Characteristic Matrix Functions and Periodic Delay Equations

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Abstract In the first part of this chapter we recall the notion of a characteristic matrix function for classes of operators as introduced in Kaashoek and Verduyn Lunel (2023). The characteristic matrix function completely describes the spectral properties of the corresponding operator. In the second part we show that the period map or monodromy operator associated with a periodic neutral delay equation has a characteristic matrix function. We end this chapter with a number of illustrative examples of periodic neutral delay equations for which we can compute the characteristic matrix function explicitly.

1 Introduction

Let X denote a complex Banach space, and let $A : \mathcal{D}(A) \to X$ be a linear operator with domain $\mathcal{D}(A)$ a subspace of X. A complex number λ belongs to the resolvent set $\rho(A)$ of A if and only if the resolvent $(zI - A)^{-1}$ exists and is bounded, i.e.,

(i) $\lambda I - A$ is one-to-one;

(ii) $\operatorname{Im} \lambda I - A = X$;

(iii) $(zI - A)^{-1}$ is bounded.

Note that for closed operators, (iii) is superfluous, since it is a direct consequence of the other assumptions by the closed graph theorem. The spectrum $\sigma(A)$ is by definition the complement of $\rho(A)$ in $\mathbb{C}$. The point spectrum $\sigma_p(A)$ is the set of those $\lambda \in \mathbb{C}$ for which $\lambda I - A$ is not one-to-one, i.e., $A\varphi = \lambda\varphi$ for some $\varphi \neq 0$. One then calls λ an eigenvalue and φ an eigenvector corresponding to λ.

The null space Ker $(\lambda I - A)$ is called the eigenspace and its dimension the *geometric multiplicity* of λ. The generalized eigenspace $\mathcal{M}_\lambda = \mathcal{M}_\lambda(A)$ is the smallest closed linear subspace that contains all Ker $(\lambda I - A)^j$ for $j = 1, 2, \ldots$ and its dimension $M(A; \lambda)$ is called the *algebraic multiplicity* of λ. If, in addition, λ is an isolated

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point in $\sigma(T)$ and $M(A;\lambda)$ is finite, then λ is called an *eigenvalue of finite type*. When $M(A;\lambda)=1$ we say that λ is a *simple* eigenvalue. A class of operators for which the eigenvalues are of finite type is formed by the compact operators. Other classes appear later in this chapter.

If λ is an eigenvalue of finite type, the operator $T = A\,|_{\mathcal{M}_\lambda}$ is a bounded operator from a finite dimensional space into itself. So the situation is reduced to the finite dimensional case, which we shall, therefore, discuss first.

Let $T:\mathbb{C}^m \to \mathbb{C}^m$ be a bounded linear operator. The eigenvalues of T are precisely given by the roots of the characteristic polynomial

$$C(z) := \det\,(zI - T)\,.$$

Over the scalar field $\mathbb{C}$ the characteristic polynomial can be factorized into a product of m linear factors

$$C(z) = \prod_{j=1}^{m}(z-\lambda_j),$$

where $\lambda_j \in \sigma(T)$. Define the multiplicity $m(\lambda_j, zI - T)$ of λ_j to be the number of times the factor $(z-\lambda_j)$ appears, or, in other words, the order of λ_j as a zero of the characteristic polynomial C. The characteristic polynomial is an annihilating polynomial of T, i.e., $C(T)=0$. The *minimal polynomial* C_m of T is defined to be an annihilating polynomial of T that divides any other annihilating polynomial. Necessarily, C_m is of the form

$$C_m(z) = \prod_{j=1}^{l}(z-\lambda_j)^{k_j},$$

where $\sigma(T) = \{\lambda_1,\ldots,\lambda_l\}$, and for $j=1,\ldots,l$, the number k_j is positive and called the *ascent* of λ_j.

Define

$$\mathcal{M}_j := \mathrm{Ker}\,(\lambda_j I - T)^{k_j}.$$

This is a T-invariant subspace, i.e., $T\mathcal{M}_j \subseteq \mathcal{M}_j$, and we can define the part of T in $\mathcal{M}_j$, i.e., $T_j = T\,|_{\mathcal{M}_j}\colon \mathcal{M}_j \to \mathcal{M}_j$. This yields $\mathbb{C}^m = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_l$. The operator T decomposes accordingly

$$T = \bigoplus_{j=1}^{l} T_j.$$

This decomposition is unique (up to the order of summands). The action of T can be broken down to the study of the action of T_j. To continue the decomposition one first studies the structure of the subspaces $\mathcal{M}_j$ more closely.

Let $\lambda \in \sigma(T)$. A vector x is called a generalized eigenvector of order r if

$$(\lambda I - T)^r x = 0 \quad \text{ while } \quad (\lambda I - T)^{r-1} x \neq 0.$$

Suppose x_{r-1} is a generalized eigenvector of order r; then there are vectors $(x_{r-2}, \ldots, x_1, x_0)$ for which $x_0 \neq 0$ and

$$\begin{aligned} Tx_0 &= \lambda x_0, \\ Tx_1 &= \lambda x_1 + x_0, \\ &\vdots \\ Tx_{r-1} &= \lambda x_{r-1} + x_{r-2} \end{aligned}$$

and hence $x_j \in \text{Ker}\,(\lambda I - T)^{j+1}$. Such a sequence is called a *Jordan chain*. Obviously, the length of the Jordan chain is less than or equal to k_λ, the ascent of λ, and a Jordan chain consists of linearly independent elements. As a consequence of this construction, the matrix representation of T_j with respect to the basis $(x_{r-2}, \ldots, x_1, x_0)$ is given by a Jordan block of order r corresponding to λ. See Diekmann et al. (1995, Chap. IV) and the next section for more information about Jordan chains for analytic matrix-valued functions.

Next consider the case that T is an operator defined on an infinite dimensional complex Banach space X, then, in general, T no longer has a matrix representation and we cannot define the characteristic polynomial of T by $\det\,(zI - T)$. Nevertheless there is a large class of operators for which one has a characteristic function whose zeros determine the spectrum of the corresponding operator. For example, this is true for the infinitestimal generator of solution semigroup corresponding to autonomous delay equations, see Diekmann et al. (1995, Chap. I). As it turned out the abstract notion of a characteristic matrix function, introduced in Kaashoek and Verduyn Lunel (1992) for unbounded operators, can be used to explain this connection. As a consequence it was possible to extend the finite dimensional theory to specific classes of unbounded operators. To briefly explain the connection between unbounded operators $A : \mathcal{D}(A) \to X$ and analytic matrix functions, as developed in Kaashoek and Verduyn Lunel (1992), let $\Delta : \Omega \to \mathcal{L}(\mathbb{C}^n)$ be an analytic $n \times n$ matrix function with $\Omega \subset \mathbb{C}$.

We call Δ a characteristic matrix function for A on Ω if there exist analytic operator functions E and F, $E : \Omega \to \mathcal{L}(\mathbb{C}^n \oplus X)$ and $F : \Omega \to \mathcal{L}(\mathbb{C}^n \oplus X)$, whose values are invertible operators, such that

$$\begin{bmatrix} \Delta(z) & 0 \\ 0 & I \end{bmatrix} = F(z) \begin{bmatrix} I_{\mathbb{C}^n} & 0 \\ 0 & zI - A \end{bmatrix} E(z), \qquad z \in \Omega.$$

The characteristic matrix function Δ completely determines the spectral properties of the unbounded operator A. See Kaashoek and Verduyn Lunel (1992) and Diekmann et al. (1995, Chap. IV) for details.

In this chapter we will follow recent work, Kaashoek and Verduyn Lunel (2023), and extend the notion of a characteristic matrix function to classes of bounded operators, and show that the period map of a periodic neutral delay equation has a characteristic matrix function.

We end the introduction with an outline of this chapter. In Sect. 2 we introduce and discuss the basic properties of Jordan chains. In Sect. 3 we introduce the notion of a characteristic matrix function for a class of bounded operators, and prove that the characteristic matrix function completely determines the spectral properties of the associated bounded operator. In Sect. 4 we show that the period map associated with a periodic neutral delay equation has a characteristic matrix function. In Sect. 5 we show that in case the period is equal to the delay, then we can compute the characteristic matrix function rather explicitly. Finally, in Sect. 6, we consider a class of periodic delay equations for which the period is two times the delay. We construct new examples for which we can compute the characteristic matrix function explicitly. In particular, we construct an example for which the period map has a finite spectrum. In the literature such examples are only known in case the period is equal to the delay, and were unknown in case the period is two times the delay.

2 Equivalence and Jordan Chains

Let X, Y, X', Y' be complex Banach spaces, and suppose that $L : \mathcal{U} \to \mathcal{L}(X, Y)$ and $M : \mathcal{U} \to \mathcal{L}(X', Y')$ are operator-valued functions, analytic on the open subset $\mathcal{U} \subset \mathbb{C}$. The two operator-valued functions L and M are called *equivalent* on $\mathcal{U}$ (see Sect. 2.4 in Bart et al. (1979)) if there exist analytic operator-valued functions $E : \mathcal{U} \to \mathcal{L}(X', X)$ and $F : \mathcal{U} \to \mathcal{L}(Y, Y')$, whose values are invertible operators, such that,

$$M(z) = F(z)L(z)E(z), \quad z \in \mathcal{U}. \tag{1}$$

Let $L : \mathcal{U} \to \mathcal{L}(X, Y)$ be an analytic operator-valued function. A point $\lambda_0 \in \mathcal{U}$ is called a *root* of L if there exists a vector $x_0 \in X$, $x_0 \neq 0$, such that,

$$L(\lambda_0)x_0 = 0.$$

An ordered set $(x_0, x_1, \ldots, x_{k-1})$ of vectors in X is called a *Jordan chain* for L at λ_0 if $x_0 \neq 0$ and

$$L(z)[x_0 + (z - \lambda_0)x_1 + \cdots + (z - \lambda_0)^{k-1}x_{k-1}] = O((z - \lambda_0)^k). \tag{2}$$

The number k is called the *length* of the chain and the maximal length of the chain starting with x_0 is called the *rank* of x_0. The analytic function

$$\sum_{l=0}^{k-1}(z - \lambda_0)^l x_l$$

in (2) is called a *root function* of L corresponding to λ_0.

