q \in [1, \infty]$ *and* $\alpha \in \mathbb{R}$, *the space* $\mathcal{B}_{p_2,q}^{\alpha,\mathrm{loc}}(U)$ *is continuously embedded in* $\mathcal{B}_{p_1,q}^{\alpha,\mathrm{loc}}(U)$.

Finally let us define the functional space where convergence will take place, the space of distributions with locally α-Hölder regularity. For that, we denote as $\mathcal{C}^r$ the set of r times continuously differentiable functions on $\mathbb{R}^d$, with $r \in \mathbb{N} \cup \{\infty\}$. We also define the $\mathcal{C}^r$ norm of a function $f \in \mathcal{C}^r$ as

$$\|f\|_{\mathcal{C}^r} := \sum_{|i|\le r} \|\partial_i f\|_{L^\infty},$$

being $i \in \mathbb{N}^d$ a multi-index.

Definition 2.5 (Hölder spaces)**.** Let $\alpha < 0, r_0 = -\lfloor \alpha \rfloor$. The space $\mathcal{C}^{\alpha}_{\mathrm{loc}}(U)$ is called the locally Hölder space with regularity $\alpha \in \mathbb{R}$ on the domain U. It is the completion of $\mathcal{C}^{\infty}_{c}(U)$ with respect to the family of semi-norms

$$f \mapsto \|\widetilde{\chi} f\|_{\mathcal{C}^{\alpha}}$$

indexed by $\widetilde{\chi} \in \mathcal{C}^{\infty}_{c}(U)$ and

$$\|f\|_{\mathcal{C}^{\alpha}} = \sup_{\lambda \in (0,1]} \sup_{x \in \mathbb{R}^d} \sup_{\eta \in B^{r_0}} \lambda^{-\alpha} \int_{\mathbb{R}^d} f(\cdot) \lambda^{-d} \eta\left(\frac{\cdot - x}{\lambda}\right),$$

where

$$B^{r_0} = \{\eta \in \mathcal{C}^{r_0} : \|\eta\|_{\mathcal{C}^{r_0}} \leq 1, \operatorname{supp} \eta \subset B(0,1)\}.$$

Note that by Furlan and Mourrat [39, Rmk. 2.18] one has $\mathcal{C}^{\alpha}_{\mathrm{loc}}(U) = \mathcal{B}^{\alpha,\mathrm{loc}}_{\infty,\infty}(U)$.

2.2. RESULTS

The first result we would like to present is an explicit computation of the k-point correlation function of the gradient squared of the dGFF field Φ_ε defined in Definition 2.1.

Theorem 2.6. *Let $\varepsilon > 0$ and $k \in \mathbb{N}$ and let the points $x^{(1)}, \dots, x^{(k)}$ in $U \subset \mathbb{R}^d$, $d \geq 2$, be given. Define $x_\varepsilon^{(j)} := \lfloor x^{(j)}/\varepsilon \rfloor$ and choose ε small enough so that $x_\varepsilon^{(j)} \in U_\varepsilon$, for all $j = 1, \dots, k$. Then*

$$\mathbb{E}\left[\prod_{j=1}^{k} \Phi_\varepsilon\left(x_\varepsilon^{(j)}\right)\right] = \sum_{\pi \in \Pi([k])} \prod_{B \in \pi} 2^{|B|-1} \sum_{\sigma \in S^0_{\mathrm{cycl}}(B)} \sum_{\eta : B \to \mathcal{E}} \prod_{j \in B} \nabla^{(1)}_{\eta(j)} \nabla^{(2)}_{\eta(\sigma(j))} G_{U_\varepsilon}\left(x_\varepsilon^{(j)}, x_\varepsilon^{(\sigma(j))}\right). \quad (2.3)$$

Moreover if $x^{(i)} \neq x^{(j)}$ for all $i \neq j$, then

$$\lim_{\varepsilon \to 0} \varepsilon^{-dk} \mathbb{E}\left[\prod_{j=1}^{k} \Phi_\varepsilon\left(x_\varepsilon^{(j)}\right)\right] = \sum_{\pi \in \Pi([k])} \prod_{B \in \pi} 2^{|B|-1} \sum_{\sigma \in S^0_{\mathrm{cycl}}(B)} \sum_{\eta : B \to \mathcal{E}} \prod_{j \in B} \partial^{(1)}_{\eta(j)} \partial^{(2)}_{\eta(\sigma(j))} g_U\left(x^{(j)}, x^{(\sigma(j))}\right). \quad (2.4)$$

Remark 2.7. It will sometimes be useful to write (2.3) as the equivalent expression

$$\mathbb{E}\left[\prod_{j=1}^{k} \Phi_\varepsilon\left(x_\varepsilon^{(j)}\right)\right] = \sum_{\substack{\pi \in \Pi([k]) \\ \text{w/o singletons}}} \prod_{B \in \pi} 2^{|B|-1} \sum_{\sigma \in S_{\mathrm{cycl}}(B)} \sum_{\eta : B \to \mathcal{E}} \prod_{j \in B} \nabla^{(1)}_{\eta(j)} \nabla^{(2)}_{\eta(\sigma(j))} G_{U_\varepsilon}\left(x_\varepsilon^{(j)}, x_\varepsilon^{(\sigma(j))}\right), \quad (2.5)$$

where the condition of σ belonging to full cycles of B without fixed points is inserted in the no-singleton condition of the permutations π.

Remark 2.8. From the above expression it is immediate to see that the 2-point function is given by

$$\mathbb{E}\left[\Phi_\varepsilon(x_\varepsilon)\Phi_\varepsilon(y_\varepsilon)\right] = 2\sum_{i,j\in[d]}\left(\nabla_i^{(1)}\nabla_j^{(2)}G_{U_\varepsilon}(x_\varepsilon,y_\varepsilon)\right)^2,$$

which will be useful later on.

The following Corollary is a direct consequence of Theorem 2.6. As already mentioned in the introduction, comparing our result with Dürre [32, Thm. 2] we obtain (1.5).

Corollary 2.9. *Let $\ell \in \mathbb{N}$. The joint cumulants $\kappa(\Phi_\varepsilon(x_\varepsilon^{(j)}) : j \in [\ell], x_\varepsilon^{(j)} \in U_\varepsilon)$ of the field Φ_ε at "level" $\varepsilon > 0$ are given by*

$$\kappa\left(\Phi_\varepsilon(x_\varepsilon^{(j)}) : j \in [\ell]\right) = 2^{\ell-1}\sum_{\sigma\in S^0_{\text{cycl}}([\ell])}\sum_{\eta:[\ell]\to\mathcal{E}}\prod_{j=1}^{\ell}\nabla^{(1)}_{\eta(j)}\nabla^{(2)}_{\eta(\sigma(j))}G_{U_\varepsilon}(x_\varepsilon^{(j)},x_\varepsilon^{(\sigma(j))}). \tag{2.6}$$

Moreover if $x^{(i)} \neq x^{(j)}$ for all $i \neq j$, then

$$\lim_{\varepsilon\to 0}\varepsilon^{-d\ell}\kappa\left(\Phi_\varepsilon(x_\varepsilon^{(j)}) : j \in [\ell]\right) = 2^{\ell-1}\sum_{\sigma\in S^0_{\text{cycl}}([\ell])}\sum_{\eta:[\ell]\to\mathcal{E}}\prod_{j=1}^{\ell}\partial^{(1)}_{\eta(j)}\partial^{(2)}_{\eta(\sigma(j))}g_U(x^{(j)},x^{(\sigma(j))}). \tag{2.7}$$

The following proposition states that in $d = 2$ the limit of the field Φ_ε is conformally covariant with scale dimension 2. This result can also be deduced for the height-one field for the sandpile model (see Dürre [32, Thm. 1]).

Proposition 2.10. *Let $U, U' \subset \mathbb{R}^2$, $k \in \mathbb{N}$, $\{x^{(j)}\}_{j\in[k]}$, and $\{x_\varepsilon^{(j)}\}_{j\in[k]}$ be as in Theorem 2.6. Furthermore let $h : U \to U'$ be a conformal mapping and call $h_\varepsilon(x^{(j)}) := \lfloor h(x^{(j)})/\varepsilon \rfloor$, for ε small enough so that $h_\varepsilon(x^{(j)}) \in U'_\varepsilon$ for all $j \in [k]$. Then*

$$\lim_{\varepsilon\to 0}\varepsilon^{-2k}\mathbb{E}\left[\prod_{j=1}^{k}\Phi_\varepsilon^U(x_\varepsilon^{(j)})\right] = \prod_{j=1}^{k}\left|h'(x^{(j)})\right|^2\lim_{\varepsilon\to 0}\varepsilon^{-2k}\mathbb{E}\left[\prod_{j=1}^{k}\Phi_\varepsilon^{U'}(h_\varepsilon(x^{(j)}))\right],$$

where now for clarity we emphasize the dependence of Φ_ε on its domain.

Finally we will show that the rescaled gradient squared of the discrete Gaussian free field will converge to white noise in some appropriate locally Hölder space with negative regularity α in $d \geq 2$ dimensions. This space is denoted as $\mathcal{C}^\alpha_{\text{loc}}(U)$ (see Definition 2.5).

Theorem 2.11. *Let $U \subset \mathbb{R}^d$ with $d \geq 2$. The gradient squared of the discrete Gaussian free field Φ_ε converges in the following sense as $\varepsilon \to 0$:*

$$\frac{\varepsilon^{-d/2}}{\sqrt{\chi}} \Phi_\varepsilon \xrightarrow{d} W,$$

where the white noise W is defined in Definition 2.2. This convergence takes place in $\mathcal{C}^\alpha_{\mathrm{loc}}(U)$ for any $\alpha < -d/2$, and the constant χ defined as

$$\chi := 2 \sum_{v \in \mathbb{Z}^d} \sum_{i,j \in [d]} \left(\nabla_i^{(1)} \nabla_j^{(2)} G_0(0, v) \right)^2 \tag{2.8}$$

is well-defined, in the sense that $0 < \chi < \infty$.

Remark 2.12. Let us remind the reader that $\mathcal{C}^\alpha_{\mathrm{loc}}(U)$ with $\alpha < -d/2$ are the optimal spaces in which the white noise lives. See for example Armstrong, Kuusi and Mourrat [4, Prop. 5.9].

2.3. FOCK SPACE STRUCTURE

Let us discuss in the following the connection to Fock spaces. We start by reminding the reader of the definition of the continuum Gaussian free field (GFF).

Definition 2.13 (Continuum Gaussian free field, Berestycki [11, Sec. 1.5])**.** The continuum Gaussian free field $\overline{\Gamma}$ with 0-boundary (or Dirichlet) conditions outside U is the unique centered Gaussian process indexed by $\mathcal{C}_c^\infty(U)$ such that

$$\mathrm{Cov}\left(\overline{\Gamma}(f), \overline{\Gamma}(g)\right) = \int_{U \times U} f(x) g(y) g_U(x, y)\, dx dy, \quad f, g \in \mathcal{C}_c^\infty(U),$$

where $g_U(\cdot, \cdot)$ was defined in Definition 1.7.

We can think of it as an isometry $\overline{\Gamma} : \mathcal{H} \to L^2(\Omega, \mathbb{P})$, for some Hilbert space $\mathcal{H}$ and some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. To fix ideas, throughout this Section let us fix $\mathcal{H} := \mathcal{H}_0^1(U)$, the order one Sobolev space with Dirichlet inner product (see Berestycki [11, Sec. 1.6]). Note that, even if the GFF is not a proper random variable, we can define its derivative as a Gaussian distributional field.

Definition 2.14 (Derivatives of the GFF, Kang and Makarov [60, p. 4])**.** The derivative of $\overline{\Gamma}$ is defined as the Gaussian distributional field $\partial_i \overline{\Gamma}$, $1 \leq i \leq d$, in the following sense:

$$\left(\partial_i \overline{\Gamma}\right)(f) := \overline{\Gamma}(\partial_i f), \quad f \in \mathcal{C}_c^\infty(U).$$

There is however another viewpoint that one can take on the GFF and its derivatives, and is that of viewing them as *Fock space fields*. This approach will be used to reinterpret the meaning of Theorem 2.6. For the reader's convenience we now recall here some basic facts about Fock spaces and their fields. Our presentation is drawn from Janson [54, Sec. 3.1] and Kang and Makarov [60, Sec. 1.2–1.4].

For $n \geq 0$, we denote $\mathcal{H}^{\odot n}$ as the n-th symmetric tensor power of $\mathcal{H}$; in other words, $\mathcal{H}^{\odot n}$ is the completion of linear combinations of elements $f_1 \odot \cdots \odot f_n$ with respect to the inner product

$$\langle f_1 \odot \cdots \odot f_n, g_1 \odot \cdots \odot g_n \rangle = \sum_{\sigma \in S_n} \prod_{i=1}^{n} \langle f_i, g_{\sigma(i)} \rangle, \quad f_i, g_i \in \mathcal{H},\ 1 \leq i \leq n.$$

The symmetric Fock space over $\mathcal{H}$ is

$$\mathrm{Fock}(\mathcal{H}) := \bigoplus_{n \geq 0} \mathcal{H}^{\odot n}.$$

We now introduce elements in $\mathrm{Fock}(\mathcal{H})$ called *Fock space fields*. We call *basic correlation functionals* the formal expressions of the form

$$\mathcal{X}_p = X_1(x_1) \odot \cdots \odot X_p(x_p),$$

for any $p \in \mathbb{N}$ and $x_1, \ldots, x_p \in U$, being $X_1, \ldots, X_p$ derivatives of $\overline{\Gamma}$. The set $\mathcal{S}(\mathcal{X}_p) := \{x_1, \ldots, x_p\}$ is called the set of *nodes* of $\mathcal{X}_p$. Basic Fock space fields are formal expressions written as products of derivatives of the Gaussian free field $\overline{\Gamma}$, for example $1 \odot \overline{\Gamma}$, $\partial\overline{\Gamma} \odot \overline{\Gamma} \odot \overline{\Gamma}$ etc. A general Fock space field X is a linear combination of basic fields. We think of any such X as a map $u \mapsto X(u)$, $u \in U$, where the values $\mathcal{X} = X(u)$ are correlation functionals with $\mathcal{S}(\mathcal{X}) = \{u\}$. Thus Fock space fields are functional-valued functions. Observe that Fock space fields may or may not be distributional random fields, but in any case we can think of them as functions in U whose values are correlation functionals.

Our goal is to define now tensor products. We will restrict our attention to tensor products over an even number of correlation functionals, even if the definition can be given for an arbitrary number of them. The reason behind this presentation is due to the set-up we will be working with.

Definition 2.15 (Tensor products in Fock spaces)**.** Let $m \in 2\mathbb{N}$. Given a collection of correlation functionals

$$\mathcal{X}_j := X_{j1}(z_{j1}) \odot \cdots \odot X_{jn_j}(z_{jn_j}), \quad 1 \leq j \leq m$$

with pairwise disjoint $\mathcal{S}(\mathcal{X}_j)$'s, the tensor product of the elements $\mathcal{X}_1, \ldots, \mathcal{X}_m$ is defined as

$$\mathcal{X}_1 \cdots \mathcal{X}_m := \sum_{\gamma} \prod_{\{u,v\} \in E_\gamma} \mathbb{E}\left[X_u(x_u) X_v(x_v)\right], \tag{2.9}$$

where the sum is taken over Feynman diagrams γ (see Subsection 2.5.2) with vertices u labeled by functionals X_{pq} in such a way that there are no contractions of vertices in the same $\mathcal{S}(\mathcal{X}_p)$. E_γ denotes the set of edges of γ. One extends the definition of tensor product to general correlation functionals by linearity.

The reader may have noticed that (2.9) is simply one version of Wick's theorem. It is indeed this formula that will allow us in Subsection 2.5.3 to prove Theorem 2.6, and that enables one to bridge Fock spaces and our cumulants in the following way. For any $j \in [k]$, $k \in \mathbb{N}$, $i_j \in [d]$, one can define the basic Fock space field $X_{i_j} := \partial_{i_j}\overline{\Gamma}$. Introduce the correlation functional

$$\mathcal{Y}_j := \sum_{i_j \in \mathcal{E}} X_{i_j}^{\odot 2}(x^{(j)})$$

for $x^{(j)} \in U$. We obtain now the statement of the next lemma.

Lemma 2.16 (k-point correlation functions as Fock space fields)**.** *Under the assumptions of Theorem 2.6,*

$$\lim_{\varepsilon \to 0} \varepsilon^{-dk}\mathbb{E}\left[\prod_{j=1}^{k} \Phi_\varepsilon\left(x_\varepsilon^{(j)}\right)\right] = \sum_{\pi \in \Pi([k])} \left(\frac{1}{2}\right)^{|\pi|} \prod_{B \in \pi} \mathcal{Y}_B\left(x^{(B)}\right)$$

where $\mathcal{Y}_B\left(x^{(B)}\right) := 2\mathcal{Y}_1 \odot \cdots \odot 2\mathcal{Y}_j$, $\mathcal{S}(\mathcal{Y}_j) = \{x^{(j)}\}$, $j \in B$. *Here* $|\pi|$ *stands for the number of blocks of the partition* π *and the tensor product on the r.h.s. is taken in the sense of Equation* (2.9)*.*

The Fock space structure is more evident from the Gaussian perspective of the dGFF, but (1.5) together with Dürre's theorem entail a corollary which we would like to highlight. We remind the reader of the definition of the constant C in (1.6).

Corollary 2.17 (Height-one field k-point functions, $d = 2$)**.** *With the same notation of Theorem 2.6 one has in* $d = 2$ *that*

$$\lim_{\varepsilon \to 0} \varepsilon^{-2k}\mathbb{E}\left[\prod_{j=1}^{k}\left(h_\varepsilon\left(x_\varepsilon^{(j)}\right) - \mathbb{E}\left[h_\varepsilon\left(x_\varepsilon^{(j)}\right)\right]\right)\right] = \sum_{\pi \in \Pi([k])} \left(-\frac{1}{2}\right)^{|\pi|} \prod_{B \in \pi} \widetilde{\mathcal{Y}}_B\left(x^{(B)}\right)$$

where $\widetilde{\mathcal{Y}}_B\left(x^{(B)}\right) := \widetilde{\mathcal{Y}}_1 \odot \cdots \odot \widetilde{\mathcal{Y}}_j$, $\mathcal{S}(\widetilde{\mathcal{Y}}_j) = \{x^{(j)}\}$ *and* $\widetilde{\mathcal{Y}}_j := C\,\mathcal{Y}_j$, $j \in B$. *As before,* $|\pi|$ *stands for the number of blocks of the partition* π*.*

Remark 2.18. Mind that our Green's functions differ from those of Dürre [32] by a factor of 2d since in their definitions we use the normalized Laplacian, whereas Dürre uses the unnormalized one. This has to be accounted for when comparing results.

2.4. OTHER PROPERTIES OF THE FIELD

2.4.1. QUASI PERMANENTAL STRUCTURE

Although our field is not a *permanental process*, it is similar to one, as we will now see. We first begin with a reminder of the definition of such processes.

Definition 2.19 (Permanental process, Eisenbaum and Kaspi [35])**.** A real-valued positive process $(\psi_x)_{x \in V}$ is a permanental process if its finite-dimensional Laplace transforms satisfy, for every $\beta_1, \ldots, \beta_n \in \mathbb{R}_+$ and every $x_1, \ldots, x_n \in V$,

$$\mathbb{E}\left[\exp\left(-\frac{1}{2}\sum_{i=1}^{n} \beta_i \psi_{x_i}\right)\right] = \det(I + \beta C)^{-1/\alpha},$$

where I is the $n \times n$-identity matrix, β is the diagonal matrix $\operatorname{diag}(\beta_i)_{1 \le i \le n}$ and $C = (C(x_i, x_j))_{1 \le i,j \le n}$ and α are fixed positive numbers. Such a process is called permanental with kernel C and index α.

Let us now define the building blocks of the permanents called *cyp*, which are sums of cyclic products, as

$$\operatorname{cyp}[C](x_1, \ldots, x_n) := \sum_{\substack{\sigma \in S_{\mathrm{cycl}}([n]) \\ |\sigma| = 1}} \prod_{i=1}^{n} C\big(x_i, x_{\sigma(i)}\big).$$

As in McCullagh and Møller [74], for $\alpha \in \mathbb{R}$, the α*-weighted permanent* (or α*-permanent*) of the matrix C at entries $x_1, \ldots, x_n$ is defined as

$$\operatorname{per}_\alpha[C](x_1, \ldots, x_n) := \sum_{\sigma \in S([n])} \alpha^{|\sigma|} \prod_{i=1}^{n} C\big(x_i, x_{\sigma(i)}\big).$$

In terms of cyps,

$$\operatorname{per}_\alpha[C](x_1, \ldots, x_n) = \sum_{\pi \in \Pi([n])} \alpha^{|\pi|} \prod_{B \in \pi} \operatorname{cyp}[C]\left((x_i)_{i \in B}\right).$$

Note that $\alpha = 1$ gives the usual permanent of the matrix, whereas $\alpha = -1$ yields $\det[-C](x_1, \ldots, x_n)$, or rather, $(-1)^n \det[C](x_1, \ldots, x_n)$. With this definition in mind, it can be shown that ψ is permanental with kernel G and index α if and only if

$$\mathbb{E}\left[\psi_{x_1} \cdots \psi_{x_n}\right] = \operatorname{per}_\alpha[C](x_1, \ldots, x_n).$$

Let us now see the following key result:

Lemma 2.20 (McCullagh and Møller [74, Lem. 1])**.** *For* $(Z(x))_x$ *a Gaussian process with mean* 0 *and covariance matrix* $C/2$*, and* $x_1, \ldots, x_n \in V$ *not necessarily different, the joint cumulants of the squared field are given by*

$$\kappa\left(Z(x_1)^2, \ldots, Z(x_n)^2\right) = \frac{1}{2}\operatorname{cyp}[C](x_1, \ldots, x_n).$$

That is, Z^2 *is a permanental process with kernel* C *and index* $1/2$*.*

In view of this result, we can then express Equation (2.3) in Theorem 2.6 in a more compact way, as

$$\mathbb{E}\left[\prod_{j=1}^{k} \Phi_\varepsilon\left(x_\varepsilon^{(j)}\right)\right] = \frac{1}{2^k} \sum_{\eta\colon [k] \to \mathcal{E}} \operatorname{per}_{(1/2)} M_\eta\left(x_\varepsilon^{(1)}, \ldots, x_\varepsilon^{(k)}\right),$$

where M_η is the submatrix of M on the edges $\left(\{x_\varepsilon^{(j)}, x_\varepsilon^{(j)} + \eta(j)\}\right)_{j\in[k]}$ (see Definition 1.9). Similarly, Equation (2.4) takes the form

$$\lim_{\varepsilon\to 0} \varepsilon^{-dk}\mathbb{E}\left[\prod_{j=1}^{k} \Phi_\varepsilon(x_\varepsilon^{(j)})\right] = \frac{1}{2^k} \sum_{\eta:[k]\to\mathcal{E}} \mathrm{per}_{(1/2)}\overline{M}_\eta(x^{(1)}, \ldots, x^{(k)}),$$

with $\overline{M}_\eta$ the limit of M_η as $\varepsilon \to 0$.

Observe that this does not imply that our field is a permanental process since the sum above is on directions, and the gradients of the dGFF in different directions are not independent of each other.

2.4.2. NON MARKOVIANITY

Although the dGFF does satisfy the spatial Markov property (see Section 1.5), the gradient squared of the dGFF does not. To see this, for simplicity let us forget the dependence on ε and work on some $\Lambda \subset \mathbb{Z}^d$, and take $\Lambda = D \cup D^c$. Let us then see if $\mathbb{E}\left[\Phi(x) \mid \Phi(D^c)\right] = \mathbb{E}\left[\Phi(x) \mid \Phi(\partial D^c)\right]$ for $x \in D$ or not. Here we use the notation $\Phi(D)$ to signify the set $\{\Phi(y), y \in D\}$. We want to show that

$$\mathbb{E}\left[\Phi(x) \mid \Phi(D^c)\right] \neq \mathbb{E}\left[\Phi(x) \mid \Phi(\partial D^c)\right].$$

Let us see a simple counter-example in dimension 1, which can be easily expanded to any d. Take the set $\{0, 1, 2, 3, 4\}$ and define a dGFF Γ there. The 0-boundary conditions imply that $\Gamma(0) = \Gamma(4) = 0$. The field Φ will then be defined on $\Lambda = \{0, 1, 2, 3\}$. As for D, take $D = \{0, 1\}$, so that $D^c = \{2, 3\}$. Instead of Φ, for simplicity and without loss of generality we will work with $|\nabla\Gamma|^2$ without subtracting the mean.

Say that $|\nabla\Gamma(2)|^2 = 1$. If $\Gamma(2) = a$ then $\Gamma(3) = a \pm 1$. If we knew for example that $|\nabla\Gamma(3)|^2 = 0$, then we would know that $\Gamma(3) = 0$, and hence $\Gamma(2) = \pm 1$, which in turn conditions $|\nabla\Gamma(1)|^2$ even further. In general, giving information on $|\nabla\Gamma(3)|^2$ constrains the values of $|\nabla\Gamma(1)|^2$, more so than if we only gave information about $|\nabla\Gamma(2)|^2$. This shows that

$$\mathbb{E}\left[|\nabla\Gamma(1)|^2 \,\middle|\, \{|\nabla\Gamma(2)|^2, |\nabla\Gamma(3)|^2\}\right] \neq \mathbb{E}\left[|\nabla\Gamma(1)|^2 \,\middle|\, \{|\nabla\Gamma(2)|^2\}\right],$$

disproving the Markov property for Φ. The same argument can be extended to any dimension.

2.5. PROOFS

2.5.1. PREVIOUS RESULTS FROM LITERATURE

Let us now expose some important results that we will refer to throughout the proofs. They refer to partially known results and partially consist of straightforward generalizations of previous results.

Our computations will rely on the fact that the distribution of the gradient field $\nabla_i\Gamma$, $i \in [d]$, is well-known. The following result is quoted from Funaki [38, Lem. 3.6].

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Lemma 2.21. *Let $\Lambda \subset \mathbb{Z}^d$ be finite, and let $(\Gamma_x)_{x\in\Lambda}$ be a 0-boundary conditions dGFF on Λ (see Definition 1.17). Then*

$$\begin{cases} \mathbb{E}\left[\nabla_i \Gamma(x)\right] = 0 & \text{if } x \in \Lambda,\ i \in [d], \\ \mathbb{E}\left[\nabla_i \Gamma(x) \nabla_j \Gamma(y)\right] = \nabla_i^{(1)} \nabla_j^{(2)} G_\Lambda(x,y) & \text{if } x, y \in \Lambda,\ i, j \in [d]. \end{cases}$$

Consequently, we can directly link the gradient dGFF to so-called transfer current matrix $M(\cdot,\cdot)$ by

$$M(e,f) = G_\Lambda(e^-, f^-) - G_\Lambda(e^+, f^-) - G_\Lambda(e^-, f^+) + G_\Lambda(e^+, f^+)$$

where e, f are oriented edges of Λ (see Kassel and Wu [61, Sec. 2]). Equivalently we can write

$$M(e,f) = \nabla_e \nabla_f G_\Lambda(e^-, f^-). \tag{2.10}$$

From Lemma 2.21, it is clear that we need to control the behaviour of double derivatives of discrete Green's function in the limit $\varepsilon \to 0$. In order to find the limiting joint moments of the point-wise field $\Phi_\varepsilon(x)$ we will need the following result about the convergence of the discrete difference of the Green's function on U_ε to the double derivative of the continuum Green's function $g_U(\cdot,\cdot)$ on a set U (see Equation (1.17)). This result follows from Theorem 1 of Kassel and Wu [61].

Lemma 2.22 (*Convergence of the Green's function differences*)**.** *Let v, w be points in the set U, with $v \neq w$. Let $\mathcal{E}$ be the set of canonical coordinate vectors. Then for all $a, b \in \mathcal{E}$,*

$$\lim_{\varepsilon\to 0} \varepsilon^{-d}\, \nabla_a^{(1)} \nabla_b^{(2)} G_{U_\varepsilon}\left(\lfloor v/\varepsilon \rfloor, \lfloor w/\varepsilon \rfloor\right) = \partial_a^{(1)} \partial_b^{(2)} g_U(v,w).$$

The next lemma is a generalization of Dürre [33, Lem. 31] for general dimensions $d \geq 2$. The proof is straightforward and will be omitted. It provides an error estimate when replacing the double difference of $G_{U_\varepsilon}(\cdot,\cdot)$ on the finite set by that of $G_0(\cdot,\cdot)$ defined on the whole lattice.

Lemma 2.23. *Let $D \subset U$ be such that the distance between D and U is non-vanishing, that is,* $\operatorname{dist}(D, \partial U) := \inf_{(x,y)\in D\times\partial U} |x-y| > 0$. *There exist $c_D > 0$ and $\varepsilon_D > 0$ such that, for all $\varepsilon \in (0, \varepsilon_D]$, for all $v, w \in D_\varepsilon := D/\varepsilon \cap \mathbb{Z}^d$ and $i, j \in [d]$,*

$$\left| \nabla_i^{(1)} \nabla_j^{(2)} G_{U_\varepsilon}(v,w) - \nabla_i^{(1)} \nabla_j^{(2)} G_0(v,w) \right| \leq c_D\, \varepsilon^d, \tag{2.11}$$

and also

$$\left| \nabla_i^{(1)} \nabla_j^{(2)} G_{U_\varepsilon}(v,w) \right| \leq c_D \cdot \begin{cases} |v-w|^{-d} & \text{if } v \neq w, \\ 1 & \text{if } v = w. \end{cases} \tag{2.12}$$

An immediate consequence of (2.12) and the expression (2.3) in Theorem 2.6 for two points gives us the following bound on the covariance of the field:

Corollary 2.24. *Let D, v and w be as in Lemma 2.23. Then*

$$\mathbb{E}\left[\Phi_\varepsilon(v)\Phi_\varepsilon(w)\right] \leq c_D \cdot \begin{cases} |v-w|^{-2d} & \text{if } v \neq w, \\ 1 & \text{if } v = w. \end{cases}$$

On the other hand, we will also make use of a straightforward extension of Lawler and Limic [67, Cor. 4.4.5] for $d = 2$ and Lawler and Limic [67, Cor. 4.3.3] for $d \geq 3$, yielding the following Lemma.

Lemma 2.25 (Asymptotic expansion of the Green's function differences)**.** *For all* $i, j \in [d]$, *as* $|v| \to +\infty$ *we have*

$$\left|\nabla_i^{(1)}\nabla_j^{(2)} G_0(0,v)\right| = \mathcal{O}\left(|v|^{-d}\right).$$

The following technical combinatorial estimate, which is an immediate extension of a corollary of Dürre [33, Lem. 37], will be important when proving tightness of the family $(\Phi_\varepsilon)_\varepsilon$, in order to bound the rate of growth of the moments of $\langle \Phi_\varepsilon, f\rangle$ for some test function f:

Lemma 2.26. *Let* $D \subset U$ *such that* $\operatorname{dist}(D, \partial U) > 0$ *and* $p \geq 2$. *Then*

$$\sum_{\substack{v_1,\dots,v_p \in D_\varepsilon \\ v_i \neq v_j \text{ for } i\neq j}} \left(\prod_{i=1}^{p-1} \frac{1}{|v_i - v_{i+1}|^d}\right) \frac{1}{|v_p - v_1|^d} = \mathcal{O}_D\left(\varepsilon^{-\frac{p}{2}-d+1}\right),$$

where $D_\varepsilon := D/\varepsilon \cap \mathbb{Z}^d$.

2.5.2. FEYNMAN DIAGRAMS

When calculating expectations of products of Gaussian variables, one often obtains expressions consisting of pairwise combinations of the variables in question. It is then useful to define a graphical representation for these objects, the so-called Feynman diagrams. For a complete exposition on the subject we refer the reader to Janson [54, Ch. 1, 3].

Definition 2.27 (Feynman diagrams, Janson [54, Def. 1.35])**.** A *Feynman diagram* γ of order $n \geq 0$ and rank $r = r(\gamma) \geq 0$ is a graph consisting of a set of n vertices and r edges without common endpoints. These are r disjoint pairs of vertices, each joined by an edge, and $n - 2r$ unpaired vertices. A Feynman diagram is said to be *complete* if $r = n/2$ and *incomplete* if $r < n/2$. Let FD_0 denote the set of all complete Feynman diagrams. A Feynman diagram labeled by n random variables $\xi_1, \dots, \xi_n$ defined on the same probability space is a Feynman diagram of order n with vertices $1, \dots, n$, where ξ_i is thought as being attached to vertex i. The *value* $v(\gamma)$ of such a Feynman diagram γ with edges (i_k, j_k), $k = 1, \dots, r$ and unpaired vertices $\{i : i \in A\}$ is given by

$$v(\gamma) = \prod_{k=1}^{r} \mathbb{E}\left[\xi_{i_k}\xi_{j_k}\right] \prod_{i\in A} \xi_i.$$

Observe that this value is in general a random variable, and it is deterministic whenever the diagram is complete.

This definition allows us to express the expectation of the product of n Gaussian random variables in terms of Feynman diagrams as follows:

Theorem 2.28 (Janson [54, Thm. 1.36]). *Let $\xi_1, \dots, \xi_n$ be centered jointly normal random variables. Then*

$$\mathbb{E}\left[\xi_1 \cdots \xi_n\right] = \sum_{\gamma} \nu(\gamma),$$

where the sum takes place over all $\gamma \in \mathrm{FD}_0$ labeled by $\xi_1, \dots, \xi_n$.

We can also decompose the Wick product of n Gaussian variables in terms of Feynman diagrams, as stated in the following theorem:

Theorem 2.29 (Janson [54, Thm. 3.4]). *Let $\xi_1, \dots, \xi_n$ be centered jointly normal random variables. Then*

$$:\xi_1 \cdots \xi_n: = \sum_{\gamma} (-1)^{r(\gamma)} \nu(\gamma),$$

being $r(\gamma)$ the rank of γ, where the sum takes place over all Feynman diagrams γ labeled by $\xi_1, \dots, \xi_n$.

An extension of Theorem 2.28 now reads:

Theorem 2.30 (Janson [54, Thm. 3.8]). *Let $\xi_1, \dots, \xi_{n+m}$ be centered jointly normal random variables, with $m, n \geq 0$. Then*

$$\mathbb{E}\left[:\xi_1 \cdots \xi_n: \xi_{n+1} \cdots \xi_{n+m}\right] = \sum_{\gamma} \nu(\gamma),$$

where the sum takes place over all complete Feynman diagrams γ labeled by $\xi_1, \dots, \xi_{n+m}$ such that no edge joins any pair ξ_i and ξ_j with $i < j \leq n$.

A formula for an even more general case can be obtained as follows:

Theorem 2.31 (Janson [54, Thm. 3.12]). *Let $Y_i = :\xi_{i1} \cdots \xi_{il_i}:$, where $\{\xi_{ij}\}_{\substack{1 \leq i \leq k \\ 1 \leq j \leq l_i}}$ are centered jointly normal variables, with $k \geq 0$ and $l_1, \dots, l_k \geq 0$. Then*

$$\mathbb{E}\left[Y_1 \cdots Y_k\right] = \sum_{\gamma} \nu(\gamma),$$

where we sum over all complete Feynman diagrams γ labeled by $\{\xi_{ij}\}_{ij}$ such that no edge joins two variables $\xi_{i_1 j_1}$ and $\xi_{i_2 j_2}$ with $i_1 = i_2$.

Remark 2.32. We said this is a formula for an even more general case than Theorem 2.30 because $:X: = X$ for any centered normal variable.

This theorem will be used for the proof of Theorem 2.6. In that case, each Y_i is the Wick product of two variables, namely $Y_i = :\xi_{i1}\xi_{i2}:$, for all $i = 1, \dots, n$. In this specific case it will hold, in fact, that $\xi_{i1} = \xi_{i2}$ for all i, but we keep a different notation for each variable in order to keep track of every possible Feynman diagram that can be made up from the variables Y_i. The value of a complete Feynman diagram γ in this setting will be given by the expression

$$\nu(\gamma) = \prod_{s=1}^{k} \mathbb{E}\left[\xi_{\alpha_s m_{\alpha_s}} \xi_{\beta_s m_{\beta_s}}\right],$$

with $\alpha_s, \beta_s \in [k]$, $\alpha_s \neq \beta_s$ for all s, and $m_{\alpha_s}, m_{\beta_s} \in \{1,2\}$.

Let us discuss a concrete example for the case with $k = 3$. One possibility is $\gamma = (V, E)$ with two copies of nodes per vertex $V = \{x_i, \tilde{x}_i : i = 1,2,3\}$ and the set of undirected edges $E = \{(x_1, x_2), (\tilde{x}_1, x_3), (\tilde{x}_2, \tilde{x}_3)\}$ which pictorially can be depicted in Figure 2.1. We have in total 8 complete Feynman diagrams in this case which can be obtained by considering the different edges resulting from pairings of the nodes $\{x_i, \tilde{x}_i : i = 1,2,3\}$ ignoring all pairings of the sort $(x_i, \tilde{x}_i)$ for all $i = 1,2,3$.

