

INCOMPLETENESS OF BOUNDEDLY AXIOMATIZABLE THEORIES

ALI ENAYAT AND ALBERT VISSER

ABSTRACT. Our main result (Theorem A) shows the incompleteness of any consistent sequential theory T formulated in a finite language such that T is axiomatized by a collection of sentences of bounded quantifier-alternation-depth. Our proof employs an appropriate reduction mechanism to rule out the possibility of completeness by simply invoking Tarski’s Undefinability of Truth theorem. We also use the proof strategy of Theorem A to obtain other incompleteness results (as in Theorems A^+ , B and B^+).

Our main result (Theorem A) was prompted by the following question¹ of Steffen Lempp and Dino Rossegger; the question arose in the context of their joint work [AGLRZ] with Uri Andrews, David Gonzalez, and Hongyu Zhu, in which they establish: *For a complete first-order theory T , the set of models of T is Π_ω^0 -complete under Wadge reducibility (i.e., reducibility via continuous functions) if and only if T does not admit a first-order axiomatization by formulae of bounded quantifier complexity.* In what follows PA^- is the well-known finitely axiomatized fragment of PA (Peano Arithmetic) whose axioms describe the non-negative substructure of discretely ordered rings (with no instance of the induction scheme, hence the minus superscript), as in Kaye’s text [K] on models of PA.

Question \diamond . *Is there a consistent completion of PA^- that is axiomatized by a set of sentences of bounded quantifier complexity?*

Remark 1. It is well-known that the answer to the above question is in the negative when PA^- is strengthened to PA. This result follows from a theorem of Rabin [Ra] that states that for each $n \in \omega$ no consistent extension of PA (in the same language) is axiomatized by a set of Σ_n -sentences.² Rabin’s result refines an earlier theorem of Ryll-Nardzewski [Ry] that states that no consistent extension of PA is finitely axiomatizable. Ryll-Nardzewski and Rabin both employed model-theoretic arguments relying on nonstandard elements to prove the aforementioned results (see Theorem 10.2 on p.132 of Kaye’s text [K] for a modern treatment). Rabin’s result can be also established with an argument that mixes proof-theoretic machinery, partial satisfaction classes, and Gödel’s second incompleteness theorem (see Theorem 2.36 of Chapter III of [HP] for an exposition). As shown by Montague [Mon], a similar result can be established for any *inductive* sequential theory T , i.e., a sequential theory that has the power to prove the full scheme of induction over its ‘natural numbers’ for all formulae in the language of T (this involves a designated interpretation of a suitable base theory of the natural numbers). In this more general setting the relevant hierarchy is based on the depth of quantifier alternations; canonical examples of inductive sequential theories include all extensions of PA, Z (Zermelo set theory), Z_2 (second order arithmetic), and KM (Kelley-Morse theory of classes).

Theorems A and A^+ are formulated for sequential theories. At first approximation, a theory is sequential if it supports a modicum of coding machinery to handle finite sequences of all objects in the domain of discourse. Sequentiality is a modest demand for theories of arithmetic and set theory; however, by a theorem of Visser [V-2], (Robinson’s) Q is not sequential. There are many equivalent definitions of sequentiality; the original definition due to Pudlák (used by Jeřábek [J-1] in his proof of sequentiality of PA^-) is as follows: A theory T is sequential if there is a formula $N(x)$, together with appropriate formulae providing interpretations of equality, and the operations of successor, addition, and multiplication for elements satisfying $N(x)$ such that T proves the translations of the axioms of Q (Robinson’s arithmetic) when relativized to $N(x)$; and additionally, there is a formula $\beta(x, i, w)$ (whose intended meaning is that x is the i -th element of a sequence w) such that T proves that every sequence can be extended by any given element of the domain of discourse, i.e., T proves:

¹Thanks to Roman Kossak for bringing this interesting question to our attention. We are also indebted to Mateusz Lelyk, Dino Rossegger, and Saeed Salehi for their helpful feedback on the preliminary drafts of this paper, and to Emil Jeřábek for his perceptive remarks on the penultimate draft.

