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HUME'S PRINCIPLE, BEGINNINGS

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ABSTRACT. In this note we derive Robinson's Arithmetic from Hume's Principle in the context of very weak theories of classes and relations.

1. Introduction

Frege's derivation of arithmetic from Hume's Principle has great beauty. In this derivation, numbers are created by an autocatalytic process starting from nothing. Moreover, in one fell swoop, we also define the arithmetical operations of addition and multiplication. In this paper we try to obtain a closer understanding of the fine structure of the employment of Hume's Principle to obtain basic arithmetical principles.

1.1. Contents of The Paper.

In the present work, I attempt a detailed and slow study to address the following questions:

a. *How much comprehension is needed to develop a weak system of arithmetic?*
b. *Can we get a more modular picture of how arithmetic follows from Hume's Principle?*

John Burgess shows, in his book [Bur05], that a predicative system of second order logic, with variables for binary relations, expanded with Hume's Principle interprets Robinson's Arithmetic Q. This result suggests the following picture. We can divide the proof of Frege's Theorem into two steps. First, in the context of a weak theory of binary relations over a domain we can develop a weak arithmetic with the help of Hume's principle and, then, by increasing the strength of the theory of relations we raise the strength of the resulting arithmetical system.

Zooming in on the development of arithmetic in a weak second order theory using Hume's principle, we see that there are, prima facie, three ingredients.

a. Hume's principle gives us an infinity of objects.
b. Hume's principle allows us to treat numbers as objects and, hence, enables us to speak about classes of numbers in a second order context.
c. Hume's principle helps us to define the arithmetical operations.

In the present paper we will concentrate on the development of *first order arithmetic* from Hume's principle, so on (a) and (c). Item (b) will only occur in so far as it is used for (c). We will show that if we start with adjunctive relation theory, a very weak second order system, then the only contribution needed of Hume's principle to develop an interpretation of Q is (a). We can define the arithmetical operations as operations on classes modulo equinumerosity without projecting these classes to

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objects. In other words, Hume's principle is not needed for (c) after all. Here, we use the presence of relations (i) to define the relation of equinumerosity and (ii) to implement the arithmetical operations on classes.

We can get a somewhat more refined and more interesting picture, if we do not work with relations but only with classes. We show that, if we start with adjunctive class theory with a *primitively given equinumerosity relation* plus the principle that says that there is no universal class, then we can develop $\mathsf{Q}_{\mathsf{add}}$, the analogue of Q for addition, without multiplication, but that we cannot develop Q.[1] To define multiplication, we need Hume's principle, so, in a class context, Hume's principle is essential for (c).

Thus, in the context of a theory of classes with equinumerosity, Hume's principle is connected to what one could call *the reverse mathematics of addition and multiplication*. Such a program involves matching 'natural' pairs of class theories with pairs of a theory of addition and a theory of addition and multiplication. We do not develop the full picture here. We present one clear result: the adjunctive theory of classes with equinumerosity and Hume's principle is mutually interpretable with Q.[2] The precise relationship between adjunctive class theory with equinumerosity and no-universe and, on the other hand, a suitable extension of $\mathsf{Q}_{\mathsf{add}}$ still has to be analyzed. We show that the first theory interprets the second. It seems plausible that, in the other direction, there is an informative reduction relation, along the lines sketched by Feferman and Vaught in their paper [FV59] for the case of true theories. We show, in Section 6, that the second theory does not interpret the first. Hence, we really do need the less restrictive Feferman-Orey relation between the theories.

1.2. **Almost Philosophical Considerations.** In the kind of foundational research exemplified by this paper, there are, very roughly, two colors, say *blue* and *yellow*.[3] *Blue* is the study of the fine structure of reasoning. Here we have *reverse mathematics*, *theory reduction*, *interpretability* and the like. *Yellow* has a tighter link with philosophy: we are looking for a serious justification of numerical reasoning, of sets, etcetera. A lot of contemporary work on Frege's *Grundgesetze* falls under *yellow*. Of course, the separation is not strict. E.g., Burgess' book [Bur05] exhibits both colors. Of course, work falling under *blue* can provide useful data for research of color *yellow*.

The primary focus of the present paper is the fine structure of reasoning, making it *blue*. Thus, the fact that we work in very weak theories does not necessarily reflect a conviction that these theories can be justified, or that we cannot justify much more.

One important aspect of both Burgess' development of Q from a predicative theory of binary relations expanded with Hume's Principle and the development in our paper is that we grant ourselves the freedom of gaining more 'good' properties of our numbers by contracting the number domain. This methodology was initiated by Edward Nelson in his book [Nel86], thus transforming Solovay's method of shortening cuts (introduced in [Sol76]) into a foundational tool.

[1]The system $\mathsf{Q}_{\mathsf{add}}$ is a subsystem of Presburger Arithmetic PresA. We show in an appendix that $\mathsf{Q}_{\mathsf{add}}$ locally interprets PresA.

[2]It would be interesting to explore the existence of tighter relations. For certain extensions of both theories we do indeed have tighter connections.

[3]The choice of colors is inspired by a famous scene from *Monty Python and the Holy Grail.*

In the context of a Frege-style development of number theory (of color *yellow*) this strategy is not unproblematic. In a Frege-style program, we are interested in defining *the natural numbers.* Not every class of objects having some desired properties usually associated with the natural numbers qualifies. After all, we want to say *what the natural numbers are.* Thus, it could be our aim to see what certain principles allow us to prove about the numbers as given by some *fixed* definition, e.g., the precise definitions employed by Frege, rather than to fiddle around with the definition to get better properties.[4] Similarly, we could be interested in a reconstruction that justifies *this* specific reasoning and not in justifying quite different reasoning with the same conclusion.

We note that the preferred definitions view is not without its own problems. The claim that such-and-such a definition is the intended one, is in need of informally rigorous argumentation. For example, Frege's definition of the natural numbers using the ancestral seems to me to be an ordinal-style definition, where Burgess' definition of proto-natural numbers (p156 of [Bur05]) has much more a cardinal flavour: it is a strengthened version of *Dedekind finiteness.* Given that Frege's development is intended to build the cardinals, which is the better one?

Let me stress that the kind of *blue* project we are involved in is not necessarily antithetical to a *yellow* view involving intended definitions. Suppose, e.g., we have a preferred definition N_0 of, say, the natural numbers, and suppose we want them to have the good property P. Suppose further we have proved P in a very weak context for a 'contracted' virtual class of natural numbers N_1. Then a good theory in which to get P for N_0 could be a theory in which N_1 and N_0 coincide. So we immediately know that a principle that implies the equality of N_0 and N_1 will have numbers with the good property P.

On the other hand, it would be interesting to see whether there is a coherent foundational idea corresponding to the methodology of shortening cuts. This would involve rejecting the claim that induction is almost analytical for the natural numbers (made by Linnebo in his paper [Lin04]).[5] When working with this method one develops the feeling that such a philosophy is just around the corner. Nelson's program as developed in [Nel86] seems to be founded on the conviction that, yes, there is a foundational idea that justifies building up the numbers by 'number system hopping'. Unfortunately, Nelson gives us no hint what such an idea could look like.

2. Preliminaries

This note will presuppose familiarity with the notions and notations introduced in [Vis09]. In this section, we provide a list of principles and theories. The best reading strategy is to glance through this section and return to it as soon as a principle is added.

We will use the following convention. Theories will be specified as e.g. $[\mathsf{a3}, \mathsf{b2}]$ where $\mathsf{a3}$ and $\mathsf{b2}$ are already specified principles. The signature of the theory will

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[4]For a paper that studies preferred definitions for a Frege-style development of Arithmetic, see Øystein Linnebo's paper [Lin04]. Linnebo provides an interesting discussion of Burgess' proof that predicative relational Frege theory interprets Q.

[5]Induction in the case discussed by Linnebo is not really induction anyway, in the light of the restriction on comprehension.

be the minimal signature that is involved in all the axioms of the theory. If a theory U is given $U[\mathsf{a3}, \mathsf{b2}]$ will be the result of adding $\mathsf{a3}$ and $\mathsf{b2}$ to U.