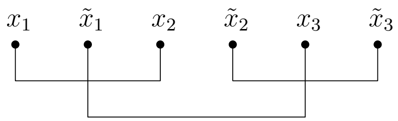

Figure 2.1 – *An example of a possible pairing of edges in a Feynman diagram.*

2.5.3. PROOF OF THEOREM 2.6

The strategy to prove the first theorem is based on decomposing the k-point functions into combinatorial expressions that involve basically covariances of Gaussian random variables. This is made possible by our explicit knowledge of the Gaussian field which underlies Φ_ε. These covariances can be estimated using the transfer matrix M (Equation (2.10)), whose scaling limit is well-known: it is the differential of the Laplacian Green's function (cf. Kassel and Wu [61, Thm. 1]).

In order to compute the k-point function we will first make use of Feynman diagrams techniques exposed in the previous subsection. In particular we will make use of Theorem 2.31.

Proof of Theorem 2.6. Let us compute the function

$$Q_k\left(x_\varepsilon^{(1)}, \ldots, x_\varepsilon^{(k)}\right) := \mathbb{E}\left[\prod_{j=1}^{k} \Phi_\varepsilon\left(x_\varepsilon^{(j)}\right)\right].$$

From Definition 2.1 of $\Phi_\varepsilon\left(x_\varepsilon^{(j)}\right)$ we know that

$$Q_k\left(x_\varepsilon^{(1)}, \ldots, x_\varepsilon^{(k)}\right) = \sum_{i_1,\ldots,i_k \in \mathcal{E}} \mathbb{E}\left[\prod_{j=1}^{k} :(\nabla_{i_j}\Gamma(x_\varepsilon^{(j)}))^2:\right],$$

with $\mathcal{E}$ the canonical basis of $\mathbb{R}^d$. In our case we have k products of the Wick product $:(\nabla_{i_j}\Gamma(x_\varepsilon^{(j)}))^2:$ (indexed by j, not i_j). So we can identify Y_j in Theorem 2.31 with $:(\nabla_{i_j}\Gamma(x_\varepsilon^{(j)}))^2:$ for any $j \in [k]$, being $\xi_{j1} = \xi_{j2} = \nabla_{i_j}\Gamma(x_\varepsilon^{(j)})$.

2

Let us denote $\overline{x_{e_i}^{(j)}} := (x^{(j)}, x^{(j)} + e_i)$, $i \in [d]$, $j \in [k]$ (we drop the dependence on ε to ease notation). Also to make notation lighter we fix the labels i_j for the moment and keep them implicit. We then define $\mathcal{U} := \{\overline{x^{(1)}}, \overline{x^{(1)}}, \ldots, \overline{x^{(k)}}, \overline{x^{(k)}}\}$, where each copy is considered distinguishable. We also define FD_0 as the set of complete Feynman diagrams on $\mathcal{U}$ such that no edge joins $\overline{x^{(i)}}$ with (the other copy of) $\overline{x^{(i)}}$. That is, a typical edge b in a Feynman diagram γ in FD_0 is of the form $(\overline{x^{(j)}}, \overline{x^{(m)}})$, with $j \neq m$ and $j, m \in [k]$. Thus by Definition 2.27 we have

$$\mathbb{E}\left[\prod_{j=1}^{k} :\left(\nabla\Gamma(x_\varepsilon^{(j)})\right)^2:\right] = \sum_{\gamma \in FD_0} \nu(\gamma),$$

which in turn is equal to

$$\sum_{\gamma \in FD_0} \prod_{b \in E_\gamma} \mathbb{E}\left[\nabla_{b^+}\Gamma\left((b^+)^-\right) \nabla_{b^-}\Gamma\left((b^-)^-\right)\right],$$

where E_γ are the edges of γ (note that the edges of γ connect edges of U_ε) and $(b^+)^-$ denotes the tail of the edge b^+ (analogously for b^-). Lemma 2.21 and Equation (2.10) yield

$$\mathbb{E}\left[\prod_{j=1}^{k} :\left(\nabla\Gamma(x_\varepsilon^{(j)})\right)^2:\right] = \sum_{\gamma \in FD_0} \prod_{b \in E_\gamma} M(b^+, b^-).$$

Now we would like to express Feynman diagrams in terms of permutations. We first note that any given $\gamma \in FD_0$ cannot join $\overline{x^{(i)}}$ with itself (neither the same nor the other copy of itself). So instead of considering permutations $\sigma \in \mathrm{Perm}(\mathcal{U})$ we consider permutations $\sigma' \in S_k$, being S_k the group of permutations of the set $[k]$. Any $\gamma \in FD_0$ is a permutation $\sigma \in \mathrm{Perm}(\mathcal{U})$, but given the constraints just mentioned, we can think of them as permutations $\sigma' \in S_k$ without fixed points; that is, $\sigma' \in S_k^0$. Thus

$$\mathbb{E}\left[\prod_{j=1}^{k} :\left(\nabla\Gamma(x_\varepsilon^{(j)})\right)^2:\right] = \sum_{\sigma' \in S_k^0} c(\sigma') \prod_{j=1}^{k} M\left(\overline{x^{(j)}}, \overline{x^{(\sigma'(j))}}\right),$$

with $c(\sigma')$ a constant that takes into account the multiplicity of different permutations σ that give rise to the same σ', depending on its number of subcycles.

Let us disassemble this expression even more. In general σ' can be decomposed in q cycles. Since $\sigma' \in S_k^0$ (in particular, it has no fixed points), there are at most

$\lfloor k/2 \rfloor$ cycles in a given σ'. Hence,

$$\mathbb{E}\left[\prod_{j=1}^{k} :\left(\nabla\Gamma(x_\varepsilon^{(j)})\right)^2:\right] = \sum_{q=1}^{\lfloor k/2 \rfloor} \sum_{\substack{\sigma' \in S_k^0 \\ \sigma' = \sigma_1' \dots \sigma_q'}} c(\sigma') \prod_{h=1}^{q} \prod_{j \in \sigma_h'} M\left(\overline{x^{(j)}}, \overline{x^{(\sigma_h'(j))}}\right),$$

where the notation $j \in \sigma_h'$ means that j belongs to the domain where σ_h' acts (non trivially). As for $c(\sigma')$, given a cycle σ_i', $i \in [q]$, it is straightforward to see that there are $2^{|\sigma_i'|-1}$ different Feynman diagrams in FD_0 that give rise to σ_i', where $|\sigma_i'|$ is the length of the orbit of σ_i'. This comes from the fact that we have two choices for each element in the domain, but swapping them gives back the original Feynman diagram, so we obtain

$$c(\sigma') = \prod_{i \in [q]} 2^{|\sigma_i'|-1}.$$

Now we note that a cyclic decomposition of a permutation of the set $[k]$ determines a partition $\pi \in \Pi([k])$ (although not injectively). This way, a sum over the number of partitions q and $\sigma' \in S_k^0$ with q cycles can be written as a sum over partitions π with no singletons, and a sum over full cycles in each block B (that is, those permutations consisting of only one cycle). Hence

$$\mathbb{E}\left[\prod_{j=1}^{k} :\left(\nabla\Gamma(x_\varepsilon^{(j)})\right)^2:\right] = \sum_{\substack{\pi \in \Pi([k]) \\ \text{w/o singletons}}} \prod_{B \in \pi} \sum_{\sigma \in S_{cycl}(B)} 2^{|B|-1} \prod_{j \in B} M\left(\overline{x^{(j)}}, \overline{x^{(\sigma(j))}}\right),$$

where we also made the switch between $\prod_{B\in\pi}$ and $\sum_{\sigma \in S_{cycl}(B)}$ by grouping by factors. Alternatively, we can express this average in terms of $S^0_{cycl}(B)$, the set of full cycles without fixed points, as

$$\mathbb{E}\left[\prod_{j=1}^{k} :\left(\nabla\Gamma(x_\varepsilon^{(j)})\right)^2:\right] = \sum_{\pi \in \Pi([k])} \prod_{B \in \pi} \sum_{\sigma \in S^0_{cycl}(B)} 2^{|B|-1} \prod_{j \in B} M\left(\overline{x^{(j)}}, \overline{x^{(\sigma(j))}}\right).$$

Finally, we need to put back the subscript i_j in the elements $\overline{x^{(j)}}$ and sum over $i_1, \dots, i_k \in \mathcal{E}$. Note that for any function $f : \mathcal{E}^k \to \mathbb{R}$ we have

$$\sum_{i_1, \dots, i_k \in \mathcal{E}} f(i_1, \dots, i_k) = \sum_{\eta : [k] \to \mathcal{E}} f\left(\eta(1), \dots, \eta(k)\right),$$

so that

$$\mathbb{E}\left[\prod_{j=1}^{k} \Phi_\varepsilon\left(x_\varepsilon^{(j)}\right)\right] = \sum_{\eta : [k] \to \mathcal{E}} \sum_{\pi \in \Pi([k])} \prod_{B \in \pi} 2^{|B|-1} \sum_{\sigma \in S^0_{cycl}(B)} \prod_{j \in B} M\left(\overline{x^{(j)}_{\eta(j)}}, \overline{x^{(\sigma(j))}_{\eta(\sigma(j))}}\right),$$

and grouping the $\eta(j)$'s according to each block $B \in \pi$ we get

$$\mathbb{E}\left[\prod_{j=1}^{k} \Phi_\varepsilon(x_\varepsilon^{(j)})\right] = \sum_{\pi \in \Pi([k])} \prod_{B \in \pi} 2^{|B|-1} \sum_{\sigma \in S^0_{\text{cycl}}(B)} \sum_{\eta: B \to \mathcal{E}} \prod_{j \in B} M\left(\overline{x^{(j)}_{\eta(j)}}, \overline{x^{(\sigma(j))}_{\eta(\sigma(j))}}\right).$$

Regarding the transfer matrix M, using Equation (2.10) we can write the above expression as

$$\mathbb{E}\left[\prod_{j=1}^{k} \Phi_\varepsilon(x_\varepsilon^{(j)})\right] = \sum_{\pi \in \Pi([k])} \prod_{B \in \pi} 2^{|B|-1} \sum_{\sigma \in S^0_{\text{cycl}}(B)} \sum_{\eta: B \to \mathcal{E}} \prod_{j \in B} \nabla^{(1)}_{\eta(j)} \nabla^{(2)}_{\eta(\sigma(j))} G_{U_\varepsilon}\left(x_\varepsilon^{(j)}, x_\varepsilon^{(\sigma(j))}\right),$$

obtaining the first result of the theorem. Finally, using Lemma 2.22 we obtain the second statement. □

2.5.4. PROOF OF COROLLARY 2.9 AND PROPOSITION 2.10

Proof of Corollary 2.9. Recall that Definition 1.8 yields

$$\mathbb{E}\left[\prod_{j=1}^{k} \Phi_\varepsilon(x_\varepsilon^{(j)})\right] = \sum_{\pi \in \Pi([k])} \prod_{B \in \pi} \kappa\left(\Phi_\varepsilon(x_\varepsilon^{(j)}) : j \in [k]\right).$$

From expressions (2.3) and (2.4) in Theorem 2.6 let us see that the equality follows factor by factor by using strong induction. For $k = 1$ it is trivially true since the mean of the field is 0. Now let now us assume that it holds for $n = 1, \dots, k-1$. From (1.18) we have that

$$\kappa\left(\Phi_\varepsilon(x_\varepsilon^{(j)}) : j \in [k]\right) = \mathbb{E}\left[\prod_{j=1}^{k} \Phi_\varepsilon(x_\varepsilon^{(j)})\right] - \sum_{\substack{\pi \in \Pi([k]) \\ |\pi| > 1}} \prod_{B \in \pi} \kappa\left(\Phi_\varepsilon(x_\varepsilon^{(j)}) : j \in B\right).$$

Using again (1.18) on the expectation term and the induction hypothesis, after cancellations we get

$$\kappa\left(\Phi_\varepsilon(x_\varepsilon^{(j)}) : j \in [k]\right) = 2^{k-1} \sum_{\sigma \in S^0_{\text{cycl}}([k])} \sum_{\eta: [k] \to \mathcal{E}} \prod_{j=1}^{k} \nabla^{(1)}_{\eta(j)} \nabla^{(2)}_{\eta(\sigma(j))} G_{U_\varepsilon}\left(x_\varepsilon^{(j)}, x_\varepsilon^{(\sigma(j))}\right).$$

Thus the proof follows by induction. □

The equality (in absolute value) between our cumulants and those of Dürre [32, Thm. 1] allows us to adapt his proof and conclude that, in the case of $d = 2$, our field is conformally covariant with scale dimension 2.

Proof of Proposition 2.10. It is known (Berestycki [11, Prop. 1.9]) that the continuum Green's function $g_U(\cdot,\cdot)$, defined in Equation (1.17), is conformally invariant against a conformal mapping $h : U \to U'$; that is, for any $v \neq w \in U$,

$$g_U(v, w) = g_{U'}(h(v), h(w)) .$$

Recalling expression (2.7) for the limiting cumulants we see that, for any integer $\ell \geq 2$,

$$\lim_{\varepsilon\to 0} \varepsilon^{-2\ell}\kappa\left(\Phi_\varepsilon^U(x_\varepsilon^{(j)}) : j \in [\ell]\right)$$
$$= 2^{\ell-1} \sum_{\sigma\in S^0_{\mathrm{cycl}}([\ell])} \sum_{\eta\colon [\ell]\to\mathcal{E}} \prod_{j=1}^{\ell} \partial^{(1)}_{\eta(j)}\partial^{(2)}_{\eta(\sigma(j))} g_{U'}(h_\varepsilon(x^{(j)}), h_\varepsilon(x^{(\sigma(j))})),$$

where the derivatives on the right hand side act on $g_{U'} \circ (h_\varepsilon, h_\varepsilon)$, not on $g_{U'}$. From the cumulants expression we deduce that, for a given permutation σ and assignment η, each point $x^{(j)}$ will appear exactly twice in the arguments of the product of differences of $g_{U'}$. Thus, using the chain rule and the Cauchy-Riemann equations, for a fixed σ we obtain an overall factor $\prod_{j=1}^{\ell} \left|h'(x^{(j)})\right|^2$ after summing over all η. We then obtain

$$\lim_{\varepsilon\to 0} \varepsilon^{-2\ell}\kappa\left(\Phi_\varepsilon^U(x_\varepsilon^{(j)}) : j \in [\ell]\right)$$
$$= \prod_{j=1}^{\ell}\left|h'(x^{(j)})\right|^2 \lim_{\varepsilon\to 0} \varepsilon^{-2\ell}\kappa\left(\Phi_\varepsilon^{U'}\left(h_\varepsilon(x^{(j)})\right) : j \in [\ell]\right).$$

The result follows plugging this expression into the moments. □

2.5.5. PROOF OF THEOREM 2.11

The proof of this Theorem will be split into two parts. First we will show that the family $(\Phi_\varepsilon)_{\varepsilon>0}$ is tight in some appropriate Besov space and then we will show convergence of finite-dimensional distributions $(\langle\Phi_\varepsilon, f_i\rangle)_{i\in[m]}$ and identify the limit.

Tightness

Proposition 2.33. *Let $U \subset \mathbb{R}^d$, $d \geq 2$. Under the scaling $\varepsilon^{-d/2}$, the family $(\Phi_\varepsilon)_{\varepsilon>0}$ is tight in $\mathcal{B}^{\alpha,\mathrm{loc}}_{p,q}(U)$ for any $\alpha < -d/2$ and $p, q \in [1, \infty]$. The family is also tight in $\mathcal{C}^\alpha_{\mathrm{loc}}(U)$ for every $p, q \in [1, \infty]$ and $\alpha < -d/2$.*

Recall that the local Besov space $\mathcal{B}^{\alpha,\mathrm{loc}}_{p,q}(U)$ was defined in Definition 2.3 and the local Hölder space $\mathcal{C}^{\alpha}_{\mathrm{loc}}(U)$ in Definition 2.5.

Finite-dimensional distributions

2

Proposition 2.34. *Let $U \subset \mathbb{R}^d$ and $d \geq 2$. There exists a normalization constant $\chi > 0$ such that, for any set of functions $\{f_i \in L^2(U) : i \in [m],\ m \in \mathbb{N}\}$, the random elements $\langle \Phi_\varepsilon, f_i \rangle$ converge in the following sense:*

$$\left(\frac{\varepsilon^{-d/2}}{\sqrt{\chi}} \langle \Phi_\varepsilon, f_i \rangle \right)_{i\in[m]} \xrightarrow{d} (\langle W, f_i \rangle)_{i\in[m]}$$

as $\varepsilon \to 0$.

PROOF OF PROPOSITION 2.33

We will use the tightness criterion given in Theorem 2.30 in Furlan and Mourrat [39]. First we need to introduce some notation. Let f and $(g^{(i)})_{1\leq i<2^d}$ be compactly supported test functions of class $\mathcal{C}^r_c(\mathbb{R}^d)$, $r \in \mathbb{N}$. Let $\Lambda_n := \mathbb{Z}^d/2^n$, and let $R > 0$ be such that

$$\operatorname{supp} f \subset B_0(R), \quad \operatorname{supp} g^{(i)} \subset B_0(R), \quad i < 2^d. \tag{2.13}$$

Let $K \subset U$ be compact and $k \in \mathbb{N}$. We say that the pair (K, k) is *adapted* if

$$2^{-k}R < \operatorname{dist}(K, U^c).$$

We say that the set $\mathcal{K}$ is a *spanning sequence* if it can be written as

$$\mathcal{K} = \{(K_n, k_n) : n \in \mathbb{N}\},$$

where (K_n) is an increasing sequence of compact subsets of U such that $\bigcup_n K_n = U$, and for every n the pair (K_n, k_n) is adapted.

Theorem 2.35 (Tightness criterion, Furlan and Mourrat [39, Thm. 2.30]). *Let the functions $f, (g^{(i)})_{1\leq i<2^d}$ in $\mathcal{C}^r_c(\mathbb{R}^d)$ with the support properties mentioned above, and fix $p \in [1,\infty)$ and $\alpha, \beta \in \mathbb{R}$ satisfying $|\alpha|, |\beta| < r, \alpha < \beta$. Let $(\Phi_m)_{m\in\mathbb{N}}$ be a family of random linear functionals on $\mathcal{C}^r_c(U)$, and let $\mathcal{K}$ be a spanning sequence. Assume that for every $(K, k) \in \mathcal{K}$, there exists a constant $c = c(K, k) < \infty$ such that for every $m \in \mathbb{N}$,*

$$\sup_{x\in\Lambda_k\cap K} \mathbb{E}\left[\left| \left\langle \Phi_m, f\left(2^k(\cdot - x)\right) \right\rangle \right|^p \right]^{1/p} \leq c \tag{2.14}$$

and

$$\sup_{x\in\Lambda_n\cap K} 2^{dn}\mathbb{E}\left[\left| \left\langle \Phi_m, g^{(i)}\left(2^n(\cdot - x)\right) \right\rangle \right|^p \right]^{1/p} \leq c\, 2^{-n\beta}, \quad i < 2^d,\ n \geq k. \tag{2.15}$$

Then the family $(\Phi_m)_m$ is tight in $\mathcal{B}^{\alpha,\mathrm{loc}}_{p,q}(U)$ for any $q \in [1,\infty]$. If moreover $\alpha < \beta - d/p$, then the family is also tight in $\mathcal{C}^{\alpha}_{\mathrm{loc}}(U)$.

Proof of Proposition 2.33. We will consider an arbitrary scaling $\varepsilon^\gamma, \gamma \in \mathbb{R}$, and then choose an optimal one to make the fields tight. We define $\widetilde{\Phi}_\varepsilon$ as the scaled version of Φ_ε, that is,

$$\widetilde{\Phi}_\varepsilon(x) := \varepsilon^\gamma \Phi_\varepsilon(x) = \varepsilon^\gamma \sum_{i=1}^{d} :(\nabla_i \Gamma(x))^2:, \quad x \in U_\varepsilon.$$

The family of random linear functionals $(\Phi_m)_{m\in\mathbb{N}}$ in Theorem 2.35 is to be identified with the fields $(\widetilde{\Phi}_\varepsilon)_{\varepsilon>0}$ taking for example ε decreasing to zero along a dyadic sequence. Now let us expand the expressions (2.14) and (2.15) in Theorem 2.35. To simplify notation, let us define $f_{k,x}(\cdot) := f(2^k(\cdot - x))$ for $k \in \mathbb{N}$ and $x \in \mathbb{R}^d$, and analogously for $g^{(i)}$.

In the proof we will set $p \in 2\mathbb{N}$. This will not affect the generality of our results because of the embedding of local Besov spaces described in Lemma 2.4. This means that we can read (2.14) and (2.15) forgetting the absolute value in the left-hand side. Let us rewrite the p-th moment of $\langle \widetilde{\Phi}_\varepsilon, f_{k,x}\rangle$ as

$$\begin{aligned} 0 &\le \mathbb{E}\left[\langle \widetilde{\Phi}_\varepsilon, f_{k,x}\rangle^p\right] \\ &= \varepsilon^{\gamma p}\, \mathbb{E}\left[\int_{U^p} \Phi_\varepsilon(\lfloor x_1/\varepsilon \rfloor) \cdots \Phi_\varepsilon(\lfloor x_p/\varepsilon \rfloor) f_{k,x}(x_1)\cdots f_{k,x}(x_p)\, dx_1 \cdots dx_p\right]. \end{aligned} \tag{2.16}$$

We will seek for a more convenient expression to work with. If we allow ourselves to slightly abuse the notation for $\widetilde{\Phi}_\varepsilon$, then we can express it in a piece-wise continuous fashion as

$$\widetilde{\Phi}_\varepsilon(x) = \varepsilon^\gamma \sum_{y\in U_\varepsilon} \mathbb{1}_{S_1(y)}(x) \sum_{i=1}^{d} :(\nabla_i \Gamma(y))^2: = \varepsilon^\gamma \sum_{y\in U_\varepsilon} \mathbb{1}_{S_1(y)}(x) \Phi_\varepsilon(y),$$

where $S_a(y)$ is the square of side-length a centered at y. Under a change of variables, if we define $U^\varepsilon := U \cap \varepsilon\mathbb{Z}^d$ (mind the superscript and the definition which is different from that of U_ε in Section 2.1) then

$$\widetilde{\Phi}_\varepsilon(x) = \varepsilon^\gamma \sum_{y\in U^\varepsilon} \mathbb{1}_{S_\varepsilon(y/\varepsilon)}(x) \Phi_\varepsilon(y/\varepsilon).$$

This way, expression (2.16) now reads

$$\begin{aligned} &\mathbb{E}\left[\langle \widetilde{\Phi}_\varepsilon, f_{k,x}\rangle^p\right] \\ &\qquad = \varepsilon^{\gamma p}\, \mathbb{E}\left[\sum_{y_1,\ldots,y_p\in U^\varepsilon} \Phi_\varepsilon(y_1/\varepsilon)\cdots\Phi_\varepsilon(y_p/\varepsilon) \prod_{j=1}^{p} \int_{S_\varepsilon(y_j)} f_{k,x}(z)\, dz\right], \end{aligned}$$

Therefore the left-hand side of expression (2.14) from Theorem 2.35 is upper-bounded by

$$\varepsilon^{\gamma} \sup_{x\in\Lambda_k\cap K} \left[\sum_{y_1,\dots,y_p\in U^{\varepsilon}} \mathbb{E}\left[\Phi_{\varepsilon}(y_1/\varepsilon)\cdots\Phi_{\varepsilon}(y_p/\varepsilon)\right] \prod_{j=1}^{p} \int_{S_{\varepsilon}(y_j)} f_{k,x}(z)\,dz \right]^{1/p}. \tag{2.17}$$

Analogously, expression (2.15) from Theorem 2.35 reads

$$\varepsilon^{\gamma}\, 2^{dn} \sup_{x\in\Lambda_n\cap K} \left[\sum_{y_1,\dots,y_p\in U^{\varepsilon}} \mathbb{E}\left[\Phi_{\varepsilon}(y_1/\varepsilon)\cdots\Phi_{\varepsilon}(y_p/\varepsilon)\right] \prod_{j=1}^{p} \int_{S_{\varepsilon}(y_j)} g^{(i)}_{n,x}(z)\,dz \right]^{1/p}. \tag{2.18}$$

Choose $\mathcal{K} = (K_n, n)_{n\in\mathbb{N}}$ with

$$K_n = \left\{x \in \mathbb{R}^d \mid \operatorname{dist}(x, U^c) \geq (2+\delta)R2^{-n}\right\},$$

for some $\delta > 0$ and R such that (2.13) holds. Let us first consider (2.18). Given that $\operatorname{supp} g^{(i)}\left(2^n(\cdot - x)\right) \subset B_x(R2^{-n})$ we can restrict the sum over y_j to the set

$$\Omega_{n,x} = \left\{y \in U^{\varepsilon} \mid d(y,x) < 2^{-n}R + \varepsilon\sqrt{d}/2\right\}.$$

We now bound (2.18) separately for the cases $2^n \geq R\varepsilon^{-1}$ and $2^n < R\varepsilon^{-1}$. If $2^n \geq R\varepsilon^{-1}$, we have

$$\sum_{y_1,\dots,y_p\in U^{\varepsilon}} \mathbb{E}\left[\Phi_{\varepsilon}(y_1/\varepsilon)\cdots\Phi_{\varepsilon}(y_p/\varepsilon)\right] \prod_{j=1}^{p} \int_{S_{\varepsilon}(y_j)} g^{(i)}_{n,x}(z)\,dz \leq$$
$$\leq \sum_{y_1,\dots,y_p\in \Omega_{n,x}} \mathbb{E}\left[\Phi_{\varepsilon}(y_1/\varepsilon)\cdots\Phi_{\varepsilon}(y_p/\varepsilon)\right] \prod_{j=1}^{p} \int_{S_{\varepsilon}(y_j)} g^{(i)}_{n,x}(z)\,dz.$$

The sum over $\Omega_{n,x}$ can be bounded by a sum over a finite amount of points independent of n, since under the condition $2^n \geq R\varepsilon^{-1}$ the set $\Omega_{x,n}$ has at most 3^d points for any x, ε and n. Let us show that the sum of these expectations is uniformly bounded by a constant.

Looking at expression (2.5) we observe the following: Any given partition with no singletons $\pi \in \Pi([p])$ can be expressed as $\pi = \{B_1, \dots, B_\ell\}$, with $1 \leq \ell \leq p$ such that $\sum_{1\leq i\leq \ell} n_i = p$, with $n_i := |B_i|$. Then the cumulant corresponding to any given B_i (see Corollary 2.9) is proportional to a sum over $\sigma \in S^0_{\text{cycl}}(B_i)$ and $\eta : B_i \to \mathcal{E}$ of terms of the form

$$\prod_{j\in B_i} \nabla^{(1)}_{\eta(j)} \nabla^{(2)}_{\eta(\sigma(j))} G_{U_\varepsilon}\left(y_j, y_{\sigma(j)}\right).$$

Using (2.12) we can bound this expression (up to a constant) by

$$\prod_{j\in B_i} \min\left\{\left|y_j - y_{\sigma(j)}\right|^{-d}, 1\right\},$$

where the minimum takes care of the case in which the set $\{y_j : j \in B_i\}$ has repeated values, so that $y_j = y_{\sigma(j)}$ for some $j \in B_i$ and some σ. So we have that

$$\mathbb{E}\big[\Phi_\varepsilon(y_1)\cdots\Phi_\varepsilon(y_p)\big] \lesssim_{K_n} \sum_{\pi\in\Pi([p])}\prod_{B\in\pi} c(|B|)\prod_{j\in B}\min\left\{\left|y_j - y_{\sigma(j)}\right|^{-d}, 1\right\} \tag{2.19}$$

for some constant $c(|B|)$ depending on B that accounts for the sum over $\sigma \in S^0_{\mathrm{cycl}}(B)$ and over $\eta : B \to \mathcal{E}$. Since $|y_i - y_j| \geq 1$ for any $y_i, y_j \in \Omega_{n,x}$ and any n and x, (2.19) is bounded by a constant depending only on p, so that

$$\sum_{y_1,\ldots,y_p\in\Omega_{n,x}} \mathbb{E}\left[\Phi_\varepsilon(y_1)\cdots\Phi_\varepsilon(y_p)\right] \lesssim_{K_n} 3^d p \sum_{\pi\in\Pi([p])}\prod_{B\in\pi} c(|B|)\prod_{j\in B} \lesssim_{K_n} 1,$$

since $|\Omega_{n,x}| \leq 3^d p$ for all n and x.

On the other hand, using the fact that

$$\int_{S_\varepsilon(y_j)} \left|g^{(i)}_{n,x}(z)\right| dz \lesssim 2^{-dn}$$

we obtain

$$\sum_{y_1,\ldots,y_p\in\Omega_{n,x}} \mathbb{E}\big[\Phi_\varepsilon(y_1/\varepsilon)\cdots\Phi_\varepsilon(y_p/\varepsilon)\big]\prod_{j=1}^{p}\int_{S_\varepsilon(y_j)} g^{(i)}_{n,x}(z)\,dz \lesssim_{K_n} 2^{-dpn},$$

which gives the bound

$$\varepsilon^\gamma\, 2^{dn} \sup_{x\in\Lambda_n\cap K}\left[\sum_{y_1,\ldots,y_p\in U^\varepsilon} \mathbb{E}\big[\Phi_\varepsilon(y_1/\varepsilon)\cdots\Phi_\varepsilon(y_p/\varepsilon)\big] \prod_{j=1}^{p}\int_{S_\varepsilon(y_j)} g^{(i)}_{n,x}(z)\,dz\right]^{1/p} \lesssim_{K_n} \varepsilon^\gamma.$$

Observe that Theorem 2.35 allows the constant c to depend on $K = K_n$, so the symbol $\lesssim_{K_n}$ is not an issue. Then, for any $\gamma \leq 0$ we can bound the above expression by a constant multiple of $2^{-\gamma n}$. On the other hand, if $2^n < R\varepsilon^{-1}$, we have

$$\int_{S_\varepsilon(y_j)} \left|g^{(i)}_{n,x}(z)\right| dz \lesssim \varepsilon^d.$$

We also note that

$$\Omega_{n,x} \subset S_{\varepsilon,x} := \left[x - 2R2^{-n}, x + 2R2^{-n}\right]^d \cap \varepsilon\mathbb{Z}^d.$$

Using this and calling $N := \lfloor 2R2^{-n}\varepsilon^{-1}\rfloor$, we obtain

$$\sum_{y_1,\ldots,y_p\in\Omega_{n,x}} \mathbb{E}\big[\Phi_\varepsilon(y_1/\varepsilon)\cdots\Phi_\varepsilon(y_p/\varepsilon)\big] \leq \sum_{y_1,\ldots,y_p\in[\![-N,N]\!]^d} \mathbb{E}\big[\Phi_\varepsilon(y_1)\cdots\Phi_\varepsilon(y_p)\big].$$

2

Let us first study the behaviour of this expression for $p = 2$. By Corollary 2.24 we get

$$\begin{aligned}\sum_{y_1,y_2\in[\![-N,N]\!]^d} &\mathbb{E}\left[\Phi_\varepsilon(y_1)\Phi_\varepsilon(y_2)\right] \\ &\lesssim_{K_n} \sum_{\substack{y_1,y_2\in[\![-N,N]\!]^d\\ y_1=y_2}} 1 + \sum_{\substack{y_1,y_2\in[\![-N,N]\!]^d\\ y_1\neq y_2}} \frac{1}{|y_1-y_2|^{2d}} \\ &\lesssim N^d + \sum_{y_1\in[\![-N,N]\!]^d} \int_1^{2\sqrt{2}N} \frac{r^{d-1}}{r^{2d}}\mathrm{d}r \\ &= N^d + \sum_{y_1\in[\![-N,N]\!]^d} \frac{1}{d}\left(1-2^{-3d/2}N^{-d}\right) \lesssim N^d.\end{aligned}$$

Let us now analyze $\mathbb{E}\left[\Phi_\varepsilon(y_1)\cdots\Phi_\varepsilon(y_p)\right]$ for an arbitrary p. In the same spirit as the case $2^n \geq R\varepsilon^{-1}$, by expression (2.19) we know that

$$\begin{aligned}\sum_{y_1,\ldots,y_p\in[\![-N,N]\!]^d} &\mathbb{E}\left[\Phi_\varepsilon(y_1)\cdots\Phi_\varepsilon(y_p)\right] \lesssim_{K_n} \\ &\sum_{\pi\in\Pi([p])}\prod_{B\in\pi} c(|B|) \sum_{y_1,\ldots,y_p\in[\![-N,N]\!]^d}\prod_{j\in B}\min\left\{\left|y_j-y_{\sigma(j)}\right|^{-d},1\right\}.\end{aligned}$$

Using Lemma 2.26 we get

$$\sum_{y_1,\ldots,y_p\in[\![-N,N]\!]^d}\prod_{j\in B}\min\left\{\left|y_j-y_{\sigma(j)}\right|^{-d},1\right\} \lesssim_{K_n} N^{\frac{n_i}{2}+d-1}$$

by identifying ε with $1/N$. So we arrive to

$$\sum_{y_1,\ldots,y_p\in[\![-N,N]\!]^d} \mathbb{E}\left[\Phi_\varepsilon(y_1)\cdots\Phi_\varepsilon(y_p)\right] \lesssim_{K_n} \sum_{\pi\in\Pi([p])}\prod_{B\in\pi} c(|B|)N^{\frac{n_i}{2}+d-1}.$$

Now we use that

$$\prod_{B\in\pi} N^{\frac{n_i}{2}+d-1} = N^{(d-1)|\pi|+\frac{p}{2}},$$

and since the sum takes place over partitions of the set $[p]$ with no singletons, putting everything back into (2.5) we see that the term with the largest value of $|\pi|$ will dominate for large N. For p even this happens when π is composed of cycles of two elements, in which case $|\pi| = p/2$. Hence,

$$\sum_{y_1,\ldots,y_p\in[\![-N,N]\!]^d} \mathbb{E}\left[\Phi_\varepsilon(y_1)\cdots\Phi_\varepsilon(y_p)\right] \lesssim_{K_n} N^{\frac{dp}{2}}$$

for p even. Finally,

$$\varepsilon^{\gamma}\, 2^{dn} \sup_{x\in\Lambda_n\cap K_n} \left[\sum_{y_1,\dots,y_p\in U^{\varepsilon}} \mathbb{E}\left[\Phi_\varepsilon(y_1/\varepsilon)\cdots\Phi_\varepsilon(y_p/\varepsilon)\right] \prod_{j=1}^{p}\int_{S_\varepsilon(y_j)} g^{(i)}_{n,x}(z)\,dz \right]^{1/p} \lesssim_{K_n} 2^{\frac{dn}{2}}\varepsilon^{\gamma+\frac{d}{2}}.$$

If $\gamma \geq -d/2$ then we can bound the above expression by a constant multiple of $2^{\frac{dn}{2}}2^{-(\gamma+\frac{d}{2})n} = 2^{-\gamma n}$. Otherwise, we cannot bound it uniformly in ε, as the bound depends increasingly on ε as it approaches 0.

Now we need to obtain similar bounds for (2.14), which applied to our case takes the expression given in (2.17). For the case $2^n \geq R\varepsilon^{-1}$ we have

$$\varepsilon^{\gamma} \sup_{x\in\Lambda_n\cap K_n} \left[\sum_{y_1,\dots,y_p\in U^{\varepsilon}} \mathbb{E}\left[\Phi_\varepsilon(y_1/\varepsilon)\cdots\Phi_\varepsilon(y_p/\varepsilon)\right] \prod_{j=1}^{p}\int_{S_\varepsilon(y_j)} f_{k,x}(z)\,dz \right]^{1/p} \lesssim \varepsilon^{\gamma}\, 2^{-dn} < \varepsilon^{\gamma+d},$$

which is bounded by some $c = c(K_n, n)$ whenever $\gamma \geq -d$. If $2^n < R\varepsilon^{-1}$ instead we get

$$\varepsilon^{\gamma} \sup_{x\in\Lambda_n\cap K_n} \left[\sum_{y_1,\dots,y_p\in U^{\varepsilon}} \mathbb{E}\left[\Phi_\varepsilon(y_1/\varepsilon)\cdots\Phi_\varepsilon(y_p/\varepsilon)\right] \prod_{j=1}^{p}\int_{S_\varepsilon(y_j)} f_{k,x}(z)\,dz \right]^{1/p} \lesssim_{K_n} \varepsilon^{\gamma+\frac{d}{2}}\, 2^{-\frac{dn}{2}}.$$

As before, only if $\gamma \geq -d/2$ we have the required bound.