²In his paper Rabin points out that this result was possibly known (but not published) by others, including Feferman, Wang, Scott, Kreisel, and Tennenbaum.

$$\forall w, x, k \exists w' \forall i, y \left[[N(k) \wedge i \leq k] \rightarrow \left[\begin{array}{c} \beta(y, i, w') \leftrightarrow \\ [i < k \wedge \beta(y, i, w)] \vee [i = k \wedge y = x] \end{array} \right] \right].$$

For more information about sequentiality, see [V-4].

In light of the aforementioned proof³ of sequentiality PA^- by Jeřábek [J-1], the following general result answers Question \diamond in the negative. Note the condition of finiteness of the language in Theorem A cannot be eliminated, as indicated in Remark 4.

- Throughout the paper Σ_n^* is the hierarchy of formulae whose measure of complexity is *depth of quantifier alternation*, as in [V-4].

Theorem A. *For any fixed $n \in \omega$, every consistent sequential theory formulated in a finite language that is axiomatized by a set of Σ_n^* -sentences is incomplete.*

Remark 2. The proofs of our other theorems are all based on the proof of Theorem A; we provide a direct proof of this theorem, employing a middle range level of abstraction, addressed to a non-specialist reader. The three ingredients of the proof are (1) partial satisfaction predicates, (2) Rosser's 'trick' used in his celebrated generalization of Gödel's first incompleteness theorem in which the technical hypothesis of ω -consistency (more precisely: 1-consistency) is removed; and (3) Tarski's Undefinability of Truth theorem⁴. Partial satisfaction predicates were first introduced by Mostowski [Mos] in his proof demonstrating that PA is essentially reflexive (i.e., PA and all of its extensions in the same language can prove the formal consistency of each of their finitely axiomatized subtheories). The technology of partial satisfaction predicates was further elaborated in the context of sequential theories by Montague [Mon], Pudlák [P-2], and finally Visser [V-4], whose variant we rely upon in our proof. As we will see, our proof shows that for complete sequential theories formulated in a finite language that are axiomatized by a set of Σ_n^* statements, the following equation holds:

$$\text{Rosser provability from True } \Sigma_n^* \text{ statements} = \text{Truth.}$$

Thus the only use of diagonalization in our proof is the rudimentary Tarskian one. It should be stressed that upon the completion of our paper we were informed by Emil Jeřábek that he struck upon the same idea in 2016 and established an abstract version of the above equation in a MathOverflow answer (as explained in Remark 6). In the subsequent discussions with Emil, he kindly expressed a preference for us to proceed with the publication of the paper with the minimal proviso that his aforementioned prior work be referenced, which is precisely the course of action the authors settled on.

There is also a conceptual/pedagogical take-away to our approach. Let \mathbf{R} be the well-known fragment of PA introduced in the Tarski-Mostowski-Robinson monograph [TMR] within which all recursive functions are representable. One can prove the (first) incompleteness theorem for a consistent computably enumerable extension T of \mathbf{R} , without any extra soundness assumptions about T , by first proving Tarski's undefinability of truth theorem with a straightforward diagonalization (with no need for the fixed point theorem, as in [TMR, p.46]), and then the incompleteness of T can be demonstrated using a reductio ad absurdum by verifying that the completeness of T implies that Rosser provability from T yields a truth definition⁵ (technically, this falls under our Theorem A⁺, by setting $A = \emptyset$ in that theorem). Note that in contrast to the usual proof of the incompleteness theorem using the fixed point theorem, our proof is not constructive, i.e., it does not yield an algorithm that takes a description of a consistent computably enumerable extension T of \mathbf{R} as input and outputs a sentence that is independent of T .

Remark 3. As an alternative to the proof of Theorem A presented below, one can also derive Theorem A (using Fact F below) from a version of Rosser's Theorem due to Saeed Salehi; see [S]. The version proposed by Salehi holds under certain abstract conditions. More specifically, viewed as an application of Salehi's result, our

³Indeed Jeřábek's result is stronger since it establishes the sequentiality of a weaker theory than the usual formulation of PA^- . This weaker theory is a universal theory and described by Jeřábek as the theory of discretely ordered commutative semirings with a least element, a theory in which the existence of predecessors is unprovable.

