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I am What I Have and I Have What I am, What am I?

Homeomerosity in Formal Concept Analysis

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Abstract. The term *homeomerosity* refers to when a whole and its parts are the same kind of thing. For instance, a computer and its processor can both be classified as machines. Homeomerosity is a prerequisite for meaningful addition and subtraction. For example, adding the area sizes of two independent regions gives another area size, but adding an area size and a number of hours yields a number with a peculiar unit. In earlier work, homeomerosity has been formalized with respect to mereological parthood, but not in concurrence with a notion of class subsumption. Both are essential to homeomerosity, as a part can only be observed to be of the same kind as the whole if they are observed to be of some kinds in the first place. In this work, we use formal concept analysis to organize conceptual representations of parts and wholes in a shared contextual model. In our doing so, we show wholes and parts can be represented by sub-concepts of a concept with respect to which they are homeomerous.

Keywords. Homeomerosity, Ontological dependence, Amounts, Mereology, Formal Concept Analysis, Extension and intension

1. Introduction

The verbs *to be* and *to have* are excellent devices for deduction. The first is a relation of both class membership and subsumption (x is a cat, a cat is an animal) and the second is a compositional relation (The cat has paws, the paws have nails). Although the two verbs are at times inaccurate and ambiguous, they seem to be at the essence of conceptual reasoning. There is a long philosophical tradition of defining terms by their intension (i.e., things are everything they have) and extension (i.e., things are everything they are) [1,15] and many things are express-able in terms of what they have and are. In mathematics, sets have elements and are elements of other sets. In physics, particles have (or are) energy. In social sciences, individuals have their identity and are members of socioeconomic classes, and in spatial sciences things have a location and are a location.

In light of this discussion, the kind of things that have what they are and are what they have forms an interesting case. Such things are called homeomerous[2] [11]. For some

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[2]This roughly translates to 'having similar parts'.

concepts, the presence of homeomerosity is evident. A partition of water is water, and if we agree that the bigger portion of water has or contains the smaller portion, then this water must be homeomerous. However, the concept can be confusing. Trees are considered to be different from the leaves they carry. However, trees are also organic things, and so are their leaves. In this sense, trees with leaves are organic things that have organic things. This poses the question; are or aren't trees the same kind of thing as their leaves? Similarly, a partition of water is generally still considered water, but what if that specific partition is the only one with salt water? Is the part of this water still the same kind of thing as the whole of this water? Or is it different because it is salty?

This kind of question becomes pertinent when the goal is to measure and reason over quantities since this notion of sameness dictates the meaning of any sum of measurements. A sum of five trees and seven flowers yields a total of twelve plants, but what does it mean to sum seven cars and five hours? This is of particular interest in geographic information science [19]. What attributes should be summed when their corresponding regions are summed? The answers to these questions are contingent on how homeomerosity is perceived. The concept has been specified, either in name or in spirit, in previous works [11,26,9,22], but has only been formalized in mereological terms. Class membership is implemented only through predicates, but the relations between different predicates are left out of consideration.

We are thus left with questions about the interaction between the relations analogous to *to be* and *to have* in things that are what they have and have what they are. Under what conditions do the intension and extension of two concepts align in such a way that they are 'the same'? Exactly when is something what it has and does it have what it is? In this work, we investigate this question, and more generally the notion of homeomerosity, with the use of formal concept analysis (FCA) [24]. FCA is a method for organizing information into a conceptual hierarchy. Objects and attributes are matched with each other through an incidence relation and aggregated into concepts. Through the intent and extent of objects and attributes, concepts can be ordered with respect to other concepts. We use this structure to make sense of what things are. We also take inspiration from mereology to imbue the incidence relation with notions of parthood to order things with respect to things they have. Because both notions are combined in the same structure, the concepts that arise can 'have' super-concepts of that concept. For example, trees have leaves and those leaves are organic things, just like trees are organic things. We show how the notions apply to an example case from geographic information science.

1.1. Related works

In the following, we discuss works related to geographic information, mereology and homeomerosity, and concept representation frameworks, with emphasis on formal concept analysis.