Theorem 2.35 now implies that under scaling $\varepsilon^{-d/2}$ the family $(\Phi_\varepsilon)_{\varepsilon>0}$ is tight in $\mathcal{B}^{\alpha,\mathrm{loc}}_{p,q}(U)$ for any $\alpha < -d/2$, any $q \in [1,\infty]$ and any $p \geq 2$ and even. Using Lemma 2.4 this holds for any $p \in [1,\infty]$. This way, the family is also tight in $\mathcal{C}^{\alpha}_{\mathrm{loc}}(U)$ for every $\alpha < -d/2$. □

Remark 2.36. Observe that the scaling ε^{-d} (the one used for the joint moments in Theorem 2.6) is outside the range of γ required for the tightness bounds, and therefore it will give a trivial scaling.

PROOF OF PROPOSITION 2.34

The proof of this proposition will be divided into three parts. Firstly, we will determine the normalizing constant χ and show that it is well-defined, in the sense that it is a strictly positive finite constant. Secondly, recalling Definition 1.8, we will demonstrate that the n-th cumulant $\kappa_n\left(\langle\Phi_\varepsilon, f\rangle\right)$ of each random variable

$\langle \Phi_\varepsilon, f\rangle$, $f \in L^2(U)$, vanishes for $n \geq 3$. Finally we show that the second cumulant $\kappa_2(\langle \Phi_\varepsilon, f\rangle, \langle \Phi_\varepsilon, g\rangle)$, $g \in L^2(U)$, which is equal to the covariance, converges to the appropriate one corresponding to that of white noise. Once we have this, we can show that any collection $(\langle \Phi_\varepsilon, f_1\rangle, \ldots, \langle \Phi_\varepsilon, f_k\rangle)$, $k \in \mathbb{N}$, is a Gaussian vector. To see this it suffices to take any linear combination $f = \sum_{i\in[k]} \alpha_i \langle \Phi_\varepsilon, f_i\rangle$, $\alpha_i \in \mathbb{R}$ for all $i \in [k]$ so that, by multilinearity, all the cumulants $\kappa_n(\langle \Phi_\varepsilon, f\rangle)$ converge to those of a centered normal with variance $\int_U f(x)^2 dx$. The ideas are partially inspired from Dürre [33, Sec. 3.6].

For the rest of this subsection we will work with test functions $f \in \mathcal{C}_c^\infty(U)$. The lifting of the results to every $f \in L^2(U)$ follows by a standard density argument (Janson [54, Ch. 1, Sec. 3]). Let us first derive a convenient representation of the action $\langle \Phi_\varepsilon, f\rangle$ defined in Equation (2.1). More precisely, defining $\langle \Phi_\varepsilon, f\rangle_S$ as

$$\langle \Phi_\varepsilon, f\rangle_S := \sum_{v\in U_\varepsilon} f(\varepsilon v)\Phi_\varepsilon(v),$$

for any test function $f \in \mathcal{C}_c^\infty(U)$ we can write

$$\langle \Phi_\varepsilon, f\rangle = \varepsilon^d \langle \Phi_\varepsilon, f\rangle_S + R_\varepsilon(f),$$

where $R_\varepsilon(f)$ denotes the reminder term that goes to 0 in L^2, as we show in the next lemma.

Lemma 2.37. *Let $U \subset \mathbb{R}^d$, $d \geq 2$. For any test function $f \in \mathcal{C}_c^\infty(U)$ as $\varepsilon \to 0$ it holds that*

$$|R_\varepsilon(f)| \xrightarrow{L^2} 0. \tag{2.20}$$

Proof. Observe that

$$\langle \Phi_\varepsilon, f\rangle = \int_U \Phi_\varepsilon(\lfloor x/\varepsilon \rfloor) f(x)dx = \sum_{x\in U_\varepsilon} \Phi_\varepsilon(x) \int_{A_x} f(y)dy,$$

where $A_x := \{a \in U : \lfloor a/\varepsilon \rfloor = x\}$. It is easy to see that $|A_x| \leq \varepsilon^d$, and given that the support of f is compact and strictly contained in U, for ε sufficiently small (depending on f), the distance between this support and the boundary ∂U will be larger than $\sqrt{d}\varepsilon$. So there is no loss of generality if we assume that $|A_x| = \varepsilon^d$.

Now, we can rewrite (2.20) as

$$\left|\sum_{x\in U_\varepsilon} \Phi_\varepsilon(x) \left(\int_{A_x} f(y)dy - \varepsilon^d f(\varepsilon x)\right)\right| = \left|\sum_{x\in U_\varepsilon} \varepsilon^d \Phi_\varepsilon(x) \left(\frac{1}{|A_x|}\int_{A_x} f(y)dy - f(\varepsilon x)\right)\right|. \tag{2.21}$$

Let us call $\mathcal{I}(x)$ the term

$$\mathcal{I}(x) := \frac{1}{|A_x|}\int_{A_x} f(y)dy - f(\varepsilon x).$$

The set A_x is not a Euclidean ball, but it has bounded eccentricity (see Stein and Shakarchi [94, Cor. 1.7]). Therefore we can apply the Lebesgue differentiation theorem to claim that $\mathcal{I}(x)$ will be of order $o(1)$, where the rate of convergence possibly depends on x and f.

To see statement (2.20), we square the expression in (2.21) and take its expectation, obtaining

$$\mathbb{E}\left[\left|\sum_{x\in U_\varepsilon} \varepsilon^d\,\Phi_\varepsilon(x)\,\mathcal{I}(x)\right|^2\right] \leq \varepsilon^{2d}\,\mathbb{E}\left[\sum_{x\in U_\varepsilon} \Phi_\varepsilon^2(x)\right]\left(\sum_{x\in U_\varepsilon} \mathcal{I}^2(x)\right) \tag{2.22}$$

where we used the Cauchy-Schwarz inequality. By Corollary 2.24 the expectation on the right-hand side can be bounded as

$$\mathbb{E}\left[\sum_{x\in U_\varepsilon} \Phi_\varepsilon^2(x)\right] \lesssim \sum_{x\in U_\varepsilon} 1 = \mathcal{O}\left(\varepsilon^{-d}\right)$$

while the second term in (2.22) is of order $o(\varepsilon^{-d})$. With the outer factor ε^{2d} (2.22) goes to 0, as we wanted to show. □

Let us remark that, by the previous lemma, proving finite-dimensional convergence of $\left\{\frac{\varepsilon^{-d/2}}{\sqrt{\chi}}\langle\Phi_\varepsilon, f_p\rangle : p\in[m]\right\}$ will be equivalent to proving finite-dimensional convergence of $\left\{\frac{\varepsilon^{d/2}}{\sqrt{\chi}}\langle\Phi_\varepsilon, f_p\rangle_S : p\in[m]\right\}$.

DEFINITION OF χ

Lemma 2.38. *Let $G_0(\cdot,\cdot)$ be the Green's function on $\mathbb{Z}^d$ defined in Section 2.1.1. The constant*

$$\chi := 2\sum_{v\in\mathbb{Z}^d}\sum_{i,j\in[d]}\left(\nabla_i^{(1)}\nabla_j^{(2)}G_0(0,v)\right)^2$$

is well-defined. In particular $\chi\in(8,+\infty)$.

Proof. Let us define κ_0 as

$$\kappa_0(v,w) := 2\sum_{i,j\in[d]}\left(\nabla_i^{(1)}\nabla_j^{(2)}G_0(v,w)\right)^2. \tag{2.23}$$

By translation invariance we notice that $\kappa_0(v,w) = \kappa_0(0,w-v)$. Moreover, using Lemma 2.25, we have that as $|v|\to+\infty$

$$\kappa_0(0,v)\lesssim |v|^{-2d}$$

so that we can bound χ from above by

$$\chi = \sum_{v\in\mathbb{Z}^d} \kappa_0(0,v) \lesssim 1 + \sum_{v\in\mathbb{Z}^d\setminus\{0\}} |v|^{-2d} < +\infty.$$

For the lower bound, since $\kappa_0(0,v) \geq 0$ for all $v \in \mathbb{Z}^d$ we can simply take $v = 0$. Choosing the directions of differentiation $i = j = 1$ in (2.23) we get the term $2\big(\nabla_1^{(1)}\nabla_1^{(2)}G_0(0,0)\big)^2$, which can be expressed as $8\big(G_0(0,0) - G_0(e_1,0)\big)^2$ using translation and rotation invariance of G_0. Now, by definition

$$\Delta G_0(0,0) = \frac{1}{2d}\sum_{x\in\mathbb{Z}^d:|x|=1} \big(G_0(x,0) - G_0(0,0)\big) = -1,$$

from which $G_0(0,0) - G_0(e_1,0) = 1$. This implies that $\chi \geq 8$, and the lemma follows. □

VANISHING CUMULANTS κ_n FOR $n \geq 3$

Lemma 2.39. *For $n \geq 3$, $f \in \mathcal{C}_c^\infty(U)$, the cumulants $\kappa_n\big(\varepsilon^{d/2}\langle\Phi_\varepsilon, f\rangle_S\big)$ go to 0 as $\varepsilon \to 0$.*

Proof. Recall that, by the multilinearity of cumulants, for $n \geq 2$ the n-th cumulant satisfies

$$\kappa_n\left(\varepsilon^{d/2}\langle\Phi_\varepsilon, f\rangle_S\right) = \varepsilon^{\frac{nd}{2}} \sum_{v_1,\dots,v_n\in D_\varepsilon} \kappa\big(\Phi_\varepsilon(v_i) : i \in [n]\big)\prod_{j=1}^n f(\varepsilon v_j), \tag{2.24}$$

with $D := \operatorname{supp} f$, which is compact inside U. The goal now is to show that

$$\varepsilon^{\frac{nd}{2}} \sum_{v_1,\dots,v_n\in D_\varepsilon} \big|\kappa\big(\Phi_\varepsilon(v_i) : i \in [n]\big)\big| \longrightarrow 0$$

as $\varepsilon \to 0$.

First, we note from the cumulants expression (2.6) and bound (2.12) in Lemma 2.23 that, for any set V of (possibly repeated) points of D_ε, with $|V| = n$, we have

$$\big|\kappa\big(\Phi_\varepsilon(v) : v \in V\big)\big| \lesssim_{D,n} \sum_{\sigma\in S^0_{\mathrm{cycl}}(V)} \prod_{v\in V} \min\left\{|v-\sigma(v)|^{-d}, 1\right\}.$$

Using the above expression and Lemma 2.26, it is immediate to see that, if V has m distinct points with $1 \leq m \leq n$,

$$\sum_{\substack{v_1,\dots,v_n\in D_\varepsilon\\ m\text{ distinct points}}} \big|\kappa\big(\Phi_\varepsilon(v_i) : i \in [n]\big)\big| = \mathcal{O}_{D,n}\left(\varepsilon^{-\frac{m}{2}-d+1}\right) \lesssim \mathcal{O}_{D,n}\left(\varepsilon^{-\frac{n}{2}-d+1}\right)$$

so that

$$\varepsilon^{\frac{nd}{2}} \sum_{v_1,\dots,v_n\in D_\varepsilon} \big|\kappa\big(\Phi_\varepsilon(v_i) : i \in [n]\big)\big| = \mathcal{O}_{D,n}\left(\varepsilon^{\frac{1}{2}(d-1)(n-2)}\right).$$

We observe in particular that for $d \geq 2$ this expression goes to 0 for any $n \geq 3$. Furthermore, going back to (2.24), since f is uniformly bounded this shows that for $n \geq 3$ the cumulants κ_n go to 0 as $\varepsilon \to 0$, as we wanted to show. □

COVARIANCE STRUCTURE κ_2

Lemma 2.40. *For any two functions $f_p, f_q \in \mathcal{C}_c^\infty(U)$, with $p, q \in [m]$ for $m \in \mathbb{N}$, we have*

$$\varepsilon^d \, \kappa\big(\langle \Phi_\varepsilon, f_p\rangle_S, \langle \Phi_\varepsilon, f_q\rangle_S\big) \xrightarrow{\varepsilon \to 0} \chi \int_U f_p(x) f_q(x) dx.$$

Proof. Without loss of generality we define the compact set $D \subset U$ as the intersection of the supports of f_p and f_q. Then

$$\varepsilon^d \, \kappa\big(\langle \Phi_\varepsilon, f_p\rangle_S, \langle \Phi_\varepsilon, f_q\rangle_S\big) = \varepsilon^d \sum_{v,w \in D_\varepsilon} f_p(\varepsilon v) f_q(\varepsilon w) \, \kappa\big(\Phi_\varepsilon(v), \Phi_\varepsilon(w)\big). \tag{2.25}$$

From Theorem 2.6, we know the exact expression of $\kappa\big(\Phi_\varepsilon(v), \Phi_\varepsilon(w)\big)$, given by

$$\kappa\big(\Phi_\varepsilon(v), \Phi_\varepsilon(w)\big) = 2 \sum_{i,j \in [d]} \Big(\nabla_i^{(1)} \nabla_j^{(2)} G_{U_\varepsilon}(v,w)\Big)^2. \tag{2.26}$$

Recall the constant $\kappa_0(v,w)$, defined in (2.23). We will approximate the covariance $\kappa\big(\Phi_\varepsilon(v), \Phi_\varepsilon(w)\big)$ by $\kappa_0(v,w)$ and then plug it in (2.25). In other words, we will approximate $G_{U_\varepsilon}(\cdot,\cdot)$ by $G_0(\cdot,\cdot)$. First we split Equation (2.25) into two parts:

$$\begin{aligned}\varepsilon^d \, \kappa\big(\langle \Phi_\varepsilon, f_p\rangle_S, \langle \Phi_\varepsilon, f_q\rangle_S\big) = \varepsilon^d & \sum_{\substack{v,w \in D_\varepsilon \\ |v-w| \leq 1/\sqrt{\varepsilon}}} f_p(\varepsilon v) f_q(\varepsilon w) \, \kappa\big(\Phi_\varepsilon(v), \Phi_\varepsilon(w)\big) \\ & + \varepsilon^d \sum_{\substack{v,w \in D_\varepsilon \\ |v-w| > 1/\sqrt{\varepsilon}}} f_p(\varepsilon v) f_q(\varepsilon w) \, \kappa\big(\Phi_\varepsilon(v), \Phi_\varepsilon(w)\big).\end{aligned} \tag{2.27}$$

The second term above can be easily disregarded. Remember that the cumulant for two random variables equals their covariance, so using Corollary 2.24 we get

$$\begin{aligned}\varepsilon^d & \sum_{\substack{v,w \in D_\varepsilon \\ |v-w| > 1/\sqrt{\varepsilon}}} f_p(\varepsilon v) f_q(\varepsilon w) \, \kappa\big(\Phi_\varepsilon(v), \Phi_\varepsilon(w)\big) \\ & \lesssim \varepsilon^d \sum_{\substack{v,w \in D_\varepsilon \\ |v-w| > 1/\sqrt{\varepsilon}}} |v-w|^{-2d} \lesssim \sum_{\substack{z \in \mathbb{Z}^d \\ |z| > 1/\sqrt{\varepsilon}}} |z|^{-2d},\end{aligned}$$

which goes to 0 as $\varepsilon \to 0$. For the first sum in (2.27), let us compute the error we are committing when replacing G_{U_ε} by G_0. We notice that

$$\max_{i,j \in [d]} \sup_{v,w \in D_\varepsilon} \sup_{\varepsilon \in (0,\varepsilon_D]} \Big|\nabla_i^{(1)} \nabla_j^{(2)} G_{U_\varepsilon}(v,w)\Big| \leq c_D$$

justified by (2.12) in Lemma 2.23, combined with

$$\max_{i,j\in[d]}\sup_{v,w\in\mathbb{Z}^d}\left|\nabla_i^{(1)}\nabla_j^{(2)}G_0(v,w)\right|\le c$$

for some $c>0$, which is a consequence of Lemma 2.25. Recalling that $|a^2-b^2|=|a-b||a+b|$ for any real numbers a,b, and setting

$$a:=\nabla_i^{(1)}\nabla_j^{(2)}G_{U_\varepsilon}(v,w)$$

and

$$b:=\nabla_i^{(1)}\nabla_j^{(2)}G_0(v,w),$$

together with (2.11) from Lemma 2.23, we obtain

$$\sum_{i,j\in[d]}\left(\nabla_i^{(1)}\nabla_j^{(2)}G_{U_\varepsilon}(v,w)\right)^2=\sum_{i,j\in[d]}\left(\nabla_i^{(1)}\nabla_j^{(2)}G_0(v,w)\right)^2+\mathcal{O}\left(\varepsilon^d\right).$$

We can use this approximation in the first summand in (2.27) and obtain

$$\begin{aligned}\varepsilon^d\sum_{\substack{v,w\in D_\varepsilon\\|v-w|\le 1/\sqrt{\varepsilon}}}f_p(\varepsilon v)f_q(\varepsilon w)\,\kappa\big(\Phi_\varepsilon(v),\Phi_\varepsilon(w)\big)\\=\varepsilon^d\sum_{\substack{v,w\in D_\varepsilon\\|v-w|\le 1/\sqrt{\varepsilon}}}f_p(\varepsilon v)f_q(\varepsilon w)\,\kappa_0(v,w)+\mathcal{O}\left(\varepsilon^{d/2}\right),\end{aligned}\tag{2.28}$$

since $\left|\left\{v,w\in D_\varepsilon:\,|v-w|<1/\sqrt{\varepsilon}\right\}\right|=\mathcal{O}\left(\varepsilon^{-\frac{3}{2}d}\right)$. Now, given that both f_p and f_q are in $\mathcal{C}_c^\infty(U)$, they are also Lipschitz continuous. Hence

$$\varepsilon^d\sum_{\substack{v,w\in D_\varepsilon\\|v-w|\le 1/\sqrt{\varepsilon}}}|f_q(\varepsilon v)-f_q(\varepsilon w)||\kappa_0(v,w)|\lesssim\varepsilon^d\sum_{\substack{v,w\in D_\varepsilon\\1\le|v-w|\le 1/\sqrt{\varepsilon}}}\frac{\sqrt{\varepsilon}}{|v-w|^{2d}}=o(1),$$

so that we can replace, up to a negligible error, $f_q(\varepsilon w)$ by $f_q(\varepsilon v)$ in (2.28), getting

$$\varepsilon^d\,\kappa\big(\langle\Phi_\varepsilon,f_p\rangle_S,\langle\Phi_\varepsilon,f_q\rangle_S\big)=\varepsilon^d\sum_{\substack{v,w\in D_\varepsilon\\|v-w|\le 1/\sqrt{\varepsilon}}}f_p(\varepsilon v)f_q(\varepsilon v)\,\kappa_0(v,w)+o(1).$$

Finally the translation invariance of κ_0 implies

$$\lim_{\varepsilon\to 0}\varepsilon^d\,\kappa\big(\langle\Phi_\varepsilon,f_p\rangle_S,\langle\Phi_\varepsilon,f_q\rangle_S\big)=\sum_{v\in\mathbb{Z}^d}\kappa_0(0,v)\int_U f_p(x)f_q(x)dx$$

as claimed. □

3

Fermionic Structure in the Abelian Sandpile Model

And AC said, "LET THERE BE LIGHT!"
And there was light.

—Isaac Asimov, *The Last Question*

In *this chapter we rigorously construct a finite volume representation for the height-one field of the Abelian sandpile model and the degree field of the uniform spanning tree in terms of the fermionic Gaussian free field. This representation can be seen as the lattice representation of a free symplectic fermion field. It allows us to compute cumulants of those fields, both in finite volume and in the scaling limit, including determining the explicit normalizing constants for fields in the corresponding logarithmic field theory. Furthermore, our results point towards universality of the height-one and degree fields, as we prove that the scaling limits of the cumulants agree (up to constants) in the square and triangular lattice. We also recover the equivalent scaling limits for the hypercubic lattice in higher dimensions, and discuss how to adapt the proofs of our results to general graphs.*

Parts of this chapter can be found in https://arxiv.org/abs/2309.08349. Awaiting peer review.

We begin the chapter with the main results in Section 3.1 for the lattice $\mathbb{Z}^d$. Section 3.2 is devoted to the proofs of the main theorems on this lattice, while Section 3.3 presents the proofs of our main results on the triangular lattice and a discussion on how to generalize the results to other lattices.

Let us now introduce the main object of study in this chapter, for which we remind the reader of the definitions of dGFF (Section 1.5) and the Grassmannian variables and the corresponding fGFF (Section 1.6).

Definition 3.1 (Gradient squared of the generators)**.** We define the "gradient squared" $\mathbf{X} = (X_v)_{v\in V}$ of the generators as

$$X_v := \frac{1}{2d}\sum_{i=1}^{2d} \nabla_{e_i}\psi(v)\nabla_{e_i}\overline{\psi}(v), \quad v \in \Lambda. \tag{3.1}$$

This "gradient squared" will be evaluated under the fGFF state. We will also need auxiliary Grassmannian observables $\mathbf{Y} = (Y_v)_{v\in V}$ defined as

$$Y_v := \prod_{i=1}^{2d} \left(1 - \nabla_{e_i}\psi(v)\nabla_{e_i}\overline{\psi}(v)\right), \quad v \in \Lambda. \tag{3.2}$$

The reader may have noticed that X_v, Y_v may not be compatible with the state $\langle\cdot\rangle^{\mathbf{0}}_{\Lambda}$ for points close to the boundary. However, our setup will allow us to work only "well inside" Λ so that we can safely work with X_v and Y_v under the Dirichlet state. In fact, Lemma 3.14 is key in this regard, showing that the fermionic Gaussian free fields defined with pinned and Dirichlet boundary conditions respectively agree on observables that only depend on Λ.

3.1. RESULTS IN $\mathbb{Z}^d$

In this section we will state our main results. The first theorem provides a representation of the expectation of the height-one field of the Abelian sandpile model in terms of the fields **X** and **Y**.

Theorem 3.2. *For $V \subseteq \Lambda^{in}$ a good set as in Definition 1.14, and* **X**, **Y** *defined in Equations (3.1) and (3.2), we have*

$$\mathbb{E}\left[\prod_{v\in V} h_\Lambda(v)\right] = \left\langle \prod_{v\in V} X_v Y_v \right\rangle^{\mathbf{p}}_{\Lambda}. \tag{3.3}$$

In the next theorem we derive a closed-form expression for the joint cumulants of the field **X**, together with their continuum scaling limit. As a reminder or notation, we let U be a connected, bounded subset of $\mathbb{R}^d$ with smooth boundary, and define $U_\varepsilon := U/\varepsilon \cap \mathbb{Z}^d$ for $\varepsilon > 0$. For any $v \in U$, let v_ε be the discrete approximation of v in U_ε; that is, $v_\varepsilon := \lfloor v/\varepsilon \rfloor$. Define also G_ε as the discrete harmonic Green's function on U_ε with 0-boundary conditions outside U_ε, and g_U the continuum harmonic Green's function on U with 0-boundary conditions outside U (recall Section 1.2). We write X^ε_v, $\mathbf{X}^\varepsilon$ and Y^ε_v, $\mathbf{Y}^\varepsilon$, with $v \in U$, to emphasize that the domain is now on U_ε, so that $X^\varepsilon_v = X_{v_\varepsilon}$, and analogously with the other variables. Cyclic permutations without fixed points of a finite set A are denoted as $S_{\text{cycl}}(A)$.

Theorem 3.3.

1. *Let $n \geq 1$ and let the set of distinct points $V := \{v_1, \dots, v_n\} \subseteq \Lambda^{\text{in}}$ be a good set. Let η denote a map from V to $E(V)$ such that $\eta(v) \in E_v$ for all $v \in V$. The joint cumulants of the field $(-X_v)_{v\in V}$ are given by*

$$\kappa^{\mathbf{0}}_\Lambda(-X_v : v \in V) = -\frac{1}{(2d)^n} \sum_{\sigma\in S_{\text{cycl}}(V)} \sum_{\eta: V\to\{e_1,\dots,e_{2d}\}} \prod_{v\in V} \nabla^{(1)}_{\eta(v)} \nabla^{(2)}_{\eta(\sigma(v))} G_\Lambda(v, \sigma(v)). \quad (3.4)$$

2. *Let $n \geq 2$ and $V := \{v_1, \dots, v_n\} \subseteq U$ be such that* $\text{dist}(V, \partial U) > 0$. *If $v_i \neq v_j$ for all $i \neq j$, then*

$$\tilde\kappa_1(v_1, \dots, v_n) := \lim_{\varepsilon\to 0} \varepsilon^{-dn} \kappa^{\mathbf{0}}_{U_\varepsilon}(-X^\varepsilon_v : v \in V)$$
$$= -\frac{1}{d^n} \sum_{\sigma\in S_{\text{cycl}}(V)} \sum_{\eta: V\to\{e_1,\dots,e_d\}} \prod_{v\in V} \partial^{(1)}_{\eta(v)} \partial^{(2)}_{\eta(\sigma(v))} g_U(v, \sigma(v)). \quad (3.5)$$

Corollary 3.4. *Let $C_2 := 2/\pi - 4/\pi^2$. Under the assumptions of Theorem 3.3 item 2 we have that*

$$\lim_{\varepsilon\to 0} \varepsilon^{-2n} \kappa\big(-C_2 X^\varepsilon_v : v \in V\big) = \lim_{\varepsilon\to 0} \varepsilon^{-2n} \kappa\big(h_{U_\varepsilon}(v_\varepsilon) : v \in V\big).$$

The expression for the limiting cumulants of the height-one field can be found in Dürre [32, Thm. 2]. The proof of the statement follows from comparing this expression to (3.5) and recalling that cumulants are homogeneous of degree n.

Another corollary of Theorem 3.3 is that the cumulants of the degree field of a uniform spanning tree are identical to the cumulants of the field $\mathbf{X}$ with respect to the fermionic GFF state.

Corollary 3.5. *Let $V \subseteq \Lambda^{\text{in}}$. Let the average degree field $(\mathcal{X}_v)_{v\in\Lambda}$ in a uniform spanning tree be defined as*

$$\mathcal{X}_v := \frac{1}{2d} \sum_{i=1}^{2d} \mathbb{1}_{\{(v, v+e_i)\in T\}} = \frac{1}{2d} \deg_T(v),$$

where T has the law $\mathbf{P}$ of the uniform spanning tree on Λ (with wired boundary conditions).

1. *Under the assumptions of Theorem 3.3 item 1, we have that*

$$\kappa\big(\mathcal{X}_v : v \in V\big) = \kappa^{\mathbf{0}}_{\Lambda}\left(X_v : v \in V\right) \tag{3.6}$$

and therefore

$$\kappa\big(\mathcal{X}_v : v \in V\big) = -\left(\frac{-1}{2d}\right)^n \sum_{\sigma \in S_{\text{cycl}}(V)} \sum_{\eta : V \to \{e_1, \ldots, e_{2d}\}} \prod_{v \in V} \nabla^{(1)}_{\eta(v)} \nabla^{(2)}_{\eta(\sigma(v))} G_{\Lambda}\left(v, \sigma(v)\right).$$

2. *Under the assumptions of Theorem 3.3 item 2, we have that*

$$\lim_{\varepsilon \to 0} \varepsilon^{-dn} \kappa\big(\mathcal{X}^{\varepsilon}_v : v \in V\big) = -\left(\frac{-1}{d}\right)^n \sum_{\sigma \in S_{\text{cycl}}(V)} \sum_{\eta : V \to \{e_1, \ldots, e_d\}} \prod_{v \in V} \partial^{(1)}_{\eta(v)} \partial^{(2)}_{\eta(\sigma(v))} g_U\left(v, \sigma(v)\right).$$

Let us now turn our attention to the cumulants of another field, namely **XY**. We remind the reader of Definition 1.9 of the transfer-current matrix, a key ingredient in many expressions we obtain in our theorems. We refer the reader to Subsection 3.2.1 for a definition of the connected permutations $S_{\text{co}}(A)$ of a finite set A.

Theorem 3.6 (Cumulants of **XY** on a graph). *For $n \geq 1$ let the set of distinct points $V := \{v_1, \ldots, v_n\} \subseteq \Lambda^{\text{in}}$ be a good set. For a set of edges $\mathcal{E} \subseteq E(V)$ and $v \in V$ denote $\mathcal{E}_v := \{f \in \mathcal{E} : f^- = v\} \subseteq E_v$. The n-th joint cumulants of the field $(X_v Y_v)_{v \in V}$ are given by*

$$\kappa^{\mathbf{0}}_{\Lambda}\left(X_v Y_v : v \in V\right) = \left(\frac{-1}{2d}\right)^n \sum_{\mathcal{E} \subseteq E(V) : |\mathcal{E}_v| \geq 1 \ \forall v} K(\mathcal{E}) \sum_{\tau \in S_{\text{co}}(\mathcal{E})} \text{sign}(\tau) \prod_{f \in \mathcal{E}} M\left(f, \tau(f)\right)$$

where $K(\mathcal{E}) := \prod_{v \in V} K(\mathcal{E}_v)$ and $K(\mathcal{E}_v) := (-1)^{|\mathcal{E}_v|} |\mathcal{E}_v|$.

Before we proceed to the next theorem, we remind the reader of the notation $\det(A)_I$, introduced before Theorem 1.24, to denote the determinant of the matrix A with its rows and columns restricted to the indexes in I.

Theorem 3.7 (Scaling limit of the cumulants of **XY**). *For any $n \geq 2$, let the set $V := \{v_1, \ldots, v_n\} \subseteq U$ be such that $\text{dist}(V, \partial U) > 0$. If $v_i \neq v_j$ for all $i \neq j$, then*

$$\tilde{\kappa}_2(v_1, \ldots, v_n) := \lim_{\varepsilon \to 0} \varepsilon^{-dn} \kappa^{\mathbf{0}}_{\Lambda}\left(X^{\varepsilon}_v Y^{\varepsilon}_v : v \in V\right) = -(C_d)^n \sum_{\sigma \in S_{\text{cycl}}(V)} \sum_{\eta : V \to \{e_1, \ldots, e_d\}} \prod_{v \in V} \partial^{(1)}_{\eta(v)} \partial^{(2)}_{\eta(\sigma(v))} g_U\left(v, \sigma(v)\right), \tag{3.7}$$

with the constant C_d *given by*

$$C_d = \frac{1}{d} \sum_{\mathcal{E}_o \subseteq E_o : \mathcal{E}_o \ni e_1} (-1)^{|\mathcal{E}_o|} |\mathcal{E}_o| \left[\det\left(\overline{M}\right)_{\mathcal{E}_o \setminus \{e_1\}} + \mathbb{1}_{\{\mathcal{E}_o \ni -e_1\}} \det(\overline{M}')_{\mathcal{E}_o \setminus \{e_1\}} \right], \tag{3.8}$$

where for any $f, g \in E_o$

$$\overline{M}(f, g) = \nabla^{(1)}_{\eta^*(f)} \nabla^{(2)}_{\eta^*(g)} G_0(f^-, g^-)$$

and

$$\overline{M}'(f, g) = \begin{cases} \overline{M}(e_1, g) & \text{if } f = -e_1, \\ \overline{M}(f, g) & \text{if } f \neq -e_1. \end{cases} \tag{3.9}$$

Remark 3.8. As expected in virtue of Theorem 3.2 and Corollary 3.4, for $d = 2$ one obtains

$$C_2 = \frac{2}{\pi} - \frac{4}{\pi^2} = \pi \, \mathbb{P}(\rho(o) = 1)$$

by using the known values of the potential kernel (see Spitzer [93, p. 149]).

Remark 3.9. In $d = 2$ the joint moments of **XY** are also conformally covariant with scale dimension 2, just like the height-one field in Dürre [32] and the squared of the gradient (bosonic) GFF in Chapter 2.

The following result establishes the limiting cumulants of the height-one field of the ASM on the hypercubic lattice in any dimension.

Corollary 3.10 (Height-one field limiting cumulants in $d \geq 2$)**.** *Under the assumptions of Theorem 3.3 item 2, for any* $d \geq 2$ *we have that*

$$\lim_{\varepsilon \to 0} \varepsilon^{-dn} \kappa\left(h_{U_\varepsilon}(v) : v \in V\right) = -(C_d)^n \sum_{\sigma \in S_{\text{cycl}}(V)} \sum_{\eta : V \to \{e_1, \ldots, e_d\}} \prod_{v \in V} \partial^{(1)}_{\eta(v)} \partial^{(2)}_{\eta(\sigma(v))} g_U\left(v, \sigma(v)\right),$$

with C_d *as in* (3.8).

Finally, we can view our field as a distribution acting on smooth test functions in the following sense. For any functions $f_1, \ldots, f_n \in \mathcal{C}_c^\infty(U)$, we define

$$\widehat{\kappa}_1\left(f_1, \ldots, f_n\right) := \int_{U^n} \kappa^{\mathbf{0}}_\Lambda\left(-X^\varepsilon_{x_1}, \ldots, -X^\varepsilon_{x_n}\right) \prod_{i \in [n]} f_i(x_i)\, dx_i,$$

respectively

$$\widehat{\kappa}_2\left(f_1, \ldots, f_n\right) := \int_{U^n} \kappa^{\mathbf{0}}_\Lambda\left(X^\varepsilon_{x_1} Y^\varepsilon_{x_1}, \ldots, X^\varepsilon_{x_n} Y^\varepsilon_{x_n}\right) \prod_{i \in [n]} f_i(x_i)\, dx_i.$$

Theorem 3.11. *For any functions $f_1, \ldots, f_n \in C_c^\infty(U)$ with pairwise disjoint supports one has*

$$\lim_{\varepsilon \to 0} \varepsilon^{-dn} \widehat{\kappa}_1 (f_1, \ldots, f_n) = \int_{U^n} \tilde{\kappa}_1(x_1, \ldots, x_n) \prod_{i \in [n]} f_i(x_i) \, dx_i, \tag{3.10}$$

with $\tilde{\kappa}_1$ defined as the limit in (3.5). An analogous result is obtained exchanging $\widehat{\kappa}_1$ and $\tilde{\kappa}_1$ by $\widehat{\kappa}_2$ and $\tilde{\kappa}_2$ respectively in (3.10).

3

Remark 3.12. We highlight the importance of the non-overlapping supports of the test functions, which allows the joint cumulants to have a non-trivial limit when scaled by ε^{-d} for each function. If instead we remove the assumption of disjoint supports, and we consider $\mathbf{X}^\varepsilon$ respectively $\mathbf{X}^\varepsilon \mathbf{Y}^\varepsilon$ as distributions acting on test functions, we need to scale the field by $\varepsilon^{-d/2}$, in which case we can show that

$$\lim_{\varepsilon \to 0} \varepsilon^{-d} \widehat{\kappa}_1 (f_1, f_2) = \langle f_1, f_2 \rangle_{L^2(U)},$$

and that

$$\lim_{\varepsilon \to 0} \varepsilon^{-\frac{dn}{2}} \widehat{\kappa}_1 (f_1, \ldots, f_n) = 0, \quad n \geq 3,$$

for any $f_1, \ldots, f_n \in C_c^\infty(U)$, and similarly for $\widehat{\kappa}_2$.

A first result in this direction was found in Kassel and Wu [61, Thm. 5], where the authors proved convergence of the law of so-called "pattern fields" for the UST. These can be seen as the random variable obtained by applying the height-one field (or a similar local observable of the UST) to a single constant test function $f_1 \equiv 1$.

However, it is possible to extend their convergence to the full setting mentioned above by following the strategy of Theorem 2.11, where the authors proved convergence in distribution in an appropriate Besov space. This proof relies on bounding the joint cumulants of the field, and since we know the decay of the double gradients of the Green's function (see Equation (3.46)) the proof can be carried through by performing exactly the same steps.

3.1.1. QUASI DETERMINANTAL STRUCTURE

In view of Section 2.4, we also observe that the joint moments of the fields $\mathbf{X}$ and $\mathbf{XY}$ are sums of permanents with index -1; that is, determinants. More specifically, expression (3.4) can be written in terms of the joint moments as

$$\left\langle \prod_{v \in V} (-X_v) \right\rangle^{\mathbf{0}} = \frac{1}{(2d)^{|V|}} \sum_{\eta : V \to \{e_1, \ldots, e_{2d}\}} \det\big(M_\eta (v_1, \ldots, v_n)\big),$$

(observe the minus sign disappears thanks to the introduction of the determinant) and (3.5) takes the form

$$\lim_{\varepsilon \to 0} \varepsilon^{-d|V|} \left\langle \prod_{v \in V} (-X_v^\varepsilon) \right\rangle_\Lambda^{\mathbf{0}} = \frac{1}{d^{|V|}} \sum_{\eta : V \to \{e_1, \ldots, e_d\}} \det\big(\overline{M}_\eta (v_1, \ldots, v_n)\big).$$

Equivalently, for **XY** expression (3.7) reads

$$\lim_{\varepsilon \to 0} \varepsilon^{-d|V|} \left\langle \prod_{v \in V} X_v^{\varepsilon} Y_v^{\varepsilon} \right\rangle_{\Lambda}^{\mathbf{0}} = (C_d)^{|V|} \sum_{\eta: V \to \{e_1, \ldots, e_d\}} \det\left(\overline{M}_{\eta}\left(v_1, \ldots, v_n\right)\right).$$

In fact, if in Chapter 2 we had also taken the conjugate counterpart of the bosonic field $\overline{\Gamma}$ and instead of squares we had taken products like $\nabla_i \Gamma \nabla_j \overline{\Gamma}$, by using the Wick theorem for complex bosons in Caracciolo, Sokal and Sportiello [18] we would obtain a field whose joint moments are sums of permanents with index $\alpha = 1$; that is, the canonical permanent. This suggests a parallelism between the square of the gradient of both the (complex) bosonic and fermionic dGFFs. In one case, its k-point function is given by sums of permanents, while in the second case it is a sum of determinants.