2.1. **Class Theories.** We start with class theories. We have a language with two sorts: $\mathfrak{o}$, the sort of objects and $\mathfrak{c}$, the sort of classes. We let $x, y, z, \ldots$ range over objects and $X, Y, Z, \ldots$ over classes. We have identity for each sort and one binary predicate $\in$ of sort $\mathfrak{oc}$. We have the following list of principles.

c1 $\vdash \exists X\, \forall x\; x \notin X$ (empty class axiom)
c2 $\vdash \exists Y\, \forall y\; (y \in Y \leftrightarrow (y \in X \vee y = x))$ (adjunction axiom),
c3 $\vdash \forall z\; (z \in X \leftrightarrow z \in Y) \to X = Y$ (extensionality)
c4 $\vdash \exists Z\, \forall z\; (z \in Z \leftrightarrow (z \in X \wedge z \in Y))$ (closure under intersection)
c5 $\vdash \exists Y\, \forall y\; (y \in Y \leftrightarrow (y \in X \wedge y \neq x))$ (subtraction axiom)
c6 $\vdash \exists Z\, \forall z\; (z \in Z \leftrightarrow (z \in X \wedge z \notin Y))$ (closure under class-subtraction)
c7 $\vdash \exists Z\, \forall z\; (z \in Z \leftrightarrow (z \in X \vee z \in Y))$ (closure under union).
c8 $\vdash \exists x\; x \notin X$ (no-universe axiom)

The theory ac is [c1,c2,c3]. We note that it has a one-sorted version, with just classes and the relation $\subseteq$, where we interpret the objects as the atoms of the $\subseteq$-ordering.[6] We also note that axiom c3 is really superfluous, since we can always interpret identity on the classes as extensional equivalence.

We note that if we have the theory with all our principles, say, ac^+, then c2 and c5 can be replaced by an axiom stating the existence of singletons. The theory ac^+ can easily be shown to be complete via quantifier elimination by a minor adaptation of the proof of the Löwenheim-Behman theorem. See [Bur05] for an exposition of the original theorem.

2.2. **Adding Equinumerosity.** We extend the language of ac with a binary relation symbol $\equiv$ of type $\mathfrak{cc}$. We have the following principles for the extended language. In the statement of some of these principles we use operations on classes. We will only use the principles in the context of axioms stating that these operations are total. We write $X \# Y$ for $\forall z\; (z \notin X \vee z \notin Y)$.

e1 $\vdash (X \equiv \emptyset \vee \emptyset \equiv X) \leftrightarrow X = \emptyset$
e2 $\vdash (x \notin X \wedge y \notin Y) \to ((X \cup \{x\}) \equiv (Y \cup \{y\}) \leftrightarrow X \equiv Y)$
e3 $\vdash X \equiv X$
e4 $\vdash X \equiv Y \to Y \equiv X$
e5 $\vdash (X \equiv Y \wedge Y \equiv Z) \to X \equiv Z$
e6 $\vdash Y \subset X \to Y \not\equiv X$ (Dedekind finiteness axiom)
e7 $\vdash (X \subseteq Y \equiv Y') \to \exists X'\; (X \equiv X' \subseteq Y')$
e8 $\vdash (X \equiv X' \wedge Y \equiv Y' \wedge X \subseteq Y \wedge X' \subseteq Y') \to (Y \setminus X) \equiv (Y' \setminus X')$
e9 $\vdash (X \# Y \wedge X' \# Y' \wedge X \equiv X' \wedge Y \equiv Y') \to (X \cup Y) \equiv (X' \cup Y')$
e10 $\vdash \exists Y'\; (Y \equiv Y' \wedge X \# Y')$

The theory eqnum is ac[e1,e2]. Feferman and Vaught in their classical paper [FV59], show that, for any cardinality κ, the theory of all classes of elements from a domain of size κ, with the subset ordering and with the relation of equinumerosity, is decidable. We can interpret eqnum into these theories by identifying the objects

[6] If we set it up correctly this one-sorted version is bi-interpretable with the original theory. This means that there are interpretations between the two theories whose compositions are definably and provably isomorphic with the identity interpretation.

with the atoms of $\subseteq$. It follows that eqnum has a decidable extension and, hence, does not interpret Q.

2.3. **Adding the Hume Relation.** To formulate Hume's principle, we add the Hume relation $\rhd$ to the language. This relation is of type $\mathfrak{co}$. We choose to work with a Hume relation as opposed to a Hume function because the functionality never seems to play any role in the arguments. Moreover, functionality is a rather strong demand. For example, in the context of ZF without choice we need Scott's trick to produce a Hume function, which depends on a lot of details of the theory. So by dropping functionality we have a substantial gain in generality. Here is a list of principles.

h1 $\vdash \exists x\, X \rhd x$
h2 $\vdash (X \rhd z \wedge Y \rhd z) \to X \equiv Y$
h3 $\vdash X \equiv Y \rhd z \to X \rhd z$

The theory Hume Light or HL is eqnum[h1,h2].

We note that we could eliminate $\equiv$ from the language by redefining $X \equiv Y$ as $\exists z\ (X \rhd z \wedge Y \rhd z)$. This would ask for a careful choice of axioms. Moreover, we would get some properties of the equinumerosity relation like reflexivity and symmetry for free.

2.4. **Relational Theories.** We introduce adjunctive relation theory ar. Adjunctive relation theory is a two-sorted theory with sorts $\mathfrak{o}$, the sort of objects, and $\mathfrak{r}$, the sort of relations. We use $x, y, z, \ldots$ to range over objects and $R, S, \ldots$ to range over relations. Our theory has identity for each sort and a ternary relation app of type $\mathfrak{roo}$. We write Rxy for $\mathsf{app}(R, x, y)$. Here is a list of principles.

r1 $\vdash \exists R\, \forall x, y\, \neg Rxy$
r2 $\vdash \exists S\, \forall u, v\ (Suv \leftrightarrow (Ruv \vee (x = u \wedge y = v)))$
r3 $\vdash \exists S\, \forall u, v\ (Suv \leftrightarrow (Ruv \wedge (x \neq u \vee y \neq v)))$

The theory ar is given by [r1,r2]. In ar we can define classes as diagonal relations, i.e., as relations such that, for all x, y, if Rxy, then $x = y$. Clearly this interpretation does yield ac for our classes. We will add principles concerning classes to ar as a matter of course under this reading. So, e.g. ar[c8] is ar plus $\vdash \exists x\, \neg Rxx$.

We remind the reader of a result of [Vis09]. There it is shown that ar plus the no-universe axiom interprets Q as a theory of cardinals over the given object domain.

2.5. **Arithmetical Theories.** We consider the following list of principles in the inclusive signature $0, \mathsf{S}, +, \times, \leq$. We often write $x \cdot y$ or xy for $x \times y$.

q1 $\vdash \mathsf{S}x \neq 0$
q2 $\vdash \mathsf{S}x = \mathsf{S}y \to x = y$
q3 $\vdash x + 0 = x$
q4 $\vdash x + \mathsf{S}y = \mathsf{S}(x + y)$
q5 $\vdash x \times 0 = 0$
q6 $\vdash x \times \mathsf{S}y = (x \times y) + x$
q7 $\vdash x = 0 \vee \exists y\ x = y + 1$
q8 $\vdash x \leq x$
q9 $\vdash (x \leq y \wedge y \leq z) \to x \leq z$
q10 $\vdash (x \leq y \wedge y \leq x) \to x = y$
q11 $\vdash x \leq y \to (x = y \vee \mathsf{S}x \leq y)$
q12 $\vdash x \leq y \vee y \leq x$

q13 $\vdash x \leq y \leftrightarrow \exists z\, (z + x = y)$
q14 $\vdash \mathsf{S}x \neq x$
q15 $\vdash (x + y) + z = x + (y + z)$
q16 $\vdash x + y = y + x$
q17 $\vdash x + z = y + z \to x = y$
q18 $\vdash x \times (y + z) = (x \times y) + (x \times z)$
q19 $\vdash x \times (y \times z) = (x \times y) \times z$

The theory $\mathsf{Q}_{\mathsf{add}}$ is the theory $[\mathsf{q1}, \mathsf{q2}, \mathsf{q3}, \mathsf{q4}, \mathsf{q7}]$, in the signature with $0, \mathsf{S}, +$. The theory Q is $\mathsf{Q}_{\mathsf{add}}[\mathsf{q5}, \mathsf{q6}]$ in the signature with $0, \mathsf{S}, +, \times$.

3. No-Universe

We first show that a version of Hume's Principle allows us to construct a totality of classes that is adjunctive, but does not contain the universe of objects. To be precise, we show that eqnum directly interprets the no-universe axiom ac8.

3.1. **Bootstrap in ac.** We start to work in ac. As a first step, define $\mathcal{X}_0$ as the virtual class of classes such that X is in $\mathcal{X}_0$ iff, for all Y, $X \cap Y$ exists. We easily see that $\mathcal{X}_0$ is closed under empty class, adjunction, and intersection. We note that this argument uses the associativity of intersection in the strong sense that if $(X \cap Y) \cap Z$ is defined, then so is $X \cap (Y \cap Z)$ and both are equal, and vice versa. The equality relies on extensionality, which we use here as a matter of course. We relativize our classes to $\mathcal{X}_0$, thereby gaining the extra axiom c4: that classes are closed under intersection.