1.1.1. Geographic information science

Geo-analysts often work with representations of continuous phenomena, such as space, time and material stuff (e.g., water). Developing a theory for this practice is a challenge, and multiple structures have been proposed. A prominent example is that of a field [2]. A (geo-)field is a spatially-continuous function. Locations can be passed to the function to retrieve thematic values (e.g., temperatures or land use qualities). [6] propose a comparable concept with continuity over time. However, a theory of fields does not explain all characteristics of continuous phenomena; retrieving a location based on a thematic value can be ambiguous, for instance if that value is 'water'. This led theorists to consider other possibilities. [16] suggests geographic masses, inspired by the linguistic notion of mass nouns. In [22], we formalized a similar concept, which we call amounts, as well as a theory of extensive quantities over amounts. Homeomerosity takes a central role in this formalization, the key idea being that if numeric qualities (e.g., area sizes) can be summed, then there should be a corresponding mereological sum where the mereological parts and the sum are of the same kind, i.e., the sum is homeomerous.

1.1.2. Mereology and homeomerosity

Mereology is the study of relations between parts and wholes. In most cases, the so-called core mereology is defined through a weak partially-ordered (reflexive, antisymmetric, transitive) parthood relation, but it can also be defined through a strict partial order [23]. Composition and decomposition principles predetermine which parts and wholes could and which should exist. For instance, the mereological sum operators describes potential wholes and the weak supplementation principle states that if a whole has a proper part, then it must have another distinct part. More specific mereologies can be defined by opting for stricter principles [20]. The philosophical-mathematical study has gained attention in the fields of ontological philosophy and ontology engineering [25,7,10,9], geographic information science [22,19], and chemistry [4,12].

As [10] notes, mereology is well-suited for the formalization of containment, but not so much for further characterizing roles, i.e., what the contained and the container are. Mereology formalizes for instance what it means that an arm can be part of a body, but not what it means that arms are not bodies. [8,9] distinguish wholes with (1) masses or continuous parts, (2) discrete homogeneous parts, and (3) heterogeneous parts, but (2) and (3) requires a notion of -geneity, which mereology does not offer. [5] provides a spatial information predication calculus for when and how thematic information is perpetuated in mereotopological structures and shows the relevance of topological relations in determining whether some region is homeomerous and whether thematic values exclude one another. For example, if a region has a certain size, then its proper parts cannot have a size greater than or equal to it. However, again, it remains unclear how the mereotopological structure interacts with the subsumption relations between the predicates.

[11] discusses and formalizes the concept of homeomerosity in pursuit of a *SubQuantityOf* relation in Web Ontology Language (OWL) and suggests a view of quantities as maximally self-connected objects. Homeomerosity concerns the mereological phenomenon of when parts and their wholes are the same kind of things. For example, a part of a spatial region is also a spatial region. However, it remains unclear how it is determined of what kind something is. This is pertinent because, for something to be of the same kind as something else, it needs to be **of a kind** in the first place. This means we need to combine conceptualization and mereology for a sufficient notion of homeomerosity.

1.1.3. Formal concept analysis

Formal concept analysis is generally considered a mathematically-grounded conceptualization approach [24]. This is because the theory is built around the idea of studying binary relations using complete lattices. Early contributions in the field focused on developing the theory, while more recent works contributed by increasing computational efficiency and developing pruning techniques[13]. Furthermore, a variety of extensions has been developed, such as triadic FCA, with which ternary relations can be studied, pattern structures, which extends FCA to graphs and numeric values, and logical concept analysis (LCA), which takes a first-order logic approach to FCA [18]. FCA finds use in mining text, web, chemistry, and bioinformatical data, as well as in ontology engineering [17].

2. Concept lattices

In this section we explain the structure of FCA and introduce additional definitions. Because the terminology of FCA may cause confusion, we first reflect on our interpretations of relevant terms. The FCA framework is centered around a binary relation called *incidence*. The domain or set of departure is called a set of *objects* and the codomain or set of destination is called a set of *attributes*. As such, incidence can be considered a subset of the Cartesian product of objects and attributes. FCA relies on the idea that the incidence relation can be decomposed into *concepts*, which are binary tuples containing a set of objects and a set of attributes. Concepts can be ordered in a complete lattice, aptly named a concept lattice, based on the subset relations between their sets of objects and dually between their sets of attributes.