⁴Tarski's theorem [TMR, p.46] is very general; it states that if T is a theory formulated in a language \mathcal{L} has the property that the diagonal function $\varphi(x) \mapsto \varphi(\ulcorner \varphi \urcorner)$ is representable in T , then there is no \mathcal{L} -formula $V(x)$ such that T proves $\psi \leftrightarrow V(\ulcorner \psi \urcorner)$ for all \mathcal{L} -sentences ψ . It is well-known that the diagonal function is representable in the theory \mathbf{R} of [TMR]; which in turn makes it clear that the diagonal function is representable in sequential theories formulated in a finite language since (Robinson's) \mathbf{Q} , and a fortiori \mathbf{R} , is interpretable in sequential theories.

⁵Using additional machinery, Gödel's second incompleteness theorem can also be derived from Tarski's Undefinability of Truth theorem; see [V-3].

proof presented gives a realization of these conditions in the case at hand and a verification that our realization works. Our proof follows an alternative route to the Rosser-style result since we employ a reduction to Tarski's undefinability of truth, where Salehi presents an argument that follows the traditional Rosser argument more directly. Our different route can also be employed at the level of abstraction of Salehi's work. Also, as kindly pointed out by Mateusz Lelyk, Theorem A follows from Proposition 31 of his joint recent work [LW] with Bartosz Wcisłó (asserting the existence of so-called (n, k) -flexible formulae for computably enumerable sequential theories).

Before presenting the proof of Theorem A, we state an important fact that plays a crucial role in the proof of Theorem A. The following result was established by Visser in [V-1], and refined in [V-4]; this result refines the work of Pudlák in [P-1] and [P-2] in which *logical depth* (length of the longest branch in the formation tree of the formula) is used as a measure of complexity instead of the depth of quantifier alternations complexity. Note that part (b) of the fact below is an immediate consequence of part (a).

Fact F. *Suppose T is a sequential theory T formulated in a finite language \mathcal{L} , and fix $n \in \omega$. Fix some interpretation \mathcal{N} of arithmetic in T satisfying $\text{I}\Delta_0$.⁶*

(a) *There is a T -provable definable cut I_n of \mathcal{N} and a formula $\text{Sat}_n(x, y)$ such that, provably in T , Sat_n satisfies the Tarskian compositional clauses for Σ_n^* -formulae in I_n (and for all variable assignments).*

(b) *There is a formula $\text{True}_n(x)$ such that, provably in T , $\text{True}_n(x)$ is extensional⁷, i.e., it respects the equivalence relation representing equality in the interpretation \mathcal{N} ; and for all models $\mathcal{M} \models T$, and for all Σ_n^* -sentences ψ , we have:*

$$\mathcal{M} \models (\psi \leftrightarrow \text{True}_n(\ulcorner \psi \urcorner)).$$

Proof of Theorem A. Suppose not, and let T be consistent completion of sequential theory formulated in a finite language \mathcal{L} . Then by the definition of sequentiality T is also sequential. Suppose to the contrary that for some $n \in \omega$, T is axiomatized by a set of Σ_n^* sentences, i.e., suppose (1) below:

(1) For some $n \in \omega$, there is a set A of Σ_n^* sentences such that for all \mathcal{L} -sentences ψ , $\psi \in T$ iff $A \vdash \psi$.

Our proof by contradiction of Theorem A will be complete once we verify Claim \heartsuit below since it contradicts Tarski's venerable Undefinability of Truth theorem.

CLAIM \heartsuit . There is a unary \mathcal{L} -formula $\varphi(x)$ such that for all \mathcal{L} -sentences ψ , $T \vdash \psi \leftrightarrow \varphi(\ulcorner \psi \urcorner)$.