We work in ac[c4]. We consider $\mathcal{X}_1$, where X is in $\mathcal{X}_1$ if, for all x, $X \setminus \{x\}$ exists. It is easy to see that $\mathcal{X}_1$ is closed under empty class, adjunction, intersection and subtraction. (We need the presence of intersection for closure under subtraction!) We relativize to $\mathcal{X}_1$, thereby gaining the axioms c4 and c5. Here c5 tells us that classes are closed under subtraction. Thus, we have directly interpreted ac[c4, c5].

3.2. **Bootstrap in eqnum.** By the result of the previous subsection, we can directly interpret eqnum[c5] in eqnum. We work in eqnum[c5]. We define $\mathcal{X}_2$ as the totality of those classes X such that:

$$X \equiv X \text{ and } \forall Y\, (X \equiv Y \to Y \equiv X) \text{ and } \forall Y, Z\, ((X \equiv Y \wedge Y \equiv Z) \to X \equiv Z).$$

It is easy to see that $\emptyset \in \mathcal{X}_2$. We show that $\mathcal{X}_2$ is closed under adjunction. We treat the case of transitivity. Suppose $(X \cup \{x\}) \equiv Y$ and $Y \equiv Z$. If x is in X, we are immediately done. Suppose $x \notin X$. We note that Y and Z cannot be empty. Let $y \in Y$ and $z \in Z$. We find that $X \equiv (Y \setminus \{y\})$ and $(Y \setminus \{y\}) \equiv (Z \setminus \{z\})$. Ergo $X \equiv (Z \setminus \{z\})$. Hence, $(X \cup \{x\}) \equiv Z$. Contracting our classes to $\mathcal{X}_2$ we gain the axioms stating that $\equiv$ is an equivalence relation. We note that this property is preserved under any further contraction of our classes. Relativizing to $\mathcal{X}_2$ gives us eqnum[e3, e4, e5].

We work in eqnum[e3, e4, e5]. We note that e3,e4,e5 are universal and, hence, preserved under contraction of the totality of classes. We repeat the procedure of the previous subsection to regain the subtraction axiom: we get eqnum[c5, e3, e4, e5].[7]

[7] We note that in in eqnum[c5] plus the no-universe principle, i.e. eqnum[c5, c8], we can directly prove that $\mathcal{X}_2$ is closed under subtraction.

We work in $\mathsf{eqnum}[\mathsf{c5}, \mathsf{e3}, \mathsf{e4}, \mathsf{e5}]$. We say that X is *Dedekind finite* iff, for every $Y \subset X$, we have $X \not\equiv Y$. Let $\mathcal{X}_3$ consist of the Dedekind finite classes. Clearly, the empty class is in $\mathcal{X}_3$.

We show that $\mathcal{X}_3$ are closed under adjunction. Suppose X is Dedekind finite. We show that $X \cup \{x\}$ is Dedekind finite. We may assume that $x \notin X$. Suppose $X_0 \subset (X \cup \{x\})$ and $X_0 \equiv (X \cup \{x\})$. It follows that X_0 is not empty. Let x_0 be x in case $x \in X_0$ and let x_0 be some arbitrary element of X_0 otherwise. We find that $(X_0 \setminus \{x_0\}) \subset X$ and $(X_0 \setminus \{x_0\}) \equiv X$. A contradiction with the assumption that X is Dedekind finite.

It is easy to see that $\mathcal{X}_3$ is closed under subtraction. We relativize to $\mathcal{X}_3$, thus gaining the axiom $\mathsf{e6}$, that all classes are Dedekind finite. So we have directly interpreted $\mathsf{eqnum}[\mathsf{c5}, \mathsf{e3}, \mathsf{e4}, \mathsf{e5}, \mathsf{e6}]$.

We note a useful fact.

Lemma 3.1 (in eqnum)**.** *Suppose Y is Dedekind finite, $X \equiv Y$ and $x \notin X$. Then, for some y, $y \notin Y$. In another formulation: suppose the universe is a Dedekind finite class, then any class equinumerous to it is again the universe.*

Proof. Suppose Y is Dedekind finite, $X \equiv Y$ and $x \notin X$. Suppose Y would be the universe, then we have $(X \cup \{x\}) \subseteq Y$, and hence $X \subset Y$. But $X \equiv Y$, contradicting the fact that Y is Dedekind finite. ❑

3.3. **Bootstrap in HL.** At this point we switch to the theory HL. Our development in this subsection is a straightforward adaptation of the treatment predicative second order logic with the Hume function in [Bur05].

By the preceding development, we can directly interpret $\mathsf{eqnum}[\mathsf{c5}, \mathsf{e3}, \mathsf{e4}, \mathsf{e5}, \mathsf{e6}]$, so we can work in $\mathsf{HL}[\mathsf{c5}, \mathsf{e3}, \mathsf{e4}, \mathsf{e5}, \mathsf{e6}]$. By replacing $X \rhd y$ by $\exists Z\ X \equiv Z \rhd y$, we gain the axiom $\mathsf{h3}$. So we may work in $\mathsf{HL}^+ := \mathsf{HL}[\mathsf{c5}, \mathsf{e3}, \mathsf{e4}, \mathsf{e5}, \mathsf{e6}, \mathsf{h3}]$.

Let $\mathcal{X}_4$ consist of all X, such that there is a class Y such that $X \equiv Y$ and, for every $y \in Y$, there is an $X_0 \subset X$ such that $X_0 \rhd y$. Clearly, $\emptyset \in \mathcal{X}_4$.

We show that $\mathcal{X}_4$ is closed under adjunction. Suppose X is in $\mathcal{X}_4$. Let Y witness that X is in $\mathcal{X}_4$. Consider $X \cup \{x\}$. We may assume that $x \notin X$. Suppose $X \rhd y$. We show that $Y \cup \{y\}$ has the desired property.

First, we note that $y \notin Y$. If it were, we would have $X \rhd y$ and $X_0 \rhd y$, for some $X_0 \subset X$. So $X_0 \equiv X$, contradicting the fact that X is Dedekind finite. Ergo, $(X \cup \{x\}) \equiv (Y \cup \{y\})$. Consider $y' \in Y \cup \{y\}$. If $y \in Y$, we can find $X_0 \subset X \subset X \cup \{x\}$ with $X_0 \rhd y$. If $y' = y$, we have $X \rhd y$ and $X \subset X \cup \{x\}$.

We prove that no element of $\mathcal{X}_4$ is the universe. Suppose X is in $\mathcal{X}_4$. Let Y be the promised witness of this fact. Let $Y \rhd y$. We find $y \notin Y$. It follows, by Lemma 3.1, that, for some x, we have $x \notin X$. Finally, we contract our classes to $\mathcal{X}_4$. This gives us $\mathsf{HP}[\mathsf{c8}]$.

3.4. **Bootstrap in adjunctive-subtractive relation theory.** We consider adjunctive-subtractive relation theory $\mathsf{ar}[\mathsf{r3}]$. In this theory we treat classes as diagonal relations. We define the equinumerosity of X and Y as the existence of a bijection between X and Y. It is immediate that $\mathsf{ar}[\mathsf{r3}]$ verifies the axioms $\mathsf{e1}$ and $\mathsf{e2}$. Thus, when we add Hume's principle in the form of $\mathsf{h1}$ and $\mathsf{h2}$ for equinumerosity-as-defined, we obtain a direct interpretation of HL. Thus, by the result of the previous subsection, we may conclude that $\mathsf{ar}[\mathsf{r3},\mathsf{h1},\mathsf{h2}]$ directly interprets ar plus the no-universe axiom $\mathsf{c8}$.

By the results of [Vis09], we find that $\mathsf{ar}[\mathsf{r3},\mathsf{h1},\mathsf{h2}]$ interprets Q as a theory of cardinals. In the converse direction, we know that Q interprets $\mathrm{I}\Delta_0 + \Omega_1$. Using familiar methods one may interpret $\mathsf{ar}[\mathsf{r3},\mathsf{h1},\mathsf{h2}]$ in this last theory. So, we find that $\mathsf{ar}[\mathsf{r3},\mathsf{h1},\mathsf{h2}]$ is mutually interpretable with Q.

