The Twin Semigroup Approach Towards Periodic Neutral Delay Equations

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Abstract In the first part of this chapter we review the recently developed theory of twin semigroups and norming dual pairs in the light of neutral delay equations. In the second part we extend the perturbation theory for twin semigroups to include time-dependent perturbations. Finally we apply this newly developed theory to neutral periodic delay equations.

1 Introduction

Consider a function x defined on the half-line $[0, \infty)$ with values in $\mathbb{R}^n$ and assume that the derivative $\dot{x}$ depends on the history of x and $\dot{x}$. More precisely, we assume that there exists $h > 0$ such that $\dot{x}(t)$ depends on $x(\tau)$ and $\dot{x}(\tau)$ for $t - h \leq \tau \leq t$. Given these restrictions we would like to consider a general linear differential equation.

To formulate precisely what type of equations we consider, we first define the segment $x_t : [-h, 0] \to \mathbb{R}^n$ by

$$x_t(\theta) := x(t+\theta), \qquad \text{for } -h \leq \theta \leq 0. \tag{1}$$

Let η and ζ be $n \times n$-matrix-valued functions of bounded variation defined on $[0, \infty)$ such that $\eta(0) = \zeta(0) = 0$, η and ζ are continuous from the right on $(0, h)$, $\eta(t) = \eta(h)$ and $\zeta(t) = \zeta(h)$ for $t \geq h$. We call such functions η and ζ of normalized bounded variation. Furthermore assume that $\eta(t)$ is continuous at $t = 0$. (See Appendix A for the precise definition and basic properties of such functions.)

The class of equations that we will study can now be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t) - \int_0^h \mathrm{d}\eta(\theta)x(t-\theta)\right] = \int_0^h \mathrm{d}\zeta(\theta)x(t-\theta). \tag{2}$$

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D. Breda (ed.), *Controlling Delayed Dynamics*, CISM International Centre for Mechanical Sciences 604, https://doi.org/10.1007/978-3-031-01129-0_1

To single out a unique solution we have to provide an initial condition at a certain time s. The initial condition should specify the values of x on the interval of length h preceding time s. Let y satisfy (2) for $t \geq s$ and the initial condition

$$y(s+\theta) = \varphi(\theta), \qquad -h \leq \theta \leq 0,$$

where $\varphi \in B\left([-h, 0]; \mathbb{R}^n\right)$, the Banach space of bounded Borel measurable functions provided with the supremum norm (see Sect. A for the precise definition and basic properties). Then x defined for $t \geq 0$ by $x(t) = y(s+t)$, satisfies (2) for $t \geq 0$ and the initial condition

$$x(\theta) = \varphi(\theta), \qquad -h \leq \theta \leq 0. \tag{3}$$

Equation (2) is time invariant and called autonomous. So we can, without loss of generality, restrict our attention to an initial condition imposed at time zero. This in contrast to time periodic equations which we will consider in Sect. 8.

Equation (2) is called a neutral functional differential equation (NFDE). A solution of the initial-value problem (2)–(3) on the half-line $[0, \infty)$ is a function $x \in B\left([0, \infty); \mathbb{R}^n\right)$ such that

(i) (3) holds;
(ii) on $(0, \infty)$, the function x is absolutely continuous and (2) holds;
(iii) the following limit exists

$$\lim_{t \downarrow 0} \frac{1}{t} \left[x(t) - \int_0^h \mathrm{d}\eta(\theta) x(t-\theta) - \varphi(0) - \int_0^h \mathrm{d}\eta(\theta)\, \varphi(-\theta) \right]$$

and equals $\int_0^h \mathrm{d}\zeta(\theta)\, \varphi(-\theta)$.

We end the introduction with an outline of this chapter. In Sect. 2 we derive a representation of the solution of a NFDE by direct methods. The main result is given in Theorem 2.4. In Sect. 3 we introduce the notions of norming dual pair and twin semigroup following Diekmann and Verduyn Lunel (2021). In Sect. 4 we introduce a concrete norming dual pair that will be used in Sect. 5 to represent the solution semigroup corresponding to a NFDE as a twin semigroup. In Sect. 6 we use the twin semigroup approach towards NFDE to prove a variation-of-constants formula, see Theorem 6.4. In Sect. 7 we develop the perturbation theory for bounded time-dependent perturbations of twin semigroups. The main result is given in Theorem 7.5. In Sect. 8 we consider periodic NFDE as an application of the perturbation theory developed in Sect. 7 and prove that periodic NFDE define a twin evolutionary system. Finally in Appendix A we review some basic properties of functions of bounded variations and complex Borel measures.

2 Introduction to NFDE

This section is concerned with the existence, uniqueness and representation of a solution of the initial-value problem (2)–(3). For $0 \le t \le h$, we can combine the two separate pieces of information given in (2) and (3) and write

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t) - \int_0^h \mathrm{d}\eta(\theta)x(t-\theta)\right] = \int_0^t \mathrm{d}\zeta(\theta)x(t-\theta) + \int_t^h \mathrm{d}\zeta(\theta)\varphi(t-\theta). \quad (4)$$

By integration and changing the order of integration we can write (4) as

$$x(t) - \int_0^h \mathrm{d}\eta(\theta)x(t-\theta) = \int_0^t \zeta(\theta)x(t-\theta)\,\mathrm{d}\theta + g(t), \quad (5)$$

where

$$g(t) := \varphi(0) - \int_0^h \mathrm{d}\eta(\theta)\varphi(-\theta) + \int_0^t \left(\int_s^h \mathrm{d}\zeta(\theta)\varphi(s-\theta)\right)\mathrm{d}s. \quad (6)$$

Next we write (5) as follows

$$x(t) = \int_0^t \mathrm{d}\eta(\theta)x(t-\theta) + \int_0^t \zeta(\theta)x(t-\theta)\,\mathrm{d}\theta + f(t), \quad (7)$$

where, using (6),

$$\begin{aligned} f(t) &:= g(t) + \int_t^h \mathrm{d}\eta(\theta)\varphi(t-\theta) \\ &= \varphi(0) + \int_0^h [\zeta(t+\sigma) - \zeta(\sigma)]\,\varphi(-\sigma)\,\mathrm{d}\sigma \\ &\qquad + \int_0^h \mathrm{d}\,[\eta(t+\sigma) - \eta(\sigma)]\,\varphi(-\sigma). \end{aligned} \quad (8)$$

Here we have used that

$$\int_0^t \left(\int_s^h \mathrm{d}\zeta(\theta)\varphi(s-\theta)\right)\mathrm{d}s = \int_0^h [\zeta(t+\sigma) - \zeta(\sigma)]\,\varphi(-\sigma)\,\mathrm{d}\sigma$$

and that

$$\int_t^h \mathrm{d}\eta(\theta)\varphi(t-\theta) = \int_0^h \mathrm{d}\eta(t+\sigma)\varphi(-\sigma).$$

It follows from Theorem A.2 that the function f defined by (8) is a bounded Borel measurable function on $[0,\infty)$ that is constant on $[h,\infty)$.

Define the function μ by

$$\mu(\theta) := \eta(\theta) + \int_0^\theta \zeta(s)\,\mathrm{d}s, \qquad 0 \le \theta \le h, \tag{9}$$

and $\mu(\theta) = \mu(h)$ for $\theta \ge h$, then μ is a $n \times n$-matrix-valued function of normalized bounded variation. Note that, since $\eta(\theta)$ is continuous at $\theta = 0$, we have that $\mu(\theta)$ is continuous at $\theta = 0$.

The convolution product of a $n \times n$-matrix-valued function of normalized bounded variation μ and a bounded Borel measurable function f is defined by

$$(\mu * f)(t) := \int_0^t \mathrm{d}\mu(\theta) f(t-\theta), \qquad t \ge 0. \tag{10}$$

From Theorem A.1, it follows that $\mu * f$ is a bounded Borel measurable function on $[0, \infty)$.

Using the convolution product defined by (10), the initial-value problem (2)–(3), i.e., (7), can be rewritten as a renewal equation for x

$$x = \mu * x + f, \tag{11}$$

where μ is given by (9) and f, given by (8), can be rewritten as

$$f(t) = \varphi(0) + \int_0^h \mathrm{d}\,[\mu(t+\sigma) - \mu(\sigma)]\,\varphi(-\sigma). \tag{12}$$

Therefore to prove existence and uniqueness of solutions of the initial-value problem (2)–(3), it suffices to prove existence and uniqueness of solutions of the renewal equation (11).

The convolution product of two $n \times n$-matrix-valued functions of normalized bounded variation μ and ν, defined by

$$(\mu * \nu)(t) := \int_0^t \mathrm{d}\mu(\theta)\nu(t-\theta), \qquad t \ge 0, \tag{13}$$

is again a function of bounded variation (see Appendix A and, in particular, Theorem A.3).

The resolvent kernel ρ of a renewal equation (11) with kernel μ and convolution product (13) is defined as the matrix solution of the resolvent equation

$$\rho = \rho * \mu + \mu = \mu * \rho + \mu. \tag{14}$$

The key property of the resolvent concerns the representation of the solution of the renewal equation (11) as

$$x = f + \rho * f. \tag{15}$$

Indeed taking to convolution with ρ on the left and right of (11) yields

$$\rho * x = (\rho * \mu) * x + \rho * f = (\rho - \mu) * x + \rho * f.$$

Hence $\mu * x = \rho * f$ and substituting this relation into (11) yields (15).

We now discuss the existence and uniqueness of the solution of (14) under the assumption that μ is a $n \times n$-matrix-valued function of normalized bounded variation. It follows from Appendix A and in particular Theorem A.1 that functions of normalized bounded variation are in one-to-one correspondence to complex Borel measures. This allows us to use measure theory to prove existence and uniqueness of the solution of (14). We start with some preparations.

Let $\mathcal{E}$ denote the Borel σ-algebra on $[0, \infty)$. The Banach space of complex Borel measures of bounded total variation is denoted by $M([0, \infty))$ (see (82)). Let $M_{loc}([0, \infty))$ denote the vector space of local measures, i.e., set functions that are defined on relatively compact Borel measurable subsets of $[0, \infty)$ and that locally behave like bounded measures: for every $T > 0$ the set function μ_T defined by

$$\mu_T(E) := \mu(E \cap [0, T]), \qquad E \in \mathcal{E},$$

belongs to $M([0, \infty))$. The elements of $M_{loc}([0, \infty))$ are called Radon measures. Since the restriction to $[0, T]$ of $\mu * \nu$ depends only on the restrictions of μ and ν to $[0, T]$, we can unambiguously extend the convolution product to $M_{loc}([0, \infty))$ (see (84)).

We continue with the existence of the resolvent ρ of a complex Borel measure μ supported on $[0, \infty)$. For details see Diekmann and Verduyn Lunel (2021, Theorem A.7) and for further information and details see Grippenberg et. al. (1990).

Theorem 2.1 *Suppose that* $\mu \in M_{loc}\left([0, \infty); \mathbb{R}^{n\times n}\right)$. *There exists a unique measure* $\rho \in M_{loc}\left([0, \infty); \mathbb{R}^{n\times n}\right)$ *satisfying either one of the following identities*

$$\rho - \mu * \rho = \mu = \rho - \rho * \mu \tag{16}$$

if and only if $\det[I - \mu(\{0\})] \neq 0$. *Furthermore, if* $\mu((0, t])$ *is continuous as* $t = 0$, *then* $\rho((0, t])$ *is continuous at* $t = 0$ *as well.*

The following theorem summarizes some relevant results for renewal equations (Diekmann and Verduyn Lunel 2021, Theorem A.9).

