The first negative Fourier coefficient of an Eisenstein series newform

Sebastián Carrillo Santana[1]

Abstract

There have been a number of papers on statistical questions concerning the sign changes of Fourier coefficients of newforms. In one such paper, Linowitz and Thompson gave a conjecture describing when, on average, the first negative sign of the Fourier coefficients of an Eisenstein series newform occurs. In this paper, we correct their conjecture and prove the corrected version.

Keywords Eisenstein series newforms · Fourier coefficients · The large sieve

Mathematics Subject Classification 11F30 · 11N36

1 Introduction

For a Dirichlet character χ and a positive integer N, we will denote by $M_k(N, \chi)$ the vector space of modular forms on $\Gamma_0(N)$ of weight k, level N and character χ. Let $E_k(N, \chi)$ be the subspace of Eisenstein series and $S_k(N, \chi)$ the subspace of cusp forms. For a prime p, we let T_p be the pth Hecke operator.

Let $H_k^*(N)$ be the subspace of $S_k(\chi_0, N)$ of newforms with trivial character χ_0. Given a newform $f \in H_k^*(N)$, let $\lambda_f(p)$ be the eigenvalue of f with respect to the Hecke operator T_p. The restriction to the trivial character ensures that the sequence $\{\lambda_f(p)\}$ is real. Many authors have studied the sequence of signs of the Hecke eigenvalues of f. For example, one could pose questions such as:

(i) Are there infinitely many primes p such that $\lambda_f(p) > 0$ (or $\lambda_f(p) < 0$)?
(ii) What is the first change of sign? More specifically, what is the smallest $n \geq 1$ (or prime p) such that $\lambda_f(n) < 0$ (or $\lambda_f(p) < 0$)? This is an analogue of the least quadratic non-residue problem.

✉ Sebastián Carrillo Santana
    s.carrillosantana@uu.nl

[1]   Mathematics Institute, Utrecht University, Hans Freudenthalgebouw, Budapestlaan 6, 3584 CD
      Utrecht, The Netherlands

(iii) Given an arbitrary sequence of signs $\varepsilon_p \in \{\pm 1\}$, what is the number of newforms f (in some family) such that $\operatorname{sgn} \lambda_f(p) = \varepsilon_p$ for all $p \leq x$?

In the cusp form setting, questions (i) and (ii) are answered in [5, 6], and [9]. In this paper, we focus on (iii). Kowalski et al. [6] obtained a lower bound for the proportion of newforms $f \in H_k^*(N)$ whose sequence of eigenvalues $\lambda_f(p)$ has signs coinciding with a prescribed sequence $\{\varepsilon_p\}$:

Theorem 1.1 (Kowalski, Lau, Soundararajan, Wu, 2010) *Let N be a squarefree number, $k \geq 2$ an even integer, and $\{\varepsilon_p\}$ a sequence of signs. Then, for any $0 < \varepsilon < \frac{1}{2}$, there exists some $c > 0$ such that*

$$\frac{1}{|H_k^*(N)|} \#\{f \in H_k^*(N) \ : \ \operatorname{sgn} \lambda_f(p) = \varepsilon_p \text{ for } p \leq z, \ p \nmid N\} \geq \left(\frac{1}{2} - \varepsilon\right)^{\pi(z)}$$

for $z = c\sqrt{\log kN \log \log kN}$ provided kN is large enough. Here $\pi(z)$ is the number of primes less than or equal to z.