We note that $\mathsf{ar}[\mathsf{c8}]$ does not directly interpret $\mathsf{ar}[\mathsf{r3},\mathsf{h1},\mathsf{h2}]$. Suppose it did. Consider a standard model $\mathcal{M}$ of $\mathsf{ar}[\mathsf{c8}]$ with an infinite domain and all finite relations over the domain. We fix a finite set of parameters of the interpretation in $\mathcal{M}$. Since, any relation in $\mathcal{M}$ is definable from finitely many objects, we may assume that the parameters are objects. The interpretation provides us with an internal model $\mathcal{N}$ of $\mathsf{ar}[\mathsf{r3},\mathsf{h1},\mathsf{h2}]$. Without loss of generality, we may assume that we also have $\mathsf{h3}$. Since every permutation of the objects in $\mathcal{M}$ extends to an automorphism of $\mathcal{M}$, and since the objects of $\mathcal{N}$ coincide with the objects of $\mathcal{M}$, any permutation of the objects of $\mathcal{N}$ that fixes the parameters extends to an automorphism of $\mathcal{N}$. Clearly, we can find a standardly finite set X in $\mathcal{N}$ and an element n in $\mathcal{N}$ such that $X \rhd_{\mathcal{N}} n$ and the elements of X (according to $\mathcal{N}$) and n are disjoint from the parameters.[8] Take any permutation σ of the domain that fixes the parameters. We have $X \equiv_{\mathcal{N}} \sigma X \rhd_{\mathcal{N}} \sigma n$. It follows that, for any non-parameter n', $X \rhd_{\mathcal{N}} n'$. Quod impossibile.

Open Question 3.1. We need the subtraction axiom $\mathsf{r3}$ in our development since the definition of equinumerosity in this context contains an existential quantifier over relations. Thus, equinumerosity is not absolute for contraction of relations. For this reason, we might loose axiom $\mathsf{h2}$ when we try to obtain subtraction of relations via contraction. Can we eliminate the use of the subtraction axiom $\mathsf{r3}$ in our result? ❏

4. Development of the Theory of Addition in $\mathsf{eqnum}[\mathsf{c8}]$

In this section we show how to interpret $\mathsf{Q}_{\mathsf{add}}$ in $\mathsf{eqnum}[\mathsf{c8}]$. We will provide a bit more of the theory of addition since this flows naturally from the development. In Appendix A, we show that we can go on to *locally* interpret full Presburger Arithmetic PresA in $\mathsf{Q}_{\mathsf{add}}$.

We begin with a lemma that works already in $\mathsf{ac}[\mathsf{c4}]$.

Lemma 4.1 ($\mathsf{ac}[\mathsf{c4}]$)**.** *Suppose $\mathcal{Y}$ is closed under empty class and adjunction. Let $\mathsf{D}(\mathcal{Y})$ consist of those classes Y such that:*

$$\forall Y'{\subseteq}Y\ \forall Z{\in}\mathcal{Y}\ (Y' \cup Z) \in \mathcal{Y}.^{9}$$

Then, $\mathsf{D}(\mathcal{Y})$ is closed under empty class, adjunction and union and downwards closed under $\subseteq$. As the consequence of the last fact, closure under intersection, class subtraction, etc., are preserved from $\mathcal{Y}$ to $\mathsf{D}(\mathcal{Y})$.

Proof. Closure under empty class, singletons, and downward closure under $\subseteq$ are trivial. We verify closure under $\cup$. Suppose Y_0 and Y_1 are in $\mathsf{D}(\mathcal{Y})$. Let $Y' \subseteq (Y_0 \cup Y_1)$. Then $(Y' \cap Y_1) \subseteq Y_1$. Consider any Z in $\mathcal{Y}$. We find that $(Y' \cap Y_1) \cup Z$ is in $\mathcal{Y}$. Since, $(Y' \cap Y_0) \subseteq Y_0$, we find that

$$(Y' \cup Z) = ((Y' \cap Y_0) \cup ((Y' \cap Y_1) \cup Z)) \in \mathcal{Y}.$$

We may conclude that $Y_0 \cup Y_1$ is in $\mathsf{D}(\mathcal{Y})$. ❏

[8]Note that X in $\mathcal{N}$ may be represented as $\vec{Y}, \vec{m}$ in $\mathcal{M}$. However, the precise representation of X is immaterial for our argument.

[9]Here '$(Y' \cup Z) \in \mathcal{Y}$' reads: $(Y' \cup Z)$ exists and is in $\mathcal{Y}$.

We start with $\mathsf{eqnum}[\mathsf{c8}]$. We first follow the bootstrap of the previous section to gain $\mathsf{eqnum}[\mathsf{c5}, \mathsf{c8}, \mathsf{e3}, \mathsf{e4}, \mathsf{e5}, \mathsf{e6}]$. Note that $\mathsf{c8}$ is universal in the class quantifiers and, hence, always preserved under contractions of the totality of classes.

To work conveniently we contract our classes to add closure under $X \cap Y$, $X \setminus Y$, $X \cup Y$. We leave this as an easy exercise to the reader. So, we may work in $\mathsf{eqnum}^+ := \mathsf{eqnum}[\mathsf{c4}, \mathsf{c6}, \mathsf{c7}, \mathsf{c8}, \mathsf{e3}, \mathsf{e4}, \mathsf{e5}, \mathsf{e6}]$. (Note that $\mathsf{c5}$ is derivable.)

We work in eqnum^+. We consider the collection of classes $\mathcal{X}_5$, such that X is in $\mathcal{X}_5$ iff, for all Y, Y', if $X \subseteq Y \equiv Y'$, then there is an X', such that $X \equiv X' \subseteq Y'$.

Clearly, the empty class is in $\mathcal{X}_5$. We show that $\mathcal{X}_5$ is closed under adjunction. Consider $X \in \mathcal{X}_5$ and any x. Without loss of generality we may assume that $x \notin X$. Suppose $X \in \mathcal{X}_5$ and $(X \cup \{x\}) \subseteq Y \equiv Y'$. Then, for some X', $X \equiv X' \subseteq Y'$. If $X' = Y'$, then $X \equiv Y$, and hence, by Dedekind finiteness, $X = Y$, contradicting the fact that $x \in Y$ and $x \notin X$. So, for some x', $x' \in Y'$ and $x' \notin X'$. We have $(X \cup \{x\}) \equiv (X' \cup \{x'\}) \subseteq Y'$.

Consider the class $\mathcal{X}_6$ of all X such that for all X', Y, Y' with $X \equiv X'$, $Y \equiv Y'$, $X \subseteq Y$, $X' \subseteq Y'$, we have $(Y \setminus X) \equiv (Y' \setminus X')$.

It is clear that $\mathcal{X}_6$ contains the empty class. We prove closure under adjunction. Consider $X \in \mathcal{X}_6$ and an arbitrary x. We may assume that $x \notin X$. Consider X', Y, Y' with $(X \cup \{x\}) \equiv X'$, $Y \equiv Y'$, $(X \cup \{x\}) \subseteq Y$, $X' \subseteq Y'$. We find that X' is not empty. Suppose $x' \in X'$. We find that $X \equiv (X' \setminus \{x'\})$, $X \subseteq Y$, $(X' \setminus \{x'\}) \subseteq Y'$. Hence $(Y \setminus X) \equiv (Y' \setminus (X' \setminus \{x'\})$. It follows that:

$$(Y \setminus (X \cup \{x\})) = ((Y \setminus X) \setminus \{x\}) \equiv ((Y' \setminus (X' \setminus \{x'\})) \setminus \{x'\}) = (Y' \setminus X').$$

Let $\mathcal{X}_7 := \mathsf{D}(\mathcal{X}_5 \cap \mathcal{X}_6)$. We find that $\mathcal{X}_7$ is closed under empty class, adjunction, union and downwards closed under $\subseteq$. Since the existential quantifier in the definition of $\mathcal{X}_5$ is $\subseteq$-bounded and subtraction of classes in the definition of $\mathcal{X}_6$ brings us to subclasses , relativization to $\mathcal{X}_7$ preserves eqnum^+ and gains us $\mathsf{e7}$ and $\mathsf{e8}$. So we may work in $\mathsf{eqnum}^\star := \mathsf{eqnum}^+[\mathsf{e7}, \mathsf{e8}]$.

We are now ready to develop the theory of the ordering of the cardinal numbers. We work in $\mathsf{eqnum}^\star$. We define $X \leq Y$ by $\exists X', Y' \ (X \equiv X' \wedge Y \equiv Y' \wedge X' \subseteq Y')$. Clearly, $\equiv$ is a congruence for $\leq$. It is immediate that $X \leq X$ and that, if $X \subseteq Y$, then $X \leq Y$.