Theorem 2.2 *Let* $\mu \in M_{loc}\left([0, \infty), \mathbb{R}^{n\times n}\right)$ *with* $\det[I - \mu(\{0\}] \neq 0$.

(i) For every $f \in B_{loc}([0, \infty), \mathbb{R}^n)$, *the renewal equation* (15) *has a unique solution* $x \in B_{loc}([0, \infty), \mathbb{R}^n)$ *given by*

$$x = f + \rho * f,$$

where ρ *satisfies* (16). *Furthermore, if* f *is locally absolutely continuous, then the solution* x *is locally absolutely continuous as well.*

(ii) If the kernel μ has no discrete part and if $f \in C([0,\infty),\mathbb{R}^n)$, then $x \in C([0,\infty),\mathbb{R}^n)$.

We now summarize the conclusions obtained so far in this section in the following theorem.

Theorem 2.3 *Let η and ζ be of normalized bounded variation. Let $\varphi \in B([-h,0];\mathbb{R}^n)$ be given. Define μ by (9). If $\det[I-\mu(0)] \neq 0$, then the NFDE (2) provided with the initial condition (3) admits a unique solution. For $t \geq 0$ this solution coincides with the unique solution of the renewal equation (11) and the solution has the representation (15) where ρ satisfies the resolvent equation (14) and f is given by (8).*

Representation (15) will be used to derive a representation of the solution of (2)–(3) directly in terms of the initial data $x_0 = \varphi$. We first need a definition. The *fundamental solution* of the delay equation (2)–(3) on $[-h,\infty)$ is defined by the $n \times n$-matrix-valued function

$$X(t) := \begin{cases} I + \rho((0,t]) & \text{for } t \geq 0, \\ 0 & \text{for } -h \leq t < 0, \end{cases} \tag{17}$$

where ρ is the resolvent of μ given by Theorem 2.1. Since $t \mapsto \mu((0,t])$ is continuous at $t=0$, it follows from Theorem 2.1 that $\rho((0,t])$ is continuous at $t=0$. Therefore we can conclude that $X(t)$ has a jump at $t=0$.

By construction, the fundamental matrix solution $X(t)$ satisfies (2) with initial data

$$X_0(\theta) = \begin{cases} I & \text{for } \theta = 0, \\ 0 & \text{for } -h \leq \theta < 0. \end{cases} \tag{18}$$

Using the fundamental matrix solution $X(t)$ given by (17) and Fubini's theorem, we can rewrite the representation formula (15) in terms of the forcing function f given by (8) directly in terms of the initial condition φ.

We summarize the result in a theorem.

Theorem 2.4 *The solution of (2)–(3) is given explicitly by*

$$x(t;\varphi) = X(t)\varphi(0) + \int_0^h \mathrm{d}\left[\int_{-h}^t \mathrm{d}X(\tau)\,(\mu(t-\tau+\sigma)-\mu(\sigma))\right]\varphi(-\sigma). \tag{19}$$

Or, equivalently, in terms of the resolvent ρ we have

$$\begin{aligned} x(t;\varphi) = (I+\rho((0,t]))\,\varphi(0) &+ \int_0^h \mathrm{d}\,[\mu(t+\sigma)-\mu(\sigma)])\varphi(-\sigma) \\ &+ \int_0^h \mathrm{d}\left[\int_0^t \rho(\mathrm{d}\tau)\,(\mu(t-\tau+\sigma)-\mu(\sigma))\right]\varphi(-\sigma). \end{aligned} \tag{20}$$

Here μ is given by (9).

3 Norming Dual Pairs and Twin Semigroups

The system of equations (2)–(3) defines an infinite-dimensional dynamical system on the state space $B([-h, 0]; \mathbb{R}^n)$, but for the qualitative study of such a dynamical system we need an adjoint theory in place (see Hale and Verduyn Lunel 1993). In the classical theory of delay equations this is the main reason to work with the state space $C([-h, 0]; \mathbb{R}^n)$ despite the fact that the initial data of the fundamental solution (see (18)) does not belong to this space. From the Riesz representation theorem it follows that the dual space of $C([-h, 0]; \mathbb{R}^n)$ has a nice characterization as the space of functions of normalized bounded variation.

The state space $B([-h, 0]; \mathbb{R}^n)$ includes the initial data of the fundamental solution but its dual space does not have a nice characterization. So although the state space $B([-h, 0]; \mathbb{R}^n)$ is a more natural space to consider, it has not yet been used because its dual space is too large to provide a useful adjoint theory. A beautiful idea to repair this discrepancy is to use the notion of a dual pair (see Aliprantis and Border 2006) made precise in Kunze (2011) for infinite-dimensional dynamical systems in the following way.

Two Banach spaces Y and $Y^\diamond$ are called a *norming dual pair* (cf. Kunze (2011)) if a bilinear map

$$\langle \cdot, \cdot \rangle : Y^\diamond \times Y \to \mathbb{R}$$

exists such that, for some $M \in [1, \infty)$,

$$|\langle y^\diamond, y\rangle| \leq M\|y^\diamond\|\|y\|$$

and, moreover,

$$\begin{aligned} \|y\| &:= \sup\left\{|\langle y^\diamond, y\rangle| \mid y^\diamond \in Y^\diamond,\ \|y^\diamond\| \leq 1\right\} \\ \|y^\diamond\| &:= \sup\left\{|\langle y^\diamond, y\rangle| \mid y \in Y,\ \|y\| \leq 1\right\}. \end{aligned}$$

So we can consider Y as a closed subspace of $Y^{\diamond *}$, the dual of $Y^\diamond$, and $Y^\diamond$ as a closed subspace of Y^* and both subspaces are necessarily weak* dense since they separate points.

The collection of linear functionals $Y^\diamond$ defines a weak topology on Y, denoted by $\sigma(Y, Y^\diamond)$. The corresponding locally convex topological vector space is denoted by $(Y, \sigma(Y, Y^\diamond))$. A crucial point in our approach is that the dual space $(Y, \sigma(Y, Y^\diamond))'$ is (isometrically isomorphic to) $Y^\diamond$ (Rudin 1991, Theorem 3.10). So if a linear functional on Y is continuous with respect to the topology induced by $Y^\diamond$, it can be (uniquely) represented by an element of $Y^\diamond$.

The next key idea to study infinite-dimensional dynamical systems on a norming dual pair is the notion of a twin operator introduced in Diekmann and Verduyn Lunel (2021).

A *twin operator* L on a norming dual pair $(Y, Y^\diamond)$ is a bounded bilinear map from $Y^\diamond \times Y$ to $\mathbb{R}$ that defines both a bounded linear map from Y to Y and a bounded linear map from $Y^\diamond$ to $Y^\diamond$. More precisely,

$$L : Y^\diamond \times Y \to \mathbb{R} \qquad (y^\diamond, y) \mapsto y^\diamond L y$$

is such that

(i) for some $C > 0$ the inequality

$$|y^\diamond L y| \le C \|y^\diamond\| \|y\|$$

holds for all $y \in Y$ and $y^\diamond \in Y^\diamond$;

(ii) for given $y \in Y$ the map $y^\diamond \mapsto y^\diamond L y$ is continuous as a map from $(Y^\diamond, \sigma(Y^\diamond, Y))$ to $\mathbb{R}$ and hence there exists $Ly \in Y$ such that

$$\langle y^\diamond, Ly \rangle = y^\diamond L y$$

for all $y^\diamond \in Y^\diamond$;

(iii) for given $y^\diamond \in Y^\diamond$ the map $y \mapsto y^\diamond L y$ is continuous as a map from $(Y, \sigma(Y, Y^\diamond))$ to $\mathbb{R}$ and hence there exists $y^\diamond L \in Y^\diamond$ such that

$$\langle y^\diamond L, y \rangle = y^\diamond L y$$

for all $y \in Y$.

So all three maps are denoted by the symbol L, but to indicate on which space L acts we write, inspired by Feller (1953) which, in turn, is inspired by matrix notation, either $y^\diamond L y$, Ly or $y^\diamond L$. As a concrete example, consider the identity operator. It maps $(y^\diamond, y)$ to $\langle y^\diamond, y \rangle$, y to y and $y^\diamond$ to $y^\diamond$.

If our starting point is a bounded linear operator $L : Y \to Y$ then there exists an associated twin operator if and only if the adjoint of L leaves the embedding of $Y^\diamond$ into Y^* invariant. We express this in words by saying that L extends to a twin operator. Likewise, if our starting point is an operator $L : Y^\diamond \to Y^\diamond$ then L extends to a twin operator if and only if the adjoint of L leaves the embedding of Y into $Y^{\diamond *}$ invariant. So a twin operator on a norming dual pair is reminiscent of the combination of a bounded linear operator on a reflexive Banach space and its adjoint, whence the adjective “twin”.

The composition of bounded bilinear maps is, in general, not defined. But for twin operators it is! Indeed, if L_1 and L_2 are both twin operators on the norming dual pair $(Y, Y^\diamond)$, we define the composition $L_1 L_2$ by

$$y^\diamond L_1 L_2 y := \langle y^\diamond L_1, L_2 y \rangle.$$

Note that this definition entails that $L_1 L_2$ acts on Y by first applying L_2 and next L_1, whereas $L_1 L_2$ acts on $Y^\diamond$ by first applying L_1 and next L_2.

Definition 3.1 A family $\{S(t)\}_{t\geq 0}$ of twin operators on a norming dual pair $(Y, Y^\diamond)$ is called a *twin semigroup* if

(i) $S(0) = I$, and $S(t+s) = S(t)S(s)$ for $t, s \geq 0$;
(ii) there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$|y^\diamond S(t)y| \leq Me^{\omega t}\|y\|\| y^\diamond\|;$$

(iii) for all $y \in Y$, $y^\diamond \in Y^\diamond$ the function

$$t \mapsto y^\diamond S(t)y$$

is measurable;
(iv) for $\operatorname{Re}\lambda > \omega$ (with ω as introduced in ii)) there exists a twin operator $\overline{S}(\lambda)$ such that

$$y^\diamond \overline{S}(\lambda)y = \int_0^\infty e^{-\lambda t} y^\diamond S(t)y \,\mathrm{d}t. \tag{21}$$

Note that the combination of ii) and iii) allows us to conclude that the right hand side of (21) defines a bounded bilinear map, but not that it defines a twin operator. Hence iv) is indeed an additional assumption.

We call $\overline{S}(\lambda)$ defined on $\{\lambda \mid \operatorname{Re}\lambda > \omega\}$ the *Laplace transform* of $\{S(t)\}$. It actually suffices to assume that the assertion of iv) holds for $\lambda = \lambda_0$ with $\operatorname{Re}\lambda_0 > \omega$. This assumption allows us to introduce the multi-valued operator

$$C = \lambda_0 I - \overline{S}(\lambda_0)^{-1} \tag{22}$$

on Y and next define the function $\lambda \mapsto \overline{S}(\lambda)$ by

$$\overline{S}(\lambda) = (\lambda I - C)^{-1} \tag{23}$$

on an open neighbourhood of λ_0.

In Definition 2.6 of Kunze (2009) an operator C is called the *generator* of the semigroup provided the Laplace transform is injective and hence C is single-valued. In Diekmann and Verduyn Lunel (2021) we adopted a more pliant position and call C the generator even when it is multi-valued and we refer to this paper for additional information.

Focusing on $\{S(t)\}_{t\geq 0}$ as a semigroup of bounded linear operators on Y, we now list some basic results from Kunze (2011).