Now, let χ_1, χ_2 be Dirichlet characters modulo N_1, N_2 and for an integer $k > 2$, define the following variant of the sum of divisors function:

$$\sigma_{\chi_1, \chi_2}^{k-1}(n) = \sum_{d|n} \chi_1\left(\frac{n}{d}\right) \chi_2(d) d^{k-1}. \tag{1.1}$$

Now assume that χ_1 and χ_2 are not simultaneously principal (mod 1). It is well known (see, for example, [3]) that if $\chi_1 \chi_2(-1) = (-1)^k$, then the function

$$E_k(\chi_1, \chi_2, z) := \frac{\delta(\chi_1)}{2} L(1 - k, \chi_2) + \sum_{n \geq 1} \sigma_{\chi_1, \chi_2}^{k-1}(n) q^n,$$

is an Eisenstein series of weight k, level $N_1 N_2$ and character $\chi_1 \chi_2$. Here $q = e^{2\pi i z}$ and

$$\delta(\chi_1) = \begin{cases} 1, & \text{if } \chi_1 \text{ is principal} \\ 0, & \text{otherwise.} \end{cases}$$

In 1977, Weisinger [11] developed a newform theory for $E_k(N, \chi)$ analogous to the one developed by Atkin and Lehner [1] for cusp forms. In this theory, we have:

- The newforms of $E_k(N, \chi)$ are functions of the form $E_k(\chi_1, \chi_2, z)$ for which $N = N_1 N_2$, $\chi = \chi_1 \chi_2$, and χ_1, χ_2 are primitive.
- The eigenvalue of $E_k(\chi_1, \chi_2, z)$ with respect to the Hecke operator T_p is $\sigma_{\chi_1, \chi_2}^{k-1}(p)$. In other words, the eigenvalues of this type of Eisenstein series coincide with its Fourier coefficients.

By exploiting the analytical properties of $\sigma_{\chi_1, \chi_2}^{k-1}(n)$, Linowitz and Thompson [8] answered the three questions mentioned at the beginning of this article for Eisenstein series newforms.

Note that by (1.1), $\sigma_{\chi_1,\chi_2}^{k-1}(n) \in \mathbb{R}$ when χ_1, χ_2 are real characters. Since we want $E_k(\chi_1, \chi_2, z)$ to be an Eisenstein series, we exclude the case when χ_1 and χ_2 are principal. We call these types of characters *quadratic* because for every fundamental discriminant D, i.e., for each discriminant arising from a quadratic number field, we can associate a real character defined by $\chi_D(m) = \left(\frac{D}{m}\right)$. Therefore, counting Eisenstein series newforms of level $N \leq x$ is equivalent to counting fundamental discriminants D_1, D_2 with $|D_1 D_2| \leq x$. Let

$$\mathscr{D} := \{(D_1, D_2) \ : \ |D_1 D_2| \leq x\}.$$

Taking all of these facts into consideration, Linowitz and Thompson [8] showed:

Theorem 1.2 (Linowitz, Thompson, 2015) *Let $\{p_1, \ldots, p_k\}$ be a sequence of primes and $\{\varepsilon_{p_1}, \ldots, \varepsilon_{p_k}\} \in \{-1, 0, 1\}$ a sequence of signs. Then,*

$$\frac{1}{|\mathscr{D}|}\#\{(D_1, D_2) \in \mathscr{D} \ : \ \operatorname{sgn} \sigma_{\chi_1,\chi_2}^{k-1}(p_i) = \varepsilon_{p_i}, \ 1 \leq i \leq k\}$$

$$\xrightarrow[x\to\infty]{} \prod_{\substack{\varepsilon_{p_i}=0 \\ 1\leq i\leq k}} \frac{1}{(p_i + 1)^2} \prod_{\substack{\varepsilon_{p_i}\neq 0 \\ 1\leq i\leq k}} \frac{p_i(p_i + 2)}{2(p_i + 1)^2}.$$

Now, let $\eta(D_1, D_2)$ represent the smallest prime p such that $\operatorname{sgn}(\sigma_{\chi_1,\chi_2}^{k-1}(p)) = -1$. Linowitz and Thompson [8] then conjectured:

Conjecture 1.3 We have

$$\frac{\sum_{|D_1 D_2|\leq x} \eta(D_1, D_2)}{\sum_{|D_1 D_2|\leq x} 1} \xrightarrow[x\to\infty]{} \theta,$$

where

$$\theta := \sum_{k=1}^{\infty} \frac{p_k^2(p_k + 2)}{2(p_k + 1)^2} \prod_{j=1}^{k-1} \frac{2 + p_j(p_j + 2)}{2(p_j + 1)^2} \approx 3.9750223902\ldots \quad (1.2)$$

They gave a heuristic argument as evidence towards their conjecture, showing:

$$\frac{\sum_{|D_1 D_2|\leq x} \eta(D_1, D_2)}{\sum_{|D_1 D_2|\leq x} 1} \xrightarrow[x\to\infty]{} \sum_{k=1}^{\infty} p_k \operatorname{Prob}(\eta(D_1, D_2) = p_k)$$

$$= \sum_{k=1}^{\infty} p_k \operatorname{Prob}(\varepsilon_{p_k} = -1) \prod_{i=1}^{k-1} \operatorname{Prob}(\varepsilon_{p_i} = 0 \text{ or } 1)$$

$$= \sum_{k=1}^{\infty} \frac{p_k^2(p_k + 2)}{2(p_k + 1)^2} \prod_{i=1}^{k-1} \left(\frac{1}{(p_i + 1)^2} + \frac{p_i(p_i + 2)}{2(p_i + 1)^2}\right),$$

where the last equality follows from Theorem 1.2. The problem with this argument is that Theorem 1.2 fixes a set of primes and then lets $x \to \infty$. In this argument we

need to allow the primes to tend to infinity with x. The authors stated: "[W]e have a good understanding of the effect of the small primes, but one would need to argue that the primes after some cutoff point do not make much of an impact on the average. Presumably, this would require using the large sieve".

The goal of the present article is to correct their conjecture by proving the following result:

Theorem 1.4 *We have*

$$\frac{\sum_{|D_1 D_2| \le x} \eta(D_1, D_2)}{\sum_{|D_1 D_2| \le x} 1} \xrightarrow[x \to \infty]{} \Theta \cdot (1 - \beta) + \alpha,$$

where

$$\Theta = \sum_{k=1}^{\infty} \frac{p_k^2}{2(p_k + 1)} \prod_{j=1}^{k-1} \frac{p_j + 2}{2(p_j + 1)},$$

$$\alpha = \sum_{k=1}^{\infty} \frac{p_k^2}{2(p_k + 1)^2} \prod_{j=1}^{k-1} \frac{p_j + 2}{2(p_j + 1)},$$

and

$$\beta = \sum_{k=1}^{\infty} \frac{p_k}{2(p_k + 1)^2} \prod_{j=1}^{k-1} \frac{p_j + 2}{2(p_j + 1)}.$$

Numerically,

$$\Theta \cdot (1 - \beta) + \alpha \approx 4.63255603509332\ldots$$

The numerical computation was done using Sage. We used RIF for interval arithmetic and we truncated at $k = 1000$.

2 Main Tools

First we will need asymptotic estimates for some sets of fundamental discriminants. It is well known (see, for example, [2]) that

$$\sum_{|D| \le x} 1 \sim \frac{x}{\zeta(2)}, \tag{2.1}$$

where D runs over all fundamental discriminants with $|D| \le x$. Here ζ is the Riemann zeta function. Now, let $n_1(m)$ be the smallest integer $n \ge 1$ relatively prime to m such that the congruence $x^2 \equiv n \pmod{m}$ has no solutions. Even though Vinogradov's

conjecture remains open, it is possible to show that large values of $n_1(p)$ are rare. More specifically, using the large sieve, Linnik [7] showed that for all $\varepsilon > 0$, we have

$$\#\{p \leq x \,:\, n_1(p) > x^\varepsilon\} \ll_\varepsilon 1.$$

Using similar ideas to the ones from Linnik's paper, Erdős [4] obtained a result concerning the average of $n_1(p)$ as p varies over prime numbers less than or equal to x:

$$\frac{1}{\pi(x)} \sum_{p \leq x} n_1(p) \xrightarrow{x \to \infty} \sum_{k=1}^{\infty} \frac{p_k}{2^k}, \tag{2.2}$$

where p_k is the kth prime and $\pi(x)$ is the prime counting function. In a similar fashion, Pollack [10] considered a variation of (2.2). We summarize his result in the following theorem:

Theorem 2.1 (Pollack, 2012) *For each fundamental discriminant D, let χ_D be the associated Dirichlet character, i.e., $\chi_D(m) := \left(\frac{D}{m}\right)$. For each character χ, let n_χ denote the least n for which $\chi(n) \notin \{0, 1\}$. Finally, let $n(D) := n_{\chi_D}$. Then*

(i) *Uniformly in k such that the kth prime satisfies $p_k \leq (\log x)^{\frac{1}{3}}$, we have*

$$\#\{|D| \leq x \,:\, n(D) = p_k\} = \frac{p_k}{2(p_k + 1)} \prod_{j=1}^{k-1} \frac{p_j + 2}{2(p_j + 1)} \frac{x}{\zeta(2)} + O(x^{\frac{2}{3}}).$$

(ii)

$$\sum_{\substack{|D| \leq x \\ n(D) > (\log x)^{\frac{1}{3}}}} n(D) = o(x).$$

Therefore, using (2.1), we have

$$\frac{\sum_{|D| \leq x} n(D)}{\sum_{|D| \leq x} 1} \xrightarrow{x \to \infty} \Theta, \tag{2.3}$$

where

$$\Theta := \sum_{k=1}^{\infty} \frac{p_k^2}{2(p_k + 1)} \prod_{j=1}^{k-1} \frac{p_j + 2}{2(p_j + 1)} \approx 4.9809473396\ldots$$

We will also need the following lemma from Linowitz and Thompson [8]:

Lemma 2.2 *Let $\mathbb{P}(\varepsilon, p)$ denote the proportion of fundamental discriminants D with $\left(\frac{D}{p}\right) = \varepsilon$. Then, we have*

$$\mathbb{P}(\varepsilon, p) = \begin{cases} \frac{p}{2p+2}, & \text{if } \varepsilon \in \{\pm 1\} \\ \frac{1}{p+1}, & \text{if } \varepsilon = 0. \end{cases}$$

3 Proof of Theorem 1.4

Let χ_1, χ_2 be Dirichlet characters associated with the fundamental discriminants D_1 and D_2. For a prime p,

$$\sigma_{\chi_1,\chi_2}^{k-1}(p) = \sum_{d|p} \chi_1\left(\frac{p}{d}\right)\chi_2(d)d^{k-1} = \chi_1(p) + \chi_2(p)p^{k-1},$$

so that

$$\operatorname{sgn}\sigma_{\chi_1,\chi_2}^{k-1}(p) = \begin{cases} \chi_1(p), & \text{if } p|D_2 \\ \chi_2(p), & \text{otherwise.} \end{cases} \tag{3.1}$$

Proof of Theorem 1.4 By (2.1),

$$\sum_{|D_1 D_2|\leq x} 1 = \sum_{|D_1|\leq x}\sum_{D_2\leq\frac{x}{|D_1|}} 1 \sim \frac{x}{\zeta(2)}\sum_{|D_1|\leq x}\frac{1}{|D_1|}.$$