Proposition 2.1 *If two analytic operator functions L and M are equivalent on $\mathcal{U}$, then there is a one-to-one correspondence between their Jordan chains.*

Proof The equivalence relation (1) is symmetric, and thus it suffices to show that Jordan chains for L yield Jordan chains for M. If $(x_0, \dots, x_{k-1})$ is a Jordan chain for L at λ_0, then

$$\begin{aligned} E(z)^{-1}(x_0 + (z-\lambda_0)x_1 + \cdots + (z-\lambda_0)^{k-1}x_{k-1}) \\ = y_0 + (z-\lambda_0)y_1 + \cdots + (z-\lambda_0)^{k-1}y_{k-1} + \text{h.o.t.} \end{aligned}$$

and $(y_0, \dots, y_{k-1})$ is a Jordan chain for M at λ_0. Here h.o.t. stands for the higher order terms. Furthermore, the equivalence yields that the null spaces Ker $L(\lambda_0)$ and Ker $M(\lambda_0)$ are isomorphic and this proves the proposition. □

Let $\Omega \subset \mathbb{C}$ and $\Delta : \Omega \to \mathcal{L}(\mathbb{C}^n)$ denote an entire $n \times n$ matrix function. If the determinant of Δ is not identically zero, then we define $m(\lambda, \Delta)$ to be the order of λ as a zero of det Δ and $k(\lambda, \Delta)$ is the order of λ as pole of the matrix function $\Delta(\cdot)^{-1}$.

Let λ_0 be an isolated root of Δ, then the Jordan chains for Δ at λ_0 have finite length, and we can organize the chains as follows. Choose an eigenvector, say $x_{1,0}$, with maximal rank, say r_1. Next, choose a Jordan chain

$$(x_{1,0}, \dots, x_{1,r_1-1})$$

of length r_1 and let N_1 be the complement in Ker $\Delta(\lambda_0)$ of the subspace spanned by $x_{1,0}$. In N_1 we choose an eigenvector $x_{2,0}$ of maximal rank, say r_2, and let

$$(x_{2,0}, \dots, x_{2,r_2-1})$$

be a corresponding Jordan chain of length r_2. We continue as follows, let N_2 be the complement in N_1 of the subspace spanned by $x_{2,0}$ and replace N_1 by N_2 in the above described procedure.

In this way, we obtain a basis $\{x_{1,0}, \dots, x_{p,0}\}$ of Ker $\Delta(\lambda_0)$ and a corresponding *canonical system* of Jordan chains

$$x_{1,0}, \dots, x_{1,r_1-1}, x_{2,0}, \dots, x_{2,r_2-1}, x_{p,0}, \dots, x_{p,r_p-1}$$

for Δ at λ_0.

It is easy to see that the rank of any eigenvector x_0 corresponding to the root λ_0 is always equal to one of the r_j for $1 \le j \le p$. Thus, the integers $r_1, \dots, r_p$ do not depend on the particular choices made in the procedure described above and are called the *zero-multiplicities* of Δ at λ_0. Their sum $r_1 + \cdots + r_p$ is called the *algebraic multiplicity* of Δ at λ_0 and will be denoted by $M(\Delta(\lambda_0))$.

To illustrate this procedure, recall the case that $\Delta = zI - A$ is a linear matrix function with A an $n \times n$ matrix as discussed in the Introduction. The Jordan chain $(x_0, \dots, x_{k-1})$ for Δ at λ_0 satisfies

$$(A - \lambda_0)x_0 = 0, \quad (A - \lambda_0)x_1 = x_0, \ \dots, \ (A - \lambda_0)x_{k-1} = x_{k-2},$$

and hence

$$\{(x_{i,0}, \dots, x_{i,r_i-1}) \mid i = 1, 2, \dots, p\}$$

is a canonical basis of eigenvectors and generalized eigenvectors for A at λ_0.

Next we recall the connection between the Jordan chains and the local Smith form for an analytic $n \times n$ matrix function $\Delta : \Omega \to \mathcal{L}(\mathbb{C}^n)$ with $\det \Delta \not\equiv 0$. Let $\lambda_0 \in \mathbb{C}$. The *local Smith form* states there exist a neighborhood $\mathcal{U}$ of λ_0 and analytic matrix functions E and F on $\mathcal{U}$ whose values are invertibe operators such that

$$\Delta(z) = F(z)D(z)E(z), \quad z \in \mathcal{U}, \tag{3}$$

where

$$D(z) = \operatorname{diag}\left[(z - \lambda_0)^{\nu_1}, \dots, (z - \lambda_0)^{\nu_n}\right], \qquad z \in \mathcal{U}. \tag{4}$$

The integers $\{\nu_1, \dots, \nu_n\}$ are uniquely determined by Δ and the diagonal matrix D in (4) is called the local Smith form for Δ at λ_0. See Gohberg et al. (1993, Theorems 1.2 and 1.3) for a proof of (3)–(4), or see Diekmann et al. (1995, Chap. IV).

The Jordan chains for the local Smith form D are easily determined, and it follows that the set of zero multiplicities is given by $\{\nu_1, \dots, \nu_n\}$. Hence, the equivalence (3) and Proposition 2.1 show that the algebraic multiplicity of Δ at λ is given by

$$M(\Delta(\lambda)) = \sum_{l=1}^{n} \nu_l.$$

On the other hand the equivalence (3) yields

$$\det \Delta(z) = \det F(z)(z - \lambda_0)^{\sum_{l=1}^{n} \nu_l} \det E(z)$$

with $\det E(\lambda_0) \neq 0$ and $\det F(\lambda_0) \neq 0$. So, $m(\lambda_0, \Delta)$ the multiplicity of λ_0 as zero of $\det \Delta$ equals

$$m(\lambda_0, \Delta) = \sum_{l=1}^{n} \nu_l$$

as well. This shows that the algebraic multiplicity of Δ at λ equals the multiplicity of λ as zero of $\det \Delta$, i.e., $m(\lambda, \Delta) = M(\Delta(\lambda))$.

The following application of the local Smith form will be used in the proof of Theorem 3.1 below. The identity in (5) below can be viewed as a matrix-valued version of the Cauchy multiplicity theorem and is due to Gohberg and Sigal, see Kaashoek and Verduyn Lunel (2023, Theorem 5.1.2).

Theorem 2.2 *Let* $\Delta : \Omega \to \mathcal{L}(\mathbb{C}^n)$ *be an analytic* $n \times n$ *matrix function on* $\Omega \subset \mathbb{C}$ *with* $\det \Delta \not\equiv 0$. *If* $\lambda_0 \in \Omega$ *is an isolated zero of* $\det \Delta$, *then*

$$m(\lambda_0, \Delta) = \mathrm{Tr}\left(\frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{\lambda_0}} \Delta(z)^{-1} \frac{\mathrm{d}}{\mathrm{d}z} \Delta(z)\, \mathrm{d}z\right), \tag{5}$$

where Γ_{λ_0} *is a small circle surrounding* λ_0 *and no other zeros of* $\det \Delta$. *Here* $\mathrm{Tr}\,(A)$ *denotes the trace of an* $n \times n$ *matrix* A.

3 Introduction to the Theory of Characteristic Matrix Functions

Let $T : X \to X$ be a bounded operator, and let $\Delta : \Omega \to \mathcal{L}(\mathbb{C}^n)$ be an analytic $n \times n$ matrix function with $\Omega \subset \mathbb{C}$. We call Δ a *characteristic matrix function* for T on Ω if there exist analytic operator functions E and F, $E : \Omega \to \mathcal{L}(\mathbb{C}^n \oplus X)$ and $F : \Omega \to \mathcal{L}(\mathbb{C}^n \oplus X)$, whose values are invertible operators, such that

$$\begin{bmatrix} \Delta(z) & 0 \\ 0 & I \end{bmatrix} = F(z) \begin{bmatrix} I_{\mathbb{C}^n} & 0 \\ 0 & I - zT \end{bmatrix} E(z), \qquad z \in \Omega. \tag{6}$$

The operator function appearing in the left hand side of (6) is called the X*-extension* of Δ.