Lemma 3.2 *The following statements are equivalent*

1. $y \in \mathcal{D}(C)$ *and* $z \in Cy$*;*
2. *there exist* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda > \omega$*, here* ω *is as introduced in ii) of Definition 3.1, and* $y, z \in Y$ *such that*

$$y = \overline{S}(\lambda)(\lambda y - z)$$

3. $y, z \in Y$ *and for all* $t > 0$

$$\int_0^t S(\tau)z\,\mathrm{d}\tau = S(t)y - y.$$

Here it should be noted that item 3. includes the assertions

- *the integral* $\int_0^t S(\tau)z\,\mathrm{d}\tau$ *defines an element of* Y *(even though at first it only defines an element of* $Y^{\diamond *}$*);*
- *the integral* $\int_0^t S(\tau)z\,\mathrm{d}\tau$ *does not depend on the choice of* $z \in Cy$ *in case* C *is multi-valued.*

Lemma 3.3 *For all* $t > 0$ *and* $y \in Y$*, we have* $\int_0^t S(\tau)y\,\mathrm{d}\tau \in \mathcal{D}(C)$ *and*

$$S(t)y - y \in C\int_0^t S(\tau)y\,\mathrm{d}\tau.$$

4 The Norming Dual Pair (B, NBV)

In the study of delay differential equations, the natural dual pair is given by

$$Y = B\left([-1, 0], \mathbb{R}^n\right) \quad \text{and} \quad Y^\diamond = NBV\left([0, 1], \mathbb{R}^n\right) \tag{24}$$

with the pairing

$$\langle y^\diamond, y\rangle = \int_{[0,1]} y^\diamond(\mathrm{d}\sigma)\cdot y(-\sigma) \tag{25}$$

(see Appendix A for the definition of NBV). Here Y is provided with the supremum norm and $Y^\diamond$ with the total variation norm (see (83)). See Diekmann and Verduyn Lunel 2021.

In the study of renewal equations, the natural dual pair is given by

$$Y = NBV\left([-1, 0], \mathbb{R}^n\right) \quad \text{and} \quad Y^\diamond = B\left([0, 1], \mathbb{R}^n\right)$$

with the pairing

$$\langle y^\diamond, y\rangle = \int_{[-1,0]} y(\mathrm{d}\sigma)\cdot y^\diamond(-\sigma).$$

Returning to (24)–(25), we first make two trivial, yet useful, observations: fix $1 \le i \le n$ and $-1 \le \theta \le 0$,

$$\int_{[0,1]} y^{\diamond}(\mathrm{d}\sigma) \cdot y(-\sigma) = y_i(\theta),$$

if $y_j^{\diamond}(\sigma) = 0$, $0 \leq \sigma \leq 1$, $j \neq i$, and $y_i^{\diamond}(\sigma) = 0$ for $0 \leq \sigma < -\theta$ and $y_i^{\diamond}(\sigma) = 1$ for $\sigma \geq -\theta$, and similarly

$$\int_{[0,1]} y^{\diamond}(\mathrm{d}\sigma) \cdot y(-\sigma) = y_i^{\diamond}(-\theta),$$

if $y_j(-\sigma) = 0$, $0 \leq \sigma \leq 1$, $j \neq i$, and $y_i(-\sigma) = 1$ for $0 \leq \sigma \leq -\theta$ and $y_i(-\sigma) = 0$ for $\sigma > -\theta$.

The point is that, consequently, in case of (24)–(25), convergence in both $(Y, \sigma(Y, Y^{\diamond}))$ and $(Y^{\diamond}, \sigma(Y^{\diamond}, Y))$ entails pointwise convergence (in, respectively, $B([-1, 0], \mathbb{R}^n)$ and $NBV([0, 1], \mathbb{R}^n)$).

In the first case, the dominated convergence theorem implies that, conversely, a bounded pointwise convergent sequence in $B([-1, 0], \mathbb{R}^n)$ converges in $(Y, \sigma(Y, Y^{\diamond}))$. For $NBV([0, 1], \mathbb{R}^n)$, this is not so clear. It is true that the pointwise limit of a sequence of functions of bounded variation is again of bounded variation (Helly's theorem), but there is no dominated convergence theorem for measures.

The following theorem is proved in Diekmann and Verduyn Lunel (2021, Theorem B.1).

Theorem 4.1 *The dual pair given by* (24) *and* (25) *is a norming dual pair, i.e.,*

$$\begin{aligned} \|y\| &= \sup\left\{ |\langle y^{\diamond}, y\rangle| \mid y^{\diamond} \in Y^{\diamond},\ \|y^{\diamond}\| \leq 1 \right\} \\ \|y^{\diamond}\| &= \sup\left\{ |\langle y^{\diamond}, y\rangle| \mid y \in Y,\ \|y\| \leq 1 \right\}. \end{aligned}$$

Furthermore

(i) $(Y, \sigma(Y, Y^{\diamond}))$ *is sequentially complete;*
(ii) *a linear map* $(Y, \sigma(Y, Y^{\diamond})) \to \mathbb{R}$ *is continuous if it is sequentially continuous.*

5 The Twin Semigroup Approach to NFDE

Consider the norming dual pair $(Y, Y^{\diamond})$ with Y and $Y^{\diamond}$ as given in Sect. 4 by (24).

By solving (2)–(3), see Theorem 2.3, we can define a Y-valued function $u : [0, \infty) \to Y$ by

$$u(t; \varphi) := x_t(\,\cdot\,; \varphi), \qquad t \geq 0, \tag{26}$$

where x_t is defined by (1), and bounded linear operators $S(t) : Y \to Y$ by

$$S(t)\varphi = u(t; \varphi). \tag{27}$$

The initial condition (2) translates into

$$S(0)\varphi = u(0;\varphi) = \varphi$$

and (27) reflects that we define a dynamical system on Y by translating along the function φ extended according to (2). Below we show that $\{S(t)\}$ is a twin semigroup and we characterize its generator C. But first we present some heuristics.

In order to motivate an abstract ODE for the Y-valued function u, we first observe that the infinitesimal formulation of the translation rule (26) amounts to the PDE

$$\frac{\partial u}{\partial t} - \frac{\partial u}{\partial \theta} = 0.$$

We need to combine this with (2), in terms of $u(t)(0) = x(t)$, and we have to specify the domain of definition of the derivative with respect to θ. The latter is actually rather subtle. An absolutely continuous function has almost everywhere a derivative and when the function is Lipschitz continuous this derivative is bounded. Thus a Lipschitz function specifies a unique L^∞-equivalence class by the process of differentiation. But *not* a unique element of Y. In fact the set

$$\begin{aligned} C\psi = \Big\{\psi' \in Y \mid \psi(\theta) = \psi(-1) + \int_{-1}^{\theta} \psi'(\sigma)\, d\sigma, \\ \psi'(0) - \int_0^h \mathrm{d}\eta(\theta)\psi'(-\theta) = \int_0^h \mathrm{d}\zeta(\theta)\psi(-\theta)\Big\} \end{aligned} \tag{28}$$

is, for a given Lipschitz continuous function ψ, very large indeed. Note that the boundary condition

$$\psi'(0) - \int_0^h \mathrm{d}\eta(\theta)\psi'(-\theta) = \int_0^h \mathrm{d}\zeta(\theta)\psi(-\theta)$$

takes care of (2). We define C as a multi-valued, unbounded, operator on Y by (28) with domain given by

$$\mathcal{D}(C) = Lip\left([-1,0],\mathbb{R}^n\right). \tag{29}$$

We claim that (2)–(3) and (26) correspond to an abstract differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} \in Cu.$$

To substantiate this claim, we shall verify that $\{S(t)\}_{t\geq 0}$ defined by (27) is a twin semigroup and, finally, that C is the corresponding generator in the sense of (23) where $\overline{S}(\lambda)$ is given by (21).

From the representation (19) of the solution of (2)–(3) we can derive an explicit representation of the semigroup $\{S(t)\}_{t\geq 0}$ defined by (27).

Theorem 5.1 *The semigroup* $\{S(t)\}_{t\geq 0}$ *defined by* (27) *is given by*

$$(S(t)\varphi)(\theta) = \int_0^h K_t(\theta, \mathrm{d}\sigma)\,\varphi(-\sigma) \tag{30}$$

with for $\sigma > 0$ *and* $-h \leq \theta \leq 0$ *the kernel* $K_t(\theta, \sigma)$ *defined by*

$$\begin{aligned} K_t(\theta, \sigma) &:= H(\sigma + t + \theta) + H(t + \theta)\rho(t + \theta) \\ &\quad + H(t + \theta) \int_0^{t+\theta} \mathrm{d}X(\xi)\,(\mu(t + \theta + \sigma - \xi) - \mu(\sigma))\,, \end{aligned} \tag{31}$$

and $K_t(\theta, 0) = 0$. *Here* ρ *denotes the resolvent of* μ *with* μ *defined in* (9), X *denotes the fundamental solution given by* (17)*, and* H *is the standard Heaviside function.*

Proof For $t + \theta < 0$ the second and third terms in the expression for K_t do not contribute, and the first term yields

$$(S(t)\varphi)(\theta) = \varphi(t + \theta)$$

which is in accordance with (27) because of (3).

Now assume that $t + \theta \geq 0$. Clearly the first term contributes a unit jump at $\sigma = 0$ and $H(t + \theta) = 1$. The second term has, as a function of σ, a jump of magnitude $\rho(t + \theta)$ at $\sigma = 0$, an absolutely continuous part with derivative given by

$$\int_0^{t+\theta} \mathrm{d}X(\xi)\,(\zeta(t + \theta + \sigma - \xi) - \zeta(\sigma))\,,$$

and a part of bounded variation given by

$$\int_0^{t+\theta} \mathrm{d}X(\xi)\,(\eta(t + \theta + \sigma - \xi) - \eta(\sigma))\,.$$

The jumps yield the first term at the right hand side of (19) (see also (20)) evaluated at $t + \theta$, the absolutely continuous part yields the second, and the bounded variation part the third term. □

Note that K_t is *bounded*, in the sense (cf. Kunze 2009, Definition 3.2) that for fixed θ in $[-1, 0]$ the function $\sigma \mapsto K_t(\theta, \sigma)$ is of normalized bounded variation, while for fixed $\sigma \in [0, 1]$ the function $\theta \mapsto K_t(\theta, \sigma)$ is bounded and measurable.

The next corollary is a general property of kernel operators.

Corollary 5.2 *The operator* $S(t)$ *extends to a twin operator.*

Proof The proof directly follows from the observation that we can represent the action of $y^\diamond S(t)$ explicitly as

$$\left(y^\diamond S(t)\right)(\sigma) = \int_0^h y^\diamond(\mathrm{d}\tau)\, K_t(-\tau, \sigma).$$

□

Theorem 5.3 *The semigroup* $\{S(t)\}_{t\geq 0}$ *defined by* (30) *is a twin semigroup.*

Proof With reference to Definition 3.1 we note that $S(0) = I$ follows directly from (30)–(31), while the semigroup property follows from the uniqueness of solutions to (2)–(3) and the fact that $S(t)$ corresponds to translation along the solution.

The exponential estimates (ii) are well-established in the theory of NFDE, see Sect. 9.3 of Hale and Verduyn Lunel (1993) or the proof of Proposition 7.3 below.

Property (iii), the measurability of $t \mapsto y^\diamond S(t)y$, is a direct consequence of the way $K_t(\theta, \sigma)$, defined in (31), depends on t.