3

3.2. PROOFS OF RESULTS IN $\mathbb{Z}^d$

Before going to the proofs of the main theorems we will give some additional definitions and auxiliary results.

3.2.1. PERMUTATIONS, GRAPHS AND PARTITIONS

In this subsection, we introduce more notation stated in Theorems 3.3 and 3.6.

General definitions. Let Λ be a finite and connected (in the usual graph sense) subset of $\mathbb{Z}^d$ and $V \subseteq \Lambda$ be a good set according to Definition 1.14. As V is good, notice that every edge in $E(V)$ is connected to exactly one vertex in V.

Recall that for any finite set A we denote the set of permutations of A by $S(A)$. Furthermore, we write $S_{\text{cycl}}(A)$ to denote the set of *cyclic* permutations of A (without fixed points). Finally recall the set $\Pi(A)$ of partitions of A.

We will use the following natural partial order between partitions (Peccati and Taqqu [82, Ch. 2]): Given two partitions $\pi, \tilde{\pi}$, we say that $\pi \le \tilde{\pi}$ if for every $B \in \pi$ there exists a $\tilde{B} \in \tilde{\pi}$ such that $B \subseteq \tilde{B}$. If $\pi \le \tilde{\pi}$, we say that π is finer than $\tilde{\pi}$ (or that $\tilde{\pi}$ is coarser than π). If $\sigma \in S(V)$, we denote as $\pi_\sigma \in \Pi(V)$ the partition given by the disjoint cycles of σ. This is the finest partition such that $\sigma(B) = B$ for all $B \in \pi_\sigma$.

Fix $A \subseteq E(V)$ such that $E_v \cap A \neq \emptyset$ for all $v \in V$, i.e. we have a set of edges with at least one edge per vertex of V. Let $\tau \in S(A)$ be a permutation of edges in A.

Permutations: connected and bare. We define the multigraph $V_\tau = (V, E_\tau(V))$ induced by τ in the following way. For each pair of vertices $v \neq w$ in V, we add one edge between v and w for each $f \in E_v, f' \in E_w$ such that either $\tau(f) = f'$ or $\tau(f') = f$. If $v = w$, we add no edge, so $\deg_{V_\tau}(v) \le |E_v|$.

Definition 3.13 (Connected and bare permutations)**.** Let $\Lambda \subseteq \mathbb{Z}^d$ finite, V good as in Definition 1.14, $|V| \ge 2$, $A \subseteq E(V)$ and $\tau \in S(A)$ be given.

- We say that τ is *connected* if the multigraph V_τ is a connected multigraph.
- We say that τ is *bare* if it is connected and $\deg_{V_\tau}(v) = 2$ for all $v \in V$ (it is immediate to see that the latter condition can be replaced by $|E_\tau(V)| = |V|$).

If $|V| = 1$, as it can happen in Theorem 3.6, we consider every permutation $\tau \in S(A)$ as both connected and bare.

We will denote by $S_{co}(A)$ the set of connected permutations in $S(A)$, and by $S_{bare}(A)$ the set of bare permutations. See Figures 3.1 and 3.2 for some examples, where the mapping $\tau(f) = f'$ is represented via an arrow $f \to f'$.

For τ bare we have, by definition, that for each v there are exactly two edges $f, f' \in A$ (possibly the same) such that $\tau(f') \notin E_v$ and $\tau^{-1}(f) \notin E_v$. We will refer to this as τ enters v through f and exits v through f'. Therefore, for any bare permutation $\tau \in S_{bare}(A)$, we can define an induced permutation on vertices $\sigma = \sigma_\tau \in S_{cycl}(V)$ given by $\sigma(v) = w$ if there there exists (a unique) $f \in E_v$ and $f' \in E_w$ such that $\tau(f) = f'$. Figure 3.3 shows an example in $d = 2$.

Any permutation of $\tau \in S(A)$ induces a partition π_τ on A given by the disjoint cycles in τ. Likewise, the partition $\pi_\tau^V \in \Pi(V)$ given by the connected components of V_τ gives the finest partition on V such that $\tau\left(\cup_{v \in B} A_v\right) = \cup_{v \in B} A_v$ for all $B \in \pi_\tau^V$. If τ is connected, then $\pi_\tau^V = V$.

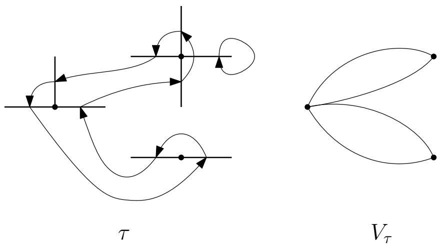

Figure 3.1 – *A connected permutation τ on edges and the multigraph V_τ associated to it, in $d = 2$. Notice that this permutation is* not *bare.*

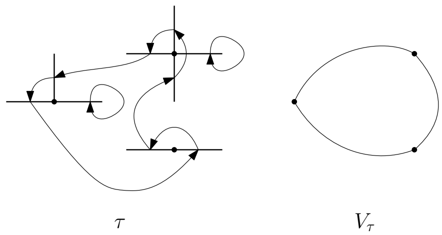

Figure 3.2 – *A bare permutation τ on edges and the multigraph V_τ associated to it, in $d = 2$.*

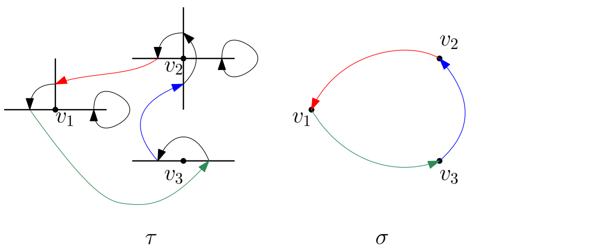

Figure 3.3 – *A bare permutation τ on edges and the induced permutation σ on points, in $d = 2$.*

3.2.2. AUXILIARY RESULTS

We start this section with a result that states $[\cdot]^{\mathbf{0}}_\Lambda$ and $[\cdot]^{\mathbf{p}}_\Lambda$ coincide over observables with support inside of Λ.

Lemma 3.14. *For all* $F \in \Omega^{2\Lambda}$ *we have that*

$$[F]^{\mathbf{0}}_\Lambda = [F]^{\mathbf{p}}_\Lambda .$$

Proof. For clarity we will distinguish between the inner product $\langle\cdot,\cdot\rangle_\Lambda$ and $\langle\cdot,\cdot\rangle_{\Lambda^g}$ with a subscript. We start by rewriting the exponent $\langle\psi, -\Delta^g\overline{\psi}\rangle_{\Lambda^g}$ and isolating the terms with either ψ_g or $\overline{\psi}_g$.

$$\langle\psi,-\Delta^g\overline{\psi}\rangle_{\Lambda^g} = \langle\psi,-\Delta_\Lambda\overline{\psi}\rangle_\Lambda \\ - \sum_{u\in\Lambda}\Delta^g(u,g)\left(\psi_g\overline{\psi}_u+\psi_u\overline{\psi}_g\right) + \underbrace{\left(1+\sum_{u\in\Lambda}\Delta^g(u,g)\right)}_{=:c_g}\psi_g\overline{\psi}_g.$$

Notice that each of the elements of the sum above commutes with all other elements. We can then compute $\exp\left(\langle\psi,-\Delta^g\overline{\psi}\rangle_{\Lambda^g}\right)$ with Definition 1.22 as

$$\begin{aligned} e^{\langle\psi,-\Delta^g\overline{\psi}\rangle_{\Lambda^g}} &= \sum_{k=0}^{\infty}\frac{1}{k!}\left(\langle\psi,-\Delta_\Lambda\overline{\psi}\rangle_\Lambda - \sum_{u\in\Lambda}\Delta^g(u,g)\left(\psi_g\overline{\psi}_u+\psi_u\overline{\psi}_g\right)+c_g\psi_g\overline{\psi}_g\right)^k \\ &= \sum_{k=0}^{\infty}\frac{1}{k!}\sum_{\substack{k_1,k_2,k_3\geq 0,\\ k_1+k_2+k_3=k}}\frac{k!}{k_1!k_2!k_3!}\left(\langle\psi,-\Delta_\Lambda\overline{\psi}\rangle_\Lambda\right)^{k_1} \\ &\qquad \left(-\sum_{u\in\Lambda}\Delta^g(u,g)\left(\psi_g\overline{\psi}_u+\psi_u\overline{\psi}_g\right)\right)^{k_2}\left(c_g\psi_g\overline{\psi}_g\right)^{k_3}. \end{aligned}$$

Multiplying this sum by $\psi_g\overline{\psi}_g$ and using (1.21), we obtain

$$\psi_g\overline{\psi}_g e^{\langle\psi,-\Delta^g\overline{\psi}\rangle_{\Lambda^g}} = \psi_g\overline{\psi}_g\sum_{k=0}^{\infty}\frac{1}{k!}\left(\langle\psi,-\Delta_\Lambda\overline{\psi}\rangle_\Lambda\right)^k = \psi_g\overline{\psi}_g e^{\langle\psi,-\Delta_\Lambda\overline{\psi}\rangle_\Lambda}.$$

Now, provided that $F\in\Omega^{2\Lambda}$, and in particular that $\psi_g,\overline{\psi}_g$ do not appear in F, we have that

$$[F]^{\mathbf{p}}_\Lambda = \left(\prod_{v\in\Lambda^g}\partial_{\overline{\psi}_v}\partial_{\psi_v}\right)\psi_g\overline{\psi}_g e^{\langle\psi,-\Delta_\Lambda\overline{\psi}\rangle_\Lambda}F = \left(\prod_{v\in\Lambda}\partial_{\overline{\psi}_v}\partial_{\psi_v}\right)e^{\langle\psi,-\Delta_\Lambda\overline{\psi}\rangle_\Lambda}F,$$

which is equal to $[F]^{\mathbf{0}}_\Lambda$. □

The previous lemma yields a corollary which will be of great importance for our work. The proof is a simple consequence of the previous lemma and the fact that, if $V \subseteq \Lambda^{\mathrm{in}}$, then for all $v \in V$ we have that $\nabla_{e_i}\psi(v), \nabla_{e_i}\overline{\psi}(v) \in \Omega^{2\Lambda}$.

Corollary 3.15. *Take* $V = \{v_1, \ldots, v_k\} \subseteq \Lambda^{\mathrm{in}}$ *and let* $\eta(\cdot), \eta'(\cdot)$ *be any two maps* $V \to \{e_1, \ldots, e_{2d}\}$*. Then*

$$\left[\prod_{j=1}^{k} \nabla_{\eta(v_j)}\psi(v_j)\nabla_{\eta'(v_j)}\overline{\psi}(v_j)\right]_{\Lambda}^{\mathbf{p}} = \left[\prod_{j=1}^{k} \nabla_{\eta(v_j)}\psi(v_j)\nabla_{\eta'(v_j)}\overline{\psi}(v_j)\right]_{\Lambda}^{\mathbf{0}}.$$

Recalling that we denote the cumulants with respect to $\langle\cdot\rangle_\Lambda^{\mathbf{p}}$ *as* $\kappa_\Lambda^{\mathbf{p}}(\cdot)$ *and similarly for* $\kappa_\Lambda^{\mathbf{0}}(\cdot)$*, with the same notation above we have*

$$\kappa_\Lambda^{\mathbf{p}}\left(\nabla_{\eta(v_j)}\psi(v_j)\nabla_{\eta'(v_j)}\overline{\psi}(v_j) : j = 1, \ldots, k\right)$$
$$= \kappa_\Lambda^{\mathbf{0}}\left(\nabla_{\eta(v_j)}\psi(v_j)\nabla_{\eta'(v_j)}\overline{\psi}(v_j) : j = 1, \ldots, k\right).$$

Remark 3.16. At least in the classical sense, we cannot talk about a Markov property in fermionic systems since it requires conditioning on certain random variables, which strictly speaking we do not have, being only able to calculate expected values. However, we can think of Lemma 3.14 as another type of Markov property for the fermionic GFF. To be more precise, for $A \subset \Lambda$ and $F \in \Omega^{2\Lambda}$ define

$$[F]_\Lambda^A := \left(\prod_{v\in\Lambda} \partial_{\overline{\psi}_v}\partial_{\psi_v}\right)\prod_{a\in A}\psi_a\overline{\psi}_a \exp\left(\langle\psi, -\Delta_\Lambda\overline{\psi}\rangle_\Lambda + \sum_{a\in A}\langle\delta_a, \psi\overline{\psi}\rangle_\Lambda\right)F,$$

and for $F \in \Omega^{2(\Lambda\setminus A)}$,

$$[F]_{\Lambda\setminus A}^{\mathbf{0}} := \left(\prod_{v\in\Lambda\setminus A} \partial_{\overline{\psi}_v}\partial_{\psi_v}\right)\exp\left(\langle\psi, -\Delta_{\Lambda\setminus A}\overline{\psi}\rangle_{\Lambda\setminus A}\right)F,$$

where for $u, v \in \Lambda\setminus A$

$$\Delta_{\Lambda\setminus A}(u,v) := \begin{cases} -|\{w \in \Lambda\setminus A : w \sim u\}| & \text{if } u = v, \\ 1 & \text{if } u \sim v, \\ 0 & \text{otherwise.} \end{cases}$$

Then the *Markov property for the fermionic GFF* says that, for any $F \in \Omega^{2(\Lambda\setminus A)}$,

$$[F]_\Lambda^A = [F]_{\Lambda\setminus A}^{\mathbf{0}}.$$

The proof follows in much the same way as that of Lemma 3.14.

3.2.3. PROOF OF THEOREM 3.2

The proof of Theorem 3.2 will consist of three steps. In the first step (Proposition 3.17), we will relate the probability of having a certain set of edges in a uniform spanning tree with the fGFF state evaluated on fermionic Grassmannian observables. The statement can be found in Bauerschmidt *et al.* [8, Cor. B.3] and we will give its full proof to stress the normalization factor. In the second step we demonstrate that these states can be written as a determinant of a matrix containing double gradients of the Green's function (Lemma 3.18) and finally, to obtain (3.3) we note that the height-one field can be expressed as the average of certain fermionic observables in the state $\langle\cdot\rangle_\Lambda^{\mathbf{0}}$.

3

For a set of oriented edges S, we abbreviate ζ_S as

$$\zeta_S := \prod_{f\in S}(\psi_{f^+}-\psi_{f^-})(\overline{\psi}_{f^+}-\overline{\psi}_{f^-}) = \prod_{f\in S}\nabla_f\psi(f^-)\nabla_f\overline{\psi}(f^-),$$

where ∇_f is an abuse of notation for $\nabla_{\eta^*(f)}$, being η^* as in (1.19). Note that $\zeta_f = \zeta_{-f}$ for any oriented edge f. Therefore, we can consider ζ as defined on *unoriented* edges altogether. For the same reason we can also write $\nabla_f\psi(f^-)\nabla_f\overline{\psi}(f^-)$ without pinning down the exact orientation of f.

Proposition 3.17. *Let $\mathcal{G} = (\Lambda, E)$ be a finite graph. For all subsets of edges $S \subseteq E$*

$$\mathbf{P}(T: S\subseteq T) = \langle\zeta_S\rangle_\Lambda^{\mathbf{p}}. \tag{3.11}$$

The following lemma will be useful to prove (3.11).

Lemma 3.18. *Let $S = \{f_1,\ldots,f_k\} \subseteq E$ be edges such that all their endpoints belong to Λ^{in}. Then*

$$\left[\prod_{j=1}^{k}\nabla_{f_j}\psi(f_j^-)\nabla_{f_j}\overline{\psi}(f_j^-)\right]_\Lambda^{\mathbf{p}} = [\zeta_S]_\Lambda^{\mathbf{p}} = \det(-\Delta_\Lambda)\det(M)_S. \tag{3.12}$$

A simple consequence of (3.12) is

$$\langle\zeta_S\rangle_\Lambda^{\mathbf{p}} = \det(M)_S.$$

Proof of Lemma 3.18. Firstly, notice that, due to Corollary 3.15, we can substitute $[\cdot]_\Lambda^{\mathbf{p}}$ by $[\cdot]_\Lambda^{\mathbf{0}}$ in the left-hand side of (3.12). It is straightforward to show that ζ_S is commuting, so that in fact

$$\begin{aligned}[\zeta_S]_\Lambda^{\mathbf{0}} &= \left(\prod_{i=1}^{n}\partial_{\overline{\psi}_i}\partial_{\psi_i}\right)\exp\left(\langle\psi,-\Delta_\Lambda\overline{\psi}\rangle\right)\zeta_S\\ &= \left(\prod_{i=1}^{n}\partial_{\overline{\psi}_i}\partial_{\psi_i}\right)\left(\prod_{i=1}^{k}\nabla_{f_i}\psi(f_i^-)\nabla_{f_i}\overline{\psi}(f_i^-)\right)\exp\left(\langle\psi,-\Delta_\Lambda\overline{\psi}\rangle\right).\end{aligned} \tag{3.13}$$

Next, observe that

$$\left(\nabla_{f_i}\psi(f_i^-) : i = 1, \ldots, k\right) = \boldsymbol{\psi}^T \widetilde{C}, \quad \left(\nabla_{f_i}\overline{\psi}(f_i^-) : i = 1, \ldots, k\right) = \widetilde{B}\overline{\boldsymbol{\psi}},$$

where $\widetilde{B} = \widetilde{C}^T$ and $\widetilde{C}$ is a $|\Lambda| \times k$ matrix such that the column corresponding to the i-th point is given by

$$\widetilde{C}(\cdot, i) = (0, \ldots, 0, -1, 0, \ldots, 0, 1, 0, \ldots, 0)^T,$$

with the -1 (resp. 1) located at the f_i^--th position (resp. that of point f_i^+). Therefore,

$$(3.13) = \left(\prod_{i=1}^{n} \partial_{\overline{\psi}_i}\partial_{\psi_i}\right)\left(\prod_{j=1}^{k}\left(\boldsymbol{\psi}^T\widetilde{C}\right)_j\left(\widetilde{B}\overline{\boldsymbol{\psi}}\right)_j\right)\exp\left(\langle\boldsymbol{\psi}, -\Delta_\Lambda\overline{\boldsymbol{\psi}}\rangle\right).$$

The matrix $-\Delta_\Lambda$ has inverse given by the Green's function $G_\Lambda(\cdot, \cdot)$. The lemma now follows from item 2 of Theorem 1.24 and the simple computation

$$\left(\widetilde{B}(G_\Lambda)\widetilde{C}\right)(f, g) = M(f, g)$$

for $f, g \in S$. □

Proof of Proposition 3.17. The proof follows from the previous Lemma 3.18 and the matrix-tree theorem (Theorem 1.11), which is this case takes the form

$$\mathbf{P}(T : S \subseteq T) = \det(M)_S. \qquad \square$$

Once we have established the previous connections between uniform spanning trees and our fermionic variables, we turn our attention to a connection between the height-one field and the uniform spanning tree.

Lemma 3.19. *Let $\mathcal{G} = (\Lambda, E)$ be a graph and $V \subseteq \Lambda^{in}$ a good set (see Definition 1.14). Let $\eta : V \to E(V)$ be a function such that $\eta(v)$ assigns to every $v \in V$ an edge incident to v. Let $\eta(V)$ be its image. Then the height-one field of the Abelian sandpile model satisfies*

$$\mathbb{E}\left[\prod_{v \in V} h_\Lambda(v)\right] = \mathbf{P}\left(e \notin T \;\forall\, e \in E(V) \setminus \eta(V)\right).$$

Proof. Recall that the burning bijection maps every recurrent configuration of the sandpile model to a spanning tree of the graph. The height-one configuration on the good set V can be considered as a special case of a minimal sandpile configuration (see Járai and Werning [56, Def. 1]). By Járai and Werning [56, Lem. 4] there is a burning sequence in the burning algorithm that burns all vertices in $\Lambda \setminus V$ before burning any vertex in V. We can use this burning sequence to understand what the tree associated to a configuration ρ on Λ such that $\rho(v) = 1, v \in V$, looks like.

Denote by $\mathcal{G}_V$ the subgraph constructed from $\mathcal{G}$ by taking only the vertices in V and wiring all edges incident to V into a ghost site (usually called "sink" in the

ASM language). Fix a spanning tree t_0 of $\mathcal{G}_V$. Since V is a good set and thus its vertices have distance at least two between each other, we know that the subgraph $\mathcal{G}_V$ consists of $|V|+1$ vertices, each of the vertices in V connected to the sink by 2d edges. We know that t_0 must contain one of those edges for each vertex. For each such t_0, we define the map $\eta : V \to E(V)$ such that $\eta(v)$ is the only edge in $E(V) \cap t_0$ which is incident to v. Hence, denoting by T_V the edges of T with one endpoint in V, we obtain

$$\mathbb{P}(\rho : \rho(v) = 1 \,\forall v \in V) = \mathbf{P}(T : T_V = \eta(V)) = \mathbf{P}\big(T : T \cap (E(V) \setminus \eta(V)) = \emptyset\big),$$

which yields the result. □

We will combine the two previous observations in the sequel and prove relation (3.3).

Proof of Theorem 3.2. Let us first prove the theorem for $V = \{o\}$, the origin (the choice of the origin is made only to avoid heavy notation as we will see), and then the general case $|V| > 1$. Consider the function $\eta : \{o\} \to E_o$ such that $\eta(o) = o + e_1$. By Lemma 3.19 we have that

$$\mathbb{P}(h_\Lambda(o) = 1) = \mathbf{P}(e_2 \notin T, \dots, e_{2d} \notin T) = \mathbf{P}(e_2 \notin T, \dots, e_{2d} \notin T, e_1 \in T).$$

By the inclusion-exclusion principle we can write

$$\mathbb{P}(h_\Lambda(o) = 1) = \mathbf{P}(e_1 \in T) - \sum_{i=2}^{2d} \mathbf{P}(e_i \in T, e_1 \in T) + \sum_{2 \le i_1 \ne i_2 \le 2d} \mathbf{P}(e_{i_1} \in T, e_{i_2} \in T, e_1 \in T) - \dots - \mathbf{P}(e_1 \in T, \dots, e_{2d} \in T).$$

We rewrite each summand above in terms of fermionic variables by using Proposition 3.17. Slightly abusing notation, we call e_i also the point $o + e_i$. Therefore, we obtain

$$\begin{aligned}\mathbb{P}(h_\Lambda(o) = 1) &= \langle (\psi_o - \psi_{e_1})(\overline{\psi}_o - \overline{\psi}_{e_1}) \rangle_\Lambda^{\mathbf{p}} \\ &\quad - \sum_{i=2}^{2d} \langle (\psi_o - \psi_{e_1})(\overline{\psi}_o - \overline{\psi}_{e_1})(\psi_o - \psi_{e_i})(\overline{\psi}_o - \overline{\psi}_{e_i}) \rangle_\Lambda^{\mathbf{p}} \\ &\quad + \dots - \left\langle \prod_{i=1}^{2d} (\psi_o - \psi_{e_i})(\overline{\psi}_o - \overline{\psi}_{e_i}) \right\rangle_\Lambda^{\mathbf{p}} . \end{aligned} \tag{3.14}$$

Now we use the formula

$$\prod_{i=1}^{n} (1 - a_i) = \sum_{A \subseteq [n]} (-1)^{|A|} \prod_{j \in A} a_j, \quad a_i \in \mathbb{R}, n \in \mathbb{N}$$

in the right-hand side of (3.14) to conclude that

$$\begin{aligned}\mathbb{P}(h_\Lambda(o)=1) &= \left\langle (\psi_o-\psi_{e_1})(\overline{\psi}_o-\overline{\psi}_{e_1})\prod_{i=2}^{2d}\left[1-(\psi_o-\psi_{e_i})(\overline{\psi}_o-\overline{\psi}_{e_i})\right]\right\rangle_\Lambda^{p} \\ &= \left\langle (\psi_o-\psi_{e_1})(\overline{\psi}_o-\overline{\psi}_{e_1})\prod_{i=1}^{2d}\left[1-(\psi_o-\psi_{e_i})(\overline{\psi}_o-\overline{\psi}_{e_i})\right]\right\rangle_\Lambda^{p}.\end{aligned}$$

In the last equality, we are using that $\left[(\psi_u-\psi_v)(\overline{\psi}_u-\overline{\psi}_w)\right]^2=0$ by the anticommutation property (1.20) for all u, v.

Recall that, by Lemma 3.19, the above probability does not depend on the choice of $\eta(\cdot)$. Hence, summing over all possible $2d$ functions $\eta : \{o\} \to E_o$, we obtain

$$\begin{aligned}&\mathbb{P}(h_\Lambda(o)=1) \\ &\quad= \left\langle \frac{1}{2d}\sum_{i=1}^{2d}(\psi_o-\psi_{e_i})\left(\overline{\psi}_o-\overline{\psi}_{e_i}\right)\prod_{j=1}^{2d}\left[1-(\psi_o-\psi_{e_j})(\overline{\psi}_o-\overline{\psi}_{e_j})\right]\right\rangle_\Lambda^{p},\end{aligned}$$

which yields the claim for $V=\{o\}$. With the appropriate change of notation the proof yields the same result for any $o \neq v \in V$.

Let us pass to the general case $|V|>1$. The event that, for all $v \in V$, the edges in $E_v \setminus \{\eta(v)\}$ are not in the spanning tree T of $\mathcal{G}$ and the edge $\{\eta(v)\}$ is in T can be decomposed as

$$\begin{aligned}&\bigcap_{v\in V}\left(\{\eta(v)\in T\}\cap\left(\bigcup_{e\in E_v\setminus\{\eta(v)\}}\{e\in T\}\right)^c\right) \\ &\qquad\qquad\qquad = \bigcap_{v\in V}\{\eta(v)\in T\}\cap\left(\bigcup_{e\in E(V)\setminus\{\eta(V)\}}\{e\in T\}\right)^c.\end{aligned}$$

We obtain, by the above and the same inclusion–exclusion principle in the second equality,

$$\begin{aligned}&\mathbf{P}\left(\bigcap_{v\in V}\left(\{\eta(v)\in T\}\cap\left(\bigcup_{e\in E_v\setminus\{\eta(v)\}}\{e\in T\}\right)^c\right)\right) \\ &\quad= \mathbf{P}\left(\bigcap_{v\in V}\{\eta(v)\in T\}\right)-\mathbf{P}\left(\bigcap_{v\in V}\{\eta(v)\in T\}\cap\bigcup_{e\in E(V)\setminus\{\eta(V)\}}\{e\in T\}\right) \\ &\quad= \sum_{S\subseteq E(V)\setminus\eta(V)}(-1)^{|S|}\mathbf{P}\left(\bigcap_{v\in V}\{\eta(v)\in T\}\cap(S\subseteq T)\right), \qquad (3.15)\end{aligned}$$

where again we sum over the probabilities that the edges of $\eta(V)$ are in the spanning tree T as well as those in $S \subseteq E(V) \setminus \eta(V)$. Again, by Proposition 3.17, Equation (3.15) becomes

$$\sum_{S \subseteq E(V)\setminus\eta(V)} (-1)^{|S|} \left\langle \prod_{f \in \eta(V)} \nabla_f \psi(f^-) \nabla_f \overline{\psi}(f^-) \prod_{g \in S} \nabla_g \psi(g^-) \nabla_g \psi(g^-) \right\rangle_\Lambda^{\mathbf{p}} . \tag{3.16}$$

We observe that the sets of edges S such that $S \cap \eta(V) \neq \emptyset$ do not contribute to (3.16), since again by the anticommutation relation $[\nabla_g \psi(g^-) \nabla_g \overline{\psi}(g^-)]^2 = 0$ for all $u, w \in \Lambda$. Moving the sum into the bracket we obtain

$$\begin{aligned}
&\sum_{S \subseteq E(V)} \left\langle \prod_{f \in \eta(V)} \nabla_f \psi(f^-) \nabla_f \overline{\psi}(f^-) \prod_{g \in S} (-1)^{|S|} \nabla_g \psi(g^-) \nabla_g \overline{\psi}(g^-) \right\rangle_\Lambda^{\mathbf{p}} \\
&= \left\langle \prod_{f \in \eta(V)} \nabla_f \psi(f^-) \nabla_f \overline{\psi}(f^-) \sum_{S \subseteq E(V)} \prod_{g \in S} (-1)^{|S|} \nabla_g \psi(g^-) \nabla_g \overline{\psi}(g^-) \right\rangle_\Lambda^{\mathbf{p}} \\
&= \left\langle \prod_{f \in \eta(V)} \nabla_f \psi(f^-) \nabla_f \overline{\psi}(f^-) \prod_{g \in E(V)} \left(1 - \nabla_g \psi(g^-) \nabla_g \overline{\psi}(g^-)\right) \right\rangle_\Lambda^{\mathbf{p}} . \qquad (3.17)
\end{aligned}$$

Recall that, by Lemma 3.19, the probability above is not depending on the choice of η. Therefore, there are $(2d)^{|V|}$ many functions $\eta(\cdot)$ giving the same expression (3.17). Normalizing the field by $(2d)^{-|V|}$ yields the result. □

3.2.4. PROOFS OF THEOREM 3.3 AND COROLLARY 3.5

In the following proof, we will write κ instead of $\kappa_\Lambda^{\mathbf{0}}$ for simplicity.

We start with a preliminary lemma which is an immediate consequence of Lemma 2.22. This general lemma will be used for the proof of Theorem 3.3 item 2 and Theorem 3.7.

Lemma 3.20. *For fixed $\varepsilon > 0$ let $V = \{v_\varepsilon, v'_\varepsilon\} \subseteq U_\varepsilon$, $E := E(U_\varepsilon)$, $f \in E_{v_\varepsilon}$, $f' \in E_{v'_\varepsilon}$ such that $\eta^*(f)$ and $\eta^*(f')$ do not depend on ε. Then*

$$\lim_{\varepsilon \to 0} \varepsilon^{-d} M(f, f') = \partial^{(1)}_{\eta^*(f)} \partial^{(2)}_{\eta^*(f')} g_U(v, v'),$$

where η^* is as in (1.19). In the right-hand side recall that ∂_e, with $e \in E$, denotes a directional derivative.

Proof of Theorem 3.3.

Part 1. Let B be a subset of $V = \{v_1, \ldots, v_n\}$, where V is a good set. For $B \subseteq V$, from

Definition 3.1, Lemma 3.18 and Lemma 3.14 we have

$$\left\langle \prod_{v\in B} X_v \right\rangle_\Lambda^0 = \frac{1}{(2d)^{|B|}} \sum_{\substack{\eta: B\to E(B)\\ \eta(v)\in E_v\ \forall v}} \left\langle \zeta_{\eta(B)} \right\rangle_\Lambda^0 = \frac{1}{(2d)^{|B|}} \sum_{\substack{\eta: B\to E(B)\\ \eta(v)\in E_v\ \forall v}} \det(M)_{\eta(B)}, \tag{3.18}$$

where ζ_S was defined as $\zeta_S := \prod_{e\in S}(\psi_{e^+} - \psi_{e^-})(\overline{\psi}_{e^+} - \overline{\psi}_{e^-})$ for some set of edges S. In the above we used that if the set V is good, then any subset $B \subseteq V$ is also good. Hence, any edge in $\eta(B)$ is incident to exactly one vertex of B. Equation (1.22) and the expansion of the determinant in terms of permutations yield

$$\kappa(X_{v_1},\ldots,X_{v_n}) = \frac{1}{(2d)^n} \sum_{\substack{\eta: V\to E(V)\\ \eta(v)\in E_v\ \forall v}} \sum_{\pi\in\Pi(V)} (|\pi|-1)!\,(-1)^{|\pi|-1} \prod_{B\in\pi} \sum_{\sigma\in S(B)} \operatorname{sign}(\sigma) \prod_{v\in B} M\big(\eta(v),\eta(\sigma(v))\big).$$

Dropping for the moment the summation over η's and the constant, the previous expression reads

$$\sum_{\pi\in\Pi(V)} (|\pi|-1)!\,(-1)^{|\pi|-1} \prod_{B\in\pi} \sum_{\sigma\in S(B)} \operatorname{sign}(\sigma) \prod_{v\in B} M\big(\eta(v),\eta(\sigma(v))\big). \tag{3.19}$$

Let $\pi = \pi_1 \cdots \pi_{|\pi|}$. We would like to swap the sum over $\sigma \in S(B)$ with the product over $B \in \pi$ in (3.19). To do this, we first note that for any function f depending on B and σ,

$$\prod_{B\in\pi} \sum_{\sigma\in S(B)} f(B,\sigma) = \sum_{\substack{\sigma_{\pi_i}\in S(\pi_i)\\ i\in[|\pi|]}} \prod_{B\in\pi} f(B,\sigma).$$

In this way, we obtain from (3.19)

$$\sum_{\pi\in\Pi(V)} (|\pi|-1)!\,(-1)^{|\pi|-1} \sum_{\substack{\sigma_{\pi_i}\in S(\pi_i),\\ i\in[|\pi|]}} \operatorname{sign}(\sigma) \prod_{B\in\pi} \prod_{v\in B} M\big(\eta(v),\eta(\sigma(v))\big),$$

where $\sigma = \sigma_{\pi_1} \cdots \sigma_{\pi_{|\pi|}}$.

Before we continue, remember the partial order between partitions introduced in Subsection 3.2.1. We will use this to sum over the different partitions.