We prove that $X \leq Y$ iff, for some X'', we have $X \equiv X'' \subseteq Y$. Suppose $X \equiv X' \subseteq Y' \equiv Y$. By $\mathsf{e8}$, we may find an X'' such that $X' \equiv X'' \subseteq Y$. Hence, $X \equiv X'' \subseteq Y$. The converse direction is trivial. It follows that:

$$\begin{aligned} X \leq Y \leq Z & \to \exists X', Y' \ (X \equiv X' \subseteq Y \equiv Y' \subseteq Z) \\ & \to \exists X'' \ (X \equiv X' \equiv X'' \subseteq Y' \subseteq Z) \\ & \to X \leq Z \end{aligned}$$

Suppose that $X \leq Y$ and $Y \leq X$. It follows that $X \equiv X' \subseteq Y \equiv Y' \subseteq X$. We find that, for some X'', $X \equiv X'' \subseteq Y' \subseteq X$. Since X is Dedekind finite, it follows that $X'' = X$ and, hence, that $Y' = X$. We may conclude that $X \equiv Y$.

Thus we see that we have derived the principles $\mathsf{q8}, \mathsf{q9}, \mathsf{q10}$, under the interpretation of numbers as classes modulo equinumerosity.

Here is a useful further insight. Suppose $X \leq (Y_0 \cup Y_1)$. We show that there are X_0, X_1, such that $X = (X_0 \cup X_1)$ and $X_0 \leq Y_0$ and $X_1 \leq Y_1$. We have

$X \equiv X' \subseteq (Y_0 \cup Y_1)$. It follows that $(Y_0 \cap X') \subseteq X' \equiv X$. Hence, for some X_0, we have $(Y_0 \cap X') \equiv X_0 \subseteq X$. It follows that:

$$X_1 := (X \setminus X_0) \equiv (X' \setminus (Y_0 \cap X')) = (X' \setminus Y_0) \subseteq Y_1.$$

It follows that $X_1 \leq Y_1$.

We prove a lemma.

Lemma 4.2 (eqnum*)**.** *Suppose $\mathcal{Y}$ is closed under empty class and adjunction. Let $\mathsf{D}^+(\mathcal{Y})$ consist of those classes Y such that:*

$$\forall Y' {\leq} Y \, \forall Z {\in} \mathcal{Y} \; (Y' \cup Z) \in \mathcal{Y}.$$

Then, $\mathsf{D}^+(\mathcal{Y})$ is closed under empty class, adjunction and union and downwards closed under $\leq$ and hence a fortiori under $\subseteq$. As the consequence of this last fact, $\mathsf{D}^+(\mathcal{Y})$ is closed under subtraction, intersection, etc.

Proof. Closure under empty class and downward closure under $\leq$ are trivial. It is also easy to see that all singletons are in $\mathsf{D}^+(\mathcal{Y})$. We verify closure under $\cup$. Suppose Y_0 and Y_1 are in $\mathsf{D}^+(\mathcal{Y})$. Let $Y' \leq (Y_0 \cup Y_1)$. We can find Y_0', Y_1', such that $(Y_0' \cup Y_1') = Y'$ and $Y_0' \leq Y_0$ and $Y_1' \leq Y_1$. Consider any $Z \in \mathcal{Y}$. Since Y_1 is in $\mathsf{D}^+(\mathcal{Y})$, we find that $(Y_1' \cup Z) \in \mathcal{Y}$. Since $Y_0 \in \mathsf{D}^+(\mathcal{Y})$, we obtain $(Y' \cup Z) = (Y_0' \cup (Y_1' \cup Z)) \in \mathcal{Y}$. We may conclude that $Y_0 \cup Y_1$ is in $\mathsf{D}^+(\mathcal{Y})$. ❑

We proceed with the development of the arithmetic of 0, successor and addition. We define $0 := \emptyset$. Clearly 0 is the minimal element of $\leq$.

We define $\mathsf{S}XY$ by $\exists x \, (x \notin X \wedge Y \equiv (X \cup \{x\}))$. It is easy to see that S is a total injective function modulo $\equiv$ and that $\emptyset$ is not in the range of S. We also have that each non-empty class has a predecessor. By Dedekind finiteness, we find that, if $\mathsf{S}XY$, then $X \not\equiv Y$. Thus we have, via the obvious interpretation, axioms q1, q2, q7, q14.

We show that, if $X \leq Y$, then either $X \equiv Y$ or, for some Z, we have $\mathsf{S}XZ$ and $Z \leq Y$. Suppose $X \leq Y$. We have $X \equiv X' \subseteq Y$, for some X'. In case $X' = Y$, we have $X \equiv Y$ and we are done. Otherwise, there is an x', such that $x' \in Y$ and $x' \notin X'$. Pick any $x \notin X$. We have $(X \cup \{x\}) \equiv (X' \cup \{x'\}) \subseteq Y$, and hence $(X \cup \{x\}) \leq Y$. Clearly $\mathsf{S}X(X \cup \{x\})$. Thus, we have proved the interpretation of q11.

Suppose $\mathsf{S}XZ$. Then, clearly $X \leq Z$. Suppose $X \leq Y \leq Z$. It follows that either $X \equiv Y$ or $Z \equiv \mathsf{S}X \leq Y \leq Z$, and, hence $Z \equiv Y$. So X is an immediate $\leq$-predecessor of Z.

To obtain the linearity of $\leq$ modulo $\equiv$, we must again contract our classes. Let $\mathcal{X}_8$ be the totality of all classes X such that, for all Y, $X \leq Y$ or $Y \leq X$. Clearly, $\emptyset$ is in $\mathcal{X}_8$. We show that $\mathcal{X}_8$ is closed under successor and, hence, under adjunction. Suppose $X \in \mathcal{X}_8$ and $\mathsf{S}XZ$. Consider any Y. In case $Y \leq X$, we immediately have $Y \leq Z$. Suppose $X \leq Y$. Then, either $X \equiv Y$, in which case $Y \leq Z$, or $Z \leq Y$. We note that S and $\leq$ are absolute under contraction to $\leq$-downwards closed totalities of classes. We contract to $\mathsf{D}^+(\mathcal{X}_8)$. We preserve eqnum* and gain linearity of $\leq$. We proceed to work in eqnum*[q12], where of course we read q12 *modulo the interpretation we developed.*

Before defining addition, we first contract our classes to gain two desirable properties. We consider the collection of classes $\mathcal{X}_9$ of all X such that for all X', Y, Y', if

$X \# Y$, $X' \# Y'$, $X \equiv X'$, $Y \equiv Y'$, then $(X \cup Y) \equiv (X' \cup Y')$. Clearly, $\emptyset \in \mathcal{X}_9$. We prove that $\mathcal{X}_9$ is closed under adjunction. Suppose $X \in \mathcal{X}_9$, $(X \cup \{x\}) \# Y$, $X' \# Y'$, $(X \cup \{x\}) \equiv X'$, $Y \equiv Y'$. If $x \in X$, we are immediately done, so suppose $x \notin X$. Clearly, X' is not empty. Let $x' \in X'$. We have $(X \# Y)$, $(X' \setminus \{x'\}) \# Y'$, $X \equiv (X' \setminus \{x'\})$, $Y \equiv Y'$. So $(X \cup Y) \equiv ((X' \setminus \{x'\}) \cup Y')$. Since $x \notin (X \cup Y)$ and $x' \notin ((X' \setminus \{x'\}) \cup Y')$, we find $((X \cup \{x\}) \cup Y) \equiv (X' \cup Y')$.

We define $\mathcal{X}_{10}$ as the totality of all classes X such that:

$$\forall Y \, \exists Y' \, (Y \equiv Y' \wedge (X \cap Y') = \emptyset).$$

Clearly, $\emptyset$ is in $\mathcal{X}_{10}$. We show that $\mathcal{X}_{10}$ is closed under adjunction. Let $X \in \mathcal{X}_{10}$ and $x \notin X$. Consider any Y. Suppose $Y^* \equiv Y$ and $(X \cap Y^*) = \emptyset$. By the no-universe principe, there is a y not in $X \cup \{x\} \cup Y^*$ and there is a y' not in $X \cup \{x, y\} \cup Y^*$. We take $Y' := (Y^* \setminus \{x\}) \cup \{y, y'\}$.

We contract our classes to $\mathsf{D}^+(\mathcal{X}_9 \cap \mathcal{X}_{10})$. Note that the existential quantifier in the definition of $\mathcal{X}_{10}$ can be $\leq$-bounded. It follows that we directly interpret $\mathsf{eqnum}^\star[\mathsf{q}12, \mathsf{e}9, \mathsf{e}10]$. We proceed in this theory.

We define:

- $\mathsf{A}XYZ :\leftrightarrow \exists X', Y' \, ((X' \cap Y') = \emptyset \wedge X \equiv X' \wedge Y \equiv Y' \wedge (X' \cup Y') \equiv Z)$.