It remains to verify that the Laplace transform defines a twin operator. By Fubini's Theorem, the Laplace transform is a kernel operator with kernel

$$\int_0^\infty e^{-\lambda t} K_t(\theta, \sigma)\, \mathrm{d}t.$$

□

Theorem 5.4 *The operator* C *defined by* (28) *and* (29) *is the generator (in the sense of* (23)*) of* $\{S(t)\}_{t\geq 0}$ *defined by* (30)*.*

Proof Assume $\varphi \in (\lambda I - C)\psi$. Then there exists $\psi' \in Y$ which is a.e. a derivative of ψ such that

$$\lambda\psi - \psi' = \varphi, \qquad -1 \leq \theta < 0,$$

satisfying the boundary condition

$$\lambda\psi(0) - \int_0^h \mathrm{d}\eta(\theta)\psi'(-\theta) - \int_0^h \mathrm{d}\zeta(\theta)\psi(-\theta) = \varphi(0).$$

Solving the differential equation yields that

$$\psi(\theta) = e^{\lambda\theta}\psi(0) + e^{\lambda\theta}\int_\theta^0 e^{-\lambda\sigma}\varphi(\sigma)\, \mathrm{d}\sigma \tag{32}$$

and accordingly the boundary condition for $\theta = 0$ boils down to

$$\psi(0) = \Delta(\lambda)^{-1} H(\lambda; \varphi), \tag{33}$$

where

$$H(\lambda;\varphi) := \varphi(0) + \lambda \int_0^h \mathrm{d}\eta(\sigma) e^{-\lambda\sigma} \int_{-\sigma}^0 e^{-\lambda\tau}\varphi(\tau)\,\mathrm{d}\tau + \int_0^h \mathrm{d}\zeta(\sigma) e^{-\lambda\sigma} \int_{-\sigma}^0 e^{-\lambda\tau}\varphi(\tau)\,\mathrm{d}\tau.$$

This requires that $\det \Delta(\lambda) \neq 0$ with

$$\Delta(\lambda) = \lambda \left[I - \int_0^h \mathrm{d}\eta(\sigma) e^{-\lambda\sigma} \right] + \int_0^h \mathrm{d}\zeta(\sigma) e^{-\lambda\sigma}.$$

Our claim is that the identity

$$(\lambda I - C)^{-1}\varphi = \int_0^\infty e^{-\lambda t} S(t)\varphi\,\mathrm{d}t \tag{34}$$

or, equivalently,

$$\psi(\theta) = \int_0^\infty e^{-\lambda t}\,(S(t)\varphi)\,(\theta)\,\mathrm{d}t$$

holds. To verify this, we first note that

$$\begin{aligned}\int_0^\infty e^{-\lambda t}\,(S(t)\varphi)\,(\theta)\,\mathrm{d}t &= \int_0^\infty e^{-\lambda t} x(t+\theta;\varphi)\,\mathrm{d}t \\ &= \int_0^{-\theta} e^{-\lambda t}\varphi(t+\theta)\,\mathrm{d}t + \int_{-\theta}^\infty e^{-\lambda t} x(t+\theta)\,\mathrm{d}t \\ &= e^{\lambda\theta}\int_\theta^0 e^{-\lambda\sigma}\varphi(\sigma)\,\mathrm{d}\sigma + e^{\lambda\theta}\bar{x}(\lambda;\varphi),\end{aligned}$$

where $\bar{x}(\lambda;\varphi) := \int_0^\infty e^{-\lambda t} x(t;\varphi)\,\mathrm{d}t$, with $x(t;\varphi)$ the solution of (2)–(3) given by (19). So, since (32) holds, to prove (34) it remains to check that

$$\psi(0) = \bar{x}(\lambda;\varphi).$$

By taking the Laplace transform on both sides of (11) we deduce that

$$\begin{aligned}\bar{x}(\lambda;\varphi) &= \left(1 - \int_0^\infty e^{-\lambda t}\,\mathrm{d}\mu(t)\right)^{-1} \bar{f}(\lambda) \\ &= \Delta(\lambda)^{-1}\lambda\bar{f}(\lambda),\end{aligned}$$

where $\bar{f}(\lambda) := \int_0^\infty e^{-\lambda t} f(t)\,\mathrm{d}t$. Therefore, using the representation of f in (12), it follows that

$$
\begin{aligned}
\lambda \bar{f}(\lambda) &= \varphi(0) + \int_0^\infty \lambda e^{-\lambda t} \int_0^t \left(\int_s^h \mathrm{d}\zeta(\theta)\varphi(s-\theta)\,\mathrm{d}s \right) \mathrm{d}t \\
&\quad + \lambda \int_0^\infty e^{-\lambda t} \int_t^h \mathrm{d}\eta(\theta)\varphi(t-\theta)\,\mathrm{d}t \\
&= \varphi(0) + \int_0^\infty e^{-\lambda t} \int_t^h \mathrm{d}\zeta(\theta)\varphi(t-\theta)\,\mathrm{d}t \\
&\quad + \lambda \int_0^\infty e^{-\lambda t} \int_t^h \mathrm{d}\eta(\theta)\varphi(t-\theta)\,\mathrm{d}t \\
&= \varphi(0) + \int_0^h \mathrm{d}\zeta(\theta) \int_0^\theta e^{-\lambda t}\varphi(t-\theta)\,\mathrm{d}t \\
&\quad + \lambda \int_0^h \mathrm{d}\eta(\theta) \int_0^\theta e^{-\lambda t}\varphi(t-\theta)\,\mathrm{d}t \\
&= \varphi(0) + \int_0^h \mathrm{d}\zeta(\theta)e^{-\lambda\theta} \int_{-\theta}^0 e^{-\lambda\sigma}\varphi(\sigma)\,\mathrm{d}\sigma \\
&\quad \lambda \int_0^h \mathrm{d}\eta(\theta)e^{-\lambda\theta} \int_{-\theta}^0 e^{-\lambda\sigma}\varphi(\sigma)\,\mathrm{d}\sigma \\
&= H(\lambda;\varphi).
\end{aligned}
$$

Therefore it follows from (33) that indeed $\psi(0) = \bar{x}(\lambda;\varphi)$ and this completes the proof of the identity (34). □

In Diekmann and Verduyn Lunel (2021), we proved Theorems 5.1, 5.3 and 5.4 for retarded functional differential equations, and gave an alternative proof of Theorem 5.3 in the neutral case using a relative bounded perturbation argument, see Diekmann and Verduyn Lunel (2021, Theorem 11.1).

6 The Variation-of-Constants Formula for NFDE

It is a direct consequence of (29) that

$$
X = \overline{\mathcal{D}(C)} = C\left([-1,0],\mathbb{R}^n\right).
$$

Clearly $C\psi \cap X$ is either empty or a singleton, cf. (28), and for the set to be nonempty we need that $\psi \in C^1$ and

$$
\psi'(0) - \int_0^h \mathrm{d}\eta(\theta)\psi'(-\theta) = \int_0^h \mathrm{d}\zeta(\theta)\psi(-\theta).
$$

So the generator A of the restriction $\{T(t)\}_{t\geq 0}$ of $\{S(t)\}_{t\geq 0}$ to X is given by

$$\mathcal{D}(A) = \left\{ \psi \in C^1 \mid \psi'(0) - \int_0^h \mathrm{d}\eta(\theta)\psi'(-\theta) = \int_0^h \mathrm{d}\zeta(\theta)\psi(-\theta) \right\}$$
$$A\psi = \psi'$$

in complete agreement with the standard theory.

As $S(t)$ maps Y into X for $t \geq 1$, one might wonder whether we gained anything at all by the extension from X to Y? Already in the pioneering work of Jack Hale he emphasized that if one adds a forcing term to (2), one needs

$$q(\theta) := \begin{cases} 1 & \text{for } \theta = 0, \\ 0 & \text{for } -1 \leq \theta < 0, \end{cases}$$

to describe the solution by way of the variation-of-constants formula.

Indeed, the solution of

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left[x(t) - \int_0^h \mathrm{d}\eta(\theta)x(t-\theta) \right] &= \int_0^h \mathrm{d}\zeta(\theta)x(t-\theta) + f(t), \qquad t \geq 0, \\ x(\theta) &= \varphi(\theta), \qquad -1 \leq \theta \leq 0, \end{aligned} \tag{35}$$

is explicitly given by

$$x_t = S(t)\varphi + \int_0^t S(t-\tau)qf(\tau)\,\mathrm{d}\tau, \tag{36}$$

where $S(t)$ is given by (30) and x_t is as defined in (1). This formally follows directly from the fact that the inhomogeneous NFDE (35) corresponds to the initial value problem

$$\frac{\mathrm{d}u}{\mathrm{d}t} \in Cu + qf, \qquad u(0) = \varphi,$$

where as before $u(t) = x_t$. Note that the solution with initial condition q is the so-called fundamental solution, cf. (18) and (17).

The integration theory presented next provides a precise underpinning of the integral in (36) and the remainder of this section is devoted to a proof of (36). In the original approach of Hale, the hidden argument θ in (36) is inserted and thus the integral reduces to the integration of an $\mathbb{R}^n$-valued function. Note that evaluation in a point corresponds to the application of a Dirac functional, so our approach yields, in a sense, a theoretical underpinning of Hale's approach.

As a final remark, we emphasize that the variation-of-constants formula (36) is the key first step towards a local stability and bifurcation theory for nonlinear problems, as shown in detail in Diekmann et. al. (1995) for retarded functional differential equations. For neutral functional differential equations this is work in progress.

Motivated by (36), we want to define an element $u(t)$ of Y by way of the action on $Y^\diamond$ expressed in the formula

$$\langle y^\diamond, u(t)\rangle = y^\diamond S(t)u_0 + \int_0^t y^\diamond S(t-\tau)q\, f(\tau)\,\mathrm{d}\tau, \tag{37}$$

where the standard assumptions are

(i) $(Y, Y^\diamond)$ is a norming dual pair;
(ii) $q \in Y$;
(iii) $f : [0, T] \to \mathbb{R}$ is bounded and measurable;
(iv) $\{S(t)\}$ is a twin semigroup,

and where u_0 (corresponding to φ in (36)) is an arbitrary element of Y.

The definition of the first term at the right hand side of (37) is no problem at all, it contributes $S(t)u_0$ to $u(t)$. The second term defines an element of $Y^{\diamond *}$, but it is not clear that this element is, without additional assumptions, represented by an element of Y. The following lemma provides a sufficient condition.

Lemma 6.1 *In addition to (i)–(iv) assume that*

$$\big(Y, \sigma(Y, Y^\diamond)\big) \quad \textit{is sequentially complete.} \tag{38}$$

Then

$$y^\diamond \mapsto \int_0^t y^\diamond S(t-\tau)q\, f(\tau)\,\mathrm{d}\tau \tag{39}$$

is represented by an element of Y, to be denoted as $\int_0^t S(t-\tau)q\, f(\tau)\,\mathrm{d}\tau$.