In its core, FCA is the analysis of structures that arise from the incidence relation. However, the terminology is overly-suggestive and may cause confusion. For example, the term ‘concept’ is still subject to discourse [14], but in FCA it refers to a specific kind of tuple. We do not intend to subject the terms to scrutiny and perpetuate the use of potentially-problematic FCA-terminology, but we do so with only its meaning within the FCA framework in mind. When we discuss objects and concepts, for example, we do not mean the philosophical notion of object or concept per se, but the specific notions of FCA-object and FCA-concept. Any subsequent discussion of objects and concepts thus concerns these FCA-versions. In the remainder of this section, terms introduced in italics are intended as FCA-specific terms.

2.1. Formal concept analysis – “to be”

In the following, we interpret the notion of being in terms of specification of concepts, such that the more specific concept ‘is’ subsumed by the more general concept.

In FCA, a *context* is defined as a triple $K = (G, M, I)$, where G is a set of *objects*, M a set of *attributes*, and $I \subseteq G \times M$ is a binary relation called *incidence* that expresses which objects are incidental with which attributes [24]. Incidence can also be understood as the graph of K. Instead of $(g, m) \in I$, we write gIm. For subsets $A \subseteq G$ of objects and subsets $B \subseteq M$ of attributes, two *derivation operators* $\uparrow$ and $\downarrow$ may be defined as follows:

$$A\uparrow = \{m \in M \mid (\forall g \in A)\ gIm\} \qquad \text{(Attributes of objects)} \qquad (1)$$

$$B\downarrow = \{g \in G \mid (\forall m \in B)\ gIm\} \qquad \text{(Objects of attributes)} \qquad (2)$$

A *formal concept* of K is a pair (A, B) such that $A \subseteq G, B \subseteq M, A\uparrow = B, B\downarrow = A$, where A is the *extent* and B is the *intent* of (A, B). The set of all formal concepts of K is denoted as $\mathscr{B}(G,M,I)$ or $\mathscr{B}(K)$. The formal concepts in K can be ordered by subset relations between extents and between intents:

$$(A_1,B_1) \leq (A_2,B_2) \iff (A_1 \subseteq A_2 \iff B_1 \supseteq B_2) \qquad \text{(Specification)} \qquad (3)$$

From this, it can be proven that $\langle \mathscr{B}(K), \leq \rangle$ forms a complete lattice [3]. In the context of FCA, $\leq$ is also known as the *specification* or *specialization* operator. Joins and meets exist and satisfy the constraints that $(A_1,B_1) \vee (A_2,B_2) = (A_2,B_2)$ iff $(A_1,B_1) \leq (A_2,B_2)$, and $(A_1,B_1) \wedge (A_2,B_2) = (A_1,B_1)$ iff $(A_2,B_2) \leq (A_1,B_1)$. Any complete lattice is bounded, which means there are a supremum and an infimum, respectively $\top, \bot \in \mathscr{B}(K)$. In order to separate these from concepts that arise from at least some incidence, we let $\top = (G, \emptyset)$ and $\bot = (\emptyset, M)$ in any $\mathscr{B}(K)$. Concepts on the left-hand side of $\leq$ may be referred to as *sub-concepts*, while those on the right-hand side may be referred to as *super-concepts*. They may also be called *more specific* and *more general*.

We add some additional notation for convenience. We let a concept (A,B) be denoted as c. Subscript specifiers of the two notations always correspond, meaning, e.g., $(A_x, B_x) = c_x$. This lets us consistently refer to the objects A_x and attributes B_x of some concept c_x without having to repeatedly define particular concepts as tuples of some objects and attributes. It also makes concise to write that a concept has or is formed over a certain object or attribute. For example, $a \in A_x$ and $b \in B_x$ means that c_x is formed over object a and attribute b. If we just mention A_x or B_x, we assume there exists some c_x. Similarly, the notation K_x always corresponds with a notation (G_x, M_x, I_x) with variable subscript x, and the appearance of G_x, M_x, and I_x suggests the existence of K_x.