We note that $\equiv$ is a congruence relation for A. Using $\mathsf{e}9$ and $\mathsf{e}10$, one can easily show that:

$$\mathsf{A}XYZ \leftrightarrow \exists Y' \; ((X \cap Y') = \emptyset \wedge Y \equiv Y' \wedge (X \cup Y') \equiv Z).$$

Moreover, by $\mathsf{e}9$ and $\mathsf{e}10$, A defines a total function modulo $\equiv$.

Trivially $\mathsf{A}X\emptyset Z$ iff $X \equiv Z$. So we have the interpretation of $\mathsf{q}3$.

We show that $\mathsf{S}XZ$ iff $\mathsf{A}X\{y\}Z$. We have $\mathsf{A}X\{y\}Z$ iff, for some Y', $\{y\} \equiv Y'$ and $(X \cap Y') = \emptyset$. It follows that $Y = \{y'\}$, for some y' with $y' \notin X$. So $Z \equiv (X \cup \{y'\})$. The converse is even easier.

It is immediate that addition is commutative. We show that it is associative. Suppose e.g. $\mathsf{A}XYZ$ and $\mathsf{A}ZUV$. We may find X', Y', Z', U', V' such that X', Y', U' are pairwise disjoint and $X' \cup Y' = Z'$ and $Z' \cup U' = V'$ and $X \equiv X'$, $Y \equiv Y'$, $U \equiv U'$, $V \equiv V'$. Let $W' := Y' \cup U'$. Then we have $\mathsf{A}Y'U'W'$ and $\mathsf{A}X'W'V'$. Ergo, $\mathsf{A}YUW'$ and $\mathsf{A}XW'V$. The converse is similar.

We note that the previous two results imply the interpretation of $\mathsf{q}4$. Moreover, we have the interpretations of $\mathsf{q}15$ and $\mathsf{q}16$.

We show $X \leq Y$ iff, for some Z, $\mathsf{A}XZY$. Suppose $X \leq Y$. Then, $X \equiv X' \subseteq Y$. It is easily seen that $\mathsf{A}X(Y \setminus X')Y$. Conversely, suppose $\mathsf{A}XZY$. Then, for some disjoint X', Z', we have $X \equiv X'$, $Z \equiv Z'$ and $(X' \cup Z') \equiv Y$. We can find an X'', with $X' \equiv X'' \subseteq Y$. So $X \equiv X'' \subseteq Y$, and hence $X \leq Y$. Thus, by commutativity, we have interpreted $\mathsf{q}13$.

We leave the easy verification of the interpretation of $\mathsf{q}17$ to the reader.

In summary: we have directly interpreted in $\mathsf{eqnum}[\mathsf{c}8]$ the principles $\mathsf{q}1$-4, and $\mathsf{q}7$-17.

5. Development of Multiplication in HL

The basic insight of this section is something very familiar to anyone who worked out his or her own arithmetization. To develop recursion for a function F such that $x < Fx$, we do not need computation sequences, since computation *sets* are sufficient. In the present context the idea is not just convenient but essential.

We start with the theory HL. By the development in the previous section, we can contract our classes to gain the principles of $\mathsf{eqnum}^\star[\mathsf{q}12, \mathsf{e}9, \mathsf{e}10]$. Thus, we also have q1-4, and q7-17 available for the interpretations of the arithmetical operations and relations specified in the previous section. Without loss of generality, we may also add h3. We call the resulting theory $\mathsf{HL}^\star$. We work in $\mathsf{HL}^\star$.

We define $\mathcal{N}$ as the virtual class of x such that $\exists X\, X \rhd x$. We define, for x, y in $\mathcal{N}$, $x \simeq y$ iff $\exists Z\, (Z \rhd x \wedge Z \rhd y)$. Clearly, $\simeq$ is an equivalence relation with domain $\mathcal{N}$. We find that $\rhd$ gives us a bijection between the classes modulo $\equiv$ and $\mathcal{N}$ modulo $\simeq$. This means that we can induce Z and and $\leq$ and S and A on $\mathcal{N}$ modulo $\simeq$. We call the resulting relations $\mathsf{Z}^\star$, $\leq^\star$, $\mathsf{S}^\star$, and $\mathsf{A}^\star$.

We turn to the definition of multiplication. We will first give the definition and then add some motivating remarks. We define $\mathsf{M}XYZ$ iff, either $X = 0$ and $Z = 0$, or $X \neq 0$ and, there is an $Y' \equiv Y$, and a $z \notin Y'$ with $Z \rhd z$, and an x with $X \rhd x$, such that:

i. $Y' \subseteq \mathcal{N}$,
ii. For all $y, y' \in (Y' \cup \{z\})$, if $y \simeq y'$, then $y = y'$.
iii. For all $y \in (Y' \cup \{z\})$, $\mathsf{Z}^\star y$ or, for some $y' \in (Y' \cup \{z\})$, $\mathsf{A}^\star y'xy$ and, for all w, if $y' <^\star w <^\star y$, then $w \notin Y'$.
iv. For all $y \in Y'$, $z >^\star y$.

The idea is as follows. First, the case that X is empty is special. So, we set that aside. If X is not empty, X times Y is Z (modulo equinumerosity) if there is a computation that witnesses this fact. The computation is a set of numbers $Y' \cup \{z\}$. We demand that the computation starts with a zero and ends with a number z representing the cardinality of Z. We have a normalizing condition (ii) that says that every number has at most one representative in the computation. This is needed since a number could have many 'copies' in our set up where we only have a Hume relation. Item (iii) tells us that our computation proceeds by adding a number of size X in each 'step'. The bit in (iii) that there are no non-intended intervening elements between two elements of a step is just there because in the weak context we can not prove that. We build it in just in order not to have to prove it. Clause (iv) is another case of 'no intervening elements'. It is added for the same reason as the extra bit in (iii).

We note that $\equiv$ is a congruence for M.

We first prove that $\mathsf{M}X0Z$ iff $Z = 0$. In case $X = 0$, we are immediately done, so suppose $X \neq 0$. We have $\mathsf{M}X0Z$ iff, for some z with $Z \rhd z$ and x with $X \rhd x$, we have $\mathsf{Z}^\star z$ or $\mathsf{A}^\star zxz$. The second case is excluded by q17. So $\mathsf{Z}^\star z$, and hence, $Z = 0$.

Suppose $\mathsf{M}XYZ$, $\mathsf{S}YU$, $\mathsf{A}ZXV$. We show that $\mathsf{M}XUV$. This is immediate when $X = 0$. So, suppose $X \neq 0$. Let Y', z, x be the promised witnesses for $\mathsf{M}XYZ$. Suppose $V \rhd v$ and $\mathsf{A}^\star zxv$. We consider $U' := Y' \cup \{z\}$. Clearly, $U \equiv U'$. Since

$v >^\star z >^\star y'$, for any y' in Y', we find that $v \not\approx u'$, for any u' in U'. It follows that U', v, x, V satisfy conditions (i,ii,iii,iv) for $\mathsf{M}XUV$.

Suppose $\mathsf{M}XUV$ and $\mathsf{S}YU$. We show that, for some Z, $\mathsf{M}XYZ$ and $\mathsf{A}ZXV$. In case $X = 0$ this is immediate. Suppose $X \neq 0$. It is easily seen that, since $U \neq 0$, we also have $V \neq 0$. Let U', v, x, V satisfy the conditions for $\mathsf{M}XUV$. By (iii) and the fact that not $\mathsf{Z}^\star(v)$, there is a $z \in U'$ such that $\mathsf{A}^\star zxv$. Let $Y' := U' \setminus \{z\}$. Clearly $Y \equiv Y'$. Let $Z \rhd z$. Then, Y', z, x, Z witness $\mathsf{M}XYZ$. Moreover $\mathsf{A}ZXV$.

We now consider $\mathcal{X}_{11}$. The class of all X such that, for all U, there is, modulo $\equiv$, a unique Z, such that $\mathsf{M}UXZ$. By the previous results, it is clear that $\mathcal{X}_{11}$ is closed under 0 and successor. We contract our classes to $\mathsf{D}^+(\mathcal{X}_{11})$, thus gaining the uniqueness clause for multiplication. The relation M is absolute for this contraction (and similar contractions). We note a subtlety here: the totality $\mathcal{N}$ is not absolute but contracts to an initial segment that is closed under addition. However, for the question whether we have $\mathsf{M}XYZ$ relativized to $\mathcal{X}_{11}$, we only need to worry about elements below z. For elements below z the old and the new numbers coincide.