We call Δ *nondegenerate* if $\det \Delta(z)$ does not vanish identically. In this case, $I - zT$ is invertible for $z \in \Omega$ if and only if $\det \Delta(z)$ is non-zero, and in that case

$$E(z) \begin{bmatrix} \Delta(z)^{-1} & 0 \\ 0 & I \end{bmatrix} F(z) = \begin{bmatrix} I_{\mathbb{C}^n} & 0 \\ 0 & (I - zT)^{-1} \end{bmatrix}, \quad \det \Delta(z) \neq 0. \tag{7}$$

Note that if $\Omega = \mathbb{C}$, then Δ is always nondegenerate (take $z = 0$ in (6)). The operator functions F and E appearing in (6) can also be described by 2×2 matrix functions with entries that are analytic operator functions on Ω. For instance, for F we have

$$F(z) \begin{pmatrix} c \\ x \end{pmatrix} = \begin{bmatrix} F_{11}(z) & F_{12}(z) \\ F_{21}(z) & F_{22}(z) \end{bmatrix} \begin{pmatrix} c \\ x \end{pmatrix} = \begin{pmatrix} F_{11}(z)c + F_{12}(z)x \\ F_{21}(z)c + F_{22}(z)x \end{pmatrix}.$$

Using these partitioning of $E(z)$ and $F(z)$, the equivalence relation (7) yields a useful representation for the resolvent operator $(I - zT)^{-1}$ of T on Ω, namely

$$(I - zT)^{-1} = E_{21}(z)\Delta(z)^{-1}F_{12}(z) + E_{22}(z)F_{22}(z), \qquad z \in \Omega. \tag{8}$$

If $Q(z) := E(z)^{-1}$ and $R(z) := F(z)^{-1}$, then it follows from (8) that

$$Q_{12}(z)(I - zT)^{-1} = \Delta(z)^{-1} F_{12}(z), \qquad z \in \Omega,$$
$$(I - zT)^{-1} R_{21}(z) = E_{21}(z)\Delta(z)^{-1}, \qquad z \in \Omega.$$

Since the zeros of det $\Delta(z)$ do not have an accumulation point in Ω, we see from (8) that the non-zero part of the spectrum of T inside Ω consists of eigenvalues of finite type only.

In this section we introduce an important class of operators T that have a characteristic matrix function Δ, i.e., there exist functions E and F such that (6) holds. Before we do this, we present a spectral resolution theorem that justifies the terminology introduced above.

The next theorem is an adapted version of Theorem 2.1 of Kaashoek and Verduyn Lunel (1992) for bounded operators and justifies the terminology introduced above. See Kaashoek and Verduyn Lunel (2023, Theorem 5.2.6) for a complete proof.

Theorem 3.1 *Let T be a bounded linear operator on a Banach space X, and let Δ be a nondegenerate characteristic matrix function for T on Ω. Then*

(i) *the set $\Omega \cap \sigma(T) \setminus \{0\}$ consists of eigenvalues of finite type and*

$$\Omega \cap \sigma(T) \setminus \{0\} = \{\lambda^{-1} \in \Omega \mid \det \Delta(\lambda) = 0\};$$

(ii) *for $\lambda_0^{-1} \in \Omega \cap \sigma(T) \setminus \{0\}$, the partial multiplicities of λ_0^{-1} as an eigenvalue of T are equal to the zero-multiplicities of Δ at λ_0;*
(iii) *for $\lambda_0^{-1} \in \Omega \cap \sigma(T) \setminus \{0\}$, the algebraic multiplicity $m(T, \lambda_0^{-1})$ of λ_0^{-1} as an eigenvalue of T equals $m(\lambda_0, \Delta)$, the order of λ_0 as a zero of* det Δ*;*
(iv) *for $\lambda_0^{-1} \in \Omega \cap \sigma(T) \setminus \{0\}$, the ascent $k(T, \lambda_0^{-1})$ of λ_0^{-1} equals $k(\lambda_0, \Delta)$, the order of λ_0 as a pole of Δ^{-1} and* dim Ker $(I - \lambda_0 T)^k = m$ *where $k = k(\lambda_0, \Delta)$ and $m = m(\lambda_0, \Delta)$.*

Next we introduce a class of operators T that have a characteristic matrix function Δ in the sense of (6). Consider an operator $T : X \to X$ that is a finite rank perturbation of a given operator, i.e., admits a representation of the form

$$T := W + R, \tag{9}$$

where $W : X \to X$ is a bounded operator and $R : X \to X$ is an operator of finite rank. Define $\Omega \subset \mathbb{C}$ such that for every $x \in X$

$$z \mapsto (I - zW)^{-1}x \text{ is analytic on } \Omega. \tag{10}$$

A specific example of such an operator W is given by a the operator of integration, i.e., for $x \in C\,([0, 1]; \mathbb{C}^n)$ define

$$(Wx)(t) := \int_0^t x(s)\,\mathrm{d}s, \quad 0 \le t \le 1. \tag{11}$$

The resolvent of the operator W defined in (11) can be computed explicitly and is given by

$$\big((I - zW)^{-1}x\big)(t) = x(t) + z\int_0^t e^{z(t-s)}x(s)\,\mathrm{d}s, \quad 0 \le t \le 1. \tag{12}$$

This shows that (10) is defined for all $z \in \mathbb{C}$ and we can take $\Omega = \mathbb{C}$ in this specific case.

The fact that R has finite rank allows us to factor R as $R = BC$, where $B : \mathbb{C}^n \to X$ and $C : X \to \mathbb{C}^n$, with $n \ge \operatorname{rank} R$. If n is equal to the rank of R we call $R = BC$ a *minimal rank factorization*.

To a pair W and R with factorization $R = BC$ we associate the $n \times n$ matrix function

$$\Delta(z) := I_{\mathbb{C}^n} - zC(I - zW)^{-1}B, \quad z \in \Omega. \tag{13}$$

The next theorem tells us that Δ defined by (13) satisfies (6) with the operator T defined by (9) on Ω. Note that in case W is given by (11) we have $\Omega = \mathbb{C}$. Hence the operator W is *quasi-nilpotent*, i.e., for every $x \in X$ we have that $z \mapsto (I - zW)^{-1}x$ is an entire function, and hence in this case Δ defined by (13) is an entire matrix function as well.

The following theorem is an adapted version of Kaashoek and Verduyn Lunel (2023, Theorem 6.1.1) and the proof is given for the convenience of the reader.

Theorem 3.2 *Let $W : X \to X$ be a bounded operator. Define $\Omega \subset \mathbb{C}$ such that* (10) *holds. The $n \times n$ entire matrix function Δ defined by* (13) *is a characteristic matrix function for the operator T defined by* (9) *on Ω. In particular, the identity* (6) *is satisfied with Δ given by* (13) *and where the analytic operator-valued functions $E(z) : \mathbb{C}^n \oplus X \to \mathbb{C}^n \oplus X$ and $F(z) : \mathbb{C}^n \oplus X \to \mathbb{C}^n \oplus X$ are given by*

$$E(z) := \begin{bmatrix} \Delta(z) & C(I - zW)^{-1} \\ -z(I - zW)^{-1}B & (I - zW)^{-1} \end{bmatrix}, \qquad z \in \Omega,$$

$$F(z) := \begin{bmatrix} \Delta(z) & -C(I - zW)^{-1} \\ zB & I_X \end{bmatrix}, \qquad z \in \Omega.$$

The inverses $E(z)^{-1} : \mathbb{C}^n \oplus X \to \mathbb{C}^n \oplus X$ and $F(z)^{-1} : \mathbb{C}^n \oplus X \to \mathbb{C}^n \oplus X$ are the operator-valued functions given by

$$E(z)^{-1} = \begin{bmatrix} I_{\mathbb{C}^n} & -C \\ zB & I - zT \end{bmatrix}, \qquad z \in \Omega,$$

$$F(z)^{-1} = \begin{bmatrix} I_{\mathbb{C}^n} & C(I - zW)^{-1} \\ -zB & I - zBC(I - zW)^{-1} \end{bmatrix}, \qquad z \in \Omega.$$

Proof Take $z \in \Omega$ fixed and apply Theorem 4.7 in Bart et al. (2008) with

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} := \begin{bmatrix} I - zW & zB \\ C & I_{\mathbb{C}^n} \end{bmatrix} : X \oplus \mathbb{C}^n \to X \oplus \mathbb{C}^n.$$

Note that both M_{11} and M_{22} are invertible operators. Hence the Schur complements of M_{11} and M_{22} in M are well defined and are given by

$$\begin{aligned} \Lambda_1 &:= M_{22} - M_{21} M_{11}^{-1} M_{12} \\ &= I_{\mathbb{C}^n} - zC(I - zW)^{-1} B = \Delta(z); \end{aligned} \tag{14}$$

$$\begin{aligned} \Lambda_2 &:= M_{11} - M_{12} M_{22}^{-1} M_{21} \\ &= I - zW - zBC = I - zT. \end{aligned} \tag{15}$$

Put

$$\begin{aligned} E_1(z) &:= \begin{bmatrix} -M_{21} M_{11}^{-1} & \Lambda_1 \\ M_{11}^{-1} & M_{11}^{-1} M_{21} \end{bmatrix} \\ &= \begin{bmatrix} -C(I - zW)^{-1} & \Delta(z) \\ (I - zW)^{-1} & z(I - zW)^{-1} B \end{bmatrix}, \\ F_1(z) &:= \begin{bmatrix} -M_{11}^{-1} M_{12} & I \\ I_{\mathbb{C}^n} - M_{22}^{-1} M_{21} M_{11}^{-1} M_{12} & M_{22}^{-1} M_{21} \end{bmatrix} \\ &= \begin{bmatrix} -z(I - zW)^{-1} B & I \\ \Delta(z) & C \end{bmatrix}. \end{aligned}$$

Then, using the identities (14) and (15), Theorem 4.7 in Bart et al. (2008) tells us that

$$\begin{bmatrix} \Delta(z) & 0 \\ 0 & I_X \end{bmatrix} = E_1(z) \begin{bmatrix} I - zT & 0 \\ 0 & I_{\mathbb{C}^n} \end{bmatrix} F_1(z). \tag{16}$$

Note that the identify (16) can be verified directly using the definitions.