Let us first sum over $\sigma \in S(V)$. Any such fixed σ can always be uniquely decomposed into m disjoint cyclic permutations without fixed points; that is, $\sigma = \sigma_1 \cdots \sigma_m$ for some $m \in [n]$. Let us call π_σ the partition induced by σ. Now the sum over elements of $\Pi(V)$ will be turned into a sum over $\Pi_\sigma(V)$, being $\Pi_\sigma(V)$ the set of partitions $\pi' \in \Pi(V)$ such that $\pi_\sigma \leq \pi'$. In this way we obtain

$$\sum_{\sigma \in S(V)} \operatorname{sign}(\sigma) \prod_{v \in V} M\big(\eta(v), \eta(\sigma(v))\big) \sum_{\pi \in \Pi_\sigma(V)} (|\pi| - 1)!\,(-1)^{|\pi|-1}.$$

Notice that any partition π such that $\pi \geq \pi_\sigma$ is given by an arbitrary union of elements of π_σ. Therefore, there is a 1-to-1 correspondence between the partitions $\Pi_\sigma(V)$ and the partitions of the set $\{1, \ldots, m_\sigma\}$, where m_σ is the number of cycles in σ. Furthermore, such correspondence preserves the size of the partitions. So we can write our expression as

$$\sum_{\sigma \in S(V)} \operatorname{sign}(\sigma) \prod_{v \in V} M\big(\eta(v), \eta(\sigma(v))\big) \sum_{\pi \in \Pi([m_\sigma])} (|\pi| - 1)!\,(-1)^{|\pi|-1}.$$

With this at hand, let us work with the sum over $\Pi\left([m_\sigma]\right)$. Notice that for any given function f of $|\pi|$ we know that

$$\sum_{\pi \in \Pi([n])} f\left(|\pi|\right) = \sum_{k=1}^{n} g(k, n) f\left(k\right), \tag{3.20}$$

with $g(k, n)$ a multiplicity factor. That is, as long as f depends only on $|\pi|$ and not on the complete permutation π, we can turn the sum in partitions of V into a sum over number of blocks each partition has, at the expense of introducing a multiplicity factor. This factor is given by the so-called Stirling numbers of the second kind, which are defined as the number of ways to partition a set of n objects into k non-empty subsets and given by

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} := \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k - i)^n.$$

Using this, we obtain

$$\sum_{\pi \in \Pi([m_\sigma])} (|\pi| - 1)!\,(-1)^{|\pi|-1} = \sum_{k=1}^{m_\sigma} \left\{ \begin{matrix} m_\sigma \\ k \end{matrix} \right\} (k-1)!\,(-1)^{k-1}. \tag{3.21}$$

Finally, due to algebraic properties of the Stirling numbers of the second kind, it holds that

$$\sum_{k=1}^{m} \left\{ \begin{matrix} m \\ k \end{matrix} \right\} (k-1)!\,(-1)^{k-1} = \begin{cases} 1 & \text{if } m = 1, \\ 0 & \text{if } m \geq 2. \end{cases} \tag{3.22}$$

Hence, we obtain

$$\sum_{\sigma\in S(V)} \mathrm{sign}(\sigma) \prod_{v\in V} M(\eta(v),\eta(\sigma(v))) \sum_{k=1}^{m_\sigma} \left\{ {m_\sigma \atop k} \right\} (k-1)!\,(-1)^{k-1}$$
$$= \sum_{\sigma\in S(V):\, m_\sigma=1} \mathrm{sign}(\sigma) \prod_{v\in V} M(\eta(v),\eta(\sigma(v)))$$
$$= \sum_{\sigma\in S_{\mathrm{cycl}}(V)} \mathrm{sign}(\sigma) \prod_{v\in V} M(\eta(v),\eta(\sigma(v))),$$

where the last equality comes from the fact that $m_\sigma = 1$ if and only if σ is a full cyclic permutation of V without fixed points. Now, for a full cycle σ of length n, we have $\mathrm{sign}(\sigma) = (-1)^{n-1}$. This way, we arrive to

$$(-1)^{n-1} \sum_{\sigma\in S_{\mathrm{cycl}}(V)} \prod_{v\in V} M(\eta(v),\eta(\sigma(v)))$$
$$= (-1)^{n-1} \sum_{\sigma\in S_{\mathrm{cycl}}(V)} \prod_{v\in V} M(\eta(v),\eta(\sigma(v))).$$

Finally, reintroducing the sum over directions of differentiation, we obtain

$$\kappa(X_{v_1},\dots,X_{v_n}) = -\left(\frac{-1}{2d}\right)^n \sum_{\sigma\in S_{\mathrm{cycl}}(V)} \sum_{\substack{\eta:\, V\to E(V)\\ \eta(v)\in E_v\ \forall v}} \prod_{v\in V} M(\eta(v),\eta(\sigma(v))).$$

By taking the field $-X_v$, and using homogeneity of cumulants, this concludes the proof of the first statement. □

Part 2. As for the second statement of the Theorem, note that in view of (3.4) and by setting $\Lambda = \Lambda_\varepsilon := U_\varepsilon$ the left-hand side of (3.5) can be equivalently written as

$$\lim_{\varepsilon\to 0}\left(-\frac{1}{(2d)^n} \sum_{\sigma\in S_{\mathrm{cycl}}(V)} \sum_{\substack{\eta:\, V\to E(V)\\ \eta(v)\in E_v\ \forall v}} \prod_{v\in V} \varepsilon^{-d} M(\eta(v_\varepsilon),\eta(\sigma(v_\varepsilon))) \right)$$
$$= \sum_{\sigma\in S_{\mathrm{cycl}}(V)} \sum_{\substack{\eta:\, V\to E(V)\\ \eta(v)\in E_v\ \forall v}} \lim_{\varepsilon\to 0}\left(-\frac{1}{(2d)^n} \prod_{v\in V} \varepsilon^{-d} M(\eta(v_\varepsilon),\eta(\sigma(v_\varepsilon))) \right), \quad (3.23)$$

where now $M := M_{U_\varepsilon}$.

Remark 3.21. Notice that, in principle, in (3.23) we should take $\sigma^\varepsilon \in S_{\mathrm{cycl}}(V_\varepsilon)$ and $\eta^\varepsilon : V_\varepsilon \to E(V_\varepsilon)$. However, there exists a natural bijection between $S_{\mathrm{cycl}}(V)$ and $S_{\mathrm{cycl}}(V_\varepsilon)$: For $\sigma \in S_{\mathrm{cycl}}(V)$, define $\sigma^\varepsilon(v_\varepsilon) := \left\lfloor \frac{\sigma(v)}{\varepsilon} \right\rfloor$. Likewise, we have a natural bijection between η's and η^ε's. Notice that this bijection works for every $\varepsilon > 0$ (which is not the case in the hexagonal lattice, as it will be discussed in Subsection 3.3.2). Therefore, we will simply write σ and η without the dependence in ε for the remainder of this proof. This also allows us to import the notion of $\eta(v)$ when $\varepsilon \to 0$ as direction rather than directed edge, since the graph structure disappears. We will use similar bijections in the proof of Theorem 3.7.

We can now continue with the proof. By (3.23) it suffices to study

$$\lim_{\varepsilon \to 0} \varepsilon^{-d} M(\eta(v_\varepsilon), \eta(\sigma(v_\varepsilon))).$$

Using Lemma 3.20, this expression converges to

$$\partial^{(1)}_{\eta^*(\eta(v))} \partial^{(2)}_{\eta^*(\eta(\sigma(v)))} g_U\left(v, \sigma(v)\right).$$

Remember that, as g_U is differentiable off-diagonal, for $x \neq y$ we have

$$\partial^{(1)}_e \partial^{(2)}_f g_U(x,y) = -\partial^{(1)}_{-e} \partial^{(2)}_f g_U(x,y) = -\partial^{(1)}_e \partial^{(2)}_{-f} g_U(x,y).$$

However, as σ is cyclic, any such negative signs above will appear twice in the product (3.23) and therefore will cancel, so

$$\prod_{v \in V} \partial^{(1)}_{\eta'(v)} \partial^{(2)}_{\eta'(\sigma(v))} g_U\left(v, \sigma(v)\right) = \prod_{v \in V} \partial^{(1)}_{\eta(v)} \partial^{(2)}_{\eta(\sigma(v))} g_U\left(v, \sigma(v)\right),$$

for any η, η' such that the scalar product $\langle \eta(v), \eta'(v) \rangle = \pm 1$ for all v. Therefore, considering 2^n choices of $\eta' : V \to \{e_1, \ldots, e_{2d}\}$ for a single $\eta : V \to \{e_1, \ldots, e_d\}$, we arrive at

$$\lim_{\varepsilon \to 0} \varepsilon^{-dn} \kappa\left(-X^\varepsilon_{v_1}, \ldots, -X^\varepsilon_{v_n}\right) = -\frac{1}{d^n} \sum_{\sigma \in S_{\mathrm{cycl}}(V)} \sum_{\eta : V \to \{e_1, \ldots, e_d\}} \prod_{v \in V} \partial^{(1)}_{\eta(v)} \partial^{(2)}_{\eta(\sigma(v))} g_U\left(v, \sigma(v)\right).$$

□

Proof of Corollary 3.5. The proof of Theorem 3.3 can be carried through as long as one has a field whose cumulants satisfy (3.18). Therefore by Proposition 3.17 we can also state the results of Theorem 3.3 for the degree field. □

3.2.5. PROOF OF THEOREM 3.6

In the following proof, we will write κ instead of $\kappa^{\mathbf{0}}_\Lambda$ for simplicity as before. This proof follows the same structure of Theorem 3.3. However, we have to account for the more intricate set of permutations of edges.

Call $Z_v := X_v Y_v$. Let $v_1, \dots, v_n$ be as in the statement of the theorem. Once again, we write the joint moments of the field of interest as a sum. Using (3.16), we have

$$\left\langle \prod_{v \in B} Z_v \right\rangle_\Lambda^{\mathbf{0}} = \frac{1}{(2d)^{|B|}} \sum_{\substack{\eta:\, B \to E(B) \\ \eta(v) \in E_v\ \forall v}} \sum_{A \subseteq E(B) \setminus \eta(B)} (-1)^{|A|} \left\langle \zeta_{\eta(B) \cup A} \right\rangle_\Lambda^{\mathbf{0}},$$

for $B \subseteq V$, where from Lemma 3.18,

$$\left\langle \zeta_{\eta(B) \cup A} \right\rangle_\Lambda^{\mathbf{0}} = \det(M)_{\eta(B) \cup A}.$$

This determinant can be written in terms of permutations of edges as

$$\det(M)_{\eta(B) \cup A} = \sum_{\tau \in S(\eta(B) \cup A)} \operatorname{sign}(\tau) \prod_{v \in \eta(B) \cup A} M(v, \tau(v)).$$

Note that, as there can be multiple edges attached to the same site, there is no longer a correspondence between permutation of edges in $\eta(B) \cup A$ and permutations in B. Using (1.22), we get

$$\kappa(Z_{v_1}, \dots, Z_{v_n}) = \left(\frac{1}{2d}\right)^n \sum_\eta \sum_A (-1)^{|A|} \sum_{\pi \in \Pi(V)} (|\pi| - 1)!\, (-1)^{|\pi|-1} \prod_{B \in \pi} \sum_{\tau \in S(\mathcal{E}_B)} \operatorname{sign}(\tau) \prod_{f \in \mathcal{E}_B} M(f, \tau(f)),$$

where, once again, the sum over η's is over all functions $\eta : V \to E(V)$ with $\eta(v) \in E_v$ for all v, the sum over A's is over the subsets of $A \subseteq E(V) \setminus \eta(V)$, and $\mathcal{E}_B = \mathcal{E}_B(\eta, A)$ is the set of edges in $\eta(V) \cup A$ that intersect sites of B. In the above, we are using that V is a good set, and therefore $\{\mathcal{E}_B(\eta, A) : B \in \pi\}$ provides a partition of $\mathcal{E} := \eta(V) \cup A$. In the following, we will write $\mathcal{E}_v$ to denote $\mathcal{E}_{\{v\}}$.

Before proceeding to analyze the sum over the partitions, as we did in the proof of Theorem 3.3, notice that $|A| = |\mathcal{E}| - n$. Therefore, the sum above only depends on η and A through $\mathcal{E}$. We notice that for a fixed $\mathcal{E}$ there are $\prod_{v \in V} |\mathcal{E}_v|$ choices for the pair $(\eta(V), A)$ yielding the same $\mathcal{E}$, so the sum above can be written as

$$\kappa(Z_{v_1}, \dots, Z_{v_n}) = \left(\frac{-1}{2d}\right)^n \sum_{\mathcal{E}:\, |\mathcal{E}_v| \geq 1\ \forall v} K(\mathcal{E}) \sum_{\pi \in \Pi(V)} (|\pi| - 1)!\, (-1)^{|\pi|-1} \prod_{B \in \pi} \sum_{\tau \in S(\mathcal{E}_B)} \operatorname{sign}(\tau) \prod_{f \in \mathcal{E}_B} M(f, \tau(f)).$$

We now concentrate on the sum over the partitions again, dropping the sum over $\mathcal{E}$ for the moment. That is, we examine

$$\sum_{\pi \in \Pi(V)} (|\pi| - 1)!\, (-1)^{|\pi|-1} \prod_{B \in \pi} \sum_{\tau \in S(\mathcal{E}_B)} \operatorname{sign}(\tau) \prod_{f \in \mathcal{E}_B} M(f, \tau(f)). \tag{3.24}$$

Once again, we write $\pi = \{\pi_1, \dots, \pi_{|\pi|}\}$ and use (3.20) to swap the sum over $\tau \in S(\mathcal{E}_B)$ with the product over $B \in \pi$ in (3.24). This way, we obtain

$$\sum_{\pi \in \Pi(V)} (|\pi| - 1)!\,(-1)^{|\pi|-1} \sum_{\substack{\tau_i \in S(\mathcal{E}_{\pi_i}),\\ i \in [|\pi|]}} \operatorname{sign}(\tau) \prod_{B \in \pi} \prod_{f \in \mathcal{E}_B} M(f, \tau(f)),$$

where $\tau = \tau_1 \cdots \tau_{|\pi|}$.

Now, we wish to swap the sum over $\pi \in \Pi(V)$ with the sum over permutations τ. Let us first sum over $\tau \in S(\mathcal{E})$. Any such τ can always be uniquely decomposed into $m(\tau)$ disjoint cyclic permutations, that is, $\tau = \tau_1 \cdots \tau_{m(\tau)}$ for some $m(\tau) \in \{1, \dots, |\mathcal{E}|\}$. Recall the definitions of π_τ^V and the set of connected permutations $S_{co}(\mathcal{E})$ of $\mathcal{E}$ as defined in Subsection 3.2.1. Denote by $\Pi_\tau(V) \subseteq \Pi(V)$ the subset of partitions of V that are coarser than π_τ^V. With this notation in place, we can rewrite the last expression as

$$\sum_{\sigma \in S(\mathcal{E})} \operatorname{sign}(\sigma) \prod_{f \in \mathcal{E}} M(f, \sigma(f)) \sum_{\pi \in \Pi_\tau(V)} (|\pi| - 1)!\,(-1)^{|\pi|-1}. \tag{3.25}$$

Notice that $\pi \in \Pi_\tau(V)$ has at most $m(\tau)$ blocks. Using again expression (3.21) in (3.25), we get

$$\sum_{\tau \in S(\mathcal{E})} \operatorname{sign}(\tau) \prod_{f \in \mathcal{E}} M(f, \tau(f)) \sum_{k=1}^{m(\pi_\tau^V)} \left\{ {m(\pi_\tau^V) \atop k} \right\} (k-1)!\,(-1)^{k-1} = \sum_{\tau \in S_{co}(\mathcal{E})} \operatorname{sign}(\tau) \prod_{f \in \mathcal{E}} M(f, \tau(f)),$$

being $m(\pi_\tau^V)$ the number of blocks of π_τ^V. The equality is due to the fact that (3.22) forces $m(\pi_\tau^V) = 1$, which in turn means that τ is connected. Reintroducing the sum over $\mathcal{E}$ (with multiplicity), we obtain

$$\kappa(Z_v : v \in V) = \left(\frac{-1}{2d}\right)^n \sum_{\mathcal{E} : |\mathcal{E}_v| \geq 1\ \forall v} K(\mathcal{E}) \sum_{\tau \in S_{co}(\mathcal{E})} \operatorname{sign}(\tau) \prod_{f \in \mathcal{E}} M(f, \tau(f)).$$

This concludes the proof of the theorem.

3.2.6. PROOF OF THEOREM 3.7

The proof is divided into four steps. In Step 1, we start from the final expression obtained in Theorem 3.6 and show that it suffices to sum over only bare permutations τ, instead of the bigger set of connected permutations. In Step 2, we simplify the expression even further, showing that only permutations that enter and exit through parallel edges on every point will give a non-zero contribution in the scaling limit as $\varepsilon \to 0$. In Step Step 3, we write the expression in terms of contributions of the permutations acting locally in the vicinity of a vertex and globally mapping an edge incident to one vertex to an edge which is incident to another vertex. Finally, in Step Step 4, we identify the global multiplicative constant of the cumulants.

Step 1. From Theorem 3.6, we start with the expression

$$\kappa\left(Z_v^\varepsilon : v \in V\right) = \left(\frac{-1}{2d}\right)^n \sum_{\mathcal{E}:|\mathcal{E}_{v_\varepsilon}|\geq 1\ \forall v} K(\mathcal{E}) \sum_{\tau \in S_{co}(\mathcal{E})} \operatorname{sign}(\tau) \prod_{f\in\mathcal{E}} M\left(f, \tau(f)\right),$$

where now $\mathcal{E}$ is a subset of edges of U_ε and $M = M_{U_\varepsilon}$. Here we have chosen ε small enough so that $v_\varepsilon \in U_\varepsilon$ for all $v \in V$. Throughout this proof we will often omit the dependence of $\mathcal{E}, \tau$ (and later of other functions as well) on ε by using the same idea as in Remark 3.21. This allows us not only to simplify the expressions, but also to use the notions of connected/bare permutations as $\varepsilon \to 0$.

We will reduce the summation over $\tau \in S_{co}$ to a summation over $\tau \in S_{bare}$ (recall the definitions in Section 3.2.1). Note that, for the edges $E_\tau(V)$ of the induced graph V_τ, we have that $|V| \leq |E_\tau(V)| \leq |E(V)|$ as τ is connected. On the other hand, for each $\tau \in S_{co}$,

$$\prod_{f\in\mathcal{E}} M\left(f, \tau(f)\right) = \mathcal{O}\left(\prod_{f\in\tilde{E}_\tau(V)} \varepsilon^d\, \partial^{(1)}_{\eta^*(f)} \partial^{(2)}_{\eta^*(\tau(f))} g_U\left(f^-, \tau(f)^-\right)\right), \tag{3.26}$$

being $\tilde{E}_\tau(V)$ those edges in E_v such that $\tau(f) \in E_w$, with $w \neq v$, and η^* is the direction induced by f resp. $\tau(f)$ on the point f^- resp. $\tau(f)^-$ as defined in Lemma 3.20. To show (3.26) notice that, by Lemma 3.20,

$$\overline{M}(f, \tau(f)) = \lim_{\varepsilon\to 0} \varepsilon^{-d} M_{U_\varepsilon}(f, \tau(f)) = \partial^{(1)}_{\eta^*(f)} \partial^{(2)}_{\eta^*(\tau(f))} g_U(v, v') \tag{3.27}$$

whenever $f^- = v_\varepsilon \neq \tau(f)^- = v'_\varepsilon$, for some $v, v' \in V$. Once again, we are disregarding the dependence of f in ε, by using the natural bijection of Remark 3.21. Using Lemma 2.23,

$$\overline{M}(f, \tau(f)) = \lim_{\varepsilon\to 0} M_{U_\varepsilon}(f, \tau(f)) = \begin{cases} \nabla^{(1)}_{e_i} \nabla^{(2)}_{e_j} G_0(o, o) & \text{if } f^- = \tau(f)^-, \\ 0 & \text{if } f^- \neq \tau(f)^-, \end{cases} \tag{3.28}$$

being $\eta^*(f) = e_i$ and $\eta^*(\tau(f)) = e_j$ for some $i, j \in [2d]$. Therefore, we can split the product $\prod_{f\in\mathcal{E}} M\left(f, \tau(f)\right)$ into one contribution where the permutation maps edges incident to one vertex v_ε to edges which are incident to another vertex $v'_\varepsilon \neq v_\varepsilon$, and another contribution with edges incident to vertices v_ε which are mapped by τ to vertices incident to the same vertex v_ε. For the former, $M(f, \tau(f))$ is of order ε^d by (3.27); for the latter, it is of order one by (3.28). This shows (3.26).

Now, remember that we rescale the cumulants by $\varepsilon^{-d|V|}$, hence the expression in (3.26) will be non-zero when taking the limit $\varepsilon \to 0$ if and only if $|V| = |E_\tau(V)|$ (it can never diverge since τ is connected and hence $|V| \leq |E_\tau(V)|$). This implies that we can consider only permutations τ which are bare. Once again, following the idea of Remark 3.21, we are ignoring the dependence of τ in ε, allowing us to take the limit as $\varepsilon \to 0$.

Step 2. Now, we examine the expression

$$\left(\frac{-1}{2d}\right)^n \sum_{\mathcal{E}:\, |\mathcal{E}_{v_\varepsilon}| \geq 1 \ \forall v_\varepsilon} K(\mathcal{E}) \sum_{\tau \in S_{\text{bare}}(\mathcal{E})} \operatorname{sign}(\tau) \prod_{f \in \mathcal{E}} M\left(f, \tau(f)\right). \tag{3.29}$$

Recall that each bare τ defines an entry and an exit edge in v_ε, as stated in Subsection 3.2.1. Consider a permutation $\tau \in S_{\text{bare}}(\mathcal{E})$ which enters v_ε through the edge f and exits through the edge f' such that $\langle f, f' \rangle \neq \pm 1$. We construct another permutation $\rho \in S_{\text{bare}}(\mathcal{E}')$ for a possibly different $\mathcal{E}'$, such that $\rho(e) = \tau(e)$ for all $e \notin \mathcal{E}_{v_\varepsilon}$, and such that it will cancel the contribution of τ in (3.29).

To construct such a permutation ρ, we take $\mathcal{E}' := \left(\cup_{v' \neq v} \mathcal{E}_{v'_\varepsilon}\right) \cup \mathcal{E}''_{v_\varepsilon}$, where

$$\mathcal{E}''_u := \{e \in \mathcal{E}_u : \langle e, f \rangle = \pm 1\} \cup \{-e \in \mathcal{E}_u : \langle e, f \rangle \neq \pm 1\}, \quad u \in U_\varepsilon.$$

Remember that for $e = (u, u + e_i)$, we write $-e$ to denote $(u, u - e_i)$. In words, $\mathcal{E}''_{v_\varepsilon}$ is the reflection of $\mathcal{E}_{v_\varepsilon}$ with respect to the direction induced by f. See Figure 3.4 for two examples in $d = 2$. Then we set $\rho(e) = \tau(e)$ for all $e \notin \mathcal{E}_{v_\varepsilon}$ and $\rho(f) = -\tau(f)$ as well as $\rho(-e) = -\tau(-e)$ for all $e \in \mathcal{E}''_{v_\varepsilon}$. That is, we invert every edge of $\mathcal{E}_{v_\varepsilon}$ that is not f. See Figure 3.5 for an example of τ and ρ in $d = 2$. Under these conditions, by (3.28) and the translation/rotation invariance of the discrete Green's function in $\mathbb{Z}^d$, we have

$$\overline{M}(e, \tau(e)) = \nabla^{(1)}_{e_2} \nabla^{(2)}_{e_1} G_0(o, o) = \nabla^{(1)}_{e_2} \nabla^{(2)}_{-e_1} G_0(o, o) \tag{3.30}$$

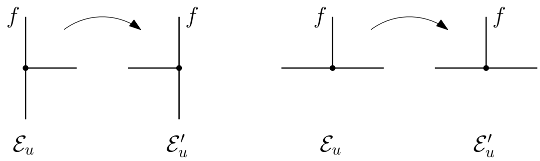

Figure 3.4 – *Two examples of $\mathcal{E}_u$ and its corresponding $\mathcal{E}'_u$ in $d = 2$.*

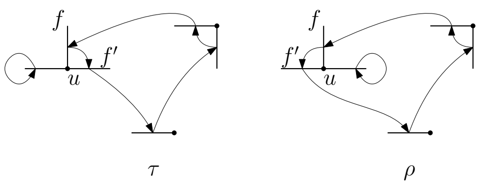

Figure 3.5 – *Example of two permutations τ (left) and ρ (right) in $d = 2$ whose exit edges from u are opposite. The contributions of these two permutations will cancel each other in the limit.*

for all $e \in \mathcal{E}_{v_\varepsilon}$ such that $e^- = \tau(e)^-$ and $\langle e, \tau(e)\rangle = 0$.

Now, remember that we assumed that τ leaves v_ε through an edge f' such that $\langle f, f'\rangle \neq \pm 1$. Therefore, ρ leaves v_ε through $-f'$. Let us call v' the point such that $\tau(f') \in \mathcal{E}_{v'_\varepsilon}$. Using Lemma 3.20, we have that

$$\begin{aligned}\overline{M}(f', \tau(f')) &= \partial^{(1)}_{\eta^*(f')}\partial^{(2)}_{\eta^*(\tau(f'))} g_U(v, v') \\ &= -\partial^{(1)}_{\eta^*(-f')}\partial^{(2)}_{\eta^*(\tau(f'))} g_U(v, v') \\ &= -\partial^{(1)}_{\eta^*(-f')}\partial^{(2)}_{\eta^*(\rho(f'))} g_U(v, v') \\ &= \overline{M}(-f', \rho(f')). \end{aligned} \tag{3.31}$$

Furthermore, notice that $K(\mathcal{E}) = K(\mathcal{E}')$ and $\text{sign}(\tau) = \text{sign}(\rho)$. Now, examining the product on the rightmost part of (3.29), we have that ρ and τ coincide for all edges outside of $\mathcal{E}_{v_\varepsilon}$. As for the contributions given by factors of edges in $\mathcal{E}_{v_\varepsilon}$, (3.30) and (3.31) imply that their product gives the same value under τ and ρ, except for the opposite sign of (3.31). Therefore, for any permutation τ which exits v_ε through an edge that is orthogonal to the entry edge, there exists a permutation ρ such that

$$K(\mathcal{E})\,\text{sign}(\tau)\prod_{f\in\mathcal{E}}\overline{M}(f, \tau(f)) = -K(\mathcal{E}')\,\text{sign}(\rho)\prod_{f\in\mathcal{E}'}\overline{M}(f, \rho(f)). \tag{3.32}$$

Thus the only bare permutations which give a contribution to the limit of (3.29) as $\varepsilon \to 0$ are those which enter v_ε through an edge f and exit v_ε through either f itself or $-f$.

For the remainder of the proof, we will use the notation

$$\overline{M}(f, \tau(f)) = \begin{cases} \nabla^{(1)}_{e_i}\nabla^{(2)}_{e_j} G_0(o, o) & \text{if } f^- = \tau(f)^-, \\ \partial^{(1)}_{e_i}\partial^{(2)}_{e_j} g_U(v, v') & \text{if } f^- = v_\varepsilon \neq v'_\varepsilon = \tau(f)^-,\ v, v' \in V \end{cases} \tag{3.33}$$

whenever $\eta^*(f) = e_i$ and $\eta^*(\tau(f)) = e_j$ for some $i, j \in [2d]$. We now need to further expand the expression

$$\left(\frac{-1}{2d}\right)^n \sum_{\mathcal{E}:\,|\mathcal{E}_v|\geq 1\ \forall v\in V} K(\mathcal{E}) \sum_{\tau\in S^*_{\text{bare}}(\mathcal{E})} \text{sign}(\tau)\prod_{f\in\mathcal{E}}\overline{M}(f, \tau(f)) \tag{3.34}$$

where $S^*_{\text{bare}}(\mathcal{E})$ indicates bare permutations that exit a point through an edge parallel to the entry one (it can be the same). Note that now $\mathcal{E}_v \subseteq E_o + v$, $v \in V$.

Step 3. As stated in Subsection 3.2.1, any bare τ induces a permutation $\sigma \in S_{\text{cycl}}(V)$ on vertices. We will extract from τ a permutation σ among vertices and a choice of edges η (as in Theorem 3.3) and we will separate it from what τ does "locally" in the edges corresponding to a given point. To do this, we need to introduce, for fixed τ, the functions

$$\eta : V \to E(V) \text{ such that } \eta(v) \in E_v \text{ for all } v$$

such that $\eta(v)$ is the edge from which τ enters v, and

$$\gamma : V \to \{-1, 1\}$$

which equals 1 at v if τ enters and exits v through the same edge $\eta(v)$, and equals -1 if τ exits v from $-\eta(v)$. In short, the exit edge from v according to τ is $\gamma(v)\eta(v)$ (from now on written as $\gamma\eta(v)$ to relieve notation). Note that η, σ and γ determine $E_\tau(V)$ and are functions of τ (we will not write this to avoid heavy notation). With the above definitions we have that (3.34) becomes

$$\left(\frac{-1}{2d}\right)^n \sum_{\mathcal{E}:|\mathcal{E}_v|\geq 1\ \forall v} \sum_{\substack{\eta:V\to E(V)\\ \eta(v)\in\mathcal{E}_v\ \forall v}} \sum_{\sigma\in S_{\mathrm{cycl}}(V)} \sum_{\gamma:V\to\{-1,1\}} \sum_{\tau\in S^*_{\mathrm{bare}}(\mathcal{E};\eta,\sigma,\gamma)} \operatorname{sign}(\tau)\left(\prod_{v\in V} K(\mathcal{E}_v)\overline{M}\big(\gamma\eta(v),\eta(\sigma(v))\big)\right) \prod_{f\in\mathcal{E}\setminus\{\gamma\eta(V)\}} \overline{M}(f,\tau(f)), \quad (3.35)$$

where $\gamma\eta(V) := \{\gamma\eta(v) : v \in V\}$, and $S^*_{\mathrm{bare}}(\mathcal{E};\eta,\sigma,\gamma)$ is the set of bare permutations as in (3.34) which now enter and exit each point v through the edges prescribed by η, σ and γ. In this case we will say that τ is compatible with $(\mathcal{E};\eta,\sigma,\gamma)$. Figures 3.6 and 3.7 give examples of compatible resp. non-compatible pairs of permutations in $d = 2$.

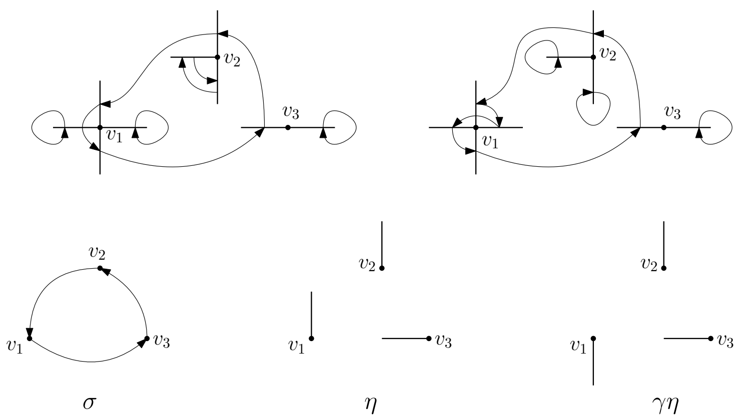

Figure 3.6 – *Top: two different compatible permutations in $d = 2$. Bottom: their corresponding σ, η and γ.*

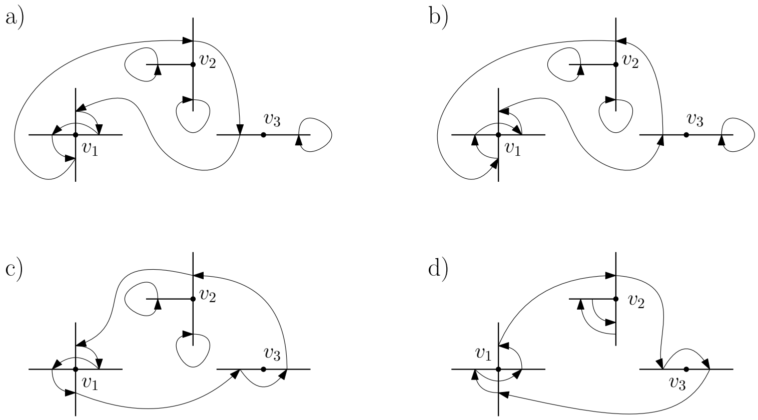

Figure 3.7 – *Four different permutations that are not compatible with those in Figure 3.6. a) Permutation that respects η and γ but not σ. b) Permutation that respects σ and γ but not $\eta(v_1)$. c) Permutation that respects σ and η but not $\gamma(w_3)$. d) Permutation that does not respect σ, nor $\eta(v_1)$, nor $\gamma(w_3)$.*

We have that (3.35) is equal to

$$\left(\frac{-1}{2d}\right)^n \sum_{\mathcal{E}:|\mathcal{E}_v|\geq 1\ \forall v} \sum_{\substack{\eta:V\to E(V)\\ \eta(v)\in\mathcal{E}_v\ \forall v}} \sum_{\sigma\in S_{\mathrm{cycl}}(V)} \sum_{\gamma:V\to\{-1,1\}} \sum_{\tau\in S^*_{\mathrm{bare}}(\mathcal{E};\eta,\sigma,\gamma)}$$

$$\operatorname{sign}(\tau)\left(\prod_{v\in V}\gamma(v)K(\mathcal{E}_v)\overline{M}\big(\eta(v),\eta(\sigma(v))\big)\right)\prod_{f\in\mathcal{E}\setminus\{\gamma\eta(V)\}}\overline{M}\,(f,\tau(f))$$

$$= -\sum_{\eta}\sum_{\sigma}\left(\prod_{v\in V}\overline{M}\big(\eta(v),\eta(\sigma(v))\big)\right)\sum_{\mathcal{E}:\mathcal{E}_v\ni\eta(v)\ \forall v} K(\mathcal{E})\sum_{\gamma}\sum_{\tau\in S^*_{\mathrm{bare}}(\mathcal{E};\eta,\sigma,\gamma)}$$

$$\frac{\operatorname{sign}(\tau)}{(-1)^{n-1}}\prod_{v\in V}\left[\frac{1}{2d}\gamma(v)\prod_{f\in\mathcal{E}_v\setminus\{\gamma\eta(v)\}}\overline{M}\,(f,\tau(f))\right]. \tag{3.36}$$

Note that $\gamma(v)$ is accounted for since, by Lemma 3.20,

$$\overline{M}\big(\gamma\eta(v),\eta(\sigma(v))\big) = \gamma(v)\,\overline{M}\big(\eta(v),\eta(\sigma(v))\big).$$

In the next step, we will fix η and σ, and prove that

$$\sum_{\mathcal{E}:\mathcal{E}_v\ni\eta(v)\ \forall v} K(\mathcal{E})\sum_{\gamma}\sum_{\tau\in S^*_{\mathrm{bare}}(\mathcal{E};\eta,\sigma,\gamma)}$$

$$\frac{\operatorname{sign}(\tau)}{(-1)^{n-1}}\prod_{v\in V}\left[\frac{1}{2d}\gamma(v)\prod_{f\in\mathcal{E}_v\setminus\{\gamma\eta(v)\}}\overline{M}\,(f,\tau(f))\right] \tag{3.37}$$

is a constant independent of ν, η and σ.

Step 4. Using σ, η and γ, we have been able to recover in (3.36) an expression that depends on permutations of vertices and directions similar to that of Theorem 3.3 item 2. To complete the proof we will perform a "surgery" to better understand expression (3.37). This surgery aims at decoupling the local behavior of τ at a vertex versus the jumps of τ between different vertices. To do this, we define

$$\omega_\nu^\tau(f) := \begin{cases} \tau(f) & \text{if } f \neq \gamma\eta(\nu) \\ \tau(\eta(\nu)) & \text{if } f = \gamma\eta(\nu), \gamma(\nu) = -1 \end{cases}, \quad f \in \mathcal{E}_\nu \setminus \{\eta(\nu)\} \tag{3.38}$$

3

and

$$\tau \setminus \omega_\nu^\tau(f) := \begin{cases} \tau(f) & \text{if } f \notin \mathcal{E}_\nu \\ \eta(\sigma(\nu)) & \text{if } f = \eta(\nu) \end{cases}, \quad f \in (\mathcal{E} \setminus \mathcal{E}_\nu) \cup \{\eta(\nu)\}.$$

In words, ω_ν^τ is the permutation induced by τ on $\mathcal{E}_\nu \setminus \{\eta(\nu)\}$ by identifying the entry and the exit edges. On the other hand, $\tau \setminus \omega_\nu^\tau(f)$ follows τ globally until it reaches the edges incident to ν_ε, from where it departs reaching the edges of the next point. In Figure 3.8 we can see some examples for $\gamma(\nu) = -1$ in $d = 2$, as the action of the surgery is trivial when $\gamma(\nu) = 1$. In the following, we state two technical lemmas the we need to complete the proof of Theorem 3.7. Their proofs will be given on page 88.

Lemma 3.22. *Let $\mathcal{E} \subseteq E(V)$, $\gamma : V \to \{-1, 1\}$, $\eta : V \to E(V)$ such that $\eta(\nu) \in \mathcal{E}_\nu$ for all $\nu \in V$, $\sigma \in S_{\text{cycl}}(V)$, and let τ be compatible with $(\mathcal{E}; \eta, \sigma, \gamma)$. For every $\nu \in V$ there is a bijection between $S(\mathcal{E}_\nu \setminus \{\eta(\nu)\})$ and $\{\omega_\nu^\tau : \tau \text{ compatible with } (\mathcal{E}; \eta, \sigma, \gamma)\}$.*

See Figure 3.9 for an instance of Lemma 3.22.