Let $A(x) := \sum_{|D|\leq x} 1$ and $f(x) := \frac{1}{x}$. Since $A(x) \sim \frac{x}{\zeta(2)}$, then by partial summation

$$\sum_{|D_1|\leq x}\frac{1}{|D_1|} = A(x)f(x) - A(1)f(1) - \int_1^x A(t)f'(t)\mathrm{d}t$$

$$\sim \frac{1}{\zeta(2)} - 1 + \int_1^x \frac{\mathrm{d}t}{\zeta(2)t}$$

$$\sim \frac{\log x}{\zeta(2)}. \tag{3.2}$$

Hence

$$\sum_{|D_1 D_2|\leq x} 1 \sim \frac{x\log x}{\zeta(2)^2}. \tag{3.3}$$

Now let us estimate the numerator. For the sake of simplicity, let $\eta := \eta(D_1, D_2)$. Then,

$$\sum_{|D_1 D_2|\leq x}\eta = \sum_{\substack{|D_1 D_2|\leq x \\ \eta|D_2}}\eta + \sum_{\substack{|D_1 D_2|\leq x \\ \eta\nmid D_2}}\eta.$$

If $\eta|D_2$, then by (3.1), η is the smallest prime p such that $\chi_1(p) \notin \{0, 1\}$, and with the notation of Theorem 2.1, this means that $\eta = n(D_1)$. Similarly, if $\eta \nmid D_2$, then $\eta = n(D_2)$. Therefore,

$$\sum_{|D_1 D_2|\leq x}\eta = \sum_{\substack{|D_1 D_2|\leq x \\ \eta|D_2}}n(D_1) + \sum_{\substack{|D_1 D_2|\leq x \\ \eta\nmid D_2}}n(D_2).$$

Now,

$$\sum_{\substack{|D_1 D_2| \le x \\ \eta \nmid D_2}} n(D_2) = \sum_{|D_1 D_2| \le x} n(D_2) - \sum_{\substack{|D_1 D_2| \le x \\ \eta | D_2}} n(D_2),$$

so that

$$\sum_{|D_1 D_2| \le x} \eta = \sum_{|D_1 D_2| \le x} n(D_2) + \sum_{\substack{|D_1 D_2| \le x \\ \eta | D_2}} n(D_1) - \sum_{\substack{|D_1 D_2| \le x \\ \eta | D_2}} n(D_2). \qquad (3.4)$$

By (2.3), we have

$$\sum_{|D_1 D_2| \le x} n(D_2) = \sum_{|D_1| \le x} \sum_{|D_2| \le \frac{x}{|D_1|}} n(D_2)$$

$$\sim \Theta \frac{x}{\zeta(2)} \sum_{|D_1| \le x} \frac{1}{|D_1|}$$

$$\sim \Theta \frac{x \log x}{\zeta(2)^2}, \qquad (3.5)$$

where the final estimate follows from (3.2). Now, by Lemma 2.2, the proportion of fundamental discriminants such that $p | D$ is $\frac{1}{p+1}$. Hence,

$$\sum_{\substack{|D_1 D_2| \le x \\ \eta | D_2}} n(D_1) = \sum_{|D_1| \le x} n(D_1) \sum_{\substack{|D_2| \le \frac{x}{|D_1|} \\ n(D_1) | D_2}} 1$$

$$= \sum_{|D_1| \le x} \frac{n(D_1)}{n(D_1) + 1} \sum_{|D_2| \le \frac{x}{|D_1|}} 1$$

$$\sim \frac{x}{\zeta(2)} \sum_{|D_1| \le x} \frac{n(D_1)}{|D_1|(n(D_1) + 1)}.$$

To find an asymptotic for the last sum we again use partial summation. Let

$$B(x) := \sum_{|D_1| \le x} \frac{n(D_1)}{n(D_1) + 1}.$$

Then, by (i) of Theorem 2.1,

$$\sum_{\substack{|D_1| \le x \\ n(D_1) \le (\log x)^{\frac{1}{3}}}} \frac{n(D_1)}{n(D_1) + 1} = \sum_{\substack{k=1 \\ p_k \le (\log x)^{\frac{1}{3}}}}^{\infty} \frac{p_k}{p_k + 1} \#\{|D_1| \le x \ : \ n(D_1) = p_k\}$$

$$\sim \alpha \frac{x}{\zeta(2)}$$

where

$$\alpha = \sum_{k=1}^{\infty} \frac{p_k^2}{2(p_k+1)^2} \prod_{j=1}^{k-1} \frac{p_j+2}{2(p_j+1)}.$$