Since T is sequential, we can find an \mathcal{L} -formula, denoted $\text{Prf}_{\text{True}_n}(\pi, x)$, that expresses “ π is (the code for) a first order proof of x from assumptions in True_n (i.e., from $\{x : \text{True}_n(x)\}$). In particular, for each *standard* \mathcal{L} -sentence ψ and *standard* π , and each model \mathcal{M} of T , we have:

(2) $\mathcal{M} \models \text{Prf}_{\text{True}_n}(\pi, \ulcorner \psi \urcorner)$ iff π is (a code for) a proof of ψ from $\text{True}_n^{\mathcal{M}} := \{\varphi : \mathcal{M} \models \text{True}_n(\ulcorner \varphi \urcorner)\}$.

Our proposed candidate of $\varphi(x)$ for establishing Claim \heartsuit is the following formula $\rho(x)$; our choice of the letter ρ indicates the fact that the formula expresses *Rosser-provability* (from the true Σ_n^* sentences).

$$\rho(x) := \exists y [\text{Prf}_{\text{True}_n}(y, x) \wedge \forall z < y \neg \text{Prf}_{\text{True}_n}(z, \neg x)].$$

Thus our goal is to show that for all \mathcal{L} -sentences ψ , $T \vdash \psi \leftrightarrow \rho(\ulcorner \psi \urcorner)$. By the completeness theorem for first order logic it suffices to show that for each model⁸ \mathcal{M} of T , $\mathcal{M} \models \psi \leftrightarrow \rho(\ulcorner \psi \urcorner)$. For the rest of the proof, let $\mathcal{M} \models T$. We will first show:

(3) For all \mathcal{L} -sentences ψ , $\mathcal{M} \models \psi \rightarrow \rho(\ulcorner \psi \urcorner)$.

To show (3), assume ψ holds in \mathcal{M} . Let A be as in (1), and note that $A \subseteq \text{True}_n^{\mathcal{M}}$. By the assumptions about T , there are finitely many sentences $\alpha_1, \dots, \alpha_n$ in A such that $\{\alpha_1, \dots, \alpha_n\} \vdash \psi$. Let $\pi_0 \in \omega$ be (the code of) a proof of ψ from $\{\alpha_1, \dots, \alpha_n\}$. Thanks to (2) we have:

(4) $\mathcal{M} \models \text{Prf}_{\text{True}_n}(\pi_0, \ulcorner \psi \urcorner)$.

The assumption of consistency of T coupled with (2) yields:

(5) $\mathcal{M} \models \forall z < \pi_0 \neg \text{Prf}_{\text{True}_n}(z, \ulcorner \psi \urcorner)$.

⁶It is well-known \mathbf{Q} has a definable cut that satisfies $\text{I}\Delta_0$ (see Theorem 5.7 of [HP]), so every sequential theory supports such an interpretation \mathcal{N} .

⁷Without this extensionality stipulation, the numeral does not necessarily work as a term.

⁸The models are just a heuristic here. In fact the whole argument can be formulated in the complete theory T .

This concludes the proof of (3).

To complete the proof of CLAIM \heartsuit , we need to show that $\mathcal{M} \models \neg\psi \rightarrow \neg\rho(\ulcorner\psi\urcorner)$ for all \mathcal{L} -sentences ψ . For this purpose assume $\mathcal{M} \models \neg\psi$.

By putting (1) and the assumption that $\mathcal{M} \models \neg\psi$, we conclude that there is a *standard* proof π_0 of $\neg\psi$ from $\text{True}_n^{\mathcal{M}}$, which by (2) implies:

(6) For some $\pi_0 \in \omega$, $\mathcal{M} \models \text{Prf}_{\text{True}_n}(\pi_0, \ulcorner\neg\psi\urcorner)$.

To see that $\mathcal{M} \models \neg\rho(\ulcorner\psi\urcorner)$ suppose to the contrary that $\mathcal{M} \models \rho(\ulcorner\psi\urcorner)$. By the choice of ρ , this means:

(7) For some $m_0 \in M$, $\mathcal{M} \models \text{Prf}_{\text{True}_n}(m, \ulcorner\psi\urcorner) \wedge \forall z < m_0 \neg\text{Prf}_{\text{True}_n}(z, \ulcorner\neg\psi\urcorner)$.