Proof There exists a sequence of step functions f_m such that $|f_m| \le |f|$ and $f_m \to f$ pointwise. Lemma 3.3 shows that

$$\int_0^t S(t-\tau)q\, f_m(\tau)\,\mathrm{d}\tau$$

belongs to Y (in fact even to $\mathcal{D}(C)$). Since (see Definition 3.1(ii))

$$\left|y^\diamond S(t-\tau)q f_m(\tau)\right| \le M e^{\omega(t-\tau)}\|q\|\|y^\diamond\| \sup_\sigma |f(\sigma)|,$$

the dominated convergence theorem implies that for every $y^\diamond \in Y^\diamond$

$$\lim_{m\to\infty} \int_0^t y^\diamond S(t-\tau)q\, f_m(\tau)\,\mathrm{d}\tau = \int_0^t y^\diamond S(t-\tau)q\, f(\tau)\,\mathrm{d}\tau.$$

The sequential completeness next guarantees that the limit too is represented by an element of Y. □

In Diekmann and Verduyn Lunel (2021) we have developed a perturbation theory to study neutral equations directly as an unbounded perturbation of a retarted equation. In order to do this, we have to replace $f(\tau)\,\mathrm{d}\tau$ by $F(\mathrm{d}\tau)$ with F of bounded variation. In this setting the approximation by step functions used in the proof of Lemma 6.1 no longer works. This observation motivates to look for an alternative sufficient condition to replace (38). This is taken care of in the following lemma.

Lemma 6.2 *In addition to (i)–(iv) assume that*

$$\begin{aligned}&\textit{a linear map } \big(Y^\diamond, \sigma(Y^\diamond, Y)\big) \to \mathbb{R} \textit{ is continuous}\\ &\textit{if it is sequentially continuous.}\end{aligned} \tag{40}$$

Then the assertion of Lemma 6.1 holds.

Proof Again we are going to make use of the dominated convergence theorem. Consider a sequence $\{y_m^\diamond\}$ in $Y^\diamond$ such that for every $y \in Y$ the sequence $\langle y_m^\diamond, y\rangle$ converges to zero in $\mathbb{R}$. Then for all relevant t and τ we have

$$\lim_{m\to\infty} y_m^\diamond S(t-\tau)q = 0$$

and consequently

$$\lim_{m\to\infty} \int_0^t y_m^\diamond S(t-\tau)q\, f(\tau)\,\mathrm{d}\tau = 0.$$

So the linear map (39) is, in the sense described in (40), sequentially continuous and therefore, by the assumption, continuous. Since

$$\big(Y^\diamond, \sigma(Y^\diamond, Y)\big)' = Y,$$

we conclude that (39) is represented by an element of Y. □

We are going to use the above results to show that a certain type of perturbation of a twin semigroup $\{S(t)\}$ yields again a *twin* semigroup. In order to do this we need a dual version of (37), i.e., we want to define an element $u^\diamond(t)$ of $Y^\diamond$ by way of the action on Y expressed in the formula

$$\langle u^\diamond(t), y\rangle = u_0^\diamond S(t)y + \int_0^t q^\diamond S(t-\tau)y\, f(\tau)\,\mathrm{d}\tau, \tag{41}$$

where the standard assumptions are as before with (ii) replaced by (ii)$'$, i.e.,

(i) $(Y, Y^\diamond)$ is a norming dual pair;
(ii)$'$ $q^\diamond \in Y^\diamond$;
(iii) $f : [0, T] \to \mathbb{R}$ is bounded and measurable;
(iv) $\{S(t)\}$ is a twin semigroup,

and where $u_0^\diamond$ is an arbitrary element of $Y^\diamond$. This implies that

$$y \mapsto \int_0^t q^\diamond S(t-\tau)y\, f(\tau)\,\mathrm{d}\tau \tag{42}$$

is represented by an element of $Y^\diamond$, to be denoted as $\int_0^t q^\diamond S(t-\tau)\, f(\tau)\,\mathrm{d}\tau$.

Applying the two lemmas above, with the role of Y and $Y^\diamond$ interchanged, we find that this is indeed the case if either

$$\big(Y^\diamond, \sigma(Y^\diamond, Y)\big) \quad \text{is sequentially complete} \tag{43}$$

or

$$\begin{aligned}&\text{a linear map } \big(Y, \sigma(Y, Y^\diamond)\big) \to \mathbb{R} \text{ is continuous}\\ &\text{if it is sequentially continuous.}\end{aligned} \tag{44}$$

Therefore to develop a perturbation theory for twin semigroups we need both (39) and (42) to be represented by elements from, respectively, Y and $Y^\diamond$. This motivates the following definition.

Definition 6.3 We say that a norming dual pair $(Y, Y^\diamond)$ is *suitable for twin perturbation* if

(a) at least one of (38) and (40) holds; and
(b) at least one of (43) and (44) holds

Recall from Theorem 4.1 that for the norming dual pair (B, NBV) introduced in Sect. 4 we have that (38) and (44) are satisfied. This shows that the norming dual pair (B, NBV) is suitable for twin perturbation.

We are now ready to give a rigorous proof of the variation-of-constants formula for NFDE.

Theorem 6.4 *The solution of the inhomogeneous NFDE* (35) *can be represented by the variation-of-constants formula* (36)*, i.e.,*

$$x_t = S(t)\varphi + \int_0^t S(t-\tau) q f(\tau)\,\mathrm{d}\tau,$$

where $S(t)$ is the twin semigroup given by (30).

Proof It follows from Theorem 4.1 that

$$Y = B([-1, 0]; \mathbb{R}^n) \quad \text{and} \quad Y^\diamond = NBV([0, 1]; \mathbb{R}^n)$$

is a norming dual pair suitable for twin perturbation. Therefore the claim follows by applying Lemma 6.1 with respect to the norming dual pair (B, NBV) and Lemma 6.2 with respect to the norming dual pair (NBV, B). □

In the treatment of renewal equations in Diekmann and Verduyn Lunel (2021) we assumed (43) and (40). In fact for delay differential equations we take as normal dual pair $(Y, Y^\diamond)$ with $Y = B([-1, 0])$ and $Y^\diamond = NBV([0, 1])$, while for renewal equations we take $(Y, Y^\diamond)$ with $Y = NBV([-1, 0])$ and $Y^\diamond = B([0, 1])$.

7 Bounded Time-Dependent Perturbation of a Twin Semigroup

In this section we assume

- $(Y, Y^\diamond)$ is a norming dual pair that is suitable for twin perturbation, cf. Definition 6.3;
- $\{S_0(t)\}$ is a twin semigroup on $(Y, Y^\diamond)$ with generator C_0;
- For $j = 1, \dots, n$ the elements $q_j \in Y$ and $t \mapsto q_j^\diamond(t) \in Y^\diamond$ are given.

Definition 7.1 A two-parameter family $U = \{U(t,s)\}_{t \geq s}$ of twin operators on a norming dual pair $(Y, Y^\diamond)$ is called a *twin evolutionary system* if

(i) $U(s,s) = I$ and $U(t,s) = U(t,r)U(r,s)$ for $s \leq r \leq t$
(ii) there exist constants $M \geq 1$ and $\omega_0 \in \mathbb{R}$ such that for all $y \in Y$, $y^\diamond \in Y^\diamond$

$$|y^\diamond U(t,s)y| \leq M e^{\omega_0 (t-s)} \|y\| \| y^\diamond \|, \qquad t \geq s;$$

(iii) Let the set $\triangle \subset \mathbb{R}^2$ be defined by $\triangle = \{(t,s) \mid -\infty < s \leq t < \infty\}$. For all $y \in Y$, $y^\diamond \in Y^\diamond$ the function

$$\triangle \ni (t,s) \mapsto y^\diamond U(t,s) y \in \mathbb{R}$$

is measurable.

Our aim is to define constructively a twin evolutionary system $\{U(t,s)\}$ corresponding to the abstract multi-valued differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} \in C(t)u, \qquad t \geq s, \qquad u(s) \text{ given,} \tag{45}$$

with

$$\mathcal{D}(C(t)) = \mathcal{D}(C_0), \qquad C(t)y = C_0 y + \sum_{j=1}^{n} \langle q_j^\diamond(t), y \rangle q_j. \tag{46}$$

The first step is to introduce a $n \times n$-matrix-valued function $k(t,s)$ on $\mathbb{R} \times \mathbb{R}$ via $k(t,s) = 0$ for $-\infty < t \leq s < \infty$ and

$$k_{ij}(t,s) := q_i^\diamond(t) S_0(t-s) q_j, \qquad -\infty < s \leq t < \infty. \tag{47}$$

Note that for each pair c_1, c_2 with $-\infty < c_1 < c_2 < \infty$ and for each $f \in L^1([c_1, c_2]; \mathbb{R}^n)$, we have

$$\sup_{\|f\| \leq 1} \int_{c_1}^{c_2} \left(\int_{c_1}^{c_2} \|k(t,s) f(s)\| \, \mathrm{d}s \right) \mathrm{d}t < \infty.$$

Here $\|f\|$ denotes the norm of f as function belonging to $L^1([c_1, c_2]; \mathbb{R}^n)$ and the map

$$f \mapsto \int_{c_1}^{t} k(t,s) f(s)\,\mathrm{d}s, \quad c_1 \le t \le c_2,$$

defines a bounded linear operator on $L^1([c_1, c_2]; \mathbb{R}^n)$ which we shall denote by K.

The linear space of lower triangular kernel functions on $[c_1, c_2] \times [c_1, c_2]$ of type L^1_{loc} endowed with the norm

$$\begin{aligned} \|k\|_1 &:= \sup_{\|f\|\le 1} \int_{c_1}^{c_2} \left(\int_{c_1}^{c_2} \|k(t,s) f(s)\|\,\mathrm{d}s \right) \mathrm{d}t \\ &= \operatorname*{ess\,sup}_{s\in[c_1,c_2]} \int_{c_1}^{c_2} \|k(t,s)\|\,\mathrm{d}t \end{aligned} \tag{48}$$

is a Banach space (see Theorem 9.2.4 and Proposition 9.2.7 of Grippenberg et. al. 1990) which we will denote by $L^1_+\left([c_1, c_2] \times [c_1, c_2]; \mathbb{R}^{n\times n}\right)$.

Now let k be a lower triangular kernel function of type L^1_{loc}. We call an $n \times n$-matrix-function $r(t,s)$ a *resolvent kernel function of* k if $r(t,s)$ is a lower triangular kernel function of type L^1_{loc} and

$$r(t,s) = k(t,s) + \int_s^t r(t,a) k(a,s)\,\mathrm{d}a, \qquad -\infty < s \le t < \infty, \tag{49}$$

$$= k(t,s) + \int_s^t k(t,a) r(a,s)\,\mathrm{d}a, \qquad -\infty < s \le t < \infty. \tag{50}$$

Define the integral operator R similar as the operator K but with the kernel $k(t,s)$ replaced by $r(t,s)$, i.e.,

$$(Rf)(t) := \int_{c_1}^{t} r(t,s) f(s)\,\mathrm{d}s, \qquad c_1 \le t \le c_2.$$

Using the integral operators K and R, it follows from the identity (50) that for $c_1 < t < c_2$ we have

$$\begin{aligned} (KRf)(t) &= \int_{c_1}^{t} k(t,s)(Rf)(s)\,\mathrm{d}s \\ &= \int_{c_1}^{t} k(t,s) \left(\int_0^s r(s,\tau) f(\tau)\,\mathrm{d}\tau \right) \mathrm{d}s \\ &= \int_{c_1}^{t} \left(\int_{\tau}^{t} k(t,s) r(s,\tau)\,\mathrm{d}s \right) f(\tau)\,\mathrm{d}\tau \\ &= \int_{c_1}^{t} (r(t,\tau) - k(t,\tau))\, f(\tau)\,\mathrm{d}\tau \\ &= (Rf)(t) - (Kf)(t), \quad c_1 \le t \le c_2. \end{aligned}$$

It follows that $KR = R - K$. Similarly, using (49), we have $RK = R - K$. This yields $KR = RK$ and

$$(I - K)(I + R) = (I + R)(I - K) = I, \tag{51}$$

where I is the identity operator on $L^1([c_1, c_2]; \mathbb{R}^n)$. Thus $I - K$ is an invertible operator on $L^1([c_1, c_2]; \mathbb{R}^n)$, and its inverse is given by $I + R$.