Moreover, the operators $E_1(z)$ and $F_1(z)$ are invertible and

$$E_1(z)^{-1} = \begin{bmatrix} -M_{12}M_{22}^{-1} & \Lambda_2 \\ M_{22}^{-1} & M_{22}^{-1}M_{21} \end{bmatrix} = \begin{bmatrix} -zB & I - zT \\ I_{\mathbb{C}^n} & C \end{bmatrix}, \tag{17}$$

$$F_1(z)^{-1} = \begin{bmatrix} -M_{22}^{-1}M_{21} & I_{\mathbb{C}^n} \\ I - M_{11}^{-1}M_{12}M_{22}^{-1}M_{21} & M_{11}^{-1}M_{12} \end{bmatrix} = \begin{bmatrix} -C & I_{\mathbb{C}^n} \\ I - z(I - zW)^{-1}BC & z(I - zW)^{-1}B \end{bmatrix}. \tag{18}$$

Finally put

$$E(z) := \begin{bmatrix} 0 & I_{\mathbb{C}^n} \\ I & 0 \end{bmatrix} F_1(z) \begin{bmatrix} I_{\mathbb{C}^n} & 0 \\ 0 & (I - zW)^{-1} \end{bmatrix},$$

$$F(z) := \begin{bmatrix} I_{\mathbb{C}^n} & 0 \\ 0 & I - zW \end{bmatrix} E_1(z) \begin{bmatrix} 0 & I \\ I_{\mathbb{C}^n} & 0 \end{bmatrix}.$$

Then the identities for $E(z)$ and $F(z)$ given in the statement of the theorem hold. For example, the identity (16) yields (6):

$$\begin{aligned} F(z) \begin{bmatrix} I_{\mathbb{C}^n} & 0 \\ 0 & I - zT \end{bmatrix} E(z) &= \\ &= \begin{bmatrix} I_{\mathbb{C}^n} & 0 \\ 0 & I - zW \end{bmatrix} E_1(z) \begin{bmatrix} 0 & I \\ I_{\mathbb{C}^n} & 0 \end{bmatrix} \begin{bmatrix} I_{\mathbb{C}^n} & 0 \\ 0 & I - zT \end{bmatrix} E(z) \\ &= \begin{bmatrix} I_{\mathbb{C}^n} & 0 \\ 0 & I - zW \end{bmatrix} E_1(z) \begin{bmatrix} I - zT & 0 \\ 0 & I_{\mathbb{C}^n} \end{bmatrix} F_1(z) \begin{bmatrix} I_{\mathbb{C}^n} & 0 \\ 0 & (I - zW)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} I_{\mathbb{C}^n} & 0 \\ 0 & I - zW \end{bmatrix} \begin{bmatrix} \Delta(z) & 0 \\ 0 & I_X \end{bmatrix} \begin{bmatrix} I_{\mathbb{C}^n} & 0 \\ 0 & (I - zW)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} \Delta(z) & 0 \\ 0 & I_X \end{bmatrix}. \end{aligned}$$

Furthermore, using (17) and (18), we see that $E(z)$ and $F(z)$ are invertible with inverses as given in the statement of the theorem. □

As a corollary to Theorem 3.2 we have the following identities involving the resolvent operator $(I - zT)^{-1}$ that will be useful in the future.

$$\begin{aligned} &(I - zT)^{-1} = z(I - zW)^{-1}B\Delta(z)^{-1}C(I - zW)^{-1} + (I - zW)^{-1}, \\ &\Delta(z)C(I - zT)^{-1} = C(I - zW)^{-1}, \\ &(I - zT)^{-1}B\Delta(z) = (I - zW)^{-1}B. \end{aligned}$$

As a first illustration of Theorem 3.2 we compute the characteristic matrix function of a rank one perturbation of the operator of integration on $C[0, 1]$, i.e., the operator

W is given by (11). More precisely, take $X = C[0,1]$ and define $T : X \to X$ by

$$(Tx)(t) := \int_0^t x(s)\,\mathrm{d}s + \int_0^1 x(s)\,\mathrm{d}\eta(s), \quad x \in C[0,1], \quad 0 \le t \le 1. \tag{19}$$

Here η is a function of bounded variation (see Appendix A of Verduyn Lunel (2023) for more information about functions of bounded variation). The operator T can be written as $T = W + R$, where W and R are operators on X defined by

$$(Wx)(t) := \int_0^t x(s)\,\mathrm{d}s, \quad (Rx)(t) := \int_0^1 x(s)\,\mathrm{d}\eta(s), \quad 0 \le t \le 1.$$

From (12) it follows that we can take $\Omega = \mathbb{C}$. It follows that T given by (19) is a finite rank perturbation of a quasi-nilpotent operator. Thus we can apply Theorem 3.2 with $\Omega = \mathbb{C}$, and $C : X \to \mathbb{C}$ and $B : \mathbb{C} \to X$ given by

$$Cx := \int_0^1 x(s)\,\mathrm{d}\eta(s), \quad x \in X, \qquad (Bc)(t) := c, \quad 0 \le t \le 1. \tag{20}$$

Then $R = BC$ is a minimal rank factorization. It follows from (12) and (20) that

$$\big((I - zW)^{-1}Bc\big)(t) = c + z\int_0^t e^{z\tau} c\,\mathrm{d}\tau = e^{zt}c, \qquad 0 \le t \le 1.$$

Using this together with (13) and (20) we derive that the corresponding characteristic matrix function Δ is the scalar function given by

$$\begin{aligned}\Delta(z) &:= I - zC(I - zW)^{-1}B \\ &= 1 - z\int_0^1 e^{zs}\,\mathrm{d}\eta(s).\end{aligned}$$

In particular, the characteristic matrix function Δ of the operator T defined by

$$(Tx)(t) := \int_0^t x(s)\,\mathrm{d}s + x(1), \quad x \in C[0,1], \tag{21}$$

is given by

$$\Delta(z) = 1 - ze^z. \tag{22}$$

Thus an application of Theorem 3.1 now yields that Δ given by (22) completely characterizes the nonzero spectrum of the operator T given by (21). In fact, the nonzero spectrum of T consists of simple eigenvalues only, and e^{μ} is an eigenvalue of T if and only if

$$\mu - e^{-\mu} = 0. \tag{23}$$

The asymptotic behavior of the roots of Equation (23) is well-known, see Diekmann et al. (1995, Sect. XI.2), and this information can be used to derive detailed estimates on the nonzero spectrum of the operator T given by (21).

4 The Period Map of a Neutral Periodic Delay Equation

In this section we consider linear periodic functional differential equations of the following type:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left[x(t) - \displaystyle\int_0^h \mathrm{d}\eta(\tau)x(t-\tau)\right] = \displaystyle\int_0^h \mathrm{d}_\tau\zeta(t,\tau)x(t-\tau), \quad t \geq s, \\ x(s+\theta) = \varphi(\theta), \qquad -h \leq \theta \leq 0. \end{cases} \tag{24}$$

Here d_τ denotes integration with respect to the τ variable and φ is a given function in $B\left([-h,0],\mathbb{C}^n\right)$, the complex Banach space of bounded Borel measurable functions provided with the supremum norm. Throughout we assume that for each $t \in \mathbb{R}$ the functions η and $\zeta(t,\cdot)$ are $n \times n$ matrices of which the entries are real functions of bounded variation on $[0,h]$ and continuous from the left on $(0,h)$, and $\eta(0) = \zeta(t,0) = 0$. See Appendix A of Verduyn Lunel (2023) for more information about functions of bounded variation. Finally, we assume periodicity of the kernel ζ, i.e., there is a non negative real number ω such that

$$\zeta(t+\omega,\cdot) = \zeta(t,\cdot), \qquad t \geq 0.$$

In Sect. 8 of Verduyn Lunel (2023) we have proved the following theorem regarding system (24).

Theorem 4.1 *Under the above conditions, equation* (24) *has a unique solution* $x = x(\cdot;s;\varphi)$ *on* $[s,\infty)$. *Furthermore the family of solution operators* $U(t,s)$, $t \geq s$, *defined by translation along the solution of* (24) *and given by*

$$\big(U(t,s)\varphi\big)(\theta) := x(t+\theta;s,\varphi), \quad -h \leq \theta \leq 0, \quad \varphi \in B\left([-h,0];\mathbb{C}^n\right),$$

is a twin evolutionary system of operators and has the following properties:

(i) $U(s,s)$ *is the identity operator for all* $s \in \mathbb{R}$,
(ii) $U(t,s)U(s,\sigma) = U(t,\sigma)$ *for all* $t \geq s \geq \sigma$.

The period map T associated with (24) is defined by $T := U(\omega,0)$. Using the results from Sect. 8 of Verduyn Lunel (2023), we have an explicit representation of the period map in terms of the fundamental matrix solution $X(t)$.

The main purpose of this section is to show that the period map T admits a characteristic matrix function Δ in the sense of (6). Before we prove this, we briefly recall the importance of the spectral properties of the period map in the study of the qualitative behaviour of the solutions of (24). See Hale and Verduyn Lunel (1993, Chap. 11) for the general qualitative theory for periodic functional differential equations.

The first result relates all period maps $U(\omega + s, s)$, $s \in \mathbb{R}$, to the operator T. For a proof of the next theorem and more information about the period maps we refer to Diekmann et al. (1995, Sect. XIV.3).

Theorem 4.2 *Assume that $\lambda \neq 0$ is an isolated eigenvalue of finite type of the operator $U(t + \omega, t)$ for each $t \geq 0$. Let $\mathcal{M}_{\lambda,t}$ denote the generalized eigenspace at λ of the operator $U(t + \omega, t)$. If $t \geq s$ then*

(i) *$\lambda \in \sigma(U(t + \omega, t))$ if and only if $\lambda \in \sigma(U(s + \omega, s))$;*
(ii) *if $\lambda \in \sigma(U(t + \omega, t))$, then $U(t, s)$ maps $\mathcal{M}_{\lambda,s}$ in a one-to-one way onto $\mathcal{M}_{\lambda,t}$.*

We continue with some more notation and terminology. If μ belongs to the non-zero point spectrum of T, then μ is called a *characteristic multiplier* of (24), and λ for which $\mu = \exp(\lambda\omega)$ (unique up to multiples of $2\pi i$) is called a *characteristic exponent* of (24).