The key observation is that putting (2) with the assumption $\mathcal{M} \models \neg\psi$ allows us to conclude that the m_0 in (7) must be a *nonstandard element* of \mathcal{M} . Thus by standardness of π_0 of (6) and the ordering properties of ‘natural numbers’ in \mathcal{M} , $\mathcal{M} \models \pi_0 < m_0$, which contradicts the second conjunct of (7). \square

The following result is the analogue of Theorem A for sufficiently strong arithmetical theories. Recall that the Σ_n -hierarchy of formulae is the usual hierarchy of arithmetical formulae in which Σ_0 -formulae are defined as formulae in which all quantifiers are bounded (recall that $\Sigma_0 = \Pi_0 = \Delta_0$ here); $\text{I}\Delta_0$ is the fragment of PA in which the induction scheme is limited to Δ_0 -formulae, and Exp is the arithmetical sentence that asserts that the exponential function 2^x is total (it is well-known that the deductive closure of $\text{I}\Delta_0$ goes well beyond that of PA^- ; moreover Exp is not provable in $\text{I}\Delta_0$).

Theorem B. *For each $n \in \omega$ every consistent extension of $\text{I}\Delta_0 + \text{Exp}$ (in the same language) that is axiomatized by a set of Σ_n -sentences is incomplete.⁹*

Proof. As shown by Gaifman and Dimitracopoulos [GD] (see Chapter V of [HP] for an exposition) for each $n \in \omega$ there is a formula Sat_{Σ_n} such that, provably in $\text{I}\Delta_0 + \text{Exp}$, Sat_{Σ_n} satisfies compositional clauses for all Σ_n -formulae. In particular there is a formula $\text{True}_{\Sigma_n}(x)$ such that for all models \mathcal{M} of $\text{I}\Delta_0 + \text{Exp}$, and for all Σ_n -sentences ψ , $\psi \in \text{True}_{\Sigma_n}^{\mathcal{M}}$ iff $\mathcal{M} \models \psi$. We can now repeat the proof strategy of Theorem A with the use of $\text{True}_{\Sigma_n}^{\mathcal{M}}$ instead of $\text{True}_n^{\mathcal{M}}$.

Alternatively, invoke the provability of the MRDP theorem on the Diophantine representability of computably enumerable sets in $\text{I}\Delta_0 + \text{Exp}$ (also shown in [GD]). By the MRDP-theorem each Σ_n -formula is equivalent to a Σ_n^* -formula in $\text{I}\Delta_0 + \text{Exp}$, so Theorem A applies. \square

With the help of Craig’s trick¹⁰ to obtain a computable axiomatization T^{Craig} of an arbitrary computably enumerable theory T , the proof strategy of Theorem A can be straightforwardly adapted by using $\text{Prf}_{T^{\text{Craig}} + \text{True}_n}$ instead of $\text{Prf}_{\text{True}_n}$ to establish the following strengthening of Theorem A:

⁹The set-theoretical analogue of Theorem B is Theorem C below concerning the well-known Levy hierarchy of formulae of set theory. Theorem C can be proved with the same strategy as in the proof of Theorem A (and the first proof of Theorem B) thanks to the availability of the relevant definable partial satisfaction classes in KP. Here KP is Kripke-Platek set theory with the scheme of foundation limited to Π_1^{Levy} -formulae (equivalently: the scheme of \in -induction for Σ_1^{Levy} -formulae). Thus in contrast to Barwise’s KP in [B], which includes the full scheme of foundation, our version of KP is finitely axiomatizable. Note that the axiom of infinity is not among the axioms of KP. The existence of definable partial satisfaction classes in KP follows from two facts: (1) KP can prove that every set is contained in a transitive set; and (2) KP can define the satisfaction predicate for all of its internal set structures (the proofs of both of these facts can be found in Barwise’s monograph [B]; the proofs therein make it clear that only Π_1^{Levy} -Foundation is invoked). See also Theorem 2.9 of [EM] (the statement of which involves KP + the axiom of infinity, but the axiom of infinity is not used in the proof). It is also worth pointing out that KP plus the negation of axiom of infinity is bi-interpretable with the fragment $\text{I}\Sigma_1$ of PA (we owe this observation to Fedor Pakhomov); indeed the two theories can be shown to be definitionally equivalent.