2.2. *Ontological dependence – ”to have”*

It can be argued that some objects depend on other objects in order to be, but not (necessarily) the other way around [21]. A house needs walls to be a house, but walls do not need a house to be walls. In logical terms, a definiendum needs its definiens, but a definiens does not need its definiendum. The `elementOf` relation $\in$ from set theory fulfils a similar role. A set needs its elements to be what it is, but elements do not need their set. We informally consider this dependence in terms of having. If a thing depends on other things in order to be what it is, then it can be said it has those things. We formalize this as a FCA-specific notion of ontological dependence.

Objects may have objects as attributes. Using this, concepts can be related to other concepts based on whether the intent of one concept depends on the extent of another. We call this relation between concepts *dependence* and denote it by $\preceq$, where $c_x \preceq c_y$ means that c_y depends on c_x. Formally:

$$c_x \preceq c_y \iff \{a \in A_x \mid \forall c_z((c_z \leq c_x \text{ and } a \in A_z) \Rightarrow c_z = c_x)\} \subseteq B_y \quad \text{(Dependence)} \quad (4)$$

For example, a bottle of water has water. The former depends on the latter because objects of the concept ”water” are at the same time attributes of the concept ”bottle of water”. If there was no water in the bottle, the bottle would no longer be an example of the concept *bottle of water*. Note that concepts that depend on the same concept can be $\leq$-ordered with one another. A bottle of water is a specific container of water. In the definition, A_x is subsetted to exclude objects that are represented by more specific concepts. Consequently, the relation can be established between, e.g., two concepts with respectively the objects 'bottle of water' and 'water' regardless of whether the concept of 'water' extends over 'salty water'. We exclude the infimum and supremum from the relation, i.e., $\neg(\top \preceq c_x)$ and $\neg(c_x \preceq \bot)$ for all c_x in $\mathscr{B}(K)$. This step makes discussion and visualization of the $\preceq$-relation simpler.

2.3. Homeomerosity

We define *homeomerosity* as a relation between a concept and another concept with respect to which the first concept is homeomerous. A concept c_x is homeomerous with respect to c_y if c_x as well as all concepts c_z that c_x depends on ($c_z \preceq c_x$) are sub-concepts of c_y, i.e.:

$$H(c_x, c_y) \iff c_x \leq c_y \text{ and } \forall c_z(c_z \preceq c_x \implies (c_z \leq c_y)) \quad \text{(Homeomerosity)} \qquad (5)$$

For example, a forest, its trees, and its leaves are all homeomerous with respect to being organic things. In earlier work we defined the concept of amount [22], and we believe homeomerosity to be its characterizing property. Therefore, if a concept is homeomerous with respect to a concept, we consider it to be an amount of that concept. For example, an amount of space has parts that are amounts of space, and it also is itself part of an amount of space. In our earlier work, we specified amounts with respect to a domain they were in, such as the domain of space or the domain of time. This domain was specified as a set that contains all amounts of the same kind. Here we can specify c_y in (5) as the domain of amounts, so instead of a set the domain is a super-concept.

Note that $H(c_x, c_y)$ does not have any requirements with regards to how the concept c_x is $\leq$-ordered with respect to the concepts it depends on. For example, a metal fork has a handle, and this handle is not a fork. However, a metal fork and all its parts may all be metal things. Therefore, $H(\mathit{Metal\ Fork}, \mathit{Metal\ thing})$. Of course, a metal fork could have parts that are not metal things, which would mean it is actually not homeomerous. In such cases, a super-concept could be specified which only has the metal parts of a metal fork as its parts.