Let $\mu \neq 0$ be an eigenvalue of finite type and let m_μ denote the algebraic multiplicity of μ. Assume that $\varphi_1, \ldots, \varphi_{m_\mu}$ in $B([-h, 0]; \mathbb{C}^n)$ is a basis of eigenvectors and generalized eigenvectors of T at μ, and let $\mathcal{M}_\mu = \text{span}\{\varphi_1, \ldots, \varphi_{m_\mu}\}$ be the corresponding generalized eigenspace. Furthermore, let Φ_0 be the m_μ-row vector defined by $\Phi_0 := [\varphi_1, \ldots, \varphi_{m_\mu}]$, viewed as a linear operator from $\mathbb{C}^{m_\mu}$ into $B([-h, 0]; \mathbb{C}^n)$. Since $\mathcal{M}_\mu$ is invariant under T, there exists a $m_\mu \times m_\mu$ matrix L with scalar entries such that

$$T\Phi_0 = \Phi_0 L,$$

and the only eigenvalue of L is $\mu \neq 0$. But then, there is an $m_\mu \times m_\mu$ matrix $\mathbb{B}$ with scalar entries such that $L = \exp(\omega\mathbb{B})$, and thus $L\exp(-\omega\mathbb{B})$ is the $m_\mu \times m_\mu$ identity matrix. Moreover, the unique eigenvalue λ of $\mathbb{B}$ satisfies the identity $\mu = \exp(\omega\lambda)$. From Theorem 4.2 it follows that if

$$\Phi(t) = \left[U(t, 0)\varphi_1 \cdots U(t, 0)\varphi_{m_\mu}\right], \qquad t \geq 0,$$

then

$$(T\Phi_0)(t) = \Phi(t)L, \quad t \geq 0.$$

Next, let $\mathbb{P}(t)$, $t \geq 0$, be the block m_μ-row vector given by

$$\mathbb{P}(t) = U(t, 0)\Phi_0 \exp(-t\mathbb{B}) = \Phi(t)\exp(-t\mathbb{B}), \quad t \geq 0.$$

Thus $\mathbb{P}(t)$ has size $1 \times m_\mu$ and its entries are in $B([-h, 0]; \mathbb{C}^n)$.

Lemma 4.3 *The function $\mathbb{P}(t)$, $t \geq 0$, is periodic with period ω. Furthermore, we have*

$$U(t, 0)\Phi_0 = \mathbb{P}(t)\exp(t\mathbb{B}), \quad t \geq 0. \tag{25}$$

Proof From item (ii) in Theorem 4.1 we know that $U(t+\omega,0) = U(t,0)T$ for $t \geq 0$. Using the latter identity we see that

$$\begin{aligned}\mathbb{P}(t+\omega) &= U(t+\omega,0)\Phi_0 e^{-(t+\omega)\mathbb{B}} \\ &= U(t,0)T\Phi_0 e^{-\omega\mathbb{B}}e^{-t\mathbb{B}} = U(t,0)\Phi_0 L e^{-\omega\mathbb{B}}e^{-t\mathbb{B}} \\ &= U(t,0)\Phi_0 e^{-t\mathbb{B}} = \mathbb{P}(t), \quad t \geq 0. \end{aligned} \tag{26}$$

Thus $\mathbb{P}(t)$ is periodic with period ω, and the final equality in (26) yields (25). □

The solution of (24) with initial value $\varphi \in \mathcal{M}_\mu$ is of *Floquet type*, i.e., of the form

$$x(t;\varphi) = p(t)\exp(t\mathbb{B})c, \tag{27}$$

where $c \in \mathbb{C}^{m_\mu}$ is such that $\varphi = \Phi_0 c$, the matrix $\mathbb{B}$ has size $m_\mu \times m_\mu$ and only one eigenvalue at λ with $\mu = e^{\lambda\omega}$, and $p(t) = p(t+\omega)$ is a periodic function. Indeed, from Lemma 4.3 it follows that

$$x(t;\varphi) = (U(t,0)\Phi_0 c)\,(0) = (\mathbb{P}(t))\,(0)\exp(t\mathbb{B})c.$$

Put $p(t) := (\mathbb{P}(t))\,(0)$, then $p(t) = p(t+\omega)$ and this shows (27). Furthermore, since λ is the only eigenvalue of $\mathbb{B}$, it follows from the Jordan decomposition of $\mathbb{B}$ that we can write

$$e^{t\mathbb{B}}c = q(t)e^{\lambda t},$$

where q is a polynomial of degree at most m_μ, and hence

$$x(t;\varphi) = p(t)q(t)\exp(\lambda t).$$

Next we prove that the period map T admits a characteristic matrix function Δ in the sense of (6). In order to apply Theorem 3.2 we have to show that T satisfies (9). In the case that $\eta = 0$ and $\omega = h$ in equation (24), we have a general result with $\Omega = \mathbb{C}$. This is the contents of the next theorem.

Theorem 4.4 *Consider equation* (24) *with* $\eta = 0$ *and period* $\omega = h$ *and let* T *on* $B\,([-h,0];\mathbb{C}^n)$ *be the corresponding period map. The period map* T *satisfies* (9)*, where* W *is quasi-nilpotent and* R *is an operator of finite rank. The operators* W *and* R *are operators on* $B\,([-h,0];\mathbb{C}^n)$ *given by*

$$(W\varphi)\,(\theta) := x(h+\theta,0;\varphi) - X(s+h+\theta,0)\varphi(0), \quad -h \leq \theta \leq 0, \tag{28}$$

$$(R\varphi)\,(\theta) := X(h+\theta,0)\varphi(0), \quad -h \leq \theta \leq 0, \tag{29}$$

where $x(t,s;\varphi)$ *denotes the solution of* (24) *with initial data* φ*, and* $X(t,s)$ *denotes the fundamental matrix solution of* (24)*, i.e., the matrix solution with initial data*

$$X_0(\theta) = \begin{cases} I & \textit{for } \theta = 0, \\ 0 & \textit{for } -h \le \theta < 0. \end{cases}$$

Proof To prove that W is quasi-nilpotent, we have to show that given ψ the equation

$$\varphi - zW\varphi = \psi. \tag{30}$$

has a unique solution φ for each $z \in \mathbb{C}$.

We will use a contraction mapping principle in a weighted norm to prove that (30) has a unique solution for each $z \in \mathbb{C}$. Define for $\gamma \in \mathbb{R}$ the weighted norm $\| \cdot \|_\gamma$ on $B[-h, 0]$ by

$$\|\varphi\|_\gamma := \max_{-h \le \theta \le 0} \|e^{\gamma(\theta - h)}\varphi(\theta)\|. \tag{31}$$

From (30) and the definition of W given in (28) it follows that

$$\psi = \varphi - z\left(x(h + \cdot, 0; \varphi) - X(h + \cdot, 0)\varphi(0)\right). \tag{32}$$

Hence $\varphi(-h) = \psi(-h)$ and φ satisfies the differential equation

$$\dot{\psi} = \dot{\varphi} - z\left(\dot{x}(h + \cdot, 0; \varphi) - \dot{X}(h + \cdot, 0)\varphi(0)\right).$$

Using (24) with $\eta = 0$ and recalling that $\zeta(h + \cdot) = \zeta(\cdot)$ we obtain

$$\begin{aligned} \dot{\psi}(\sigma) = \dot{\varphi}(\sigma) - z \int_0^h & \mathrm{d}\zeta(\sigma, \theta) x(h + \sigma - \theta, 0; \varphi) \\ & - z \int_0^{h+\sigma} \mathrm{d}\zeta(\sigma, \theta) X(h + \sigma - \theta, 0)\varphi(0). \end{aligned}$$

Using (32) we can rewrite this equation as follows:

$$\begin{aligned} \dot{\psi}(\sigma) = \dot{\varphi}(\sigma) - \int_0^{h+\sigma} & \mathrm{d}\zeta(\sigma, \theta)\left(\varphi(\sigma - \theta) - \psi(\sigma - \theta)\right) \\ & - z \int_{h+\sigma}^h \mathrm{d}\zeta(\sigma, \theta)\varphi(h + \sigma - \theta). \end{aligned} \tag{33}$$

To show that given ψ the equation (33) has a unique solution φ for each $z \in \mathbb{C}$, we first write the convolution part for φ (and similarly for ψ) in (33) which is given by

$$\dot{\varphi}(\sigma) - \int_0^{h+\sigma} \mathrm{d}\zeta(\sigma, \theta)\varphi(\sigma - \theta),$$

as follows:

$$\dot{\varphi}(\sigma) - \zeta(\sigma, h+\sigma)\varphi(-1) - \int_{-h}^{\sigma} \zeta(\sigma, \sigma-\tau)\dot{\varphi}(\tau)\,\mathrm{d}\tau.$$

Put $k(\sigma, \tau) := \zeta(\sigma, \sigma - \tau)$ and rewrite equation (33) now as follows:

$$\dot{\varphi}(\sigma) - \int_{-h}^{\sigma} k(\sigma, \tau)\dot{\varphi}(\tau)\,\mathrm{d}\tau = F(\sigma; \psi), \tag{34}$$

where

$$F(z, \sigma; \varphi, \psi) := \dot{\psi}(\sigma) - \int_{h}^{\sigma} k(\sigma, \tau)\psi(\tau)\,\mathrm{d}\tau + z\int_{h+\sigma}^{h} \mathrm{d}\zeta(\sigma, \theta)\varphi(h+\sigma-\theta).$$