Lemma 3.23 (Surgery of τ). *Fix $\nu \in V$, $\mathcal{E}$, η, σ, γ as above. Let τ be compatible with $(\mathcal{E}; \eta, \sigma, \gamma)$. Then for $\mathcal{E}_\nu \ni \eta(\nu)$ we have that*

$$\text{sign}(\tau) = \gamma(\nu)\,\text{sign}(\tau \setminus \omega_\nu^\tau)\,\text{sign}(\omega_\nu^\tau) \tag{3.39}$$

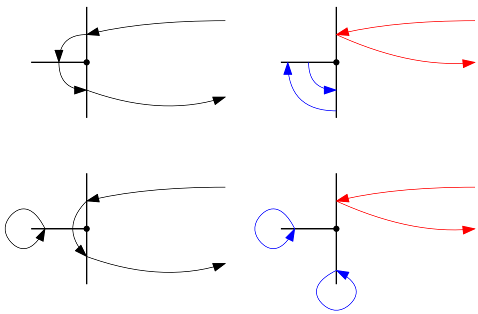

Figure 3.8 – *Example of two permutations τ on the left and their respective ω_ν^τ (in blue) and $\tau \setminus \omega_\nu^\tau$ (in red) on the right, both for the case $\gamma(\nu) = -1$, in $d = 2$.*

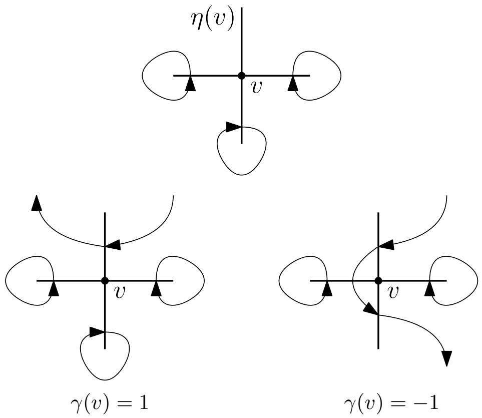

Figure 3.9 – *Top: a permutation* $\omega \in S(\mathcal{E}_v \setminus \{\eta(v)\})$. *Bottom: there is only one* τ *compatible with* η, γ *that can induce* $\omega_v^\tau = \omega$*:* τ *is depicted on the left for* $\gamma(v) = 1$ *and on the right for* $\gamma(v) = -1$.

and

$$\prod_{f \in \mathcal{E}_v \setminus \{\gamma\eta(v)\}} \overline{M}(f, \tau(f)) = \frac{\overline{M}(\eta(v), \omega_v^\tau(\gamma\eta(v)))}{\overline{M}(\gamma\eta(v), \omega_v^\tau(\gamma\eta(v)))} \prod_{f \in \mathcal{E}_v \setminus \{\eta(v)\}} \overline{M}(f, \omega_v^\tau(f)) . \tag{3.40}$$

Equivalently we can write

$$\prod_{f \in \mathcal{E}_v \setminus \{\gamma\eta(v)\}} \overline{M}(f, \tau(f)) = \prod_{f \in \mathcal{E}_v \setminus \{\eta(v)\}} \overline{M}^\gamma(f, \omega_v^\tau(f)) , \tag{3.41}$$

where for any $g \in \mathcal{E}_v \setminus \{\eta(v)\}$

$$\overline{M}^\gamma(f, g) = \begin{cases} \overline{M}(\eta(v), g) & \text{if } f = \gamma\eta(v), \\ \overline{M}(f, g) & \text{if } f \neq \gamma\eta(v). \end{cases}$$

Note that $\overline{M}^\gamma$ is not necessarily a symmetric matrix anymore. In Figure 3.10, we can see an example in $d = 2$ of the surgery from τ to $\omega_{v_2}^\tau$, $\omega_{v_1}^\tau$, $\omega_{v_3}^\tau$ and $((\tau \setminus \omega_{v_1}^\tau) \setminus \omega_{v_2}^\tau) \setminus \omega_{v_3}^\tau$.

We will now use these lemmas to rewrite (3.37) in a more compact form. Using (3.39) recursively, we get

$$\operatorname{sign}(\tau) = \left(\prod_{v \in V} \gamma(v) \operatorname{sign}(\omega_v^\tau) \right) \operatorname{sign}((((\tau \setminus \omega_{v_1}^\tau) \setminus \omega_{v_2}^\tau) \setminus \ldots) \setminus \omega_{v_n}^\tau).$$

Note that the permutation $(((\tau \setminus \omega_{v_1}^\tau) \setminus \omega_{v_2}^\tau) \setminus \ldots) \setminus \omega_{v_n}^\tau$ equals the permutation

$$(\eta(v_1), \eta(\sigma(v_1)), \eta(\sigma(\sigma(v_1))), \ldots, \eta(\sigma^{n-1}(v_1)))$$

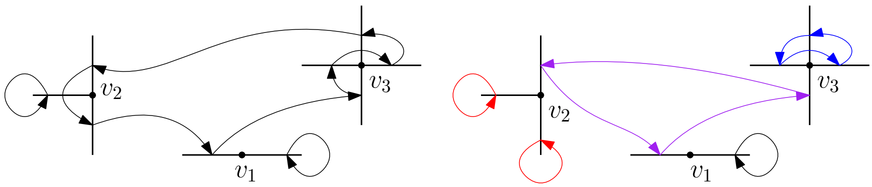

Figure 3.10 – *Example of a permutation* τ *on the left and its respective* $\omega^\tau_{v_1}$ *(in black),* $\omega^\tau_{v_2}$ *(in red),* $\omega^\tau_{v_3}$ *(in blue) and* $((\tau \setminus \omega^\tau_{v_1}) \setminus \omega^\tau_{v_2}) \setminus \omega^\tau_{v_3}$ *(in purple) on the right, in* $d = 2$.

and, as such, it constitutes a cyclic permutation on n edges in $\mathcal{E}$, so that

$$\frac{\operatorname{sign}((((\tau \setminus \omega^\tau_{v_1}) \setminus \omega^\tau_{v_2}) \setminus \ldots) \setminus \omega^\tau_{v_n})}{(-1)^{n-1}} = 1.$$

With this in mind, applying (3.41) at every v we can rewrite (3.37) as

$$\sum_{\mathcal{E}:\mathcal{E}_v \ni \eta(v)\,\forall v} K(\mathcal{E}) \sum_{\gamma} \sum_{\tau \in S^*_{\text{bare}}(\mathcal{E};\eta,\sigma,\gamma)} \left(\prod_{v \in V} \frac{1}{2d} \gamma^2(v) \operatorname{sign}(\omega^\tau_v) \right) \prod_{v \in V} \mathbb{1}_{\{\mathcal{E}_v \ni \gamma\eta(v)\}} \prod_{f \in \mathcal{E}_v \setminus \{\eta(v)\}} \overline{M}^\gamma (f, \tau(f)) .$$

Furthermore, recall that, given $\gamma(v)$, $\omega^\tau_v(\gamma\eta(v)) = \tau(\eta(v))$, which means that now the dependence on τ is only through ω^τ_v and $\gamma(v)$. This, together with Lemma 3.22, allows us to obtain

$$\sum_{\mathcal{E}:\mathcal{E} \ni \eta(V)} K(\mathcal{E}) \sum_{\gamma} \sum_{\omega_v \in S(\mathcal{E}_v \setminus \{\eta(v)\}), v \in V} \left(\prod_{v \in V} \frac{1}{2d} \operatorname{sign}(\omega_v) \right) \prod_{v \in V} \mathbb{1}_{\{\mathcal{E}_v \ni \gamma\eta(v)\}} \prod_{f \in \mathcal{E}_v \setminus \{\eta(v)\}} \overline{M}^\gamma (f, \omega_v(f)) . \tag{3.42}$$

At this point, we note that the expression above does not depend on v, σ or η anymore. In fact, as $\omega_v(f)^- = f^- = v$, we have that $\overline{M}(f, \omega_v(f))$ is a constant by definition (see (3.33)). Therefore, without loss of generality we can take $v = o$, $\eta(v) = e_1$ to get that (3.42) is equal to the n-th power of

$$\frac{1}{2d} \sum_{\mathcal{E}_o:\mathcal{E}_o \ni e_1} K(\mathcal{E}_o) \sum_{\gamma \in \{-1,1\}} \Bigg[\mathbb{1}_{\{\gamma=1\}} \sum_{\omega \in S(\mathcal{E}_o \setminus \{e_1\})} \operatorname{sign}(\omega) \prod_{f \in \mathcal{E}_o \setminus \{e_1\}} \overline{M}^\gamma (f, \omega(f)) + \mathbb{1}_{\{\gamma=-1\}} \mathbb{1}_{\{\mathcal{E}_o \ni -e_1\}} \sum_{\omega \in S(\mathcal{E}_o \setminus \{e_1\})} \operatorname{sign}(\omega) \prod_{f \in \mathcal{E}_o \setminus \{e_1\}} \overline{M}^\gamma (f, \omega(f)) \Bigg] . \tag{3.43}$$

Using the definition of determinant of a matrix, after applying the sum on $\gamma \in \{-1,1\}$ the first term in the square brackets above is equal to $\det(\overline{M})_{\mathcal{E}_o\setminus\{e_1\}}$, while for $\gamma=-1$ the second one yields $\mathbb{1}_{\{\mathcal{E}_o\ni -e_1\}}\det(\overline{M}')_{\mathcal{E}_o\setminus\{e_1\}}$, with $\overline{M}'$ as in (3.9). Summing these contributions we obtain

$$\tilde{\kappa}_2(v_1,\ldots,v_n) = -(C_d)^n \sum_{\sigma\in S_{\mathrm{cycl}}(V)} \sum_{\eta: V\to\{e_1,\ldots,e_d\}} \prod_{v\in V} \partial^{(1)}_{\eta(v)}\partial^{(2)}_{\eta(\sigma(v))} g_U(v,\sigma(v)), \tag{3.44}$$

with

$$C_d = \frac{1}{d}\sum_{\mathcal{E}_o:\mathcal{E}_o\ni e_1} (-1)^{|\mathcal{E}_o|}|\mathcal{E}_o| \left[\det\left(\overline{M}\right)_{\mathcal{E}_o\setminus\{e_1\}} + \mathbb{1}_{\{\mathcal{E}_o\ni -e_1\}}\det\left(\overline{M}'\right)_{\mathcal{E}_o\setminus\{e_1\}}\right], \tag{3.45}$$

where the factor $1/2$ was canceled from (3.43) since we now take in (3.44) the sum over d directions η, introducing a multiplicity of 2 for each point $v \in V$. Using (3.33) this concludes the proof of Theorem 3.7. □

In the final part of the section we give the proofs for Lemmas 3.22 and 3.23.

Proof of Lemma 3.22. The fact that every τ induces a permutation on $\mathcal{E}_v \setminus \{\eta(v)\}$ follows from its definition in Equation (3.38). For the converse, consider a permutation $\omega \in S(\mathcal{E}_v\setminus\{\eta(v)\})$. Using the fact that γ is fixed in our assumptions, we can reconstruct τ locally, and in turn ω^τ_v, according to the values of γ. If $\gamma(v)=1$ then the only τ which satisfies $\omega^\tau_v=\omega$ in $\mathcal{E}_v\setminus\{\eta(v)\}$ is $\tau(f)=\omega(f)$. If instead $\gamma(v)=-1$, the only τ which satisfies $\omega^\tau_v=\omega$ is $\tau(f)=\omega(f)$ for $f\in\mathcal{E}_v\setminus\{\eta(v)\}$ and $\tau(\eta(v))=\omega(\gamma\eta(v))$. □

Proof of Lemma 3.23. We will assume without loss of generality that $\gamma=-1$, as for $\gamma=1$ we have trivially that $\tau=\omega^\tau_v\circ(\tau\setminus\omega^\tau_v)$. For $\eta(v)\neq\gamma\eta(v)$, we expand the left-hand side of (3.40) to get

$$\begin{aligned}
\prod_{f\in\mathcal{E}_v\setminus\{\gamma\eta(v)\}} \overline{M}(f,\tau(f)) &= \overline{M}(\eta(v),\tau(\eta(v))) \prod_{f\in\mathcal{E}_v\setminus\{\gamma\eta(v),\eta(v)\}} \overline{M}(f,\tau(f)) \\
&= \overline{M}(\eta(v),\omega^\tau_v(\gamma\eta(v))) \prod_{f\in\mathcal{E}_v\setminus\{\eta(v),\gamma\eta(v)\}} \overline{M}(f,\omega^\tau_v(f)) \\
&= \frac{\overline{M}(\eta(v),\omega^\tau_v(\gamma\eta(v)))}{\overline{M}(\gamma\eta(v),\omega^\tau_v(\gamma\eta(v)))} \prod_{f\in\mathcal{E}_v\setminus\{\eta(v)\}} \overline{M}(f,\omega^\tau_v(f)).
\end{aligned}$$

In the second line we used that $\omega^\tau_v(\gamma\eta(v))=\tau(\eta(v))$ and $\tau(f)=\omega^\tau_v(f)$ for all $f\in\mathcal{E}_v\setminus\{\eta(v)\}$. Expression (3.41) follows from (3.40) and the definition of $\overline{M}^\gamma$.

As for the signs, the permutation $(\tau\setminus\omega^\tau_v)\circ\omega^\tau_v$ can be written in terms of the decomposition (in transpositions) of τ by suppressing the transposition $(\eta(v),\tau(-\eta(v)))$. Therefore, their parities differ by a negative sign, and we get

$$\mathrm{sign}(\tau) = -\,\mathrm{sign}(\tau\setminus\omega^\tau_v)\,\mathrm{sign}(\omega^\tau_v).$$

□

3.2.7. PROOF OF THEOREM 3.11

From Lemma 2.23 we know that

$$\left|\nabla^{(1)}_{\delta_1}\nabla^{(2)}_{\delta_2} G_{U_\varepsilon}(v_\varepsilon, w_\varepsilon)\right| \leq C \cdot \begin{cases} |v_\varepsilon - w_\varepsilon|^{-d} & \text{if } v_\varepsilon \neq w_\varepsilon, \\ 1 & \text{if } v_\varepsilon = w_\varepsilon, \end{cases} \tag{3.46}$$

for some $C = C(D) > 0$, any $\delta_1, \delta_2 \in \mathcal{E}$ and $v_\varepsilon, w_\varepsilon \in D_\varepsilon$, with $D \subseteq U$ such that $\operatorname{dist}(D, \partial U) > 0$. The functions $f_1, \ldots, f_n$ are bounded and have disjoint and compact supports, respectively referred to as $D_1, \ldots, D_n$. Let

$$\mathfrak{D} := \min_{i \neq j \in [n]} \operatorname{dist}(D_i, D_j) > 0, \qquad \mathfrak{F} := \max_{i \in [n]} \sup_{x \in D} \|f_i(x)\| < \infty.$$

Now, the integrand in (3.10), namely

$$\tilde{\kappa}_1\left(\lfloor x_1/\varepsilon \rfloor, \ldots, \lfloor x_n/\varepsilon \rfloor\right) \prod_{i \in [n]} f_i(x_i),$$

can be bounded by (3.46) by some constant multiple of

$$\left(\max_{x_1 \in D_1, \ldots, x_n \in D_n} \left|\lfloor x_n/\varepsilon \rfloor - \lfloor x_1/\varepsilon \rfloor\right|^{-d} \prod_{i=1}^{n-1} \left|\lfloor x_i/\varepsilon \rfloor - \lfloor x_{i+1}/\varepsilon \rfloor\right|^{-d}\right) \mathfrak{F}^n \leq \varepsilon^{dn} \mathfrak{D}^{-dn} \mathfrak{F}^n$$

whenever ε is small enough (so that $\lfloor x_i/\varepsilon \rfloor \neq \lfloor x_j/\varepsilon \rfloor$ for all $i \neq j$). Using dominated convergence we can introduce the limit inside the integral in the left-hand side of (3.10), obtaining the desired result from (3.5).

3.3. TOWARDS UNIVERSALITY

In this section, we will discuss generalizations of our results regarding the cumulants and the scaling limits for the height-one and degree fields to other lattices. This indicates universal behavior of those two fields. In particular, we will prove the analogous results of Corollary 3.5 and Theorem 3.6 for the triangular lattice.

We will mostly focus on the differences of the proofs and discuss the key assumptions that we believe to be sufficient to extend our results to certain general families of graphs. In particular, all these assumptions also apply to the hexagonal lattice. This will be further discussed in Section 3.3.2.

3.3.1. TRIANGULAR LATTICE

Let us first define the coordinate directions of the triangular lattice, that is,

$$\tilde{e}_j = \left(\cos\left(\frac{\pi(j-1)}{3}\right), \sin\left(\frac{\pi(j-1)}{3}\right)\right), \quad j = 1, \ldots, 6.$$

Notice that $\tilde{e}_{j+3} = -\tilde{e}_j$ for $j = 1, 2, 3$. We also consider the set of directions

$$\tilde{E}_o := \{\tilde{e}_j : j = 1, \ldots, 6\}.$$

Similarly to the hypercubic lattice case, we will use $\tilde{E}_o$ to denote the directed edges leaving the origin as well as the undirected edges containing the origin, or simply the unit vectors in their respective directions. The set $\tilde{E}_v$ will be analogously defined as the set of edges incident to a site v and $\tilde{E}(V) := \cup_{v \in V} \tilde{E}_v$. The triangular lattice in dimension 2 (see Figure 3.11) is then given by

$$\mathbf{T} := \{a_1 \tilde{e}_1 + a_2 \tilde{e}_2 : \, a_1, a_2 \in \mathbb{Z}\}.$$

For any finite connected set $\Lambda \subseteq \mathbf{T}$ we can define the discrete Laplacian Δ_Λ analogously to (1.15), with $\Delta_\Lambda(u, v) = -6$ if $u = v$, for $u, v \in \Lambda$. Likewise, we can define the Green's function (or potential kernel) via Definition 1.5. We can also extend the notion of good set of points Definition 1.14 to subsets of the triangular lattice by simply using its graph distance, and we can define both the UST and the ASM in $\Lambda \subseteq \mathbf{T}$, noticing that the burning algorithm and Lemma 3.19 still hold in this lattice. The fGFF with Dirichlet boundary conditions can be defined again by simply using the graph Laplacian of $\Lambda \subseteq \mathbf{T}$. Trivially, Lemma 3.14 still holds in this setting.

The fermionic observables can be taken as

$$X_v := \frac{1}{\deg_{\mathbf{T}}(v)} \sum_{e:\, e^- = v} \nabla_e \psi(v) \nabla_e \overline{\psi}(v), \quad v \in \Lambda,$$

and

$$Y_v := \prod_{e:\, e^- = v} \left(1 - \nabla_e \psi(v) \nabla_e \overline{\psi}(v)\right), \quad v \in \Lambda.$$

It follows that if the set $V \subseteq \Lambda \subseteq \mathbf{T}$ is good, then trivially Equation (3.6) and Theorem 3.2 (with Dirichlet boundary conditions, rather than pinned) still hold. In fact, Theorem 3.2 is valid for any finite subset of a translation invariant graph.

We will now state the results for cumulants of the degree field and the height-one field on the triangular lattice $\mathbf{T}$. Recall the definition of the average degree field of the UST in Corollary 3.5.

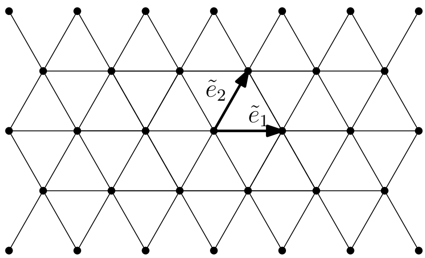

Figure 3.11 – *Triangular lattice and its spanning vectors $\tilde{e}_1$ and $\tilde{e}_2$.*

Theorem 3.24.

1. *Let $n \in \mathbb{N}$ and let the set of points $V := \{v_1, \dots, v_n\} \subseteq \Lambda$ be a good set. The joint cumulants of the average degree field of the UST with wired boundary conditions $(\mathcal{X}_v)_{v\in V}$ are given by*

$$\kappa^{\mathbf{0}}_{\Lambda}(\mathcal{X}_v : v \in V) = \kappa^{\mathbf{0}}_{\Lambda}(X_v : v \in V)$$
$$= -\left(\frac{-1}{6}\right)^n \sum_{\sigma\in S_{\mathrm{cycl}}(V)} \sum_{\eta : V\to \tilde{E}_o} \prod_{v\in V} \nabla^{(1)}_{\eta(v)} \nabla^{(2)}_{\eta(\sigma(v))} G_{\Lambda}(v, \sigma(v)) . \quad (3.47)$$

2. *Let $n \geq 1$, $V := \{v_1, \dots, v_n\} \subseteq \Lambda^{\mathrm{in}}$ be given. For a set of edges $\mathcal{E} \subseteq \tilde{E}(V)$ and $v \in V$ denote $\mathcal{E}_v := \{f \in \mathcal{E} : f^- = v\} \subseteq \tilde{E}_v$. The n-th joint cumulants of the field $(X_v Y_v)_{v\in V}$ are given by*

$$\kappa_{\Lambda}(h_{\Lambda}(v) : v \in V) = \kappa^{\mathbf{0}}_{\Lambda}(X_v Y_v : v \in V)$$
$$= \left(\frac{-1}{6}\right)^n \sum_{\mathcal{E}\subseteq\tilde{E}(V): |\mathcal{E}_v|\geq 1\ \forall v} K(\mathcal{E}) \sum_{\tau\in S_{\mathrm{co}}(\mathcal{E})} \mathrm{sign}(\tau) \prod_{f\in\mathcal{E}} M(f, \tau(f))$$

where $K(\mathcal{E}) := \prod_{v\in V} K(\mathcal{E}_v)$ *and* $K(\mathcal{E}_v) := (-1)^{|\mathcal{E}_v|}|\mathcal{E}_v|$.

In the above we use $\nabla_{\tilde{e}_j} f(v) := f(v + \tilde{e}_j) - f(v)$ for $v \in \mathbf{T}$ and $j \in \{1, \dots, 6\}$. The proof of Theorem 3.24 follows in an analogous way to the proof of Corollary 3.5, item 1, and the proof of Theorem 3.6, so we will skip it. In the following we will compute the scaling limits.

Theorem 3.25. *Let $n \geq 2$ and $V := \{v_1, \dots, v_n\} \subseteq U$ a good set such* $\mathrm{dist}(V, \partial U) > 0$, *where $U \subset \mathbb{R}^2$ is smooth, connected and bounded.*

1. *For the average degree field $(\mathcal{X}_v)_{v\in\Lambda}$ of the UST, we have that*

$$\lim_{\varepsilon\to 0} \varepsilon^{-2n} \kappa^{\mathbf{0}}_{\Lambda}(\mathcal{X}^{\varepsilon}_v : v \in V)$$
$$= -\left(-\frac{1}{2}\right)^n \sum_{\sigma\in S_{\mathrm{cycl}}(V)} \sum_{\eta : V\to\{e_1, e_2\}} \prod_{v\in V} \partial^{(1)}_{\eta(v)} \partial^{(2)}_{\eta(\sigma(v))} g_U(v, \sigma(v)) . \quad (3.48)$$

2. *For the ASM height-one field $(h^{\varepsilon}_v)_{v\in\Lambda}$, we have that*

$$\lim_{\varepsilon\to 0} \varepsilon^{-2n} \kappa_{\Lambda}(h^{\varepsilon}_v : v \in V) = \lim_{\varepsilon\to 0} \varepsilon^{-2n} \kappa^{\mathbf{0}}_{\Lambda}(X^{\varepsilon}_v Y^{\varepsilon}_v : v \in V)$$
$$= -(C_{\mathbf{T}})^n \sum_{\sigma\in S_{\mathrm{cycl}}(V)} \sum_{\eta : V\to\{e_1, e_2\}} \prod_{v\in V} \partial^{(1)}_{\eta(v)} \partial^{(2)}_{\eta(\sigma(v))} g_U(v, \sigma(v)) ,$$

with

$$C_{\mathbf{T}} = -\frac{25}{36} + \frac{162}{\pi^4} - \frac{99\sqrt{3}}{\pi^3} + \frac{99}{2\pi^2} - \frac{5}{4\sqrt{3}\pi} \approx 0.2241. \quad (3.49)$$

3

Notice that the expression in (3.48) for the degree field appears with the same constant as in the square lattice, although this is not the case for the height-one field in (3.49). Moreover, the sum over the directions of derivation is over e_1 and e_2, and not $\tilde{e}_1$, $\tilde{e}_2$ and $\tilde{e}_3$.

Proof of Theorem 3.25 item 1. Using Lemma 3.20 for the triangular lattice (see Kassel and Wu [61, Thm. 1]) and expression (3.47), we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0} -\varepsilon^{2n}\left(\frac{-1}{6}\right)^n \sum_{\sigma\in S_{\mathrm{cycl}}(V)} \sum_{\eta: V\to\tilde{E}_o} \prod_{v\in V} \nabla^{(1)}_{\eta(v)} \nabla^{(2)}_{\eta(\sigma(v))} G_\Lambda(v,\sigma(v)) \\
&= -\left(\frac{-1}{6}\right)^n \sum_{\sigma\in S_{\mathrm{cycl}}(V)} \sum_{\eta: V\to\tilde{E}_o} \prod_{v\in V} \partial^{(1)}_{\eta(v)} \partial^{(2)}_{\eta(\sigma(v))} g_U(v,\sigma(v)) \\
&= -\left(\frac{-1}{6}\right)^n \sum_{\sigma\in S_{\mathrm{cycl}}(V)} \sum_{\eta: V\setminus\{v_1\}\to\tilde{E}_o} \left(\prod_{v\in V\setminus\{v_1,\sigma^{-1}(v_1)\}} \partial^{(1)}_{\eta(v)} \partial^{(2)}_{\eta(\sigma(v))} g_U(v,\sigma(v)) \right) \\
&\qquad \sum_{\eta(v_1)\in\tilde{E}_o} \partial^{(1)}_{\eta(\sigma^{-1}(v_1))} \partial^{(2)}_{\eta(v_1)} g_U(v_1,\sigma(v_1))\, \partial^{(1)}_{\eta(v_1)} \partial^{(2)}_{\eta(\sigma(v_1))} g_U(v_1,\sigma(v_1)),
\end{aligned}$$

where in the last expression we simply isolated the factors that depend on $\eta(v_1)$. For any two differentiable functions $f_1, f_2 : U \to \mathbb{R}$ and vectors $\eta = (\eta_1,\eta_2) = (\cos((j-1)\pi/3), \sin((j-1)\pi/3))$ with $j = 1,\ldots,6$, we trivially have that

$$\sum_{\eta\in\tilde{E}_o} \partial_\eta f_1(x,y)\partial_\eta f_2(x,y) = 3 \sum_{\eta'\in\{e_1,e_2\}} \partial_{\eta'} f_1(x,y)\partial_{\eta'} f_2(x,y), \tag{3.50}$$

By iterating and combining the last two expressions, we are able to change the sum from the directions $\tilde{E}_o$ to $\{e_1, e_2\}$, the usual axis directions. It follows that

$$\begin{aligned}
&-\left(\frac{-1}{6}\right)^n \sum_{\sigma\in S_{\mathrm{cycl}}(V)} \sum_{\eta: V\to\tilde{E}_o} \prod_{v\in V} \partial^{(1)}_{\eta(v)} \partial^{(2)}_{\eta(\sigma(v))} g_U(v,\sigma(v)) \\
&\qquad = -\left(\frac{-1}{2}\right)^n \sum_{\sigma\in S_{\mathrm{cycl}}(V)} \sum_{\eta: V\to\{e_1,e_2\}} \prod_{v\in V} \partial^{(1)}_{\eta(v)} \partial^{(2)}_{\eta(\sigma(v))} g_U(v,\sigma(v)),
\end{aligned}$$

which concludes the claim of the first statement. □

Proof of Theorem 3.25 item 2.

Step 1. Setting Z_v as in the proof of Theorem 3.6, Theorem 3.24 item 2 yields the expression

$$\kappa\left(Z^\varepsilon_v : v \in V\right) = \left(\frac{-1}{6}\right)^n \sum_{\mathcal{E}\subseteq\tilde{E}(V):|\mathcal{E}_{v_\varepsilon}|\geq 1\ \forall v_\varepsilon} K(\mathcal{E}) \sum_{\tau\in S_{\mathrm{co}}(\mathcal{E})} \mathrm{sign}(\tau) \prod_{f\in\mathcal{E}} M(f,\tau(f)).$$

The set $S_{co}(\mathcal{E})$ is defined analogously to the square lattice case. Once again, using the equivalent of Lemma 3.20 for the triangular lattice, we get that only bare permutations make contributions to the limiting expression, that is

$$\lim_{\varepsilon\to 0}\varepsilon^{-2n}\kappa\left(Z_v^\varepsilon : v\in V\right)=\left(\frac{-1}{6}\right)^n\sum_{\mathcal{E}:|\mathcal{E}_v|\geq 1\ \forall v}K(\mathcal{E})\sum_{\tau\in S_{\mathrm{bare}}(\mathcal{E})}\mathrm{sign}(\tau)\prod_{f\in\mathcal{E}}\overline{M}\left(f,\tau(f)\right). \tag{3.51}$$

The definitions of $S_{\mathrm{bare}}(\mathcal{E})$ and $\overline{M}$ are also analogous to the square lattice case. We will again abuse notation here by referring to $\mathcal{E}$ as edges instead of directions of derivation.

3

Step 2. As we will see, this step is more delicate than its square counterpart, as in the triangular lattice we will have fewer cancellations.

Given $\tau\in S_{\mathrm{bare}}(\mathcal{E})$, fix $v\in V$, and let $\eta(v)=\eta(v,\tau)$ be the edge through which τ enters v. Let $\alpha(v)\in\{0,\ldots,5\}$. We define $\eta^\alpha(v)$ as the edge through which τ exists v, where $\alpha(v)\pi/3$ denotes the angle between the entry and exit edges. Let $\gamma_\alpha(v):=\cos\left(\alpha(v)\pi/3\right)$, so that $\gamma_\alpha(v)\in\{-1,-1/2,0,1/2,1\}$ and overall

$$\langle\eta(v),\eta^\alpha(v)\rangle=\|\eta(v)\|\,\|\eta^\alpha(v)\|\,\gamma_\alpha(v).$$

As benchmark recall that in the square lattice the angles between entry and exit edges are multiples of $\pi/2$, hence their cosines belong to $\{-1,0,1\}$.

Define $R_{v,\eta}:\mathbb{R}^2\to\mathbb{R}^2$ to be the reflection on the line given by $\eta(v)$. We then define

$$\mathcal{E}':=R_{v,\eta}(\mathcal{E}):=\left(\bigcup_{v'\neq v}\mathcal{E}_{v'}\right)\cup\{R_{v,\eta}(e):\ e\in\mathcal{E}_v\}$$

and, for $\tau\in S_{\mathrm{bare}}(\mathcal{E})$, define $\rho\in S_{\mathrm{bare}}(\mathcal{E}')$ as

$$\rho(e)=\begin{cases}\tau(e) & \text{if } e\in\cup_{v'\neq v}\mathcal{E}_{v'},\\ \tau(\eta^\alpha(v)) & \text{if } e=R_{v,\eta}(\eta^\alpha(v)),\\ R_{v,\eta}(\tau(e')) & \text{if } e=R_{v,\eta}(e') \text{ for some } e'\in\mathcal{E}_v\setminus\{\eta^\alpha(v)\}.\end{cases}$$

See Figure 3.12 for an example of the reflected permutation ρ. We can then see that, once again, $K(\mathcal{E})=K(\mathcal{E}')$ and $\mathrm{sign}(\tau)=\mathrm{sign}(\rho)$. Furthermore, with simple calculations of inner products we have

$$\overline{M}\left(\eta^\alpha(v),\eta(\sigma(v))\right)+\overline{M}\left(R_{v,\eta}(\eta^\alpha(v)),\eta(\sigma(v))\right)=2\cos\left(\frac{\alpha(v)\pi}{3}\right)\overline{M}\big(\eta(v),\eta(\sigma(v))\big).$$

This equation will play the role of (3.32), because now (3.51) is equal to (again we

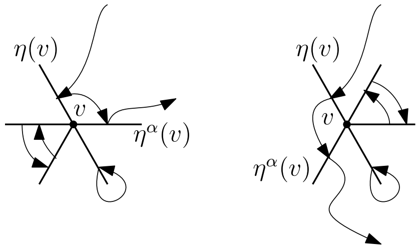

Figure 3.12 – *Left: a permutation* τ *on* v. *Right: its reflection* ρ.

remind of our abuse of notation by using $\eta(v)$ to denote directions)

$$\left(\frac{-1}{6}\right)^n \sum_{\mathcal{E}:|\mathcal{E}_v|\geq 1\ \forall v} K(\mathcal{E}) \sum_{\substack{\eta: V\to\tilde{E}(V)\\ \eta(v)\in\mathcal{E}_v\ \forall v}} \sum_{\sigma\in S_{\mathrm{cycl}}(V)} \sum_{\alpha: V\to\{0,\dots,5\}} \sum_{\tau\in S_{\mathrm{bare}}(\mathcal{E};\eta,\sigma,\alpha)} \mathrm{sign}(\tau)$$
$$\prod_{f\in\mathcal{E}\setminus\eta^\alpha(V)} \overline{M}\left(f,\tau(f)\right) \prod_{v\in V}\gamma_\alpha(v) \underbrace{\prod_{v\in V} \partial^{(1)}_{\eta(v)}\partial^{(2)}_{\eta(\sigma(v))} g_U\left(\eta(v),\eta(\sigma(v))\right)}_{(\star)}. \tag{3.52}$$

Here $S_{\mathrm{bare}}(\mathcal{E};\eta,\sigma,\alpha)$ is the set of permutations compatible with $\mathcal{E},\eta,\sigma,\alpha$.

The factor $(\star)$, which accounts for the interactions between different points, only depends on the entry directions given by η, not on the exit directions η^α. This is the key cancellation to obtain expressions of the form (3.7), up to constant.

Step 3. We rewrite expression (3.52) as

$$\left(\frac{-1}{6}\right)^n \sum_{\substack{\eta: V\to\tilde{E}(V)\\ \eta(v)\in E_v\ \forall v}} \sum_{\sigma\in S_{\mathrm{cycl}}(V)} \prod_{v\in V} \partial^{(1)}_{\eta(v)}\partial^{(2)}_{\eta(\sigma(v))} g_U\left(\eta(v),\eta(\sigma(v))\right)$$
$$\underbrace{\sum_{\mathcal{E}:\mathcal{E}\supseteq\eta(V)} \sum_{\alpha}\sum_{\tau} \mathrm{sign}(\tau) \prod_{v\in V}\left(K(\mathcal{E}_v)\gamma_\alpha(v) \prod_{f\in\mathcal{E}_v\setminus\{\eta^\alpha(v)\}} \overline{M}\left(f,\tau(f)\right)\right)}_{(\star\star)}. \tag{3.53}$$

Remark that if $\eta^\alpha(v)\notin\mathcal{E}$, the set $S_{\mathrm{bare}}(\mathcal{E};\eta,\sigma,\alpha)$ is empty, and therefore not contributing to the sum. Notice that all entries of the type $\overline{M}(e,\tau(e))$ in $(\star\star)$ are discrete double gradients of the Green function of the full triangular lattice $\mathbf{T}$ (see Equation (3.33)). In the following we will prove that $(\star\star)$ does not depend on the choice of η nor σ. The value of the term $(\star\star)$, together with $-1/6$, will give the n-th power of the constant $C_{\mathbf{T}}$.

Step 4. Following the approach we used in the hypercubic lattice, we once again proceed with "surgeries" that will help us evaluate the local constant. For this, given $\eta : V \to \tilde{E}(V)$, $\alpha : V \to \{0,\dots,5\}$, $\mathcal{E} \subseteq \tilde{E}(V)$ with $\eta(v), \eta^\alpha(v) \in \mathcal{E}_v$, and $\tau \in S_{\text{bare}}(\mathcal{E}; \eta, \sigma, \alpha)$, we define $\omega_v^\tau(\mathcal{E}_v \setminus \{\eta(v)\})$ and $\tau \setminus \omega_v^\tau((\mathcal{E} \setminus \mathcal{E}_v) \cup \{\eta(v)\})$ as

$$\omega_v^\tau(f) := \begin{cases} \tau(f) & \text{if } f \neq \eta^\alpha(v) \\ \tau(\eta(v)) & \text{if } f = \eta^\alpha(v), \alpha(v) \neq 0 \end{cases}, \quad f \in \mathcal{E}_v \setminus \{\eta(v)\}$$

and

$$\tau \setminus \omega_v^\tau(f) := \begin{cases} \tau(f) & \text{if } f \notin \mathcal{E}_v \\ \eta(\sigma(v)) & \text{if } f = \eta(v) \end{cases}, \quad f \in (\mathcal{E} \setminus \mathcal{E}_v) \cup \{\eta(v)\}.$$

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An example of ω_v^τ can be found in Figure 3.13.

In this context the analogous statement of Lemma 3.22 still holds. However, there is a subtle difference in the analogous statement of Lemma 3.23.