Now, by (ii) of Theorem 2.1,

$$\sum_{\substack{|D_1|\leq x \\ n(D_1)>(\log x)^{\frac{1}{3}}}} \frac{n(D_1)}{n(D_1)+1} \leq \sum_{\substack{|D_1|\leq x \\ n(D_1)>(\log x)^{\frac{1}{3}}}} n(D_1) = o(x).$$

Hence,

$$B(x) = \sum_{\substack{|D_1|\leq x \\ n(D_1)\leq(\log x)^{\frac{1}{3}}}} \frac{n(D_1)}{n(D_1)+1} + \sum_{\substack{|D_1|\leq x \\ n(D_1)>(\log x)^{\frac{1}{3}}}} \frac{n(D_1)}{n(D_1)+1} \sim \alpha \frac{x}{\zeta(2)}.$$

Therefore,

$$\sum_{|D_1|\leq x} \frac{n(D_1)}{|D_1|(n(D_1)+1)} = B(x)f(x) - B(1)f(1) - \int_1^x B(t)f'(t)\mathrm{d}t \sim \alpha \frac{\log x}{\zeta(2)},$$

so that

$$\sum_{\substack{|D_1 D_2|\leq x \\ \eta|D_2}} n(D_1) \sim \alpha \frac{x \log x}{\zeta(2)^2}. \tag{3.6}$$

Finally,

$$\sum_{\substack{|D_1 D_2|\leq x \\ \eta|D_2}} n(D_2) = \sum_{|D_2|\leq x} n(D_2) \sum_{|D_1|\leq \frac{x}{|D_2|}} \frac{1}{n(D_1)+1}.$$

To get an estimate for the inner sum, let

$$C(x) := \sum_{|D_1|\leq \frac{x}{|D_2|}} \frac{1}{n(D_1)+1}.$$

Then, by (i) of Theorem 2.1,

$$\sum_{\substack{|D_1| \le \frac{x}{|D_2|} \\ n(D_1) \le (\log x)^{\frac{1}{3}}}} \frac{1}{n(D_1) + 1} = \sum_{\substack{k=1 \\ p_k \le (\log x)^{\frac{1}{3}}}}^{\infty} \frac{1}{p_k + 1} \#\{|D_1| \le \frac{x}{|D_2|} : n(D_1) = p_k\}$$

$$\sim \beta \frac{x}{\zeta(2)|D_2|}$$

where

$$\beta = \sum_{k=1}^{\infty} \frac{p_k}{2(p_k + 1)^2} \prod_{j=1}^{k-1} \frac{p_j + 2}{2(p_j + 1)}.$$

On the other hand,

$$\sum_{\substack{|D_1| \le \frac{x}{|D_2|} \\ n(D_1) > (\log x)^{\frac{1}{3}}}} \frac{1}{n(D_1) + 1} \le \sum_{|D_1| \le \frac{x}{|D_2|}} \frac{1}{(\log x)^{\frac{1}{3}} + 1}$$

$$\sim \frac{x}{|D_2|\zeta(2)((\log x)^{\frac{1}{3}} + 1)}$$

$$\le \frac{x}{(\log x)^{\frac{1}{3}} + 1}$$

$$= o(x).$$

Hence,

$$C(x) = \sum_{\substack{|D_1| \le \frac{x}{|D_2|} \\ n(D_1) \le (\log x)^{\frac{1}{3}}}} \frac{1}{n(D_1) + 1} + \sum_{\substack{|D_1| \le \frac{x}{|D_2|} \\ n(D_1) > (\log x)^{\frac{1}{3}}}} \frac{1}{n(D_1) + 1} \sim \beta \frac{x}{\zeta(2)|D_2|}.$$