Solving equation (34) yields

$$\dot{\varphi}(\sigma) = F(z, \sigma; \varphi, \psi) + \int_{s}^{\sigma} r(\sigma, \tau)F(z, \tau; \varphi, \psi)\,\mathrm{d}\tau, \tag{35}$$

where $r(t, s)$ denotes the resolvent of $k(t, s)$, see Theorem 7.2 of Verduyn Lunel (2023). Integration of equation (35) yields

$$\varphi(\sigma) = G(z, \sigma; \varphi, \psi), \tag{36}$$

where

$$\begin{aligned} G(z, \sigma; \varphi, \psi) := \psi(-1) &+ \int_{-1}^{\sigma} F(z, \xi; \varphi, \psi)\,\mathrm{d}\xi \\ &+ \int_{-1}^{\sigma}\int_{s}^{\xi} r(\xi, \tau)F(z, \tau; \varphi, \psi)\,\mathrm{d}\tau\,\mathrm{d}\xi. \end{aligned}$$

Using the exponential estimate for the resolvent $r(t, s)$, see Proposition 7.3 of Verduyn Lunel (2023) and the weighted norm (31), we can estimate

$$\|G(z, \sigma; \varphi_1, \psi) - G(z, \sigma; \varphi_2, \psi)\|_\gamma \le \frac{C}{|\gamma|}\|\varphi_1 - \varphi_2\|_\gamma,$$

where $C > 0$ and $\gamma > 0$ is sufficiently large. So the map $\varphi \mapsto G(z, \sigma; \varphi, \psi)$ is a contraction for $\gamma > 0$ sufficiently large. This shows that equation (36) has a unique solution. Therefore equation (33) has a unique solution and this completes the proof that (30) has a unique solution for every $z \in \mathbb{C}$ and $\psi \in B([-h, 0], \mathbb{C}^n)$. □

In case $\eta \neq 0$, it turns out that $\Omega \subset \mathbb{C}$ with $\Omega \neq \mathbb{C}$. See the examples in Sect. 5. By choosing Ω appropriately, we can extend the proof of Theorem 4.4 to include the case that $\eta \neq 0$ and $\omega = h$ using a time-dependent version of Theorem 2.1 of Verduyn Lunel (2023) (instead of using Theorem 7.2 of Verduyn Lunel (2023) as we did in the proof of Theorem 4.4). The theory of Grippenberg et al. (1990) can be used to prove a time-dependent version of Theorem 2.1 of Verduyn Lunel (2023). To make the present work self-contained we have decided to focus on the examples in Sect. 5 and not to aim for a general abstract result.

It is also possible to extend the proof of Theorem 4.4 to include the case that the period ω is an integer multiple of the delay. The construction however becomes more involved, see Kaashoek and Verduyn Lunel (2023, Chap. 11). Again in the present work we have decided to focus on the examples in Sect. 6 and not to aim for a general abstract result.

As an application of Theorem 3.2 we can, in case $\eta = 0$ and $\omega = h$, compute the characteristic matrix function Δ of the period map T associated with (24) explicitly in terms of the fundamental matrix solution.

Corollary 4.5 *Consider equation* (24) *with* $\eta = 0$ *and period* $\omega = h$ *and let* T *on* $B[-h, 0]$ *be the corresponding period map. The characteristic matrix function* $\Delta(z)$ *of the period map* T *is given by*

$$\Delta(z) = I - zC(I - zW)^{-1}B,$$

where the operator W *is given by* (28) *and the operators* $B : \mathbb{C}^n \to B[-h, 0]$ *and* $C : B[-h, 0]\mathbb{C}^n$ *are defined by*

$$(Bu)(\theta) := X(h + \theta, 0)u, \quad -h \le \theta \le 0, \quad \textit{and} \quad C\varphi := \varphi(0). \tag{37}$$

Proof From the definitions of B and C in (37) we see that, using (29),

$$(BC\varphi)(\theta) = (B\varphi(0)(\theta) = X(h + \theta, 0)\varphi(0) = (R\varphi)(\theta), \quad -h \le \theta \le 0.$$

Thus $R = BC$, and Theorem 3.2 yields that $\Delta(z)$ is a characteristic matrix function for T. □

As a first illustration of the results of this section we consider the delay equation

$$\dot{x}(t) = b(t)x(t - 1),$$

where $b(t) = b(t + 1)$. The period map T on $B\left([-1, 0]; \mathbb{C}^n\right)$ is given by

$$(T\varphi)(\theta) = \varphi(0) + \int_{-1}^{\theta} b(\sigma)\varphi(\sigma)\, \mathrm{d}\sigma, \qquad -1 \le \theta \le 0.$$

The operator W on $B\left([-1, 0]; \mathbb{C}^n\right)$, defined in (28), is given by

$$(W\varphi)(\theta) = \int_{-1}^{\theta} b(\sigma)\varphi(\sigma)\, \mathrm{d}\sigma, \qquad -1 \le \theta \le 0. \tag{38}$$

and the resolvent of W defined by (38) is given by

$$\left((I - zW)^{-1}\varphi\right)(\theta) = \varphi(\theta) + z\int_{-1}^{\theta} G(\theta - s; z)b(s)\varphi(s)\,\mathrm{d}s,\ -1 \le \theta \le 0.$$

Here $G(t; z)$ is the fundamental solution of the homogeneous ordinary differential equation

$$\dot{x}(t) = zb(t)x(t), \qquad t \ge -1,$$

normalized to 1 at $t = -1$. Therefore $G(t; z)$ is given by

$$G(t; z) = \exp\left(\int_{-1}^{t} zb(s)\,\mathrm{d}s\right), \quad t \ge -1.$$

Moreover, $B : \mathbb{C}^n \to B([-1, 0]; \mathbb{C}^n)$ and $C : B([-1, 0]; \mathbb{C}^n) \to \mathbb{C}^n$ are defined by

$$(Bu)(\theta) := u, \quad -1 \le \theta \le 0, \quad \text{and} \quad C\varphi := \varphi(0).$$

Furthermore

$$\left((I - zW)^{-1}B\right)(\theta) = G(\theta; z), \quad -1 \le \theta \le 0,$$

and hence

$$\begin{aligned}\Delta(z) &:= 1 - zC_0(I - zW)^{-1}B \\ &= 1 - zG(0; z) = 1 - ze^{zm(b)},\end{aligned}$$

where

$$m(b) := \int_{-1}^{0} b(s)\,\mathrm{d}s.$$

An application of Theorem 3.1 now yields that all nonzero eigenvalues of T are algebraically simple eigenvalues. Furthermore, if b is such that $m(b) = 0$, then the nonzero spectrum of T consists of the single point $\{1\}$ only.

5 Scalar Periodic Delay Equations of Period One

Consider the scalar periodic delay equation

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left[x(t) - cx(t-1)\right] = b(t)x(t-1), & t \ge s, \\ x(\theta) = \varphi(\theta), & -1 \le \theta \le 0. \end{cases} \tag{39}$$

Here b is a complex-valued continuous periodic function of period one defined on the full real line, $c \in \mathbb{C}$, and $\varphi \in B\,([-1,0];\mathbb{C}^n)$.

The period map T on $B\,([-1,0];\mathbb{C}^n)$ defined by the periodic delay equation (39) is given by

$$(T\varphi)(\theta) = \varphi(0) - c\varphi(-1) + c\varphi(\theta) + \int_{-1}^{\theta} b(s)\varphi(s)\,\mathrm{d}s, \quad -1 \le \theta \le 0.$$

The operator T consists of two parts: a finite rank operator R defined by

$$R\varphi := \varphi(0), \qquad \varphi \in B\left([-1,0];\mathbb{C}^n\right), \tag{40}$$

and an operator W on $B\,([-1,0];\mathbb{C}^n)$ defined by

$$(W\varphi)(\theta) := c\varphi(\theta) - c\varphi(-1) + \int_{-1}^{\theta} b(s)\varphi(s)\,\mathrm{d}s, \quad -1 \le \theta \le 0. \tag{41}$$

The minimal rank factorization of R is given by $R = BC$ with B and C, for each $u \in \mathbb{C}$ and $\varphi \in B\,([-1,0];\mathbb{C}^n)$, defined by

$$(Bu)(\theta) := u, \quad -1 \le \theta \le 0, \quad \text{and} \quad (C\varphi) := \varphi(0).$$

The resolvent $(I - zW)^{-1}$ of the operator W defined by (41) is analytic for $z \in \Omega$ with $\Omega = \{z \in \mathbb{C} \mid 1 - cz \neq 0\}$ and explicitly given by

$$\left((I - zW)^{-1}\varphi\right)(\theta) = \varphi(\theta) + z\int_{-1}^{\theta} G(\theta - s; z)b(s)\varphi(s)\,\mathrm{d}s, \quad -1 \le \theta \le 0.$$

Here $G(t;z)$ is the fundamental solution of the ordinary differential equation

$$(1 - cz)\dot{x}(t) = zb(t)x(t), \qquad t \ge -1,$$

normalized to 1 at $t = -1$. So $G(t;z)$ is given by

$$G(t;z) = \frac{1}{1 - cz}\exp\left(\int_{-1}^{t} \frac{z}{1 - cz} b(\sigma)\,\mathrm{d}\sigma\right), \quad t \ge -1.$$

Moreover,

$$\left((I - zW)^{-1}B\right)(\theta) = (1 - cz)G(\theta;z), \quad -1 \le \theta \le 0,$$

and hence according to Theorem 3.2, the characteristic function Δ defined by (13) is given by

$$\begin{aligned}\Delta(z) &:= 1 - zC(I - zW)^{-1}B \\ &= 1 - z(1 - cz)G(0; z).\end{aligned}$$

Summarizing we have proved the following result.