Lemma 3.26 (Surgery of τ). *Fix $v \in V$ and $\mathcal{E} \subseteq \tilde{E}(V)$, $\eta : V \to \tilde{E}(V)$ with $\eta(v) \in \mathcal{E}_v$ for all $v \in V$, $\sigma \in S_{\text{cycl}}(V)$, and $\alpha : V \to \{0,\dots,5\}$. Let τ be compatible with $\mathcal{E}, \eta$, σ and α. Then*

$$\operatorname{sign}(\tau) = (-1)^{\mathbb{1}_{\{\alpha(v)\neq 0\}}} \operatorname{sign}(\tau \setminus \omega_v^\tau(f)) \operatorname{sign}(\omega_v^\tau).$$

Furthermore,

$$\prod_{f \in \mathcal{E}_v \setminus \{\eta^\alpha(v)\}} \overline{M}(f, \tau(f)) = \frac{\overline{M}(\eta(v), \omega_v^\tau(\eta^\alpha(v)))}{\overline{M}(\eta^\alpha(v), \omega_v^\tau(\eta^\alpha(v)))} \prod_{f \in \mathcal{E}_v \setminus \{\eta(v)\}} \overline{M}(f, \tau(f)).$$

Equivalently, we can write that

$$\prod_{f \in \mathcal{E}_v \setminus \{\eta^\alpha(v)\}} \overline{M}(f, \tau(f)) = \prod_{f \in \mathcal{E}_v \setminus \{\eta(v)\}} \overline{M}^\alpha(f, \omega_v^\tau(f)),$$

where for any $g \in \mathcal{E}_v$

$$\overline{M}^\alpha(f, g) := \begin{cases} \overline{M}(\eta(v), g) & \text{if } f = \eta^\alpha(v), \\ \overline{M}(f, g) & \text{if } f \neq \eta^\alpha(v). \end{cases}$$

Remark that the matrix $\overline{M}^\alpha$ is not symmetric anymore. Lemma 3.26 reads almost the same as its hypercubic counterpart Lemma 3.23, and its proof follows in the

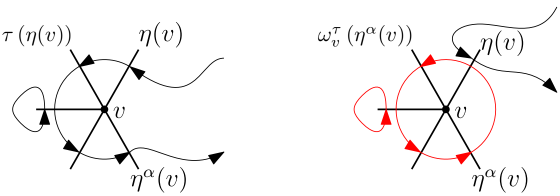

Figure 3.13 – *Left: a permutation τ at the point v. Right: the surgery applied to τ, with ω_v^τ denoted in red.*

3

same manner, so it will be omitted. Using the translation invariance of $\overline{M}$, and setting $\eta(v) = \tilde{e}_1$, $\eta^\alpha(v) = \tilde{e}_{1+\alpha}$, $\omega_v^\tau = \omega$, it follows that $(\star\star)$ in Step 3 is equal to the n-th power of

$$\frac{1}{6}\sum_{\mathcal{E}_o:\mathcal{E}_o\ni\tilde{e}_1} K(\mathcal{E}_o)\sum_{\alpha=0}^{5}\Bigg[\mathbb{1}_{\{\alpha=0\}}\sum_{\omega\in S(\mathcal{E}_o\setminus\{\tilde{e}_1\})}\text{sign}(\omega)\prod_{f\in\mathcal{E}_o\setminus\{\tilde{e}_1\}}\overline{M}(f,\omega(f)) - \gamma_\alpha\mathbb{1}_{\{\alpha\neq 0\}}\mathbb{1}_{\{\mathcal{E}_o\ni\tilde{e}_{1+\alpha}\}}\sum_{\omega\in S(\mathcal{E}_o\setminus\{\tilde{e}_1\})}\text{sign}(\omega)\prod_{f\in\mathcal{E}_o\setminus\{\tilde{e}_1\}}\overline{M}^\alpha(f,\omega(f))\Bigg].$$

Using Equation 3.50, we obtain the cumulants

$$-(C_\mathbf{T})^n\sum_{\sigma\in S_{\text{cycl}}(V)}\sum_{\eta:V\to\{e_1,e_2\}}\prod_{v\in V}\partial^{(1)}_{\eta(v)}\partial^{(2)}_{\eta(\sigma(v))}g_U(v,\sigma(v)),$$

with

$$C_\mathbf{T} = \frac{1}{2}\sum_{\mathcal{E}_o:\mathcal{E}_o\ni\tilde{e}_1}(-1)^{|\mathcal{E}_o|}|\mathcal{E}_o|\Bigg[\det\left(\overline{M}\right)_{\mathcal{E}_o\setminus\{\tilde{e}_1\}} - \sum_{\alpha=1}^{5}\gamma_\alpha\mathbb{1}_{\{\mathcal{E}_o\ni\tilde{e}_{1+\alpha}\}}\det\left(\overline{M}^\alpha\right)_{\mathcal{E}_o\setminus\{\tilde{e}_1\}}\Bigg]. \quad (3.54)$$

Plugging in the values of the potential kernel of the triangular lattice (see e.g. Kenyon and Wilson [65] or Poncelet and Ruelle [85]), this concludes the proof. □

Remark 3.27. Note that, for the triangular lattice, we have that

$$C_\mathbf{T} = \left(\frac{1}{18}+\frac{1}{\sqrt{3}\pi}\right)^{-1}\mathbb{P}\left(h(o)=1\right),$$

where $\mathbb{P}\left(h(o)=1\right)$ was computed in Poncelet and Ruelle [85, Eq. (4.3)].

Remark 3.28. As a safety check, calculating expression (3.54) with $\gamma_\alpha = \cos(\alpha\pi/2)$ and $\alpha\in\{0,1,2,3\}$ yields the same value of the square lattice in $d=2$ as in (3.45).

3.3.2. GENERAL GRAPHS

A natural question is whether our approach would work on general graphs $\mathcal{G}$ embedded in $\mathbb{R}^d$. In this section, we would like to highlight the key ingredients we needed working on $\mathbb{Z}^d$ or $\mathbf{T}$ to prove our results.

Ingredient 1 – Matrix-Tree Theorem: as previously mentioned, Theorem 3.2, Theorem 3.3 item 1 and Theorem 3.6 work on any finite graph for which the Matrix-Tree Theorem and the burning algorithm are valid, which includes subsets of any transitive, regular graph with bounded degree.

Ingredient 2 – Good approximation of the Green's function: for the scaling limits, we need that the equivalent of Lemma 3.20 holds. For example, this is the case for graphs $\mathcal{G}$ such that the sequence $(\mathcal{G}_\varepsilon)_\varepsilon$ with $\mathcal{G}_\varepsilon = \varepsilon\mathcal{G}$ is a "good approximation of $\mathbb{R}^d$" in the sense of Kassel and Wu [61, Thm. 1]. They give a sufficient criterion to obtain such convergence, which in dimension 2 includes transient isoradial graphs (and therefore the triangular and hexagonal lattices).

Ingredient 3 – Isotropic neighborhoods: we believe that the neighborhood of each vertex needs to be "isotropic", in the sense that for $v \in \mathcal{G}$, we need

$$\sum_{u: u\sim v} (u_i - v_i)(u_j - v_j) = c_{\mathcal{G}}\, \delta_{i,j}, \quad \forall\, i,j \in [d],$$

for some constant $c_{\mathcal{G}}$, $v = (v_1, \ldots, v_d)$, $u = (u_1, \ldots, u_d)$, and $\delta_{i,j}$ the Kronecker's delta function. This is needed to substitute (3.50) and replace the reflection cancellations used in Lemma 3.26.

With these ingredients in place, we believe that the scaling limit of the cumulants of the degree field of the UST should have the same form we obtained, up to constants that only depend on $c_{\mathcal{G}}$. The same applies to the height-one of the ASM field. However in this case the global constant in front of the cumulants would also depend on the values of the double discrete derivative of the Green's function on $\mathcal{G}$ in a neighborhood of the origin. Furthermore, such a constant will be very similar to the expression for $C_{\mathbf{T}}$ given in (3.54).

The reader might have noticed that all the conditions above are satisfied by the hexagonal lattice $\mathbf{H}$, on which the height-one of the ASM has also been studied (see Poncelet and Ruelle [85]). The main difficulty for such a lattice is the lack of translation invariance, leading to the set of space "directions" depending on the points. This means that the sum in η's, say for example in (3.23), will depend on ε, so that the convergence of the cumulants (either for the UST or the ASM) cannot be done as we perform in this article. However with minor technical modifications to our proofs we should recover similar results. In fact the UST degree field should have the same global constant $c_{\mathbf{H}} = -1/2$ as in (3.48), whereas for the ASM $C_{\mathbf{T}}$ in (3.49) should be replaced by $C_{\mathbf{H}} = 1/8$.

Finally, we also expect that these results can be extended to other graphs that are not translation-invariant, although an extra site-dependent scaling might be necessary in this case.

4

Fermions and Uniform Spanning Trees

Isn't it funny how day by day nothing changes,
but when we look back everything is different.

—-Clive S. Lewis

IN *this last chapter we establish a clear correspondence between probabilities of certain edges belonging to a realization of the uniform spanning tree (UST), and the states of a fermionic Gaussian free field. Namely, we express the probabilities of given edges belonging or not to the UST in terms of fermionic Gaussian expectations. This allows us to explicitly calculate joint probability mass functions of the degree of the UST on a general finite graph, as well as obtain their scaling limits for certain regular lattices.*

Parts of this chapter can be found in https://arxiv.org/abs/2312.14992. Awaiting peer review.

THROUGHOUT this chapter we will work with a general graph $\mathcal{G} = (\Lambda, E)$. Section 4.1 is devoted to recapitulate on the fermionic formalism used throughout the chapter, as well as stating the first general results linking fermionic Gaussian states and UST probabilities. The moments/cumulants, both for a finite graph and the limiting case, are in Section 4.2. At the end of that section we also discuss the case of the complete graph K_n and its limit $n \to \infty$. Finally, Section 4.3 is devoted to the proofs of the main theorems.

4.1. FERMIONS AND THE UNIFORM SPANNING TREE

Let us see how the fermionic variables defined in the previous chapter allow us to study probabilistic behaviors of the edges of a realization of the UST measure.

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As before, we consider gradients of the generators in the following sense:

Definition 4.1 (Gradient of the generators)**.** The gradient of the generators in the i-th direction is given by

$$\nabla_{e_i}\psi(v) = \psi_{v+e_i} - \psi_v, \quad \nabla_{e_i}\overline{\psi}(v) = \overline{\psi}_{v+e_i} - \overline{\psi}_v, \quad v \in \Lambda,\ i = 1, \dots, \deg_{\mathcal{G}}(v).$$

Define $\zeta(e)$ as

$$\zeta(e) := \nabla_e\psi(e^-)\nabla_e\overline{\psi}(e^+),$$

and observe that the elements $\zeta(\cdot)$ are commutative, that is,

$$\zeta(a)\zeta(b) = \zeta(b)\zeta(a), \quad \forall\, a, b \in E,$$

but we still have that $\zeta(a)^2 = 0$. These objects are key when analyzing probabilities of edges showing up in the UST, in the sense of the result that follows.

Proposition 4.2. *Let the tree* T *be a realization of the UST measure. For* $F, G \subseteq E$, $F \cap G = \emptyset$ *it holds that*

$$\mathbf{P}(F \subseteq T, G \cap T = \emptyset) = \left\langle \prod_{f \in F} \zeta(f) \prod_{g \in G} (1 - \zeta(g)) \right\rangle = \det\left(M^{(|F|)}\right),$$

where

$$M^{(|F|)}(i,j) = \begin{cases} M(i,j) & \text{if } i \le |F|, \\ -M(i,j) & \text{if } |F|+1 \le i \le |F|+|G|, i \neq j, \\ 1 - M(i,j) & \text{if } |F|+1 \le i \le |F|+|G|, i = j. \end{cases}$$

Proof. Observe that

$$\prod_{g \in G} (1 - \zeta(g)) = \sum_{\gamma \subseteq G} (-1)^{|\gamma|} \prod_{g \in \gamma} \zeta(g),$$

so that

$$\left\langle \prod_{f\in F} \zeta(f) \prod_{g\in G} (1-\zeta(g)) \right\rangle$$
$$= \sum_{\gamma\subseteq G} (-1)^{|\gamma|} \left\langle \prod_{g\in\gamma} \zeta(g) \prod_{f\in F} \zeta(f) \right\rangle = \sum_{\gamma\subseteq G} (-1)^{|\gamma|} \mathbf{P}(F\subseteq T, \gamma\subseteq T).$$

Using the inclusion-exclusion principle, we obtain the first equality. The equality between the first and third members follows from Pemantle [83, Thm. 4.3] (noting that there is a typo in their definition of $M^{(|F|)}$). □

Remark 4.3. In view of Proposition 4.2 we have the following recipe to cook up a field whose expectation matches that of the UST: for each edge f we want in the UST, add a factor $\zeta(f)$, and for each edge g we do not want, add a factor $1-\zeta(g)$. Observe that once we add an edge e by adding the factor $\zeta(e)$, adding another factor $1-\zeta(e)$ does nothing. This can be easily seen from the fact that

$$\langle \zeta(e)\,(1-\zeta(e))\rangle = \langle \zeta(e)\rangle - \left\langle \zeta(e)^2 \right\rangle = \langle \zeta(e)\rangle .$$

Remark 4.4. In view of the spatial Markov property of the UST described in Section 1.3, Proposition 4.2 allows us to see that ζ, viewed as a field, satisfies another type of Markov property as well. Take four mutually disjoint sets A, B, C, D, all subsets of E, with

$$\mathbf{P}(A\subset T, B\cap T=\emptyset)\neq 0.$$

Then from Proposition 4.2,

$$\begin{aligned}
&\mathbf{P}_{\mathcal{G}}(C\subseteq T, D\cap T=\emptyset \mid A\subseteq T, B\cap T=\emptyset)\\
&\quad= \frac{\mathbf{P}_{\mathcal{G}}(C\subseteq T, D\cap T=\emptyset, A\subseteq T, B\cap T=\emptyset)}{\mathbf{P}_{\mathcal{G}}(A\subseteq T, B\cap T=\emptyset)}\\
&\quad= \frac{\langle \prod_{a\in A}\zeta(a)\prod_{c\in C}\zeta(c)\prod_{b\in B}(1-\zeta(b))\prod_{d\in D}(1-\zeta(d))\rangle_{\mathcal{G}}}{\langle \prod_{a\in A}\zeta(a)\prod_{b\in B}(1-\zeta(b))\rangle_{\mathcal{G}}},
\end{aligned}$$

which because of the Markov property of the UST (Section 1.3) is equal to

$$\mathbf{P}_{(\mathcal{G}\setminus B)/A}(C\subseteq T, D\cap T=\emptyset) = \left\langle \prod_{c\in C}\zeta(c)\prod_{d\in D}(1-\zeta(d)) \right\rangle_{(\mathcal{G}\setminus B)/A}.$$

The *Markov property for* ζ then says that

$$\begin{aligned}
&\left\langle \prod_{a\in A}\zeta(a)\prod_{b\in B}(1-\zeta(b))\prod_{c\in C}\zeta(c)\prod_{d\in D}(1-\zeta(d)) \right\rangle_{\mathcal{G}}\\
&\quad= \left\langle \prod_{a\in A}\zeta(a)\prod_{b\in B}(1-\zeta(b)) \right\rangle_{\mathcal{G}} \left\langle \prod_{c\in C}\zeta(c)\prod_{d\in D}(1-\zeta(d)) \right\rangle_{(\mathcal{G}\setminus B)/A}.
\end{aligned}$$

In words, this property tells us that we can decompose the joint probability of two events $E_1 = \{A \subseteq T, B \cap T = \emptyset\}$ and $E_2 = \{C \subseteq T, D \cap T = \emptyset\}$ into a product of two probabilities: one is the probability of E_1 in the original graph $\mathcal{G}$, and the other one is the probability of E_2 in a modified graph $(\mathcal{G}\backslash B)/A$. In short, we can always decouple a joint probability into a product of two marginal probabilities, at the expense of modifying the graph.

4.1.1. Degree of the uniform spanning tree

So far we have seen the relationships between fermionic variables and particular edges on a spanning tree. We will now use those results to study the behavior of the degree of a realization of the UST measure at given points on the graph.

4

Remark 4.5. Until now we have defined edges on graphs to be oriented. However, in the following definitions the orientation play no rôle, so we will consider edges as non-oriented.

Let $\mathcal{G} = (\Lambda, E)$ be any graph. For each $v \in \Lambda$ and $k_v \in \{1, \dots, \deg_{\mathcal{G}}(v)\}$, we define the field $\mathbf{X}^{(\mathbf{k})} = (X_v^{(k_v)})_{v \in V}$ as

$$X_v^{(k_v)} := \sum_{\mathcal{E} \subseteq E_v : |\mathcal{E}| = k_v} \prod_{e \in \mathcal{E}} \zeta(e), \quad v \in \Lambda. \tag{4.1}$$

In view of Remark 4.3, this is equivalent to asking that the degree of the UST at a point v is at least k_v; that is,

$$\sum_{\mathcal{E} \subseteq E_v : |\mathcal{E}| = k_v} \prod_{e \in \mathcal{E}} \mathbb{1}_{\{e \in T\}}, \quad v \in \Lambda.$$

If $k_v = 1$ for all v, this is just the field $(X_v)_v$ defined in Chapter 3. Observe also that, because of the nilpotency property of fermions,

$$X_v^{(k_v)} = (X_v)^{k_v},$$

so we will sometimes indistinctly denote it as $X_v^{k_v}$ The same applies for $\mathbf{X}^{(\mathbf{k})}$ written as $\mathbf{X}^{\mathbf{k}}$. We will also need auxiliary Grassmannian observables $\mathbf{Y} = (Y_v)_{v \in V}$ given by

$$Y_v := \prod_{e \in E_v} (1 - \zeta(e)), \quad v \in \Lambda.$$

Define the degree field of the UST $(D_v)_{v \in \Lambda}$ as

$$D_v := \sum_{e \in E_v} \mathbb{1}_{\{e \in T\}}, \tag{4.2}$$

which is "equal" (in the sense of its finite-dimensional distributions) to $(X_v)_v$, as it was seen in Chapter 3. More precisely, for $V \subseteq \Lambda$ a good set (neighboring points will be dealt with in Section 4.2),

$$\mathbf{E}\left[\prod_{v \in V} D_v\right] = \left\langle \prod_{v \in V} X_v \right\rangle.$$

For $k_v \in \{1, \dots, \deg_{\mathcal{G}}(v)\}$, define the degree-$k_v$ field as

$$\delta_v^{(k_v)} = \mathbb{1}_{\{D_v = k_v\}}$$

As a consequence of Proposition 4.2, we can express the probability that the degree of the UST at different not neighboring points has a certain value as in the theorem that follows.

Theorem 4.6. *Let $V \subset \Lambda$ be a good set. For any $k_v \in \{1, \dots, \deg_{\mathcal{G}}(v)\}$, with $v \in V$, it holds that*

$$\mathbf{P}\left(D_v = k_v, v \in V\right) = \mathbf{E}\left[\prod_{v \in V} \delta_v^{(k_v)}\right] = \left\langle \prod_{v \in V} X_v^{k_v} Y_v \right\rangle .$$

Note that this is a generalization of Theorem 3.2, where we obtain the same result for $k_v = 1$ for all $v \in V$, even though in that case our main focus was the height-one field of the Abelian sandpile model.

Remark 4.7. Observe that points in V need to be different. In fact, for $v \in V$,

$$\mathbf{E}\left[D_v^2\right] \neq \langle X_v^2 \rangle,$$

and of course neither does it hold for larger powers. This is because the square of an indicator function (see (4.2)) is the same indicator, whereas the square of $\zeta(e)$, $e \in E$, is 0 (see (4.1)). However, using Proposition 4.2 we observe that

$$\langle X_v (X_v + 1) \rangle = \langle X_v^2 \rangle + \langle X_v \rangle = \sum_{\substack{e,f \in E_v \\ e \neq f}} \det\left(M\right)_{e,f} + \sum_{e \in E_v} M(e,e) = \mathbf{E}\left[D_v^2\right].$$

Following the same reasoning,

$$\mathbf{E}\left[D_v^m\right] = \sum_{i \in [m]} a_i \langle X_v^i \rangle,$$

where the coefficients a_i correspond to a modification of the binomial coefficients. More precisely, for $i = 1, \dots, \lfloor m/2 \rfloor$

$$a_i = \binom{m}{i-1},$$

while for $i = \lfloor m/2 \rfloor + 1, \dots, m$

$$a_i = \binom{m}{i}.$$

We could also find the reverse expression, that is, $\langle X_v^m \rangle$ as a function of $\mathbf{E}\left[D_v^i\right]$, $i \in [m]$. We can use the results on Pemantle [83, Sec. 5.2] to obtain

$$\langle X_v^m \rangle = m!\, \mathbf{E}\left[\binom{D_v}{m}\right] = \mathbf{E}\left[\prod_{i=0}^{m-1} (D_v - i)\right]$$

for any $m \in \mathbb{N}$.

4.2. CUMULANTS OF THE UST DEGREE

We will now study the cumulants of the fields $\mathbf{X}^{\mathbf{k}}\mathbf{Y}$ on an arbitrary graph, and then obtain limiting expressions for some particular lattices. The next theorem is a generalization of Theorem 3.6 when $k_v = 1$ for all points $v \in V$.

Theorem 4.8 (Cumulants of $\mathbf{X}^{\mathbf{k}}\mathbf{Y}$ on a graph). *Let $\mathcal{G} = (\Lambda, E)$ be any graph. Let $n \geq 1$, $V := \{v_1, \dots, v_n\} \subseteq \Lambda^{in}$ be a good set, with $v_i \neq v_j$ for all $i \neq j$. For a set of edges $\mathcal{E} \subseteq E$ and $v \in V$ denote $\mathcal{E}_v := \{f \in \mathcal{E} : f^- = v\} \subseteq E_v$. The n-th joint cumulants of the fields $(X_v^{k_v} Y_v)_{v \in V}$ are given by*

$$\kappa\left(X_v^{k_v} Y_v : v \in V\right) = (-1)^{\sum_v k_v} \sum_{\mathcal{E} \subseteq E : |\mathcal{E}_v| \geq k_v \, \forall v} K(\mathcal{E}) \sum_{\tau \in S_{co}(\mathcal{E})} \mathrm{sign}(\tau) \prod_{f \in \mathcal{E}} M\left(f, \tau(f)\right)$$

where

$$K(\mathcal{E}) := \prod_{v \in V} K(\mathcal{E}_v), \quad K(\mathcal{E}_v) := (-1)^{|\mathcal{E}_v|} \binom{|\mathcal{E}_v|}{k_v},$$

$M = M_{E(V)}$*, and* $k_v \in \mathbb{N}$ *for all* $v \in V$.

Remark 4.9. The reader might be wondering why we work with cumulants instead of moments in this case, which in view of Theorem 4.6 it seems to only introduce complications. The reason for this is that cumulants are independent of the mean, which allows us to obtain a limiting result in the next theorem without the need of renormalizing.

Let $\alpha \in \{0, \dots, p-1\}$, where p is the number of edges contained in any two dimensional plane generated by any two non-parallel edges incident on any $v \in V$; that is, 4 for the hypercubic lattice in d dimensions, 6 for the triangular lattice and 3 for the hexagonal one. Let $\gamma_\alpha := \cos(2\pi\alpha/p)$. This next theorem is a generalization of Theorem 3.6 when $k_v = 1$ for all $v \in V$. We unify their statements and proofs in one theorem.

Theorem 4.10 (Scaling limit of the cumulants of $\mathbf{X}^{\mathbf{k}}\mathbf{Y}$). *Let $n \geq 2$, $\mathbf{L}$ the lattice $\mathbb{Z}^d$ or $\mathbf{T}$, and $V := \{v_1, \dots, v_n\} \subseteq U$ be such that $\mathrm{dist}(V, \partial U) > 0$. Let $\left(\left(X_v^{k_v}\right)^\varepsilon Y_v^\varepsilon\right)_v$ be defined on $U_\varepsilon = U/\varepsilon \cap \mathbf{L}$. If $v_i \neq v_j$ for all $i \neq j$, then*

$$\tilde{\kappa}(v_1, \dots, v_n) := \lim_{\varepsilon \to 0} \varepsilon^{-dn} \kappa\left(\left(X_v^{k_v}\right)^\varepsilon Y_v^\varepsilon : v \in V\right)$$
$$= -\left[\prod_{v \in V} C_L^{(k_v)}\right] \sum_{\sigma \in S_{cycl}(V)} \sum_{\eta : V \to \{\tilde{e}_1, \dots, \tilde{e}_d\}} \prod_{v \in V} \partial^{(1)}_{\eta(v)} \partial^{(2)}_{\eta(\sigma(v))} g_U\left(v, \sigma(v)\right), \quad (4.3)$$

where the constants $C_{\mathbf{L}}^{(k_v)}$ *are given by*

$$C_{\mathbf{L}}^{(k_v)} = (-1)^{k_v+1}\, c_{\mathbf{L}} \sum_{\substack{\mathcal{E}\in E_o:\mathcal{E}\ni e_1\\ |\mathcal{E}|\geq k_v}} (-1)^{|\mathcal{E}_v|}\binom{|\mathcal{E}|}{k_v}\Bigg[\det\left(\overline{M}\right)_{\mathcal{E}\setminus\{e_1\}} - \sum_{\alpha=1}^{p-1}\gamma_\alpha \mathbb{1}_{\{e_{1+\alpha}\in\mathcal{E}\}}\det(\overline{M}^{\alpha})_{\mathcal{E}\setminus\{e_1\}}\Bigg],$$

with $c_{\mathbb{Z}^d} = 2$ *for all* $d \geq 2$, $c_{\mathbf{T}} = 3$, *and for any* $f, g \in E_v$

$$\overline{M}(f,g) = \nabla^{(1)}_{\eta^*(f)}\nabla^{(2)}_{\eta^*(g)} G_0(f^-, g^-)$$

and

$$\overline{M}^{\alpha}(f,g) = \begin{cases} \overline{M}(e_1, g) & \text{if } f = e_{1+\alpha},\\ \overline{M}(f,g) & \text{if } f \neq e_{1+\alpha}. \end{cases} \tag{4.4}$$

Remark 4.11. As we will see in the proof, the same techniques are immediately generalizable to the hexagonal lattice; that is, for $\mathbf{L} = \mathbf{H}$. However, that case requires more care, since we have to account for the two types of vertices in that lattice. We believe an adaptation of the proof to that case only adds obscurity to the matter, but nonetheless it can still be done, yielding the same expression with $p = 3$ and $c_{\mathbf{H}} = 3/2$.

Remark 4.12. After the proof of this theorem, on page 115 we provide a table with the explicit values of $C_{\mathbf{L}}^{(k)}$ for $\mathbb{Z}^2$, $\mathbf{T}$ and $\mathbf{H}$. The reader will observe that $C_{\mathbf{H}}^{(2)} = 0$, which means that any cumulant involving $k_v = 2$ at any v automatically vanishes on the hexagonal lattice.

What about neighboring points? A natural question that arises is whether we can relax the good set condition on the set V in theorems 4.6 and 4.8. The answer is yes, as we explain below.

Let $\mathcal{G} = (\Lambda, E)$ be any graph, T a realization of the UST distribution, and $v \sim w \in \Lambda$. Then

$$\begin{aligned}\mathbf{P}(D_v = k_v, D_w = k_w) \\ &= \mathbf{P}(D_v = k_v, D_w = k_w, \{v,w\}\in T) + \mathbf{P}(D_v = k_v, D_w = k_w, \{v,w\}\notin T).\end{aligned}$$

As we saw in Remark 4.3, the condition $\{v,w\} \in T$ translates, in the fermionic language, to introducing the multiplicative factor $\zeta(\{v,w\})$, whereas for $\{v,w\}\notin T$ we need to introduce $1-\zeta(\{v,w\})$. In view of Theorem 4.8, we have

$$\kappa\left(X_v^{k_v}Y_v, X_w^{k_w}Y_w, \zeta(\{v,w\})\right) = (-1)^{k_v+k_w}\sum_{\substack{|\mathcal{E}_v|\geq k_v-1\\ \{v,w\}\notin\mathcal{E}_v}}\sum_{\substack{|\mathcal{E}_w|\geq k_w-1\\ \{v,w\}\notin\mathcal{E}_w}}(-1)^{|\mathcal{E}_v|+|\mathcal{E}_w|}\times \binom{|\mathcal{E}_v|}{k_v-1}\binom{|\mathcal{E}_w|}{k_w-1}\sum_{\tau\in S_{co}(\mathcal{E})}\operatorname{sign}(\tau)\prod_{f\in\mathcal{E}} M(f,\tau(f)).$$

Equivalently,

$$\kappa\left(X_v^{k_v}Y_v, X_w^{k_w}Y_w, 1-\zeta(\{v,w\})\right) = (-1)^{k_v+k_w} \sum_{\substack{|\mathcal{E}_v|\geq k_v \\ \{v,w\}\notin\mathcal{E}_v}} \sum_{\substack{|\mathcal{E}_w|\geq k_w \\ \{v,w\}\notin\mathcal{E}_w}} (-1)^{|\mathcal{E}_v|+|\mathcal{E}_w|} \times \binom{|\mathcal{E}_v|}{k_v}\binom{|\mathcal{E}_w|}{k_w} \sum_{\tau\in S_{co}(\mathcal{E})} \operatorname{sign}(\tau) \prod_{f\in\mathcal{E}} M(f,\tau(f)).$$

With these expressions we can calculate the moments that give the sought-after probabilities. This is immediately generalized to the case of an arbitrary finite amount of points.

4

Complete graphs It is shown in Pemantle [83, Thm. 1.3] that, for any complete graph K_n with n vertices, as n goes to infinity the degree of the UST at any vertex v converges in distribution to a random variable $1+\mathcal{P}(1)$, being $\mathcal{P}(1)$ a Poisson variable with parameter 1. This can also be obtained as a corollary from our Theorem 4.8 in a much shorter way, as follows:

Theorem 4.13 (Pemantle [83, Thm. 1.3])**.** *Let K_n be a complete graph with n vertices, and let $V(K_n)$ be its vertex set. For any $v \in V(K_n)$ it holds that*

$$D_v \xrightarrow[n\to\infty]{\text{dist}} 1+\mathcal{P}(1),$$

with $\mathcal{P}(1)$ a Poisson random variable with parameter 1.

Alternative simpler proof. From Theorem 4.8, for $k=1,\dots,n$ we have that

$$\mathbf{P}(D_v = k) = (-1)^k \sum_{\mathcal{E}\in E_v:\mathcal{E}\geq k} (-1)^{|\mathcal{E}|}\binom{|\mathcal{E}|}{k}\det(M)_{\mathcal{E}}.$$

According to Pemantle [83], the matrix M for a complete graph K_n is given by

$$M(e,f) = \begin{cases} 2/n & \text{if } e=f, \\ 1/n & \text{if } e\neq f. \end{cases}$$

Straightforward calculations then yield

$$\det(M)_{\mathcal{E}} = \frac{1+|\mathcal{E}|}{n^{|\mathcal{E}|}}.$$

This way,

$$\begin{aligned}\mathbf{P}(D_v = k) &= (-1)^k \sum_{\mathcal{E}\in E_v:\mathcal{E}\geq k} (-1)^{|\mathcal{E}|}\binom{|\mathcal{E}|}{k}\frac{1+|\mathcal{E}|}{n^{|\mathcal{E}|}} \\ &= (-1)^k \sum_{k'=k}^{n-1}\binom{n-1}{k'}(-1)^{k'}\binom{k'}{k}\frac{1+k'}{n^{k'}}.\end{aligned}$$

After algebraic manipulations,

$$\mathbf{P}(D_v = k) = (1+k)(n-1)^{-(2+k)} \left(\frac{n-1}{n}\right)^n n \left[n\binom{n-1}{k} - \binom{n}{1+k} \right].$$

Taking the limit $n \to \infty$,

$$\mathbf{P}(D_v = k) \xrightarrow{n\to\infty} \frac{e^{-1}}{(k-1)!}, \quad k \geq 1,$$

which exactly matches the distribution of a random variable $1 + \mathcal{P}(1)$. □

Remark 4.14. As Pemantle [83, Sec. 5.2] mentions, this result holds for a more general set of graphs, which the author calls *Gino-regular graphs*, and the proof follows in the same way. A sequence of graphs $(\mathcal{G}_n)_n$ is called Gino-regular if there exists a sequence of positive integers $(D_n)_n$ such that

(i) as $n \to \infty$ the maximum and minimum degree of any vertex in $\mathcal{G}_n$ behave as $(1 + o(1))D_n$, and

(ii) the maximum and minimum over vertices $x, y, z, x \neq y$ of $\mathcal{G}_n$ of the probability that a symmetric random walk on $\mathcal{G}_n$ started at x hits y before z behaves as $1/2 + o(1)$ as $n \to \infty$,

where by $o(1)$ we intend a quantity that vanishes as $n \to \infty$. The set of complete graphs $(K_n)_n$ satisfy these conditions, and so do the n-cubes.

This type of graphs allow for an asymptotic calculation of the determinant of M, so that in the limit we obtain the same results as in the case of the complete graph.

4.3. PROOFS

4.3.1. PROOF OF THEOREM 4.6

The first equality is trivial from the fact that $\mathbb{P}(X \in A) = \mathbb{E}[\mathbb{1}_{\{X\in A\}}]$ for any random variable X and any measurable set A. Let us then prove the second equality, starting with a simple lemma.

Lemma 4.15. *The degree-k fields satisfy*

$$\mathbf{E}\left[\prod_{v\in V} \delta_v^{(k_v)}\right] = \sum_{\substack{\eta:\, V \to 2^{E_o} \\ |\eta(v)| = k_v\ \forall v \in V}} \mathbf{P}\Big(\{e \in T\ \forall\, e \in \eta(V)\} \cap \\ \cap \left\{e' \notin T\ \forall\, e' \in E(V) \setminus \eta(V)\right\}\Big),$$

where $\eta(V)$ is an abuse of notation for $\cup_{v\in V}\eta(v)$.

Proof. This is immediate from the fact that, for any random variable X, any $\mathcal{I} \subset \mathbb{N}$, and measurable sets A_i with $i \in \mathcal{I}$, $\mathbf{E}\left[\prod_{i\in\mathcal{I}} \mathbb{1}_{\{X\in A_i\}}\right] = \mathbf{P}\left(\bigcap_{i\in\mathcal{I}} A_i\right)$. □

Proof of Theorem 4.6. In view of Lemma 4.15, take any $\eta : V \to 2^{E(V)}$, with $|\eta(v)| = k_v$, $k_v \in \{1, \ldots, \deg_{\mathcal{G}}(v)\}$, for each $v \in V$. First we observe that

$$\bigcap_{v\in V}\left(\{\eta(v)\subseteq T\}\cap\left(\bigcup_{e\in E_v\setminus\{\eta(v)\}}\{e\in T\}\right)^c\right)$$
$$=\bigcap_{v\in V}\{\eta(v)\subseteq T\}\cap\left(\bigcup_{e\in E(V)\setminus\{\eta(V)\}}\{e\in T\}\right)^c .$$

By the inclusion–exclusion principle,

$$\mathbf{P}\left(\bigcap_{v\in V}\left(\{\eta(v)\subseteq T\}\cap\left(\bigcup_{e\in E_v\setminus\{\eta(v)\}}\{e\in T\}\right)^c\right)\right)$$
$$=\mathbf{P}\left(\bigcap_{v\in V}\{\eta(v)\subseteq T\}\right)-\mathbf{P}\left(\bigcap_{v\in V}\{\eta(v)\subseteq T\}\cap\bigcup_{e\in E(V)\setminus\{\eta(V)\}}\{e\in T\}\right)$$
$$=\sum_{S\subseteq E(V)\setminus\eta(V)}(-1)^{|S|}\mathbf{P}\left(\bigcap_{v\in V}\{\eta(v)\subseteq T\}\cap(S\subseteq T)\right),$$

where we sum over the probabilities that the edges of $\eta(V)$ are in the spanning tree T as well as those in $S \subseteq E(V)\setminus\eta(V)$. By Proposition 3.17, this becomes

$$\sum_{S\subseteq E(V)\setminus\eta(V)}(-1)^{|S|}\left\langle\prod_{\{r,s\}\in\eta(V)}\zeta(\{r,s\})\prod_{\{u,w\}\in S}\zeta(\{u,w\})\right\rangle. \tag{4.5}$$

By the anticommutation relation, the sets of edges S such that $S\cap\eta(V)\neq\emptyset$ do not contribute to (4.5). This way,

$$\sum_{S\subseteq E(V)}\left\langle\prod_{\{r,s\}\in\eta(V)}\zeta(\{r,s\})\prod_{\{u,w\}\in S}(-1)^{|S|}\zeta(\{u,w\})\right\rangle$$
$$=\left\langle\prod_{\{r,s\}\in\eta(V)}\zeta(\{r,s\})\sum_{S\subseteq E(V)}\prod_{\{u,w\}\in S}(-1)^{|S|}\zeta(\{u,w\})\right\rangle$$
$$=\left\langle\prod_{\{r,s\}\in\eta(V)}\zeta(\{r,s\})\prod_{\{u,w\}\in E(V)}(1-\zeta(\{u,w\}))\right\rangle.$$

Observing that the first product is

$$\prod_{\{r,s\}\in\eta(V)}\zeta(\{r,s\})=\prod_{v\in V}\prod_{e\in\eta(v)}\zeta(e)$$

and summing over all possible such η's, we obtain the result. □

4.3.2. PROOF OF THEOREM 4.8

Proof of Theorem 4.8. Call $Z_v^{(k_v)} := X_v^{k_v} Y_v$. Using the same arguments as in the proof of Theorem 3.6 we get

$$\kappa\left(Z_{v_1}^{(k_{v_1})}, \dots, Z_{v_n}^{(k_{v_n})}\right) = \sum_{\eta} \sum_{A} (-1)^{|A|} \sum_{\pi \in \Pi(V)} (|\pi|-1)!\,(-1)^{|\pi|-1} \prod_{B \in \pi} \sum_{\tau \in S(E_B)} \text{sign}(\tau) \prod_{f \in E_B} M\left(f, \tau(f)\right),$$

where the sum over η's is over all functions $\eta : V \to E(V)$ with $\eta(v) \in E_v$ for all v, the sum over A's is over the subsets of $A \subseteq E(V) \setminus \eta(V)$, and $E_B = E_B(\eta, A)$ is the set of edges in $\eta(V) \cup A$ that intersect sites of B.