Theorem C. *For each $n \in \omega$ every consistent completion of KP (in the same language) that is axiomatized by a set of Σ_n^{Levy} -sentences is incomplete.*

¹⁰Suppose T is computably enumerable. Fix an instance of a tautology τ in the language of T , and recursively define $\tau^0 := \tau$ and $\tau^{n+1} := \tau^n \wedge \tau$. Then define T^{Craig} as the result of replacing each $\varphi \in T$ with $\tau^n \wedge \varphi$, where n is a witness for $\varphi \in T$. It can be readily checked that T^{Craig} is computable (indeed it is primitive recursive).

Theorem A⁺. *Let T be a computably enumerable sequential theory formulated in a finite language \mathcal{L} and suppose A is a collection of \mathcal{L} -sentences such that $A \subseteq \Sigma_n^*$ for some $n \in \omega$ and $T \cup A$ is consistent. Then $T \cup A$ is incomplete.¹¹*

Similarly, we can obtain the following strengthening of Theorem B:

Theorem B⁺. *Let T be a computably enumerable extension of $\text{I}\Delta_0 + \text{Exp}$ (in the same language) and suppose A is a collection of arithmetical sentences such that $A \subseteq \Sigma_n$ for some $n \in \omega$ and $T \cup A$ is consistent. Then $T \cup A$ is incomplete.¹²*

Remark 4. The assumption of finiteness of language cannot be lifted in Theorem A. For example, consider the theory $U = \text{CT}_\omega^-[\text{I}\Sigma_1]$ of ω -iterated compositional truth over $\text{I}\Sigma_1$ (without any induction for formulae using nonarithmetical symbols, hence the minus superscript) formulated in an extension of the language \mathcal{L}_A of arithmetic with infinitely many predicates $\{\text{T}_{n+1} : n \in \omega\}$, and Tarski-style compositional axioms that stipulate that T_{n+1} is compositional for all \mathcal{L}_n -formulae, with $\mathcal{L}_0 = \mathcal{L}_A$ and $\mathcal{L}_{n+1} = \mathcal{L}_n \cup \{\text{T}_{n+1}\}$. Then since bi-conditionals of form $\varphi \longleftrightarrow \text{T}_{n+1}(\ulcorner \varphi \urcorner)$ are provable in U for every \mathcal{L}_n -sentence (thanks to the available composition axioms) ANY complete extension V of U is axiomatized by U (which is of bounded complexity) together with atomic sentences of form $\text{T}_{n+1}(\ulcorner \varphi \urcorner)$ where $\varphi \in V$ and φ is an \mathcal{L}_n -sentence, thus U axiomatizable by a set of axioms of bounded quantifier complexity. Note that U is a sequential theory since it is an extension of the sequential theory $\text{I}\Sigma_1$. By strengthening U with a single axiom, namely the so-called axiom Int of internal induction that uses T_1 to state that all arithmetical instances of induction hold, we obtain *a theory in an infinite language, and axiomatized by a set of sentences of bounded quantifier complexity, whose deductive closure extends PA and every completion of which is axiomatizable by a set of sentences of bounded complexity.* Let us also note that it is well-known that the technique of ‘ \mathcal{M} -logic’ of Krajewski-Kotlarski-Lachlan (see Kaye’s text [K] for an exposition), or the more recent robust model-theoretic technique introduced in [EV] allow one to show that $\text{CT}_\omega^-[\text{I}\Sigma_1] + \text{Int}$ is conservative over PA.

Alternatively, by starting with any theory T formulated in a language \mathcal{L} , we can apply a process known in model theory as *Morleyization*¹³ or *atomization* to obtain an extension T^+ of T , formulated in an extension \mathcal{L}^+ of \mathcal{L} , such that T^+ is axiomatized by adding a collection of sentences of bounded quantifier depth to T , and T^+ has elimination of quantifiers in the sense that for each \mathcal{L}^+ -formula $\varphi(x_1, \dots, x_n)$, there is an n -ary predicate $P_\varphi \in \mathcal{L}^+$ such that the equivalence $\varphi(x_1, \dots, x_n) \leftrightarrow P_\varphi(x_1, \dots, x_n)$ is provable in T^+ . The advantage of this second construction is that it does not require the resources to build conservative truth predicates. The atomization of a theory is well-known to be model-theoretically conservative, whereas a truth-predicate of the type T_1 in the previous example is already not model-theoretically conservative since it imposes recursive saturation on models of arithmetic supporting it (by a remarkable theorem of Lachlan; see Kaye’s text [K] for an exposition).