Theorem 5.1 *Let W and R be the operators given by (41) and (40), respectively. Then $T = W + R$ is the period map associated with (39), and T has a characteristic matrix function on $\Omega = \{z \in \mathbb{C} \mid 1 - cz \neq 0\}$, namely the function $\Delta(z)$ given by*

$$\Delta(z) = 1 - z \exp\left(\frac{z}{1 - cz} \int_{-1}^{0} b(s)\, \mathrm{d}s\right)$$

Furthermore $\lambda = 1/c$ belongs to the essential spectrum of T.

Note that in the retarded case ($c = 0$), the set Ω defined in Theorem 5.1 equals $\mathbb{C}$, and the operator W is a quasi-nilpotent operator. See also the example at the end of the previous section.

6 Scalar Periodic Delay Equations (Two Periodic)

In this section we consider the special class of scalar periodic delay equations

$$\begin{cases} \dot{x}(t) = b(t)x(t-1), & t \geq s, \\ x(t) = \varphi(t), & -1 \leq t \leq 0, \end{cases}$$

where b is of the form

$$b(t) := \begin{cases} b_0(t) & 0 \leq t \bmod 2 < 1 \\ \alpha b_0(t) & 1 \leq t \bmod 2 < 2, \end{cases} \tag{42}$$

where $\alpha \in \mathbb{R} \setminus \{0\}$ and b_0 is a complex-valued continuous periodic function of period 1. So b is a complex-valued continuous periodic function of period two for $\alpha \neq 1$. The situation is similar to the one periodic case considered in the previous section, but the computations of the period map become more involved. The special class of equations considered in this section present a rich class of new examples.

The period two map $T : B([-1, 0]; \mathbb{C}^n) \to B([-1, 0]; \mathbb{C}^n)$ for the periodic delay equation $\dot{x}(t) = b(t)x(t-1)$ with b given by (42) becomes

$$
\begin{aligned}
(T\varphi)(\theta) = \varphi(0) + \int_{-1}^{0} b_0(s)\varphi(s)\,\mathrm{d}s \\
+ \alpha \int_{-1}^{\theta} b_0(s) \left(\varphi(0) + \int_{-1}^{s} b_0(\sigma)\varphi(\sigma)\,\mathrm{d}\sigma \right) \mathrm{d}s \\
= \varphi(0) + \int_{-1}^{0} b_0(s)\varphi(s)\,\mathrm{d}s + \alpha\varphi(0) \int_{-1}^{\theta} b_0(s)\,\mathrm{d}s \\
+ \alpha \int_{-1}^{\theta} b_0(s) \int_{-1}^{s} b_0(\sigma)\varphi(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}s. \qquad (43)
\end{aligned}
$$

From the representation (43) for T we conclude that T satisfies (9), where W and R are operators acting on $C[-1, 0]$ given by

$$
(W\varphi)(\theta) := \alpha \int_{-1}^{\theta} b_0(s) \int_{-1}^{s} b_0(\sigma)\varphi(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}s, \qquad (44)
$$

$$
(R\varphi)(\theta) := \varphi(0) + \int_{-1}^{0} b_0(s)\varphi(s)\,\mathrm{d}s + \alpha\varphi(0) \int_{-1}^{\theta} b_0(s)\,\mathrm{d}s.
$$

Furthermore, the rank two operator R admits a minimal rank factorization $R = BC$, where

$$
B : \mathbb{C}^2 \to B\left([-1, 0]; \mathbb{C}^n\right), \qquad B \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} (\theta) := c_1 + c_2 \int_{-1}^{\theta} b_0(s)\,ds, \qquad (45)
$$

$$
C : B\left([-1, 0]; \mathbb{C}^n\right) \to \mathbb{C}^2, \qquad C\varphi := \begin{pmatrix} \varphi(0) + \int_{-1}^{0} b_0(s)\varphi(s)\,ds \\ \alpha\varphi(0) \end{pmatrix}. \qquad (46)
$$

Lemma 6.1 *The operator W defined by* (44) *is a quasi-nilpotent operator, and the resolvent of W is given by*

$$
\left((I - zW)^{-1}\varphi\right)(\theta) = \varphi(\theta) + \int_{-1}^{\theta} \frac{\partial g}{\partial s}(\alpha, z; \theta, s)\varphi(s)\,ds, \quad -1 \le \theta \le 0,
$$

where

$$
\begin{aligned}
g(\alpha, z; \theta, s) := \frac{1}{2} \exp\left(\sqrt{\alpha z} \int_{s}^{\theta} b_0(\sigma)\,\mathrm{d}\sigma \right) + \\
+ \frac{1}{2} \exp\left(-\sqrt{\alpha z} \int_{s}^{\theta} b_0(\sigma)\,\mathrm{d}\sigma \right), \quad -1 \le s \le \theta \le 0. \qquad (47)
\end{aligned}
$$

Proof Put $\psi = (I - zW)^{-1}\varphi$, then we need to solve the equation

$$
\psi - zW\psi = \varphi \qquad (48)
$$

with φ given and ψ as the unknown. By differentiating (48) we arrive at the following initial value problem

$$\psi'(\theta) - \alpha z b_0(\theta) \int_{-1}^{\theta} b_0(\sigma)\psi(\sigma)\,\mathrm{d}\sigma = \varphi'(\theta), \quad -1 \le \theta \le 0 \tag{49}$$

with initial condition $\psi(-1) = \varphi(-1)$. The general solution of the homogeneous part of the differential integral equation in (49) is given by

$$\psi(\theta) = g(\alpha, z; \theta, -1)\varphi(-1), \qquad -1 \le \theta \le 0,$$

where $g(\alpha, z; \theta, -1)$ is given by (47) with $s = -1$. A particular solution of the differential equation in (49) with $\psi(-1) = 0$ is given by

$$\psi_p(\theta) = \int_{-1}^{\theta} g(\alpha, z; \theta, s)\varphi'(s)\,\mathrm{d}s,$$

where $g(\alpha, z; \theta, s)$ is given by (47) and we have used that $g(\alpha, z; \theta, \theta) = 1$ and $g'(\alpha, z; s, s) = 0$.

This shows that the solution of the initial value problem (49) with initial condition $\psi(-1) = \varphi(-1)$ is given by

$$\begin{aligned} \psi(\theta) &= g(\alpha, z; \theta, -1)\varphi(-1) + \int_{-1}^{\theta} g(\alpha, z; \theta, s)\varphi'(s)\,ds \\ &= \varphi(\theta) + \int_{-1}^{\theta} \frac{\partial g}{\partial s}(\alpha, z; \theta, s)\varphi(s)\,ds, \end{aligned} \tag{50}$$

where we have used integration by parts in the last identity. This completes the proof of the lemma. □

An application of Theorem 3.2 now yields that the period map T given by (43) has a characteristic matrix function Δ given by (13). In the next theorem we will compute Δ explicitly.

Theorem 6.2 *The characteristic matrix function $\Delta(z) : \mathbb{C}^2 \to \mathbb{C}^2$ associated with the operator T defined by (43) is given by*

$$\Delta(z) = \begin{bmatrix} 1 - z\gamma_1(z) - \frac{1}{\alpha}\gamma_2(z) & -\frac{1}{\alpha}\left(\gamma_1(z) + \gamma_2(z) - 1\right) \\ -z\alpha\gamma_1(z) & 1 - \gamma_2(z) \end{bmatrix}, \tag{51}$$

where

$$\gamma_1(z) := \frac{1}{2}\left(\exp\left(\sqrt{\alpha z}m(b_0)\right) + \exp\left(-\sqrt{\alpha z}m(b_0)\right)\right), \qquad -1 \le \theta \le 0.$$

$$\gamma_2(z) := \frac{\sqrt{\alpha z}}{2}\left(\exp\left(\sqrt{\alpha z}m(b_0)\right) - \exp\left(-\sqrt{\alpha z}m(b_0)\right)\right), \qquad -1 \le \theta \le 0,$$

where $m(b_0) := \int_{-1}^{0} b_0(\sigma)\,\mathrm{d}\sigma$. *Moreover,*

$$\det \Delta(z) = 1 - \left(\frac{1+\alpha}{\alpha}\right)\gamma_2(z) - z.$$

Proof Observe using (45) and (50) that

$$\left((I - zW)^{-1}B\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\right)(\theta) = g(\alpha, z; \theta, -1)c_1 + c_2\int_{-1}^{\theta} b_0(s)g(\alpha, z; \theta, s)\,\mathrm{d}s. \tag{52}$$

Using (47) we can rewrite the last term in (52) as follows

$$\int_{-1}^{\theta} b_0(s)g(\alpha, z; \theta, s)\,\mathrm{d}s = \frac{1}{2}x + \frac{1}{2}y, \tag{53}$$

where

$$x := \int_{-1}^{\theta} b_0(s)\exp\left(\sqrt{\alpha z}\int_{s}^{\theta} b_0(\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}s,$$
$$y := \int_{-1}^{\theta} b_0(s)\exp\left(-\sqrt{\alpha z}\int_{s}^{\theta} b_0(\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}s.$$

Now put $k(s) := \int_s^\theta b_0(\sigma)\,\mathrm{d}\sigma$, $c := \sqrt{\alpha z}$, and $\varphi(s) := ck(s)$. Note that both $k(\theta)$ and $\varphi(\theta)$ are zero. Furthermore, we have

$$\begin{aligned} x &= -\int_{-1}^{\theta} k'(s)\exp\left(ck(s)\right)\mathrm{d}s = -\frac{1}{c}\int_{-1}^{\theta} \varphi'(s)\exp\left(\varphi(s)\right)\mathrm{d}s \\ &= -\frac{1}{c}\exp(\varphi(s))\Big|_{-1}^{\theta} = -\frac{1}{c} + \frac{1}{c}\exp\left(ck(-1)\right). \end{aligned}$$