Notice that $|A| = |\eta(B) \cup A| - \sum_v k_v$. Therefore, the sum above only depends on η and A through $\eta(B) \cup A$. We then denote $\mathcal{E} = E(\eta, A) := \eta(V) \cup A$ and recall $\mathcal{E}_B = \{f \in \mathcal{E} : \{f^-\} \cap B \neq \emptyset\}$. For $v \in V$ we will simplify notation by writing $\mathcal{E}_v$ rather than $\mathcal{E}_{\{v\}}$.

We notice that for a fixed $\mathcal{E}$ there are $\prod_{v \in V} \binom{|\mathcal{E}_v|}{k_v}$ choices for $\eta(V)$ and A yielding the same $\mathcal{E}$, so the sum above can be written as

$$\kappa\left(Z_v^{(k_v)} : v \in V\right) = (-1)^{\sum_v k_v} \sum_{\mathcal{E} : |\mathcal{E}_v| \geq k_v\ \forall v} K(\mathcal{E}) \sum_{\pi \in \Pi(V)} (|\pi|-1)!\,(-1)^{|\pi|-1} \prod_{B \in \pi} \sum_{\tau \in S(\mathcal{E}_B)} \text{sign}(\tau) \prod_{f \in \mathcal{E}_B} M\left(f, \tau(f)\right).$$

The sum over partitions $\Pi(V)$ can again be treated in much the same way as in Theorem 3.6, yielding

$$\kappa\left(Z_v^{(k_v)} : v \in V\right) = (-1)^{\sum_v k_v} \sum_{\mathcal{E} : |\mathcal{E}_v| \geq k_v\ \forall v} K(\mathcal{E}) \sum_{\tau \in S_{co}(\mathcal{E})} \text{sign}(\tau) \prod_{f \in \mathcal{E}} M\left(f, \tau(f)\right)$$

as we wanted to show. □

4.3.3. PROOF OF THEOREM 4.10

Proof of Theorem 4.10. We will do a general proof that works for both $\mathbf{L} = \mathbb{Z}^d$ and $\mathbf{L} = \mathbf{T}$ (and $\mathbf{H}$ with an exception that we will mention below). The proof is divided into four steps. In Step 1, we start from the final expression obtained in Theorem 4.8 and show that it suffices to sum over only bare permutations τ, instead of the bigger set of connected permutations. In Step 2, we write the expression in terms of contributions of the permutations acting locally in the vicinity of a vertex and globally mapping an edge incident to one vertex to an edge which is incident to another vertex. In Step 3 we argue that, given a permutation τ on edges and an entry edge for any given point $v \in V$, only the projection of the exit edge onto the entry edge will contribute to the final expression, so we can treat the former as a new edge in the direction of the entry one, weighed by its projection. Finally, in Step 4 we identify the global multiplicative constant of the cumulants.

Step 1. From Theorem 4.8 we start with the expression

$$\kappa\left(\left(Z_v^{(k_v)}\right)^{\varepsilon} : v \in V\right) = (-1)^{\sum_v k_v} \sum_{\mathcal{E}: |\mathcal{E}_{v_\varepsilon}| \geq k_v \ \forall v} K(\mathcal{E}) \sum_{\tau \in S_{co}(\mathcal{E})} \operatorname{sign}(\tau) \prod_{f \in \mathcal{E}} M\left(f, \tau(f)\right).$$

This step is practically identical to Step 1 in the proof of Theorem 3.7, since it does not depend on k_v, so we omit the whole derivation. It is obtained that, in the limit $\varepsilon \to 0$, only *bare* permutations contribute to the final result, obtaining the expression

$$(-1)^{\sum_v k_v} \sum_{\mathcal{E}: |\mathcal{E}_v| \geq k_v \ \forall v} K(\mathcal{E}) \sum_{\tau \in S_{bare}(\mathcal{E})} \operatorname{sign}(\tau) \prod_{f \in \mathcal{E}} \overline{M}\left(f, \tau(f)\right), \tag{4.6}$$

where we use the notation

$$\overline{M}(f, \tau(f)) = \begin{cases} \nabla_{e_i}^{(1)} \nabla_{e_j}^{(2)} G_0(o, o) & \text{if } f^- = \tau(f)^-, \\ \partial_{e_i}^{(1)} \partial_{e_j}^{(2)} g_U(v, v') & \text{if } f^- = v_\varepsilon \neq v'_\varepsilon = \tau(f)^-,\ v, v' \in V \end{cases} \tag{4.7}$$

whenever $\eta^*(f) = e_i$ and $\eta^*(\tau(f)) = e_j$ for some $e_i, e_j \in E_o$.

Remark 4.16. In the hexagonal lattice there are two types of points: those with edges at 0, $2\pi/3$ and $4\pi/3$ degrees, and those with edges at $\pi/3$, π and $5\pi/6$ degrees. Following the proof in Chapter 3, this step needs extra care when dealing with the hexagonal lattice, since as $\varepsilon \to 0$, v_ε alternates between the two different types of points. Nevertheless, regardless of the point, the contribution will be the same and the result holds for **H** as well, but we omit this technical detail.

Step 2. Given $\tau \in S_{bare}(\mathcal{E})$, fix $v \in V$, and let $\eta(v) = \eta(v, \tau)$ be the edge through which τ enters v. Let $\alpha(v) \in \{0, \ldots, p-1\}$, where p is the number of edges contained in any two dimensional plane generated by any two edges incident on any $v \in V$; that is, 4 for the hypercubic lattice in d dimensions and 6 for the triangular lattice. We define $\eta^\alpha(v)$ as the edge through which τ exists v, and $2\pi\alpha(v)/p$ denotes the angle between the entry and exit edges. Let $\gamma_\alpha(v) := \cos\left(2\pi\alpha(v)/p\right)$, so that

$$\langle \eta(v), \eta^\alpha(v) \rangle = \gamma_\alpha(v).$$

In the case of the hypercubic lattice the angles between entry and exit edges are multiples of $\pi/2$, hence their cosines belong to $\{-1, 0, 1\}$, whereas in the triangular lattice in $d = 2$ angles are multiples of $\pi/3$, and their cosines belong to $\{-1, -1/2, 0, 1/2, 1\}$.

As stated in Subsection 3.2.1, any bare τ induces a permutation $\sigma \in S_{cycl}(V)$ on vertices. We will extract from τ a permutation σ among vertices and a choice of edges η, and we will separate it from what τ does "locally" in the edges corresponding to a given point. Note that η, σ and α determine $E_\tau(V)$ and are functions of τ (we will not write this to avoid heavy notation). With the above definitions we have that (4.6) becomes

$$(-1)^{\sum_v k_v} \sum_{\mathcal{E}:|\mathcal{E}_v|\geq k_v\ \forall v} \sum_{\substack{\eta: V\to E(V)\\ \eta(v)\in\mathcal{E}_v\ \forall v}} \sum_{\sigma\in S_{\mathrm{cycl}}(V)} \sum_{\alpha: V\to\{0,\dots,p-1\}} \sum_{\tau\in S_{\mathrm{bare}}(\mathcal{E};\eta,\sigma,\alpha)} \mathrm{sign}(\tau) \left(\prod_{v\in V} K(\mathcal{E}_v)\overline{M}(\eta^\alpha(v),\eta(\sigma(v)))\right) \prod_{f\in\mathcal{E}\setminus\{\eta^\alpha(V)\}} \overline{M}(f,\tau(f)), \tag{4.8}$$

where $\eta^\alpha(V) \coloneqq \{\eta^\alpha(v) : v \in V\}$, and $S_{\mathrm{bare}}(\mathcal{E}; \eta, \sigma, \alpha)$ is the set of bare permutations which now enter and exit each point v through the edges prescribed by η, σ and α. In this case we will say that τ is compatible with $(\mathcal{E}; \eta, \sigma, \alpha)$.

4

Step 3. Define $R_{v,\eta,\alpha} : \mathbb{R}^d \to \mathbb{R}^d$ to be the reflection perpendicular to the line given by $\eta(v)$, parallel to the plane generated by $\eta(v)$ and $\eta^\alpha(v)$ (in case they are co-linear the reflection is the identity). More precisely, let us call $\mathfrak{S}$ the plane generated by $\eta(v)$ and $\eta^\alpha(v)$, assuming they are not co-linear. Any edge $e \in \mathcal{E}$ can always be decomposed as

$$e = \mathcal{P}^{\mathfrak{S}}(e) + \mathcal{P}^{\mathfrak{S}^\perp}(e),$$

being $\mathcal{P}^{\mathfrak{S}}$ (resp. $\mathcal{P}^{\mathfrak{S}^\perp}$) the orthogonal projection operator on $\mathfrak{S}$ (resp. $\mathfrak{S}^\perp$, that is, the orthogonal complement of $\mathfrak{S}$ on $\mathbb{R}^d$). In turn, this can be further decomposed as

$$e = \mathcal{P}^{\mathfrak{S}}(e)_{\eta(v)} + \mathcal{P}^{\mathfrak{S}}(e)_{\eta(v)^\perp} + \mathcal{P}^{\mathfrak{S}^\perp}(e),$$

being $\mathcal{P}^{\mathfrak{S}}(e)_{\eta(v)}$ the component of $\mathcal{P}^{\mathfrak{S}}(e)$ in the direction of $\eta(v)$, and $\mathcal{P}^{\mathfrak{S}}(e)_{\eta(v)^\perp}$ its orthogonal complement. Of course, $\mathcal{P}^{\mathfrak{S}}(e)_{\eta(v)} = (e)_{\eta(v)}$, that is, the component (or projection) of e in the direction of $\eta(v)$. Let us rewrite this as

$$e = \mathcal{P}^{\mathfrak{S}}(e)_{\eta(v)^\perp} + e'$$

for some unique $e' \in \mathbb{R}^d$. We then define $R_{v,\eta,\alpha} : \mathbb{R}^d \to \mathbb{R}^d$ as

$$R_{v,\eta,\alpha}(e) \coloneqq -\mathcal{P}^{\mathfrak{S}}(e)_{\eta(v)^\perp} + e'.$$

We then define

$$\mathcal{E}' \coloneqq R_{v,\eta,\alpha}(\mathcal{E}) \coloneqq \left(\bigcup_{v'\neq v} \mathcal{E}_{v'}\right) \cup \{R_{v,\eta,\alpha}(e) : e \in \mathcal{E}_v\}$$

and, for $\tau \in S_{\mathrm{bare}}(\mathcal{E})$, define $\rho \in S_{\mathrm{bare}}(\mathcal{E}')$ as

$$\rho(e) = \begin{cases} \tau(e) & \text{if } e \in \cup_{v'\neq v}\mathcal{E}_{v'}, \\ \tau(\eta^\alpha(v)) & \text{if } e = R_{v,\eta,\alpha}(\eta^\alpha(v)), \\ R_{v,\eta,\alpha}(\tau(e')) & \text{if } e = R_{v,\eta,\alpha}(e') \text{ for some } e' \in \mathcal{E}_v \setminus \{\eta^\alpha(v)\}. \end{cases}$$

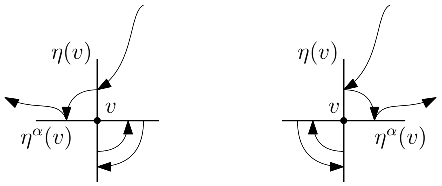

Figure 4.1 – *Square lattice in* $d = 2$*. Left: a permutation* τ *on* v*. Right: its reflection* ρ*.*

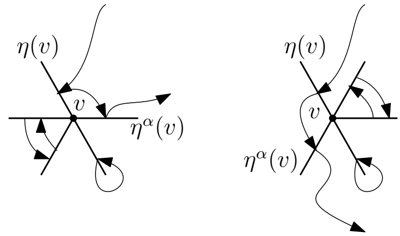

Figure 4.2 – *Triangular lattice in* $d = 2$*. Left: a permutation* τ *on* v*. Right: its reflection* ρ*.*

See Figure 4.1 for an example of the reflected permutation ρ in the square lattice, and Figure 4.2 for the triangular lattice. We can then see that $K(\mathcal{E}) = K(\mathcal{E}')$ and $\text{sign}(\tau) = \text{sign}(\rho)$. Furthermore, with simple calculations of inner products we have

$$\overline{M}\big(\eta^{\alpha}(v), \eta(\sigma(v))\big) + \overline{M}\big(R_{v,\eta}(\eta^{\alpha}(v)), \eta(\sigma(v))\big) \\ = 2\cos\left(\frac{2\pi\alpha(v)}{p}\right)\overline{M}\big(\eta(v), \eta(\sigma(v))\big). \quad (4.9)$$

Observe that these cancellations happen in the hypercubic, triangular and hexagonal lattices due to their high symmetries.

With 4.9 in mind, Equation (4.8) becomes

$$(-1)^{\sum_v k_v} \sum_{\mathcal{E}: |\mathcal{E}_v| \geq k_v \ \forall v} \ \sum_{\substack{\eta: V \to E(V) \\ \eta(v) \in \mathcal{E}_v \ \forall v}} \ \sum_{\sigma \in S_{\text{cycl}}(V)} \ \sum_{\alpha: V \to \{0, \ldots, p-1\}} \ \sum_{\tau \in S_{\text{bare}}(\mathcal{E}; \eta, \sigma, \alpha)} \text{sign}(\tau) \\ \prod_{f \in \mathcal{E} \setminus \eta^{\alpha}(V)} \overline{M}(f, \tau(f)) \prod_{v \in V} K(\mathcal{E}_v) \gamma_{\alpha}(v) \underbrace{\prod_{v \in V} \partial^{(1)}_{\eta(v)} \partial^{(2)}_{\eta(\sigma(v))} g_U\big(\eta(v), \eta(\sigma(v))\big)}_{(\star)}. \quad (4.10)$$

The factor $(\star)$, which accounts for the interactions between different points, only depends on the entry directions given by η, not on the exit directions η^{α}. This is the key cancellation to obtain expressions of the form (4.3), up to constant.

We rewrite expression (4.10) as

$$\sum_{\substack{\eta: V \to E(V) \\ \eta(v) \in E_v \ \forall v}} \ \sum_{\sigma \in S_{\text{cycl}}(V)} \prod_{v \in V} \partial^{(1)}_{\eta(v)} \partial^{(2)}_{\eta(\sigma(v))} g_U\big(\eta(v), \eta(\sigma(v))\big) \\ \prod_{v \in V} \underbrace{(-1)^{k_v} \sum_{\substack{\mathcal{E}_v: \mathcal{E}_v \ni \eta(v) \\ |\mathcal{E}_v| \geq k_v}} K(\mathcal{E}_v) \sum_{\alpha=0}^{p-1} \gamma_{\alpha}(v) \sum_{\tau} \text{sign}(\tau) \prod_{f \in \mathcal{E}_v \setminus \{\eta^{\alpha}(v)\}} \overline{M}(f, \tau(f))}_{(\star\star)}. \quad (4.11)$$

Remark that if $\eta^\alpha(v) \notin \mathcal{E}$, the set $S_{\text{bare}}(\mathcal{E}; \eta, \sigma, \alpha)$ is empty, and therefore not contributing to the sum.

Notice that all entries of the type $\overline{M}(e, \tau(e))$ in $(\star\star)$ are discrete double gradients of the Green function of the full lattice $\mathbf{L}$ (see Equation (4.7)). In the following we will prove that $(\star\star)$ does not depend on the choice of η nor σ. The value of the term $(\star\star)$ will give the constants $C_{\mathbf{L}}^{(k_v)}$ (up to an overall minus sign).

Step 4. Using σ, η and α, we have been able to isolate in (4.11) an expression that depends only on permutations of vertices. To complete the proof we will perform a "surgery" to better understand expression (4.11). This surgery aims at decoupling the local behavior of τ at a vertex versus the jumps of τ between different vertices.

To do this, given $\eta : V \to E(V)$, $\alpha : V \to \{0, \dots, p-1\}$, $\tau \in S_{\text{bare}}(\mathcal{E}; \eta, \sigma, \alpha)$, and $\mathcal{E} \subseteq E(V)$ with $\eta(v), \eta^\alpha(v) \in \mathcal{E}_v$, we define the permutations $\omega_v^\tau(\mathcal{E}_v \setminus \{\eta(v)\})$ and $\tau \setminus \omega_v^\tau\big((\mathcal{E} \setminus \mathcal{E}_v) \cup \{\eta(v)\}\big)$ as

$$\omega_v^\tau(f) \coloneqq \begin{cases} \tau(f) & \text{if } f \neq \eta^\alpha(v) \\ \tau(\eta(v)) & \text{if } f = \eta^\alpha(v), \alpha(v) \neq 0 \end{cases}, \quad f \in \mathcal{E}_v \setminus \{\eta(v)\},$$

and

$$\tau \setminus \omega_v^\tau(f) \coloneqq \begin{cases} \tau(f) & \text{if } f \notin \mathcal{E}_v \\ \eta(\sigma(v)) & \text{if } f = \eta(v) \end{cases}, \quad f \in (\mathcal{E} \setminus \mathcal{E}_v) \cup \{\eta(v)\}.$$

In words, ω_v^τ is the permutation induced by τ on $\mathcal{E}_v \setminus \{\eta(v)\}$ by identifying the entry and the exit edges. On the other hand, $\tau \setminus \omega_v^\tau(f)$ follows τ globally until it reaches the edges incident to v_ε, from where it departs reaching the edges of the next point. An example of ω_v^τ for the triangular lattice can be found in Figure 4.3.

In the following we state the two technical lemmas that we used in Chapter 3.

Lemma 4.17. *Let* $\mathcal{E} \subseteq E(V)$, $\alpha : V \to \{0, \dots, p-1\}$, $\eta : V \to E(V)$ *such that* $\eta(v) \in \mathcal{E}_v$ *for all* $v \in V$, $\sigma \in S_{\text{cycl}}(V)$, *and let* τ *be compatible with* $(\mathcal{E}; \eta, \sigma, \alpha)$. *For every* $v \in V$ *there is a bijection between* $S(\mathcal{E}_v \setminus \{\eta(v)\})$ *and* $\{\omega_v^\tau : \tau \text{ compatible with } (\mathcal{E}; \eta, \sigma, \alpha)\}$.

Lemma 4.18 (Surgery of τ). *Fix* $v \in V$ *and* $\mathcal{E}, \eta, \sigma, \alpha$ *as above. Let* τ *be compatible with* $\mathcal{E}, \eta, \sigma$ *and* α. *Then*

$$\operatorname{sign}(\tau) = (-1)^{\mathbb{1}_{\{\alpha(v) \neq 0\}}} \operatorname{sign}(\tau \setminus \omega_v^\tau(f)) \operatorname{sign}(\omega_v^\tau). \tag{4.12}$$

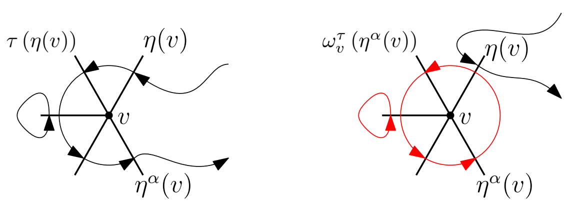

Figure 4.3 – *Left: a permutation* τ *at the point* v. *Right: the surgery applied to* τ, *with* ω_v^τ *denoted in red.*

Furthermore,

$$\prod_{f\in\mathcal{E}_v\setminus\{\eta^\alpha(v)\}} \overline{M}(f,\tau(f)) = \frac{\overline{M}(\eta(v),\omega_v^\tau(\eta^\alpha(v)))}{\overline{M}(\eta^\alpha(v),\omega_v^\tau(\eta^\alpha(v)))} \prod_{f\in\mathcal{E}_v\setminus\{\eta(v)\}} \overline{M}(f,\tau(f)).$$

Equivalently, we can write that

$$\prod_{f\in\mathcal{E}_v\setminus\{\eta^\alpha(v)\}} \overline{M}(f,\tau(f)) = \prod_{f\in\mathcal{E}_v\setminus\{\eta(v)\}} \overline{M}^\alpha(f,\omega_v^\tau(f)), \tag{4.13}$$

where for any $g\in\mathcal{E}_v$

$$\overline{M}^\alpha(f,g) := \begin{cases}\overline{M}(\eta(v),g) & \text{if } f=\eta^\alpha(v),\\ \overline{M}(f,g) & \text{if } f\neq\eta^\alpha(v).\end{cases}$$

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Remark that the matrix $\overline{M}^\alpha$ is not symmetric anymore. We will now use these lemmas to rewrite (4.11) in a more compact form. Using (4.12) recursively, we get

$$\mathrm{sign}(\tau) = \left(\prod_{v\in V}(-1)^{\mathbb{1}_{\{\alpha(v)\neq 0\}}}\,\mathrm{sign}(\omega_v^\tau)\right)\mathrm{sign}((((\tau\setminus\omega_{v_1}^\tau)\setminus\omega_{v_2}^\tau)\setminus\ldots)\setminus\omega_{v_n}^\tau).$$

Note that the permutation $(((\tau\setminus\omega_{v_1}^\tau)\setminus\omega_{v_2}^\tau)\setminus\ldots)\setminus\omega_{v_n}^\tau$ equals the permutation

$$(\eta(v_1),\eta(\sigma(v_1)),\eta(\sigma(\sigma(v_1))),\ldots,\eta(\sigma^{n-1}(v_1)))$$

and, as such, it constitutes a cyclic permutation on n edges in $\mathcal{E}$, so that

$$\mathrm{sign}((((\tau\setminus\omega_{v_1}^\tau)\setminus\omega_{v_2}^\tau)\setminus\ldots)\setminus\omega_{v_n}^\tau) = (-1)^{n-1}.$$

With this in mind, applying (4.13) at every v we can rewrite $\prod_{v\in V}(\star\star)$ as

$$(-1)^{n-1}\prod_{v\in V}(-1)^{k_v}\sum_{\substack{\mathcal{E}_v:\mathcal{E}_v\ni\eta(v)\\|\mathcal{E}_v|\geq k_v}} K(\mathcal{E}_v)\sum_{\alpha=0}^{p-1}\gamma_\alpha(v)\,\mathbb{1}_{\{\eta^\alpha(v)\in\mathcal{E}_v\}}$$
$$\sum_{\tau\in S_{\mathrm{bare}}(\mathcal{E};\eta,\sigma,\alpha)}(-1)^{\mathbb{1}_{\{\alpha(v)\neq 0\}}}\,\mathrm{sign}(\omega_v^\tau)\prod_{f\in\mathcal{E}_v\setminus\{\eta(v)\}}\overline{M}^\alpha(f,\omega_v^\tau(f)).$$

Recall that, given $\alpha(v)$, $\omega_v^\tau(\eta^\alpha(v)) = \tau(\eta(v))$, which means that now the dependence on τ is only through ω_v^τ and $\alpha(v)$. This, together with Lemma 4.17, allows us to obtain

$$-\prod_{v\in V}(-1)^{1+k_v}\sum_{\substack{\mathcal{E}_v:\mathcal{E}_v\ni\eta(v)\\|\mathcal{E}_v|\geq k_v}} K(\mathcal{E}_v)\sum_{\alpha=0}^{p-1}\gamma_\alpha(v)\,\mathbb{1}_{\{\eta^\alpha(v)\in\mathcal{E}_v\}}$$
$$\sum_{\omega_v\in S(\mathcal{E}_v\setminus\{\eta(v)\})}(-1)^{\mathbb{1}_{\{\alpha(v)\neq 0\}}}\,\mathrm{sign}(\omega_v)\prod_{f\in\mathcal{E}_v\setminus\{\eta(v)\}}\overline{M}^\alpha(f,\omega_v(f)). \tag{4.14}$$

At this point, we note that the expression above does not depend on σ or η anymore, and only depends on v through k_v. In fact, as $\omega_v(f)^- = f^- = v$, we have that $\overline{M}(f, \omega_v(f))$ is a constant by definition (see (4.7)). Therefore, without loss of generality, we can take $v = o$, $\eta(v) = e_1$ to get that (4.14) is equal to minus the product over v of

$$(-1)^{1+k_v} \sum_{\substack{\mathcal{E}_o : \mathcal{E}_o \ni e_1 \\ |\mathcal{E}_o| \geq k_v}} K(\mathcal{E}_o) \sum_{\alpha=0}^{p-1} \Bigg[\mathbb{1}_{\{\alpha=0\}} \sum_{\omega \in S(\mathcal{E}_o \setminus \{e_1\})} \operatorname{sign}(\omega) \prod_{f \in \mathcal{E}_o \setminus \{e_1\}} \overline{M}(f, \omega(f))$$
$$- \gamma_\alpha(v)\, \mathbb{1}_{\{e_{1+\alpha} \in \mathcal{E}_o\}} \mathbb{1}_{\{\alpha \neq 0\}} \sum_{\omega \in S(\mathcal{E}_o \setminus \{e_1\})} \operatorname{sign}(\omega) \prod_{f \in \mathcal{E}_o \setminus \{e_1\}} \overline{M}^\alpha(f, \omega(f)) \Bigg].$$

Using the definition of determinant, after applying the sum on $\alpha \in \{0, \ldots, p-1\}$ the first term in the square brackets above is equal to $\det(\overline{M})_{\mathcal{E}_o \setminus \{e_1\}}$, while for $\alpha \neq 0$ the second one yields $\mathbb{1}_{\{e_{1+\alpha} \in \mathcal{E}_o\}} \det(\overline{M}^\alpha)_{\mathcal{E}_o \setminus \{e_1\}}$, with $\overline{M}^\alpha$ as in (4.4). Summing these contributions we obtain the cumulants

$$- \left[\prod_{v \in V} C_{\mathbf{L}}^{(k_v)} \right] \left(\frac{1}{c_{\mathbf{L}}} \right)^n \sum_{\sigma \in S_{\text{cycl}}(V)} \sum_{\eta : V \to E_o} \prod_{v \in V} \partial^{(1)}_{\eta(v)} \partial^{(2)}_{\eta(\sigma(v))} g_U(v, \sigma(v)) =$$
$$- \left[\prod_{v \in V} C_{\mathbf{L}}^{(k_v)} \right] \sum_{\sigma \in S_{\text{cycl}}(V)} \sum_{\eta : V \to \{\tilde{e}_1, \ldots, \tilde{e}_d\}} \prod_{v \in V} \partial^{(1)}_{\eta(v)} \partial^{(2)}_{\eta(\sigma(v))} g_U(v, \sigma(v)),$$

where the last change of coordinates is identical to the one in Chapter 3, being

$$C_{\mathbf{L}}^{(k_v)} = (-1)^{k_v+1} c_{\mathbf{L}} \sum_{\substack{\mathcal{E}_o \ni e_1 \\ |\mathcal{E}_o| \geq k_v}} (-1)^{|\mathcal{E}_o|} \binom{|\mathcal{E}_o|}{k_v} \Bigg[\det\left(\overline{M}\right)_{\mathcal{E}_o \setminus \{e_1\}}$$
$$- \sum_{\alpha=1}^{p-1} \gamma_\alpha \mathbb{1}_{\{e_{1+\alpha} \in \mathcal{E}_o\}} \det\left(\overline{M}^\alpha\right)_{\mathcal{E}_o \setminus \{e_1\}} \Bigg],$$

with $c_{\mathbb{Z}^d} = 2$ for all $d \geq 2$, and $c_{\mathbf{T}} = 3$. □

Remark 4.19. We highlight once again that, with the technical exception of Step 1, all the other steps follow in much the same way for $\mathbf{H}$, in which case $p = 3$, and the value of $c_{\mathbf{H}}$ can also be calculated, obtaining $c_{\mathbf{H}} = 3/2$.

Using the potential kernel values of the lattices (see e.g. Kenyon and Wilson [65] or Poncelet and Ruelle [85]), some values of $C_{\mathbf{L}}^{(k_v)}$ in two dimensions are

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$$C^{(1)}_{\mathbb{Z}^2} = \frac{8}{\pi} - \frac{16}{\pi^2} \approx 0.9253 \qquad C^{(2)}_{\mathbb{Z}^2} = 18 - \frac{72}{\pi} + \frac{96}{\pi^2} \approx 4.8085$$

$$C^{(3)}_{\mathbb{Z}^2} = 2 + \frac{16}{\pi} \approx 7.0930 \qquad C^{(4)}_{\mathbb{Z}^2} = -2$$

$$C^{(1)}_{\mathbf{T}} = -\frac{25}{6} - \frac{5\sqrt{3}}{2\pi} + \frac{297}{\pi^2} - \frac{594\sqrt{3}}{\pi^3} + \frac{972}{\pi^4} \approx 1.3443$$

$$C^{(2)}_{\mathbf{T}} = -\frac{35}{8} + \frac{611\sqrt{3}}{4\pi} - \frac{4077}{2\pi^2} + \frac{3159\sqrt{3}}{\pi^3} - \frac{4860}{\pi^4} \approx -0.1296$$

$$C^{(3)}_{\mathbf{T}} = \frac{239}{4} - \frac{537\sqrt{3}}{\pi} + \frac{5031}{\pi^2} - \frac{6696\sqrt{3}}{\pi^3} + \frac{9720}{\pi^4} \approx -0.8286$$

$$C^{(4)}_{\mathbf{T}} = -\frac{599}{6} + \frac{1433\sqrt{3}}{2\pi} - \frac{5832}{\pi^2} + \frac{7074\sqrt{3}}{\pi^3} - \frac{9720}{\pi^4} \approx -0.3339$$

$$C^{(5)}_{\mathbf{T}} = \frac{247}{4} - \frac{841\sqrt{3}}{2\pi} + \frac{3240}{\pi^2} - \frac{3726\sqrt{3}}{\pi^3} + \frac{4860}{\pi^4} \approx -0.0497$$

$$C^{(6)}_{\mathbf{T}} = -\frac{105}{8} + \frac{363\sqrt{3}}{4\pi} - \frac{1395}{2\pi^2} + \frac{783\sqrt{3}}{\pi^3} - \frac{972}{\pi^4} \approx -0.0026$$

$$C^{(1)}_{\mathbf{H}} = \frac{3}{4} \qquad C^{(2)}_{\mathbf{H}} = 0 \qquad C^{(3)}_{\mathbf{H}} = -\frac{3}{4}$$

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Discussions

Ālea iacta est ("The die is cast").

—Julius Caesar

In this thesis we managed to study a special function of both bosonic and fermionic versions of the discrete Gaussian free field, namely the gradient squared, and we related them to the uniform spanning tree model and the Abelian sandpile model. We believe to have exhausted the study of their joint moments properties, at least in what respects the basic ingredients for their exact calculations, both in the discrete setting and in the limit. However, there are still many possible extensions and research paths for the future, as we will discuss in the following.

In Chapter 2, the idea of the proof of tightness in Proposition 2.33 is based on the application of a criterion by Furlan and Mourrat [39] for local Hölder and Besov spaces. The proof requires a precise control of the summability of k-point functions, which is provided by Theorem 2.6 and explicit estimates for double derivatives of the Green's function in a domain. Observe that the proof is based only on the growth of sums of moments at different points. Thus this technique can be generalized to prove tightness of other fields just by having information on these bounds, which is usually easier to obtain than the whole expression on the joint moments.

Regarding the convergence of finite-dimensional distributions in Proposition 2.34, note that this strategy can be generalized to prove convergence to white noise of other families of fields, given the relatively mild conditions that we used from the field in question. Among them, one only requires knowledge on bounds of sums of joint cumulants, the existence of an infinite volume measure, and the finiteness of the susceptibility constant. Note that similar scaling results were given for random fields on the lattice satisfying the FKG inequality in Newman [81].

As for the subsequent two chapters, as already mentioned, the connection between critical lattice models and CFTs is not fully understood even in the simpler setting (without log divergences), in which there is a range of natural candidates for the limiting CFTs. In the logCFT setting much less is understood, as fewer explicit candidates are identified. This thesis opens up the avenue for a deeper mathematical investigation of the possible connections between ASM, UST, fGFF and the associated logCFTs.

One important challenge is to determine the logarithmic fields describing higher heights and general observables of the ASM rigorously. There is strong evidence (Ruelle [88]) that the logarithmic field describing higher heights in the ASM does not belong to the free symplectic fermion theory (1.8). It would be interesting to push our methods in this direction to see whether they can provide some light on the logCFTs describing higher heights. So far, to the author's knowledge there has not been any exact computation regarding the joint moments of heights other than one.

In general, a much broader goal is at stake. In the words of Ruelle [87], the ultimate objective would be to prove the following statement:

The scaling limit of the measure infinite volume measure of the stationary state of the Abelian sandpile model is the field-theoretic measure of a two-dimensional logarithmic conformal field theory with central charge $c = -2$.

Given the recent advances in conformal field theory, we hope this conjecture is not far from being proved.

The possibility of universality is something we also consider worth pushing forward. As we have seen, our limiting joint moments results for the height-one field and the uniform spanning tree hold not only in $\mathbb{Z}^d$ but also in other lattices with very specific symmetry properties. Our cancellation techniques fail when those symmetries are absent, so we would need different tools to tackle the problem of more general lattices. In particular, isoradial graphs are a setting worth exploring given their nice properties for statistical mechanics systems (see Chelkak and Smirnov [22]). Proving our results in a more general lattice would give us the important property of universality.

The Fock space structure, especially in the fermionic case, is not something well understood from our perspective, and little literature is available. We believe that a proper understanding of convergence of fermionic fields in Fock spaces will shed some light on the nature of the fields studied in the limit.

Finally, we would like to mention the possibility of using *supersymmetry* techniques (see Swan [95]). Supersymmetry is a concept widely used in physics, which could in principle allow us to "convert" bosonic systems into fermionic ones and vice versa. A proper employment of this theory might, we suspect, aid us to obtain fermionic results "for free" only by studying its bosonic counterpart, and conversely.

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I would also like to dedicate part of this space to thank all members of the doctoral committee for the time dedicated to read and assess this thesis.

It has already been four years for me living in this beautiful country. But as gorgeous as tulips might be, what makes a country beautiful are the people that inhabit it. They say it is not easy to make friends and meaningful connections in a place that far away from one's original home (much less in the middle of a pandemic!). So I must have done something wrong, because I was always surrounded by people, in different cities, from different countries, met for different reasons, sharing different things, soms ook een beetje meer Nederlands aan het leren! I will not mention all the many names here, but those who I am talking to already know. Thank you for accompanying me. You didn't make me feel like home; rather, you made this place my home.

My friends. Los muchachos. Vuelvo a escribirles después de cinco años. Aquella vez escribí *"Algunos aún siguen acá presentes"*, y si bien esa frase ya quedó obsoleta, cuando interpretamos la palabra "acá" de manera relativa a la situación presente, la frase es más válida que nunca. Muchas veces me preguntan: *"¿Viviendo acá, no extrañás a tus amigos?"*, y automáticamente respondo *"¡Los extrañaría si estuviera en Argentina! Ahora los tengo más cerca que nunca."* Gracias por los viajes, por las risas, por los soportes mutuos para encarar nuestras vidas adultas de la manera más fructífera posible. Por estar.

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Finalmente, gracias a los que hoy ya no están: Bianca, mi abuela, Francisco.

Curriculum Vitæ

Alan Rapoport was born on November 15, 1994, in the city of Puerto Madryn, Argentina. Soon after he moved to Buenos Aires when he was three, where he spent his whole life before moving to the Netherlands for his Ph.D.

He studied both Physics and Electrical Engineering at the University of Buenos Aires, obtaining the degrees of *Licentiate* and *Engineer*. He completed his Physics thesis in the topic of quantum field theory under the supervision of Prof. Fidel Schaposnik†, the result of which ended in a publication in the journal Physics Letters B. He completed his Electrical Engineering thesis in stochastic traffic models under the supervision of Prof. Pablo A. Ferrari and Dr. Sebastián P. Grynberg.

In October 2020 he commenced his Ph.D. in TU Delft, under the supervision of Dr. Wioletta M. Ruszel and Dr. Alessandra Cipriani. In December 2021 he was transferred to Utrecht University, and stayed there until the completion of the degree. He did two stays of research in London, hosted by AC at University College London. In November 2024 he will be joining the private sector.

List of Publications

3. A. Rapoport. “Correlations in uniform spanning trees: a fermionic approach”. In: *arXiv* (2023). arXiv: 2312.14992

2. A. Cipriani, L. Chiarini, A. Rapoport and W. M. Ruszel. “Fermionic Gaussian free field structure in the Abelian sandpile model and uniform spanning tree”. In: *arXiv* (2023). arXiv: 2004.04720

1. A. Cipriani, R. S. Hazra, A. Rapoport and W. M. Ruszel. “Properties of the Gradient Squared of the Discrete Gaussian Free Field”. In: *Journal of Statistical Physics* 190.11 (2023), p. 171