Theorem 7.2 *If $k(t, s)$ is a lower triangular kernel function of type L^1_{loc}, then $k(t, s)$ has a unique resolvent kernel function $r(t, s)$ of type L^1_{loc}. In particular, the integral equation $x = Kx + f$ has a unique solution given by $x = f + Rf$.*

Proof The proof will be done in three steps. Throughout $k(t, s)$ is a lower triangular kernel function of type L^1_{loc}.

STEP 1. First note that if k_1 and k_2 are lower triangular kernel functions on $\mathbb{R} \times \mathbb{R}$, then the same holds true for the functions

$$(t, s) \mapsto \int_s^t k_1(t, a)k_2(a, s)\, da \quad \text{and} \quad (t, s) \mapsto \int_s^t k_2(t, a)k_1(a, s)\, da.$$

Furthermore, from the discussion in the paragraph preceding the present theorem it follows that a resolvent kernel function of type L^1_{loc} is unique whenever it exists.

STEP 2. Because of uniqueness of the resolvent kernel of type L^1_{loc}, it suffices to prove existence of a resolvent kernel on $[c_1, c_2]$ for every $c_1, c_2 \in (0, \infty)$ with $c_1 < c_2$. Assume first that $\|k\|_1 \leq 1$ with $\|k\|_1$ given by (48), then the map

$$r(t, s) \mapsto \int_s^t k(t, a)r(a, s)\, \mathrm{d}a + k(t, s)$$

is a contraction on $L^1_+\left([c_1, c_2] \times [c_1, c_2]; \mathbb{R}^{n\times n}\right)$. This shows that (50) (and, using (51), similarly (49)) has a unique solution, and this solution is a resolvent kernel of type L^1_{loc}.

STEP 3. Since $k(t, s)$ is a lower triangular kernel function of type L^1_{loc}, we define a scaled lower triangular kernel function of type L^1_{loc} by

$$\widehat{k}(t, s) := e^{-\gamma(t-s)}k(t, s).$$

Since the norm of $\widehat{k}$ is defined by (see (48))

$$\|\widehat{k}\|_1 := \operatorname*{ess\,sup}_{s\in[c_1,c_2]} \int_{c_1}^{c_2} \|\widehat{k}(t, s)\|\, \mathrm{d}t = \operatorname*{ess\,sup}_{s\in[c_1,c_2]} \int_{c_1}^{c_2} e^{-\gamma(t-s)}\|k(t, s)\|\, \mathrm{d}t,$$

we can choose γ so large that $\|\widehat{k}\|_1 < 1$. From Step 2, it follows that the equation

$$\widehat{r}(t, s) = \widehat{k}(t, s) + \int_s^t \widehat{k}(t, a)\widehat{r}(a, s)\, \mathrm{d}a$$

has a unique solution $\widehat{r} \in L^1_+\left([c_1, c_2] \times [c_1, c_2]; \mathbb{R}^{n\times n}\right)$. Therefore, we have

$$\widehat{r}(t,s) = e^{-\gamma(t-s)}k(t,s) + \int_s^t e^{-\gamma(t-a)}k(t,a)\widehat{r}(a,s)\,\mathrm{d}a,$$

and hence

$$e^{\gamma(t-s)}\widehat{r}(t,s) = k(t,s) + \int_s^t k(t,a)e^{\gamma(a-s)}\widehat{r}(a,s)\,\mathrm{d}a.$$

Thus

$$r(t,s) = k(t,s) + \int_s^t k(t,a)r(a,s)\,\mathrm{d}a,$$

where $r(t,s) = e^{\gamma(t-s)}\widehat{r}(t,s)$. This completes the proof. □

Proposition 7.3 *If $k(t,s)$ is a lower triangular kernel function that satisfies the estimate $\|k(t,s)\| \le m(t)$ for $0 \le s \le t$ and $r(t,s)$ denotes the corresponding resolvent kernel function, then*

$$\|r(t,s)\| \le m(t)\exp\left[\int_s^t m(\sigma)\,\mathrm{d}\sigma\right], \quad 0 \le s \le t < \infty.$$

Proof From the estimate $\|k(t,s)\| \le m(t)$ for $0 \le s \le t$ we obtain the following integral inequality for the function $u(t,s) := \|r(t,s)\|$ on $0 \le s \le t$:

$$u(t,s) \le m(t) + m(t)\int_s^t u(a,s)\,\mathrm{d}a, \quad 0 \le s \le t < \infty. \tag{52}$$

Now fix $s \in [0,\infty)$, and put

$$q(t) := \exp\left[-\int_s^t m(\sigma)\,d\sigma\right]\int_s^t u(a,s)\,\mathrm{d}a. \quad t \ge s. \tag{53}$$

Differentiation of q with respect to t yields

$$\begin{aligned}\frac{\mathrm{d}q}{\mathrm{d}t}(t) &= -m(t)q(t) + \exp\left[-\int_s^t m(\sigma)\,\mathrm{d}\sigma\right]u(t,s)\\ &= \left(u(t,s) - m(t)\int_s^t u(a,s)\,\mathrm{d}a\right)\exp\left[-\int_s^t m(\sigma)\,\mathrm{d}\sigma\right]\\ &\le m(t)\exp\left[-\int_s^t m(\sigma)\,\mathrm{d}\sigma\right],\end{aligned}$$

where we have used (52). Integration from s to t yields the inequality

$$q(t) \leq \int_s^t m(a) \exp\left[-\int_s^a m(\sigma)\,\mathrm{d}\sigma\right] \mathrm{d}a = 1 - \exp\left[-\int_s^t m(\sigma)\,\mathrm{d}\sigma\right].$$

Together with the definition of q in (53) we arrive at

$$\begin{aligned} m(t)\int_s^t u(a,s)\,\mathrm{d}a &= m(t)\exp\left[\int_s^t m(\sigma)\,\mathrm{d}\sigma\right] q(t) \\ &\leq -m(t) + m(t)\exp\left[\int_s^t m(\sigma)\,\mathrm{d}\sigma\right]. \end{aligned}$$

Substitution into (52) yields

$$u(t,s) \leq m(t)\exp\left[\int_s^t m(\sigma)\,\mathrm{d}\sigma\right], \quad 0 \leq s \leq t < \infty,$$

which completes the proof. □

In the context of the variation-of-constants spirit (46) motivates us to presuppose that $U(t,s)$ and $S_0(t)$ should be related to each other by the equation

$$U(t,s) = S_0(t-s) + \int_s^t S_0(t-\tau)B(\tau)U(\tau,s)\,\mathrm{d}\tau, \qquad t \geq s, \tag{54}$$

where

$$B(t)y := \sum_{j=1}^n \langle q_j^\diamond(t), y\rangle q_j, \qquad t \geq s. \tag{55}$$

By letting $B(t)$ act on (54) we obtain, for a given initial point $y \in Y$, a finite dimensional renewal equation.

To derive this renewal equation, we first write (55) as

$$B(t)y = \langle q^\diamond(t), y\rangle \cdot q, \qquad t \geq s, \tag{56}$$

where $t \mapsto q^\diamond(t)$ is the n-vector-valued function with $Y^\diamond$-valued components $q_j^\diamond(t)$ and q is the n-vector-valued with Y-valued components q_j. Here we use $\cdot$ to denote the inner product in $\mathbb{R}^n$.

We can factor (a rank factorization) B as $B = B_2B_1$ with $B_1 : Y \to \mathbb{R}^n$ and $B_2 : \mathbb{R}^n \to Y$ defined by

$$B_1(t)y := \langle q^\diamond(t), y\rangle, \qquad B_2x := \sum_{j=1}^n x_j q_j, \qquad t \geq s. \tag{57}$$

Now let (54) act on $y \in Y$ and use (56) to obtain

$$U(t,s)y = S_0(t-s)y + \int_s^t S_0(t-\tau)q^\diamond(\tau)U(\tau,s)y \cdot q\,\mathrm{d}\tau, \quad t \ge s. \tag{58}$$

Next act on both sides of (58) with the operator $B_1(t)$ as defined in (57) to arrive at

$$v(t,s)y = q^\diamond(t)S_0(t-s)y + \int_s^t k(t,\tau)v(\tau,s)y\,\mathrm{d}\tau, \qquad t \ge s, \tag{59}$$

where

$$v(t,s)y := B_1(t)U(t,s)y = q^\diamond(t)U(t,s)y, \qquad t \ge s,$$

and the lower triangular kernel function $k(t,s)$ is given by (47). Using Theorem 7.2 we can express the solution of (59) in terms of the resolvent $r(t,s)$ of the kernel $k(t,s)$ and the forcing function $t \mapsto q^\diamond(t)S_0(t-s)y$ by the formula

$$v(t,s)y = q^\diamond(t)S_0(t-s)y + \int_s^t r(t,\tau)q^\diamond(\tau)S_0(\tau-s)y\,\mathrm{d}\tau, \quad t \ge s. \tag{60}$$

And now that the function $v(t,s)y$, representing $q^\diamond(t)U(t,s)y$, can be considered as known, Eq. (54) becomes an explicit formula for $U(t,s)$:

$$U(t,s) = S_0(t-s) + \int_s^t S_0(t-\tau)q \cdot v(\tau,s)\,\mathrm{d}\tau, \qquad t \ge s. \tag{61}$$

Please note that, with this definition of $U(t,s)$, we do indeed have that

$$v(t,s)y = q^\diamond(t)U(t,s)y$$

(compare (61) to (59)).

Formula (61) is well suited for proving, on the basis of Lemma 6.1 or Lemma 6.2, that $U(t,s)$ maps Y into Y. But not for proving that $U(t,s)$ maps $Y^\diamond$ into $Y^\diamond$. So even though this may seem superfluous, we now provide an alternative dual constructive definition starting from the following equation

$$U(t,s) = S_0(t-s) + \int_s^t U(t,\tau)B(\tau)S_0(\tau-s)\,\mathrm{d}\tau, \qquad t \ge s, \tag{62}$$

which is the variant of (54) in which the roles of $U(t,s)$ and $S_0(t)$ are interchanged. Let (62) act (from the right) on $y^\diamond \in Y^\diamond$ and next let the resulting identity act on the vector q. Using (56) this yields the equation

$$y^\diamond w(t,s) = y^\diamond S_0(t-s)q + \int_s^t y^\diamond w(t,\tau)k(\tau,s)\,\mathrm{d}\tau, \qquad t \ge s, \tag{63}$$

where $y^\diamond w(t,s) := y^\diamond U(t,s)q$ and $k(t,s)$ is given by (47). The formula

$$y^\diamond w(t,s) = y^\diamond S_0(t-s)q + \int_s^t y^\diamond S_0(t-\tau)q\, r(\tau,s)\,\mathrm{d}\tau, \quad t \geq s, \tag{64}$$

expresses the solution of (63) in terms of the forcing function in (63) and the resolvent $r(t,s)$ of the kernel $k(t,s)$. Next use (56) to rewrite (62) in the form

$$U(t,s) = S_0(t-s) + \int_s^t w(t,\tau) \cdot q^\diamond S_0(\tau - s)\,\mathrm{d}\tau, \qquad t \geq s. \tag{65}$$

Please note that indeed $y^\diamond w(t,s) = y^\diamond U(t,s)q$ (compare (65) to (63)).