Remark 5.1. At this point we could take a shortcut and obtain the desired interpretation of Q by employing the result of [Šve07]. However, since we are so close to the the end of our development, we will finish it. Moreover, we get a little bit more in this way, since we will also preserve full $\mathsf{HL}^\star$. ❏

We let $\mathcal{X}_{12}$ be the totality of classes X, such that:

$$\forall Y{\leq}X\, \forall U, V\ (\mathsf{M}UXV \to \exists W \leq V\ \mathsf{M}UYW).$$

Clearly, 0 is in $\mathcal{X}_{12}$. Suppose X is in $\mathcal{X}_{12}$. We show that any X' with $\mathsf{S}XX'$ is in $\mathcal{X}_{12}$. Suppose that $\mathsf{M}UX'V$. In case $U = 0$, we are immediately done, so we may assume that $U \neq 0$. Suppose $Y \leq X'$. We can easily prove that $Y \leq X$ or $Y = X'$. In the second case we are immediately done. In the first case, we can find a Z, such that $\mathsf{M}UXZ$ and $\mathsf{A}ZUV$. It follows that, for some $W \leq Z$, we have $\mathsf{M}UYW$, since $Z \leq V$, we find $W \leq V$.

We relativize to $\mathsf{D}^+(\mathcal{X}_{12})$, thus obtaining the principle

$$(\mathsf{p1}) \quad \vdash \forall Y{\leq}X\, \forall U, V\ (\mathsf{M}UXV \to \exists W \leq V\ \mathsf{M}UYW).$$

Let $\mathcal{X}_{13}$ consist of those X such that, for all U, X', X'', Z, Z', Z'', if $\mathsf{A}XX'X''$, $\mathsf{A}ZZ'Z''$, $\mathsf{M}UXZ$ and $\mathsf{M}UX'Z'$, then $\mathsf{M}UX''Z''$. It is easy to see that 0 is in $\mathcal{X}_{13}$.

We prove that $\mathcal{X}_{13}$ is closed under successor. Suppose $X \in \mathcal{X}_{13}$, $\mathsf{S}XY$, $\mathsf{A}YY'Y''$, $\mathsf{A}ZZ'Z''$, $\mathsf{M}UYZ$ and $\mathsf{M}UY'Z'$. Let $\mathsf{A}XY'X''$. We find that, for some W, we have $\mathsf{M}UXW$ and $\mathsf{A}UWZ$. Let $\mathsf{A}WZ'W''$. We may conclude that $\mathsf{M}UX''W''$. We note that $\mathsf{S}X''Y''$, since $\mathsf{S}XY$ and $\mathsf{A}XY'X''$ and $\mathsf{A}YY'Y''$. Also $\mathsf{A}W''UZ''$, since $\mathsf{A}WZ'W''$ and $\mathsf{A}WUZ$ and $\mathsf{A}ZZ'Z''$. We may conclude that $\mathsf{M}UY''Z''$.

We relativize to $\mathsf{D}^+(\mathcal{X}_{13})$. Thus obtaining the principle:

$$(\mathsf{p2}) \quad \vdash (\mathsf{A}XX'X'' \wedge \mathsf{A}ZZ'Z'' \wedge \mathsf{M}UXZ \wedge \mathsf{M}UX'Z') \to \mathsf{M}UX''Z''.$$

Let $\mathcal{X}_{14}$ be the totality of all X, such that, $\mathsf{M}WYU$ and $\mathsf{M}UXV$ implies that there is a Z, with MYXZ and $\mathsf{M}WZV$.

It is easily seen that 0 is in $\mathcal{X}_{15}$. Let $X \in \mathcal{X}_{14}$. Suppose $\mathsf{S}XX'$. $\mathsf{M}WYU$ and $\mathsf{M}UX'V'$. Then, for some V, $\mathsf{M}UXV$ and $\mathsf{A}VUV'$. It follows that for some Z, $\mathsf{M}YXZ$ and $\mathsf{M}WZV$. We find that, for some Z', $\mathsf{M}YX'Z'$ and $\mathsf{A}ZYZ'$. We have

$\mathsf{A}ZYZ'$, $\mathsf{A}VUV'$, $\mathsf{M}WZV$, $\mathsf{M}WYU$. Hence, $\mathsf{M}WZ'V'$. If we contract our classes to $\mathsf{D}^+(\mathcal{X}_{14})$, we gain the principle:

$$(\mathsf{p3}) \quad \vdash (\mathsf{M}WYU \wedge \mathsf{M}UXV) \wedge \mathsf{M}YXZ) \to \mathsf{M}WZV).$$

Finally, let $\mathcal{X}_{15}$ consist of all X, such that, for all U, there is a Z with $\mathsf{M}UXZ$. We can easily see that $\mathcal{X}_{14}$ is downwards closed w.r.t. $\leq$ and is closed under 0, singletons and addition. It follows that $\mathcal{X}_{14}$ is closed under union, since if $\mathsf{A}(X \setminus Y)YZ$, then $(X \cup Y) \equiv Z$. We show that multiplication is defined on $\mathcal{X}_{14}$ and that it is closed under multiplication. Suppose that $X_0, X_1 \in \mathcal{X}_{14}$. We have $\mathsf{M}X_0X_1X$, for some X. We also have, for some Z_0 and Z, $\mathsf{M}UX_0Z_0$ and $\mathsf{M}Z_0X_1Z$. By $\mathsf{p3}$, we have: $\mathsf{M}UXZ$.

When we contract our classes to X_{15}, we preserve $\mathsf{HP}^\star$ and gain $\mathsf{q5}$, $\mathsf{q6}$, $\mathsf{q18}$ and $\mathsf{q19}$. Thus we end with $\mathsf{HP}^\star[\mathsf{q5}, \mathsf{q6}, \mathsf{q18}, \mathsf{q19}]$, which contains Q.

Conversely, since Q interprets $\mathrm{I}\Delta_0 + \Omega_1$, we can show that Q interprets HL.

6. An Application

We show that the presence of a certain preorder on our domain is equivalent (in the sense of mutual direct interpretability) to Hume's principle.

We expand the signature of eqnum with new symbols $\mathcal{N}$ of type $\mathfrak{o}$, $\mathsf{Z}^\star$ of type $\mathfrak{o}$, and $\leq^\star$ of type $\mathfrak{oo}$. We add axioms stating that $\leq^\star$ is a linear preorder on $\mathcal{N}$ with $\mathsf{Z}^\star$ the non-empty virtual class of its initial elements. We demand that every element of $\mathcal{N}$ has strict $\leq^\star$-successors. We call the resulting theory $\mathsf{eqnum}_{\mathsf{ord}}$. Let the induced equivalence relation of $\leq^\star$ be $\simeq$.

We show that HL is directly interpretable in $\mathsf{eqnum}_{\mathsf{ord}}$. We work in $\mathsf{eqnum}_{\mathsf{ord}}$. We first contract our classes so that we gain subtraction of elements plus the fact that $\equiv$ is an equivalence relation. This enables us to work in $\mathsf{eqnum}_{\mathsf{ord}}[\mathsf{c5}, \mathsf{e3}, \mathsf{e4}, \mathsf{e5}]$.

We say that X is an x-class iff, $x \in \mathcal{N}$, and, for all $x' \in X$, $x' <^\star x$, and, for all $x' <^\star x$, there is precisely one $x'' \simeq x'$, such that $x'' \in X$.

We consider the totality $\mathcal{N}_0$ of all x in $\mathcal{N}$, such that for all x-classes X' and X'', we have $X' \equiv X''$. Clearly, all elements of $\mathsf{Z}^\star$ are in $\mathcal{N}_0$.

Suppose that x is in $\mathcal{N}_0$ and that y is a direct successor of x. We show that y is in $\mathcal{N}_0$. Suppose Y' and Y'' are y-classes. Since x is is a direct predecessor of y, we will have direct predecessors of y in Y' and Y'', say these are x', respectively x''. We note that $x \simeq x'$ and $x \simeq x''$. Clearly, $Y' \setminus \{x'\}$ and $Y'' \setminus \{x''\}$ are x-classes. Hence, $(Y' \setminus \{x'\}) \equiv (Y'' \setminus \{x''\})$. But then also $Y' \equiv Y''$.

Let $\mathcal{X}^\star$ be the totality of classes X for which there is a y in $\mathcal{N}_0$ and a Y, such that Y is a y-class and $X \equiv Y$. It is easy to see that $\mathcal{X}^\star$ is closed under empty class and adjunction. Moreover, a Y that witnesses that X is in $\mathcal{X}^\star$, will be itself in $\mathcal{X}^\star$.

We contract our classes to $\mathcal{X}^\star$. We gain the following principle: for every X there is a $y \in \mathcal{N}_0$, and a y-class Y, such that $X \equiv Y$.