We may further distinguish cases where the attributes are required to be homeomerous with respect to the same concept as well. We denote this by H_E and call it *extensive homeomerosity*:

$$H_E(c_x, c_y) \iff \forall c_z(c_z \preceq c_x \implies H(c_x, c_y) \text{ and } H(c_z, c_y)) \quad \text{(Extensive hom.)} \qquad (6)$$

For example, an amount of water is extensively-homeomerous if all its parts are considered amounts of water as well. However, note that the parts of those parts do not need to be homeomerous with respect to water for the first amount of water to be extensively-homeomerous. If we want to specify that all parts of an amount of water are homeomerous with respect to water, we can use recursion. We denote such cases by H_ρ and call it *continuous homeomerosity*:

$$H_\rho(c_x, c_y) \iff \forall c_z(c_z \preceq c_x \implies H(c_x, c_y) \text{ and } H_\rho(c_z, c_y)) \quad \text{(Continuous hom.)} \qquad (7)$$

Continuous homeomerosity of x with respect to an amount of water could imply, for example, that if you split x in half, then divide one half into seven parts, and split one part in half again, you will get an amount of water. This continues until the point where parts themselves do not have parts anymore, i.e., the atomic level. In terms of concepts, a concept c_x could be considered atomic if it has no relation such that $c_y \preceq c_x$. It could be that there is no atomic level, for example if the underlying incidence relation is infinite.

3. Homeomerosity in geographic information: A simple example case

Concept lattices can be used as structures for metadata to determine whether data qualities are transferable from inputs to outputs through mereological operations over geographic data. For example, if one were to take a segment from 'industrial area', that segment could be called 'industrial area' as well. However, such reasoning does not apply for all attributes; a segment of a city is no longer a city. In the following, we show how aggregations of data representations of geographic phenomena can be represented using FCA.

3.1. Merging Het IJsselmeer, Het Markermeer, and de Waddenzee

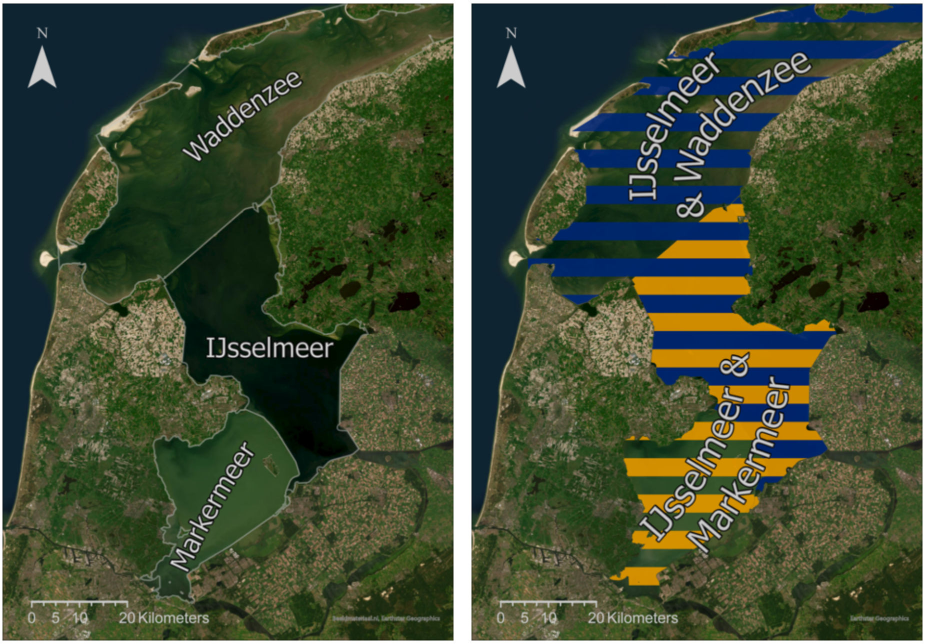
The Netherlands embrace two lakes and are crowned by an inland sea to the North. The first of the lakes, het IJsselmeer, is separated from de Waddenzee by a well-known dam named de Afsluitdijk[3]. To the south it borders the second lake, het Markermeer, with only de Houtribdijk, a dam spanning 26 kilometers, in-between them. Over time, after the dams were constructed during the 20th century, the water in the lakes turned sweet, while the water in de Waddenzee stayed salty. Figure 1a shows a map of the three water bodies. The three water bodies are represented as polygon vector data, with one polygon for each water body, in a geographic information system (GIS).