From this we see that

$$\sum_{\substack{|D_1 D_2| \le x \\ \eta | D_2}} n(D_2) \sim \sum_{|D_2| \le x} \beta \frac{n(D_2)x}{\zeta(2)|D_2|} \sim \Theta\beta \frac{x \log x}{\zeta(2)^2}, \tag{3.7}$$

where the last estimate follows from partial summation and applying Theorem 2.1. Therefore, plugging (3.5), (3.6) and (3.7) into (3.4) shows that

$$\sum_{|D_1 D_2| \le x} \eta \sim (\Theta + \alpha - \Theta\beta) \frac{x \log x}{\zeta(2)^2}.$$

This together with (3.3) completes the proof. □

Remark 3.1 We can give the following explanation of why Linowitz and Thompson's Conjecture 1.3 was slightly off from the correct number: the result from Theorem 1.2 is not uniform in k for the choice of the p_k (we fix a set of primes beforehand), while the result from Theorem 2.1 is uniform in k satisfying $p_k \leq (\log x)^{\frac{1}{3}}$. In order to make Linowitz and Thompson's heuristic argument rigorous we would first need to show that Theorem 1.2 holds uniformly in k such that $p_k \leq f(x)$ for some function f with $f(x) \xrightarrow[x \to \infty]{} \infty$. Then,

$$
\frac{\sum_{|D_1 D_2| \leq x} \eta(D_1, D_2)}{\sum_{|D_1 D_2| \leq x} 1} = \sum_{p_k \leq f(x)} p_k \operatorname{Prob}(\eta(D_1, D_2) = p_k)
$$
$$
+ \sum_{p_k > f(x)} p_k \operatorname{Prob}(\eta(D_1, D_2) = p_k)
$$
$$
\xrightarrow[x \to \infty]{} \theta + \mu,
$$

where θ is the conjectured constant (1.2) and

$$
\mu = \lim_{x \to \infty} \sum_{p_k > f(x)} p_k \operatorname{Prob}(\eta(D_1, D_2) = p_k).
$$

Linowitz and Thompson conjectured that $\mu = 0$, but according to Theorem 1.4, μ does make a small contribution.

Declarations

References

1. Atkin, A.O.L., Lehner, J.: Hecke operators on $\Gamma_0(m)$. Math. Ann. **185**, 134–160 (1970)
2. Cohen, H., Diaz, F.D., Olivier, M.: Counting discriminants of number fields. J. Théor. Nombres Bordeaux **18**(3), 573–593 (2006)
3. Diamond, F., Shurman, J.: A First Course in Modular Forms (Graduate Texts in Mathematics), vol. 228. Springer, New York (2005)

4. Erdős, P.: Remarks on number theory. I. Mat. Lapok. **12**, 10–17 (1961)
5. Kohnen, W., Sengupta, J.: On the first sign change of Hecke eigenvalues of newforms. Math. Z. **254**(1), 173–184 (2006)
6. Kowalski, E., Lau, Y.-K., Soundararajan, K., Wu, J.: On modular signs. Math. Proc. Cambridge Philos. Soc. **149**(3), 389–411 (2010)
7. Linnik, U.V.: A remark on the least quadratic non-residue. C. R. (Doklady) Acad. Sci. URSS (N.S.) **36**, 119–120 (1942)
8. Linowitz, B., Thompson, L.: The sign changes of Fourier coefficients of Eisenstein series. Ramanujan J. **37**(2), 223–241 (2015)
9. Matomäki, K.: On signs of Fourier coefficients of cusp forms. Math. Proc. Cambridge Philos. Soc. **152**(2), 207–222 (2012)
10. Pollack, P.: The average least quadratic nonresidue modulo m and other variations on a theme of Erdős. J. Number Theory **132**(6), 1185–1202 (2012)
11. Weisinger, J.: Some Results on Classical Eisenstein Series and Modular Forms over Function Fields. ProQuest LLC, Ann Arbor, MI (1977) (Thesis (Ph.D.)-Harvard University)