Of course we should now verify that the integrals in (61) and (65) do indeed define the same object. Writing the integral in (61) as $w_0 * v$ and the integral in (65) as $w * v_0$, equality follows from (60) written in the form

$$v = v_0 + r * v_0$$

and (64) written in the form

$$w = w_0 + w_0 * r$$

since

$$\begin{aligned} w_0 * v &= w_0 * (v_0 + r * v_0) = w_0 * v_0 + w_0 * r * v_0 \\ &= (w_0 + w_0 * r) * v_0 = w * v_0. \end{aligned}$$

Before we can prove Theorem 7.5 below we first need an auxiliary result.

Lemma 7.4 *The solution $v(t,s)y$ of* (59) *has the property*

$$v(t,s)y = v(t,r)U(r,s)y, \qquad t \geq r \geq s. \tag{66}$$

Proof From (59) it follows that

$$\begin{aligned} v(t,s)y = q^\diamond(t)S_0(t-r)S_0(r-s)y &+ \int_s^r k(t,\tau)v(\tau,s)y\,\mathrm{d}\tau \\ &+ \int_r^t k(t,\tau)v(\tau,s)y\,\mathrm{d}\sigma, \qquad t \geq r \geq s, \end{aligned}$$

and, by uniqueness, (66) follows provided the following identity holds

$$q^\diamond(t)S_0(t-r)S_0(r-s)y + \int_s^r k(t,\tau)v(\tau,s)y\,\mathrm{d}\tau = q^\diamond(t)S_0(t-r)U(r,s)y.$$

Recall from (47) that

$$k(t,s) = q^\diamond S_0(t-s)q = q^\diamond S_0(t-r)S_0(r-s)q, \qquad t \geq r \geq s,$$

so we conclude from (61) that this identity does indeed hold. ☐

Theorem 7.5 *Equation* (61) *in combination with* (60)*, or Eq.* (65) *in combination with* (64)*, defines a twin evolutionary system* $\{U(t,s)\}$ *corresponding to the abstract differential equation* (45)*.*

Proof Fix $t \geq s$. Since $(Y, Y^\diamond)$ is suitable for twin perturbation, we can use (61) and either Lemma 6.1 or Lemma 6.2 to deduce that $U(t,s)$ maps Y into Y. Similarly we can use (65) and the observation concerning (42) to deduce that $U(t,s)$ maps $Y^\diamond$ into $Y^\diamond$. So $U(t,s)$ is a twin operator.

Next we use Lemma 7.4 to derive the property

$$U(t,s) = U(t,r)U(r,s), \qquad t \geq r \geq s, \tag{67}$$

To verify (67), we start from (61) and use Lemma 7.4 to write

$$\begin{aligned}
U(t,s)y &= S_0(t-r)\left[S_0(r-s)y + \int_s^r S_0(r-\tau)q \cdot v(\tau,s)y\,d\tau\right] \\
&\quad + \int_r^t S_0(t-\tau)q \cdot v(\tau,r)U(r,s)y\,d\tau \\
&= S_0(t-r)U(r,s)y + \int_r^t S_0(t-\tau)q \cdot v(\tau,r)U(r,s)y\,d\tau \\
&= U(t,r)U(r,s)y.
\end{aligned}$$

Both the property $S(s,s) = I$ and the measurability, for all $y \in Y$, $y^\diamond \in Y^\diamond$, of $t \mapsto y^\diamond S(t)y$ follow from (61) and the corresponding properties of $\{S_0(t)\}$.

Finally, the exponential estimate for $y^\diamond S_0(t)y$ yields exponential estimates for both the kernel $k(t,s)$ and the forcing function $t \mapsto q^\diamond(t)S_0(t-s)y$, $t \geq s$, in the renewal equation (59). Therefore, using Proposition 7.3 we obtain an exponential estimate for the resolvent $r(t,s)$ and hence via (60) an exponential bound for $v(t,s)y$. Finally, using (61) we obtain an exponential bound for $y^\diamond U(t,s)y$ for $t \geq s$.

This completes the proof of Theorem 7.5. ☐

8 A Perturbation Approach Towards Periodic NFDE

We shall be dealing with linear periodic neutral functional differential equations of the following type:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left[x(t) - \displaystyle\int_0^h [\mathrm{d}\eta(\tau)]x(t-\tau)\right] = \displaystyle\int_0^h [\mathrm{d}_\tau\zeta(t,\tau)]x(t-\tau), \ t \geq s, \\ x(s+\theta) = \varphi(\theta), \qquad -h \leq \theta \leq 0. \end{cases} \tag{68}$$

Here d_τ denotes integration with respect to the τ variable and φ is a given function in $B([-h, 0], \mathbb{R}^n)$. Throughout we assume that for each $t \in \mathbb{R}$ the functions η and $\zeta(t, \cdot)$ are $n \times n$ matrices of which the entries are real functions of bounded variation on $[0, h]$ and continuous from the left on $(0, h)$, and $\eta(0) = \zeta(t, 0) = 0$. Moreover, it is assumed that there is a nondecreasing bounded function $m \in L^1_{\mathrm{loc}}[-h, \infty)$ such that

$$\mathrm{Var}_{[-h,0]}\, \zeta(t, \cdot) \leq m(t), \qquad t \geq 0.$$

Theorem 8.1 *Under the above conditions, Eq.* (68) *defines a well-posed dynamical system, that is, Eq.* (68) *has a unique solution* x *on* $[0, \infty)$ *such that* $x_t \in B([-h, 0], \mathbb{R}^n)$ *for* $t \geq 0$.

The above theorem is an extension of Theorem 6.1.1 in Hale and Verduyn Lunel (1993) to the neutral case. In this section we shall derive Theorem 8.1 as a corollary of Theorem 7.5 using the perturbation approach developed in the previous section.

Consider as the unperturbed problem the special case $\zeta = 0$ in (68). Let y denote the solution of the autonomous NFDE

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left[y(t) - \displaystyle\int_0^h [\mathrm{d}\eta(\tau)]y(t-\tau)\right] = 0, \quad t \geq 0, \\ y(\theta) = \varphi(\theta), \qquad -h \leq \theta \leq 0. \end{cases} \tag{69}$$

From the theory developed in Sect. 2, it follows that the solution y of (69) satisfies the autonomous renewal equation

$$y(t) - \int_0^t \mathrm{d}\eta(\theta)y(t-\theta) = f_0(t), \quad t \geq s, \tag{70}$$

where

$$f_0(t) := \varphi(0) - \int_0^h \mathrm{d}\eta(\theta)\varphi(-\theta) + \int_t^h \mathrm{d}\eta(\theta)\varphi(t-\theta), \quad t \geq s. \tag{71}$$

The solution of (70) is given by

$$y(t) = f_0(t) + \int_0^t \mathrm{d}\rho_0(\theta) f_0(t-\theta), \quad t \geq 0, \tag{72}$$

where ρ_0 denotes the resolvent of η, i.e., it satisfies the resolvent equation

$$\rho_0 = \eta * \rho_0 + \eta, \tag{73}$$

see Theorem 2.2. Denote by $X(t) = I + \rho_0(t)$ the fundamental matrix solution of (69) so that we can write the solution y given by (72) as

$$y(t) = \int_0^t \mathrm{d}X(\tau) f_0(t-\tau), \quad t \geq 0. \tag{74}$$

It follows from Theorem 5.3 that the semigroup $\{S_0(t)\}$ defined by translation along the solution of (69), i.e.,

$$(S_0(t)\varphi)(\theta) = y(t+\theta; \varphi), \quad -h \leq \theta \leq 0, \quad t \geq 0,$$

is a twin semigroup.

Define for $i = 1, \ldots, n$ elements $q_i \in Y$ and functions $t \mapsto q_i^\diamond(t) \in Y^\diamond$ by

$$q_i(\theta) := \begin{cases} 0 & \text{for } -h \leq \theta < 0, \\ e_i & \text{for } \theta = 0, \end{cases} \tag{75}$$

where e_i is the i-th unit vector in $\mathbb{R}^n$ and the maps $t \mapsto q_i^\diamond(t)$ are defined by

$$\left(q_i^\diamond(t)\right)(\theta) := \zeta_i(t, \theta), \quad -h \leq \theta \leq 0, \quad t \geq 0, \tag{76}$$

where ζ_i is the i-th row of the $n \times n$-matrix-valued function ζ.

For the matrix kernel $k(t, s)$ introduced in (47) we have, using (75) and (76), the representation

$$\begin{aligned} k_{ij}(t, s) &= q_i^\diamond(t) S_0(t-s) q_j \\ &= \int_0^{t-s} \mathrm{d}_\tau \zeta_i(t, \tau) X_j(t-s-\theta), \quad t \geq s, \end{aligned} \tag{77}$$

where $X_j(t)$ is the j-th column of the fundamental matrix solution $X(t)$. Furthermore for $\varphi \in Y$, using (76),

$$\begin{aligned} q^\diamond(t) S_0(t-s)\varphi &= q^\diamond(t) y(t-s; \varphi) \\ &= \int_0^h \mathrm{d}\zeta(t, \theta) y(t-s-\theta; \varphi), \quad t \geq s. \end{aligned}$$

Let $v(t, s)\varphi$ be the solution to the renewal equation (59), i.e.,

$$v(t, s)\varphi = q^\diamond(t) S_0(t-s)\varphi + \int_s^t k(t, \tau) v(\tau, s)\varphi \, \mathrm{d}\tau, \quad t \geq s,$$

where the kernel $k(t, s)$ is given by (77). We claim that the solution $v(t, s)\varphi$ is given by

$$v(t, s)\varphi = \int_0^h \mathrm{d}\zeta(t, \theta) x(t-\theta; \varphi), \quad t \geq s, \tag{78}$$

where $x(\cdot; \varphi)$ is the solution of (68).

Define

$$\bar{v}(t,s)\varphi := \int_0^h \mathrm{d}\zeta(t,\theta)x(t-\theta;\varphi). \tag{79}$$

To prove that $v(t,s) = \bar{v}(t,s)$ it suffices to show that $\bar{v}(t,s)\varphi$ is also a solution of the renewal equation (59).

Let $x(\cdot;\varphi)$ be the solution of (68). Similar as before we can rewrite equation (68) to obtain that x is a solution of the renewal equation

$$x(t) - \int_0^t \mathrm{d}\eta(\theta)x(t-\theta) = \int_s^t \bar{v}(\sigma,s)\varphi\,\mathrm{d}\sigma + f_0(t), \tag{80}$$

where f_0 is given by (71). Note that the left hand side of (80) can be written as $x - \eta * x$. Using the resolvent equation (73) we obtain

$$\begin{aligned}(1+\rho_0)*(x-\eta*x) &= x - \eta*x + \rho_0*x - \rho_0*\eta*x \\ &= x - \eta*x + \rho_0*x - (\rho_0-\eta)*x \\ &= x.\end{aligned}$$

Thus if we take on both sides of (80) the convolution with the fundamental solution $X(t) = I + \rho_0(t)$ of (69) then

$$\begin{aligned}x(t) &= y(t;\varphi) + \int_s^t \mathrm{d}X(t-\tau)\int_s^\tau \bar{v}(\sigma,s)\varphi\,\mathrm{d}\sigma \\ &= y(t;\varphi) + \int_s^t X(t-\tau)\bar{v}(\tau,s)\varphi\,\mathrm{d}\tau,\end{aligned} \tag{81}$$

where y is given by (74). Finally take the convolution with $q^\diamond(t)$ on both sides of (81) to arrive at

$$\begin{aligned}\bar{v}(t,s)\varphi &= q^\diamond(t)y(t;\varphi) + \int_s^t q^\diamond(t)X(t-\tau)\bar{v}(\tau,s)\varphi\,\mathrm{d}\tau \\ &= q^\diamond(t)S_0(t-s)\varphi + \int_s^t \left[\int_0^h \mathrm{d}\zeta(t,\theta)X(t-\tau-\theta)\right]\bar{v}(\tau,s)\varphi\,\mathrm{d}\tau \\ &= q^\diamond(t)S_0(t-s)\varphi + \int_s^t k(t,\tau)\bar{v}(\tau,s)\varphi\,\mathrm{d}\tau,\end{aligned}$$

where we have used (77) and (78). Therefore $\bar{v}(t,s)\varphi$ given by (79) satisfies the identity

$$\bar{v}(t,s)\varphi = q^\diamond(t)S_0(t-s)\varphi + \int_s^t k(t,\tau)\bar{v}(\tau,s)\varphi\,\mathrm{d}\tau.$$

This shows that $\bar{v}(t,s)\varphi$ is a solution to the renewal equation (59) and completes the proof of the claim (78).