We define $X \rhd y$ iff, $y \in \mathcal{N}_0$ and, for some y-class Y, we have $X \equiv Y$. We have already seen that we must have $\mathsf{h1}$. Suppose $X \rhd y$ and $X' \rhd y$. Let Y and Y' witness this fact. Since, $y \in \mathcal{N}_0$, we have $Y \equiv Y'$. Hence, $X \equiv X'$. This gives us $\mathsf{h2}$.

We apply the above insight to show that Presburger Arithmetic PresA does not interpret $\mathsf{eqnum}[\mathsf{c8}]$. Our argument is is quick and easy. Probably inspection of the quantifier elimination for PresA could yield much stronger results. To exclude any possible interpretation we must take into account that our interpretations could have special features like the presence of parameters or being piecewise. Fortunately, we can work in a structure in which every element is definable and that has infinitely definable elements. This means that interpretations with parameters and pieces can always be replaced by interpretations without parameters and pieces.

Suppose PresA does interpret $\mathsf{eqnum}[\mathsf{c8}]$, say via K. We may assume that K is a parameter-free, one-piece interpretation. However, K may still be more-dimensional. We consider the internal model $\mathcal{K}$ given by K in the standard model $\mathbb{N}_{\mathsf{add}} := \langle \omega, 0, \mathsf{S}, +, \leq \rangle$.

The object domain of $\mathcal{K}$ will consist of a definable infinite totality of tuples $(n_0, \ldots, n_{k-1})$. We can define, in $\mathbb{N}_{\mathsf{add}}$, the following ordering on these tuples: $(n_0, \ldots, n_{k-1}) \preceq (m_0, \ldots, m_{k-1})$ iff $(n_0 + \ldots + n_{k-1}) < (m_0 + \ldots + m_{k-1})$ or $((n_0 + \ldots + n_{k-1}) = (m_0 + \ldots + m_{k-1})$ and $(n_0 < m_0$ or $(n_0 = m_0$ and $n_1 < m_1)$ or $\ldots$ $(n_0 = m_0$ and $\ldots$ $n_{k-1} \leq m_{k-1}))$. This gives our domain order type ω. Thus we can extend our interpretation of $\mathsf{eqnum}[\mathsf{c8}]$ to an interpretation of $\mathsf{eqnum}_{\mathsf{ord}}$. This gives us an interpretation of HL and, hence, of Q in PresA. Quod impossibile, since Q is essentially undecidable and PresA is decidable.

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Appendix A. Presburger Arithmetic

Presburger Arithmetic, PresA, for the natural numbers is axiomatized by the following principles.[10] The signature consists of 0, 1 and +.

$\mathsf{PresA1}$ $\vdash x + 1 \neq 0$,
$\mathsf{PresA2}$ $\vdash x + z = y + z \to x = y$,
$\mathsf{PresA3}$ $\vdash x + 0 = x$,
$\mathsf{PresA4}$ $\vdash x + (y + z) = (x + y) + z$,
$\mathsf{PresA5}$ $\vdash x = 0 \vee \exists y\, x = y + 1$,
$\mathsf{PresA6}$ $\vdash x + y = y + x$,
$\mathsf{PresA7}$ $\vdash \exists z\, (x + z = y \vee x = y + z)$,
$\mathsf{PresA8}$ for any $n \geq 2$, we have $\vdash \exists y\, (x = \underline{n}\, y \vee x = \underline{n}\, y + 1 \vee \ldots x = \underline{n}\, y + (\underline{n-1}))$.

[10] I took this axiomatization principles from Clemens Grabmayer's Master's Thesis. See Clemens' site `http://www.phil.uu.nl/~clemens/`. I only changed the order a bit and omitted a superfluous axiom.

We call the instances of PresA8: PresA8n. We easily show that, for $n, m \geq 2$, PresA8n and PresA8m are equivalent, over the other axioms, to PresA8(nm). It follows that any finite set of instances of PresA8 is equivalent to a single instance. We call the theory PresA1,2,3,4,5,6,7,8n: PresAn. One can show that, for $n > 1$, PresA8($n+1$) is not derivable from PresA8n. See e.g. [Smo91].

We show how to interpret PresAn (for $n \geq 2$) in $\mathsf{Q}_{\mathsf{add}}$. We work in $\mathsf{Q}_{\mathsf{add}}$. First let $\mathcal{J}_0$ be the virtual class of all x such that for all y, z, $(y+z)+x = y+(z+x)$ and $y+x = z+x \rightarrow y = z$ and $0+x = x$ and $1+x = x+1$. We easily prove that $\mathcal{J}_0$ contains 0, 1 and is closed under addition and predecessor. Relativizing to $\mathcal{J}_0$ gives us axioms PresA$1,2,3,4,5$, $\vdash 0+x = x$ and $\vdash x+1 = 1+x$. This theory clearly extends $\mathsf{Q}_{\mathsf{add}}$. We work in the extended theory. We will follow the usual convention of omitting brackets, which is justified by associativity.

Let $\mathcal{J}_1$ be the class of all x such that, for all y, $y+x = x+y$. We easily see that $\mathcal{J}_1$ is closed under 0, 1, + and predecessor. Relativizing to $\mathcal{J}_1$, gives us axioms PresA$1,2,3,4,5,6$. We proceed to work in this system.

We define $x \leq y$ by: $\exists z\, x+z = y$. It is easy to verify that $\leq$ is a partial order with minimum 0 and that addition is monotonic w.r.t. $\leq$.

Let $\mathcal{J}_2$ be the class of all x such that, for all $z \leq x$ and for all y, $z \leq y$ or $y \leq z$. Clearly, $\mathcal{J}_2$ is downwards closed under $\leq$ and 0 and 1 are in $\mathcal{J}_2$. Suppose x_0 and x_1 are in $\mathcal{J}_2$. We show that $(x_0+x_1) \in \mathcal{J}_2$. Consider any $z \leq (x_0+x_1)$ and any y. In case $z \leq x_0$ we are immediately done. Suppose $x_0 \leq z$, say $z = x_0+u$. It follows that $u \leq x_1$. In case $y \leq u$, we have $y \leq z$. Suppose $u \leq y$, say $u+v = y$. If $v \leq x_0$, we have $y \leq z$; if $v \geq x_0$, we have $y \geq z$. Relativizing to $\mathcal{J}_2$, gives us axioms PresA1,2,3,4,5,6,7. For PresA7, note that it is equivalent to $\forall x, y\, (x \leq y \vee y \leq x)$ and that $\leq$ is absolute under relativization to downwards closed sets, since $x \leq y \leftrightarrow \exists z{\leq}y\; x+z = y$.

Finally, let $\mathcal{J}_3$ be the class of all x, such that, for all $z \leq x$, we have:

$$\exists y\, (z = \underline{n}\, y \vee z = \underline{n}\, y + 1 \vee \ldots z = \underline{n}\, y + (\underline{n{-}1})).$$

It is easy to see that $\mathcal{J}_3$ is downward closed under $\leq$ and that 0, 1 are in J_3. We show that $\mathcal{J}_3$ is closed under addition. Suppose $x_0, x_1 \in J_3$ and $z \leq (x_0+x_1)$. In case $z \leq x_0$, we are immediately done. Otherwise, $z = x_0+u$, for some u. It follows that $u \leq x_1$. We have, for some y_0, y_1 and for some standard k_0, k_1, $x_0 = \underline{n}\, y_0 + \underline{k}_0$, $u = \underline{n}\, y_1 + \underline{k}_1$. Hence,

$$z = x_0 + u = \underline{n}\, y_0 + \underline{k}_0 + \underline{n}\, y_1 + \underline{k}_0 = \underline{n}\,(y_0+y_1) + \underline{k_0+k_1}.$$

In case $(k_0+k_1) < n$, we are done. Otherwise $0 \leq (k_0+k_1-n) < n$, and we find $z = \underline{n}\,(y_0+y_1+1) + \underline{k_0+k_1{-}n}$.

We note that the quantifier $\exists y$ in axiom PresA8n can be bounded by x. It follows that relativization to $\mathcal{J}_3$, gives us PresAn.

Appendix B. Questions

(1) Is ac[c8] interpretable in PresA?
(2) Is the complexity of deciding the theory of finite classes with equinumerosity over an infinite domain higher than the complexity of deciding PresA?
(3) Is PresA interpretable in $\mathsf{Q}_{\mathsf{add}}$? I conjecture: *no*.

(4) Is an extension of $\mathsf{Q}_{\mathsf{add}}$ interpretable in $\mathsf{Q}_{\mathsf{add}}$ that contains a faster-than-linear function? I conjecture: *no*.

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