Remark 5. In Theorem B, the theory $\text{I}\Delta_0 + \text{Exp}$ cannot be weakened to PA^- , i.e., for some $n \in \omega$ there is a consistent completion of PA^- (in the same language) that is axiomatized by a set of Σ_n -sentences. The proof of this and related results will appear in our upcoming paper with Mateusz Łełyk. It will be hard to prove an analogous result about $\text{I}\Delta_0$ since the proof strategy of Theorem B makes it clear that the analogous result for $\text{I}\Delta_0$ implies that $\text{I}\Delta_0$ does not prove that there is a Σ_1 -satisfaction predicate, which is known to be a tall order, as indicated in [AKP].

Remark 6. The proof of Theorem A/A⁺ can be used to establish:

(∇) *There is no consistent complete theory T interpreting \mathbf{R} that is of the form $T = T_0 + A$, where T_0 is computably enumerable, and A is a subset of some decidable set Γ of sentences for which T has a truth predicate.*

Theorem A/A⁺ then follows by putting (∇), with the choice of $\Gamma := \Sigma_n^*$, together with Fact F. Similarly, Theorems B/B⁺ and Theorem C can be obtained from (∇) by invoking the corresponding well-known analogues

¹¹Note that if $A = \emptyset$, then the proof strategy of Theorem A, when applied to the setting of Theorem A⁺, goes through for all computably enumerable consistent extensions T of the Tarski-Mostowski-Robinson theory \mathbf{R} , without the assumption of sequentiality of T .

¹²Theorem C of footnote 9 also readily lends itself to an analogous strengthening.

¹³According to Hodges, atomization was introduced by Skolem in the 1920s, and has “nothing to do with Morley”. It is classically known that the atomization of a theory can be axiomatized by sentences of the form $\forall x_1 \dots \forall x_m \exists y_1 \dots \exists y_k \delta$, where δ is quantifier-free. See pp. 62-64 (especially Theorem 2.6.6) of Hodges’ majestic text [H].

of Fact F. This more general approach is further explored in [V-5], but it should be noted that upon the completion of our paper we were kindly informed by Emil Jeřábek that (∇) was formulated and established in his MathOverflow answer [J-2].

We conclude with a question that is motivated by Theorem A and the fact that \mathbf{Q} is not a sequential theory [V-2].

Question #. *Is it possible for a consistent completion of \mathbf{Q} to be axiomatized by a collection of sentences of bounded quantifier-depth?*