Finally apply to (61) the element of $Y^\diamond$ that corresponds to the Dirac measure in $-\theta \in [0, 1]$ to obtain

$$\begin{aligned}(U(t,s)\varphi)(\theta) &= y(t-s+\theta) + \int_s^t X(t-\tau+\theta)\cdot v(\tau,s)\varphi \,\mathrm{d}\tau \\ &= x(t-s+\theta;\varphi),\end{aligned}$$

where in the last identity we have used (81).

Thus we conclude that the the perturbation approach based on the abstract variation-of-constants formula developed in the previous section precisely yields the twin evolutionary system defined by translation along the solution of (68).

We summarize this result in a theorem.

Theorem 8.2 *Under the above conditions, translation along the solution of equation* (68) *defines a twin evolutionary system* $\{U(t,s)\}_{t\geq s}$ *given by* (61).

A Review of Functions of Bounded Variation

In this appendix $\mathcal{E}$ denotes the Borel σ-algebra on $[0,\infty)$. For $E \in \mathcal{E}$, we call a sequence of disjoint sets $\{E_j\}$ in $\mathcal{E}$ a *partition* of E if $\cup_{j=1}^{\infty} E_j = E$. A complex Borel measure is a map $\mu : \mathcal{E} \to \mathbb{C}$ such that $\mu(\emptyset) = 0$ and

$$\mu(E) = \sum_{j=1}^{\infty} \mu(E_j),$$

for every partition $\{E_j\}$ of E with the series converging absolutely. In the following we will often omit the adjective 'bounded'. The *total variation* measure $|\mu|$ of a complex Borel measure μ is given by

$$|\mu|(E) = \sup\left\{\sum_{j=0}^{n} |\mu(E_j)| \mid n \in \mathbb{N},\ \{E_j\} \text{ a partition of } E \text{ in } \mathcal{E}\right\}.$$

The vector space of complex Borel measures of bounded total variation is denoted by $M([0,\infty))$. Provided with the total variation norm given by

$$\|\mu\|_{TV} = |\mu|\,([0,\infty))\,, \tag{82}$$

the vector space $M([0,\infty))$ becomes a Banach space.

Let $f : [0,\infty) \to \mathbb{C}$ be a given function. For a partition $\{E_j\}$ of $[0,t]$ with $E_j = [t_{j-1}, t_j)$ and $0 = t_0 < t_1 < \cdots < t_n = t$ we define the function $T_f : [0,\infty) \to [0,\infty]$ by

$$T_f(t) := \sup \sum_{j=1}^{n} |f(t_j) - f(t_{j-1})|,$$

where the supremum is taken over $n \in \mathbb{N}$ and all such partitions of $[0, t]$. The extended real function T_f is called the *total variation function* of f. Note that if $0 \leq a < b$, then $T_f(b) - T_f(a) \geq 0$ and hence T_f is an increasing function.

If $\lim_{t\to\infty} T_f(t)$ is finite, then we call f a function of bounded variation. We denote the space of all such functions by BV. The space $NBV([0,\infty))$ of *normalized* functions of bounded variation is defined by

$$\begin{aligned} NBV([0,\infty)) := \{f \in BV \mid f \text{ is continuous from the right on } (0,\infty) \\ \text{and } f(0) = 0\}. \end{aligned}$$

Provided with the norm

$$\|f\|_{TV} := \lim_{t\to\infty} T_f(t) \tag{83}$$

the space $NBV([0,\infty))$ becomes a Banach space. More generally, we define for $-\infty < a < b < \infty$, the vector space $NBV([a,b])$ to be the space of functions $f : [a,b] \to \mathbb{C}$ such that $f(a) = 0$, f is continuous from the right on the open interval (a,b), and whose total variation on $[a,b]$, given by $T_f(b) - T_f(a) = T_f(b)$, is finite. Provided with the norm $\|f\|_{TV} := T_f(b)$, the space $NBV([a,b])$ becomes a Banach space. We extend the domain of definition of a function of bounded variation by defining $f(t) = 0$ for $t < 0$ if $f \in NBV([0,\infty))$ and $f(t) = 0$ for $t < a$ and $f(t) = f(b)$ for $t > b$ if $f \in NBV([a,b])$.

The following fundamental result (see Folland 1999, Theorem 3.29) provides the correspondence between functions of bounded variation and complex Borel measures.

Theorem A.1 *Let μ be a complex Borel measure on $\mathbb{R}$. If $f : [0,\infty) \to \mathbb{C}$ is defined by $f(t) = \mu((0,t])$, then $f \in NBV([0,\infty))$. Conversely, if $f \in NBV([0,\infty))$ is given, then there is a unique complex Borel measure μ_f such that $\mu_f((0,t]) = f(t)$. Moreover $|\mu_f| = \mu_{T_f}$.*

Given a function $f \in NBV([a,b])$ with corresponding measure μ_f, we define the Lebesgue-Stieltjes integral $\int g\,\mathrm{d}f$ or $\int g(x)\,f(\mathrm{d}x)$ to be $\int g\,\mathrm{d}\mu_f$. Thus, a Lebesgue-Stieltjes integral is a special Lebesgue integral and the theory for the Lebesgue integral applies to the Lebesgue-Stieltjes integral. We embed $L^1([0,\infty))$ into $M([0,\infty))$ by identifying $f \in L^1([0,\infty))$ with the measure μ defined by

$$\mu(E) = \int_E f(x)\,\mathrm{d}x \quad \text{or, in short,} \quad \mu(\mathrm{d}x) = f(x)\,\mathrm{d}x.$$

In this section we collect some results about the convolution of a measure and a function and the convolution of two measures needed to study renewal equations.

For details we refer to Appendix A of Diekmann and Verduyn Lunel (2021) and for further results we refer to Folland (1999); Grippenberg et. al. (1990).

Let $B\left([0,\infty)\right)$ denote the vector space of all bounded, Borel measurable functions $f : [0,\infty) \to \mathbb{R}$. Provided with the supremum norm (denoted by $\|\cdot\|$), the space $B\left([0,\infty)\right)$ becomes a Banach space. With $B\left([a,b]\right)$ we denote the Banach space of all bounded, Borel measurable functions $f : [a,b] \to \mathbb{R}$ provided with the supremum norm.

The half-line convolution $\mu * f$ of a measure $\mu \in M([0,\infty))$ and a Borel measurable function $f \in B\left([0,\infty)\right)$ is the function

$$(\mu * f)(t) := \int_{[0,t]} \mu(\mathrm{d}s) f(t-s)$$

defined for those values of t for which $[0,t] \ni s \mapsto f(t-s)$ is $|\mu|$-integrable.

The following result can be found in Grippenberg et. al. (1990, Theorem 3.6.1(ii)).

Theorem A.2 *If $f \in B\left([0,\infty)\right)$ and $\mu \in M\left([0,\infty)\right)$, then the convolution of f and μ satisfies $\mu * f \in B\left([0,\infty)\right)$ and*

$$\|\mu * f\| \leq \|\mu\|_{TV} \|f\|.$$

The half-line convolution $\mu * \nu$ of two measures $\mu, \nu \in M\left([0,\infty)\right)$ is defined by the complex Borel measure that to each Borel set $E \in \mathcal{E}$ assigns the value

$$(\mu * \nu)(E) := \int_{[0,\infty)} \mu(\mathrm{d}s)\nu\left((E-s)_+\right), \tag{84}$$

where $(E-s)_+ := \{e - s \mid e \in E\} \cap [0,\infty)$ (cf. Grippenberg et. al. 1990, Definition 4.1.1)).

If χ_E is the characteristic function of the set E, then

$$\nu((E-s)_+) = \int_{[0,\infty)} \chi_E(\sigma + s)\nu(\mathrm{d}\sigma),$$

where $[0,\infty) \ni \sigma \mapsto \chi_E(\sigma + s)$ is the characteristic function of $(E-s)_+$. It follows from Theorem A.2 that $s \mapsto \nu(E-s)_+)$ belongs to $B\left([0,\infty)\right)$ and hence the definition of the convolution of two measures $\mu * \nu : \mathcal{E} \to \mathbb{C}$ given in (84) makes sense. Furthermore, using Fubini's Theorem, we have the following useful identity

$$
\begin{aligned}
\mu * \nu(E) &= \int_{[0,\infty)} \mu(\mathrm{d}s)\nu\left((E-s)_+\right) \\
&= \int_{[0,\infty)} \int_{[0,\infty)} \chi_E(\sigma+s)\mu(\mathrm{d}s)\nu(\mathrm{d}\sigma) \\
&= \int_{[0,\infty)} \mu\left((E-s)_+\right)\nu(\mathrm{d}s).
\end{aligned}
$$

The following result can be found in Grippenberg et. al. (1990, Theorem 4.1.2(ii)).

Theorem A.3 *Let $\mu, \nu \in M([0,\infty))$ and let the convolution $\mu * \nu$ be defined by* (84).

(i) The convolution $\mu * \nu$ belongs to $M([0,\infty))$ and

$$
\|\mu * \nu\|_{TV} \le \|\mu\|_{TV}\|\nu\|_{TV}.
$$

(ii) For any bounded Borel function $h \in B([0,\infty))$, we have

$$
\int_{[0,\infty)} h(t)\,(\mu * \nu)\,(\mathrm{d}t) = \int_{[0,\infty)} \int_{[0,\infty)} h(t+s)\,\mu(\mathrm{d}t)\nu(\mathrm{d}s).
$$

Using the one-to-one correspondence between complex Borel measures and functions of bounded variation, see Theorem A.1, we can combine the above results to obtain the following theorem (see Diekmann and Verduyn Lunel 2021, Theorem A.5).

Theorem A.4 *If $f \in NBV([0,\infty))$ and $\mu \in M([0,\infty))$, then the convolution of μ and f satisfies $\mu * f \in NBV([0,\infty))$ and*

$$
\|\mu * f\|_{TV} \le \|\mu\|_{TV}\|f\|_{TV}.
$$

We also need the following result (see Diekmann and Verduyn Lunel 2021, Theorem A.6).

Theorem A.5 *Let $\mu \in M([0,\infty))$ and let $f : [0,\infty) \to \mathbb{C}$ be a bounded continuous function. If μ has no discrete part, then $\mu * f$ is a bounded continuous function and*

$$
\|\mu * f\| \le \|\mu\|_{TV}\|f\|.
$$

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