Similarly

$$\begin{aligned} y &= \int_{-1}^{\theta} k'(s)\exp\left(-ck(s)\right)\mathrm{d}s = \frac{1}{c}\int_{-1}^{\theta} -\varphi'(s)\exp\left(-\varphi(s)\right)\mathrm{d}s \\ &= \frac{1}{c}\exp(-\varphi(s))\Big|_{-1}^{\theta} = \frac{1}{c} - \frac{1}{c}\left(v - ck(-1)\right). \end{aligned}$$

Summarizing and using (53) we have

$$\int_{-1}^{\theta} b_0(s)g(\alpha, z; \theta, s)\,\mathrm{d}s = \frac{1}{2c}\exp\left(ck(-1)\right) - \frac{1}{2c}\exp\left(-ck(-1)\right)$$
$$= \frac{1}{2\sqrt{\alpha z}}\Bigg(\exp\left(\sqrt{\alpha z}\int_{-1}^{\theta} b_0(\sigma)\,\mathrm{d}\sigma\right)$$
$$- \exp\left(-\sqrt{\alpha z}\int_{-1}^{\theta} b_0(\sigma)\,\mathrm{d}\sigma\right)\Bigg).$$

In particular,

$$\int_{-1}^{0} b_0(s)g(\alpha, z; 0, s)\,\mathrm{d}s = \frac{1}{\alpha z}\gamma_2(z).$$

Since $g(\alpha, z; 0, -1) = \gamma_1(z)$ this shows that

$$\left((I - zW)^{-1}B\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\right)(0) = \gamma_1(z)c_1 + \frac{1}{\alpha z}\gamma_2(z)c_2.$$

Similarly using (47) with $\theta = t$ and $s = -1$ we obtain

$$\int_{-1}^{0} b_0(t)g(\alpha, z; t, -1)\,dt = \frac{1}{2}\tilde{x} + \frac{1}{2}\tilde{y},$$

where

$$\tilde{x} := \int_{-1}^{0} b_0(t)\exp\left(\sqrt{\alpha z}\int_{-1}^{t} b_0(\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}t,$$
$$\tilde{y} := \int_{-1}^{0} b_0(t)\exp\left(-\sqrt{\alpha z}\int_{-1}^{t} b_0(\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}t.$$

Now put $\ell(t) := \int_{-1}^{t} b_0(\sigma)\,\mathrm{d}\sigma$, and let $c := \sqrt{\alpha z}$. Then $\ell'(t) = b_0(t)$, and hence

$$\tilde{x} = \int_{-1}^{0} \ell'(t)\exp\big(cl(t)\big)\,dt = \frac{1}{c}\exp\big(c\ell(t)\big)\Big|_{-1}^{0}$$
$$= \frac{1}{c}\exp\left(c\int_{-1}^{0} b_0(\sigma)\,\mathrm{d}\sigma\right) - \frac{1}{c}.$$

An analogous calculation with $\tilde{y}$ in place of $\tilde{x}$ yields

$$\tilde{y} = \int_{-1}^{0} \ell'(t)\exp\left(-cl(t)\right)\mathrm{d}t = -\frac{1}{c}\exp\left(-c\ell(t)\right)\Big|_{-1}^{0}$$
$$= -\frac{1}{c}\exp\left(-c\int_{-1}^{0} b_0(\sigma)\,\mathrm{d}\sigma\right) + \frac{1}{c}.$$

It follows that

$$\begin{aligned}\int_{-1}^{0} b_0(t) g(\alpha, z; t, -1)\,\mathrm{d}t &= \\ &= \frac{1}{2\sqrt{\alpha z}} \left(\exp\left(\sqrt{\alpha z}\int_{-1}^{0} b_0(\sigma)\,\mathrm{d}\sigma\right) - \exp\left(-\sqrt{\alpha z}\int_{-1}^{0} b_0(\sigma)\,\mathrm{d}\sigma\right)\right) \\ &= \frac{1}{\alpha z}\gamma_2(z).\end{aligned}$$

Furthermore

$$\int_{-1}^{0} b_0(t) \int_{-1}^{s} b_0(\sigma) g(\alpha, z; t, \sigma)\,\mathrm{d}\sigma\,\mathrm{d}t = \frac{1}{\alpha z}\left(\gamma_1(z) - 1\right).$$

Thus using (46) it follows that $C(I - zW)^{-1}B$ can be written as

$$C(I - zW)^{-1}B = \begin{pmatrix} \gamma_1(z) + \frac{1}{\alpha z}\gamma_2(z) & \frac{1}{\alpha z}\left(\gamma_1(z) + \gamma_2(z) - 1\right) \\ \alpha\gamma_1(z) & \frac{1}{z}\gamma_2(z) \end{pmatrix}.$$

This proves that $\Delta(z)$ is given by (51). Moreover

$$\begin{aligned}\det \Delta(z) &= \left(1 - z\gamma_1(z) - \alpha^{-1}\gamma_2(z)\right)\left(1 - \gamma_2(z)\right) \\ &\qquad - z\gamma_1(z)\left(\gamma_1(z) + \gamma_2(z) - 1\right) \\ &= 1 - \alpha^{-1}\gamma_2(z) - \gamma_2(z) + \alpha^{-1}\gamma_2^2 - z\gamma_1^2.\end{aligned}$$

Next observe that

$$\begin{aligned}\alpha^{-1}\gamma_2^2 - z\gamma_1^2 &= \alpha^{-1}\left(\frac{\sqrt{\alpha z}}{2}\left(e^{\sqrt{\alpha z}m(b_0)} - e^{-\sqrt{\alpha z}m(b_0)}\right)\right)^2 \\ &\qquad - z\left(\frac{1}{2}\left(e^{\sqrt{\alpha z}m(b_0)} + e^{-\sqrt{\alpha z}m(b_0)}\right)\right)^2 \\ &= -z\end{aligned}$$

Thus

$$\det \Delta(z) = 1 - \left(\frac{1+\alpha}{\alpha}\right)\gamma_2(z) - z,$$

and this completes the proof of the theorem. □

To illustrate the applications of Theorem 6.2, we first consider the case $\alpha = 1$ and $b_0 \equiv 1$ and the case $\alpha = -1$ and $b_0 \equiv 1$.

In case $\alpha = 1$ and $b_0 \equiv 1$, the operator T defined by (43) can be written as $T = T_1^2$, where $T_1 : B\left([-1, 0]; \mathbb{C}^n\right) \to B\left([-1, 0]; \mathbb{C}^n\right)$ is given by

$$(T_1\varphi)(\theta) := \varphi(0) + \int_{-1}^{\theta} \varphi(s)\, ds.$$

It follows from Theorem 5.1 that the operator T_1 admits a characteristic matrix function Δ_1 defined by $\Delta_1(z) := 1 - ze^{-z}$. Furthermore, it follows from Theorem 6.2 that the characteristic matrix function Δ corresponding to period map T defined by (43) satisfies

$$\begin{aligned}\det \Delta(z) &= 1 - \sqrt{z}\left(e^{\sqrt{z}} - e^{-\sqrt{z}}\right) - z \\ &= \left(1 - \sqrt{z}e^{\sqrt{z}}\right)\left(1 + \sqrt{z}e^{-\sqrt{z}}\right) \\ &= \Delta_1(\sqrt{z})\Delta_1(-\sqrt{z}).\end{aligned}$$

Since $T = T_1^2$, this result is in agreement with the fact that

$$\lambda \in \sigma(T) \setminus \{0\} \text{ if and only if } \sqrt{\lambda} \text{ or } -\sqrt{\lambda} \text{ belongs to } \sigma(T_0) \setminus \{0\}.$$

In general, if we define the operator $T_\alpha : B\left([-1, 0]; \mathbb{C}^n\right) \to B\left([-1, 0]; \mathbb{C}^n\right)$ by

$$(T_\alpha\varphi)(\theta) := \varphi(0) + \alpha \int_{-1}^{\theta} b_0(s)\varphi(s)\, \mathrm{d}s, \qquad \alpha \neq 0,$$

then the operator T defined by (43) can be written as

$$T = T_\alpha T_1.$$

In case $\alpha = -1$ and $b_0 \equiv 1$, the operator T becomes $T = T_{-1}T_1$. In this case

$$\det \Delta(z) = 1 - z \quad \text{and} \quad \sigma(T_{-1}T_1) \setminus \{0\} = \{1\}.$$

As a next example consider the 2-periodic delay equation

$$\dot{x}(t) = \cos(\pi t)x(t-1). \tag{54}$$

Define

$$b_0(t) := \begin{cases} \cos(\pi t) & 0 \le t \bmod 2 < 1, \\ -\cos(\pi t) & 1 \le t \bmod 2 < 2. \end{cases} \tag{55}$$

Then b_0 is 1-periodic and

$$m(b_0) = -\int_{-1}^{0} \cos(\pi s)\, ds = 0.$$

Furthermore

$$\cos(\pi t) = \begin{cases} b_0(t) & 0 \le t \bmod 2 < 1, \\ -b_0(t) & 1 \le t \bmod 2 < 2. \end{cases}$$

Thus it follows that $b(t) = \cos(\pi t)$ satisfies (42) with $b_0(t)$ defined by (55) and $\alpha = -1$. Consequently, the spectrum of the period map associated with (54) consists of a single point only.

Examples of periodic delay equations with period a multiple of the delay for which the spectrum of the period map is finite was a open problem in the literature. With the class of equations considered in this section, we can now construct many periodic delay equations for which one still can compute the characteristic matrix function rather explicitly.

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