REFERENCES

- [AKP] Z. Adamowicz, L. Kołodziejczyk, and J. Paris, *Truth definitions without exponentiation and the Σ_1 collection scheme*, **Journal of Symbolic Logic**, 77, pp. 649–655 (2012).
- [AGLRZ] U. Andrews, D. Gonzalez, S. Lempp, D. Rossegger, and H. Zhu, *The Borel complexity of the class of models of first-order theories*, submitted for publication.
- [B] J. Barwise, **Admissible Sets and Structures**, Springer-Verlag, Berlin (1975).
- [EM] A. Enayat and Z. McKenzie, *Initial embeddings of models of set theory*, **Journal of Symbolic Logic**, 86, pp. 1584–1611 (2021).
- [EV] A. Enayat and A. Visser, *New constructions of satisfaction classes*. In Theodora Achourioti, Henri Galinon, José Martínez Fernández, and Kentaro Fujimoto, editors, **Unifying the Philosophy of Truth**, pp. 321–325. Springer-Verlag, 2015.
- [GD] H. Gaifman and C. Dimitracopoulos, *Fragments of Peano’s arithmetic and the MRDP theorem*, in **Logic and algorithmic** (Zurich, 1980), pp. 187–206, Monogr. Enseign. Math., 30 [Monographs of L’Enseignement Mathématique], Université de Genève, L’Enseignement Mathématique, Geneva, 1982.
- [HP] P. Hájek and P. Pudlák, **Metamathematics of First-Order Arithmetic**, Perspectives in Mathematical Logic, Springer, Berlin, 1993.
- [H] W. Hodges, **Model Theory**, Cambridge University, 1993.
- [J-1] E. Jeřábek, *Sequence encoding without induction*, **Mathematical Logic Quarterly** 58(3), pp. 244–248 (2012).
- [J-2] E. Jeřábek, *A “completion” of ZFC^-* , **MathOverflow**, URL: <https://mathoverflow.net/q/257044> (version: 2016-12-12).
- [K] R. Kaye, **Models of Peano Arithmetic**, Oxford University Press, Oxford, 1991.
- [LW] M. Lelyk and B. Wcisło, *Universal properties of truth*, **arXiv:2304.00370** [math.LO] (2023).
- [Mon] R. Montague, *Semantical closure and non-finite axiomatizability. I*, in **Infinitistic Methods** (Proc. Sympos. Foundations of Math., Warsaw, 1959). Pergamon, Oxford; Państwowe Wydawnictwo Naukowe, Warsaw, pp. 45–69, 1961.
- [Mos] A. Mostowski, *On models of axiomatic systems*, **Fundamenta Mathematicae** vol. 39 (1952), pp. 133–158.
- [P-1] P. Pudlák, *On the lengths of proofs of finitistic consistency statements in first order theories*, in: **Logic Colloquium ’84**, J. B. Paris, A.J. Wilkie, and G. M. Wilmers, eds., North-Holland, Amsterdam, pp. 165–196. 1986.
- [P-2] P. Pudlák, *The lengths of proofs*, in: **Handbook of Proof Theory**, edited by S. Buss, Studies in Logic and the Foundations of Mathematics Vol. 137, Elsevier, pp. 547–637, 1998.
- [Ra] M. Rabin, *Nonstandard models and the independence of the induction axiom*, **Essays on the foundations of mathematics**, pp. 287–299, Magnes Press, The Hebrew University, Jerusalem, 1961.
- [Ry] C. Ryll-Nardzewski, *The role of the axiom of induction in elementary arithmetic*, **Fundamenta Mathematicae**, 39, pp. 239–63 (1952).
- [S] S. Salehi, *A Reunion of Gödel, Tarski, Carnap, and Rosser*, **Journal of Logic and Computation**, exad001, <https://doi.org/10.1093/logcom/exad001>
- [TMR] A. Tarski, A. Mostowski, and R. Robinson, **Undecidable Theories**, North Holland, Amsterdam, 1953.
- [V-1] A. Visser, *The unprovability of small inconsistency, A study of local and global interpretability*, **Archive for Mathematical Logic** 32, no.4, pp. 275–298 (1993).
- [V-2] A. Visser, *On \mathbf{Q}* , **Soft Computing** 21, pp. 39–56 (2017). <https://doi.org/10.1007/s00500-016-2341-5>
- [V-3] A. Visser, *From Tarski to Gödel—or how to derive the second incompleteness theorem from the undefinability of truth without self-reference*, **Journal of Logic and Computation** 29(5), pp. 595–604 (2019)
- [V-4] A. Visser, *The small-is-very-small principle*, **Mathematical Logic Quarterly**, 65, no.4, pp. 453–478 (2019).
- [V-5] A. Visser, *Rosser and Kreisel-provability in a general setting* (in preparation).

ALI ENAYAT, DEPARTMENT OF PHILOSOPHY, LINGUISTICS, AND THEORY OF SCIENCE, UNIVERSITY OF GOTHENBURG, SWEDEN.

ali.enayat@gu.se

ALBERT VISSER, DEPARTMENT OF PHILOSOPHY, FACULTY OF HUMANITIES, UTRECHT UNIVERSITY, THE NETHERLANDS.

albert.visser@uu.nl