A geo-analyst may want to consider scenarios where either or both of the dams were to be removed. To achieve this, they may merge – also termed dissolve or amalgamate – het IJsselmeer with either or both het Markermeer and de Waddenzee. Two merges of the two sets of contiguous water bodies are shown in Figure 1b. A common problem with this kind of merging procedure is that in a GIS it is not specified which attributes the resulting polygon may inherit from the inputs. It is up to the analyst to decide whether the resulting polygons have sweet water, salty water, or whether this dichotomy should be left unspecified.

The scenario can be described by a set of incidence tuples and consequently by concepts formed over these tuples. For example, a sea is incidental with salt water and a lake is incidental with sweet water, so they are – in part – some sort of container of respectively salt water and sweet water. Het IJsselmeer, het Markermeer, and de Waddenzee are also such containers, but they also contain more specific things by which they can be identified. Three generic attributes *IJsselmeerID*, *MarkermeerID*, and *WaddenzeeID* may serve as amalgamations of these specific things. A merge of two bodies of water can also be considered a container of those two bodies of water. For example, het IJsselmeer and het Markermeer can be considered attributes of some merge *IJ & M*.

Let x be an intransitive and irreflexive set of incidence tuples describing the scenario. The transitive closure of this set is denoted as x^+ and the reflexive closure as x^{ref}. Let $T = x^+ \setminus x$ and $R = x^{\text{ref}} \setminus x$. Table 1 shows the incidences of x, T, and R, with objects

[3] We use the Dutch particles *de* and *het* for Dutch placenames.



(a) het IJsselmeer, het Markermeer and de Waddenzee in the Netherlands

(b) Merges of het IJsselmeer with Waddenzee (Blue) and with Markermeer (Orange).

Figure 1.: Merge of three lakes

Table 1.: Contexts of the three water bodies. Objects are row headers, attributes are columns headers. A symbol (x, R, or T) at a cell denotes an element of the symbolized set. For example, at the intersection of 'Water' and 'Seas' there is an element of T.

	Water	Sweet water	Salt water	IJsselmeerID	MarkermeerID	WaddenzeeID	Water body	Sea	Lake	IJsselmeer (IJ)	Markermeer (M)	Waddenzee (W)	Water bodies	Seas	Lakes	IJ & M	IJ & W
Water	R																
Sweet water		R															
Salt water			R														
IJsselmeerID				R													
MarkermeerID					R												
WaddenzeeID						R											
Water body	x						R										
Sea	x		x					R									
Lake	x	x							R								
IJsselmeer (IJ)	x	x		x						R							
Markermeer (M)	x	x			x						R						
Waddenzee (W)	x		x			x						R					
Water bodies	T						x						R				
Seas	T		T				x	x						R			
Lakes	T	T					x		x						R		
IJ & M	T	T		T	T		x		x	x	x					R	
IJ & W	T	T	T	T		T	x	x	x	x		x					R

along the y-axis and attributes along the x-axis. We build three contexts from the incidences, namely one which is intransitive and irreflexive ($I = x$), one which is transitive and irreflexive ($I = x \cup T$), and one which is transitive and reflexive ($I = x \cup T \cup R$). In each of the contexts, the sets of objects and attributes stay the same.

For convenience, we address concepts by the objects or the attributes shown on their corresponding nodes in the concept lattice. We address concepts through objects with the notation C_o*(object)* and through attributes with the notation C_a*(attribute)*. For example, the most specific concept of the object *Markermeer* would be denoted as C_o*(Markermeer)* and the most general concept of the attribute *Sweet water* would be denoted as C_a*(Sweet water)*.

3.2. Concepts from an irreflexive, intransitive incidence

The concept lattice of the context over the irreflexive, intransitive incidence is shown in figure 2. The most salient characteristic of the lattice is that the $\leq$-order and $\preceq$-relation are disjoint. This observation already indicates that no example of homeomerosity can be found.

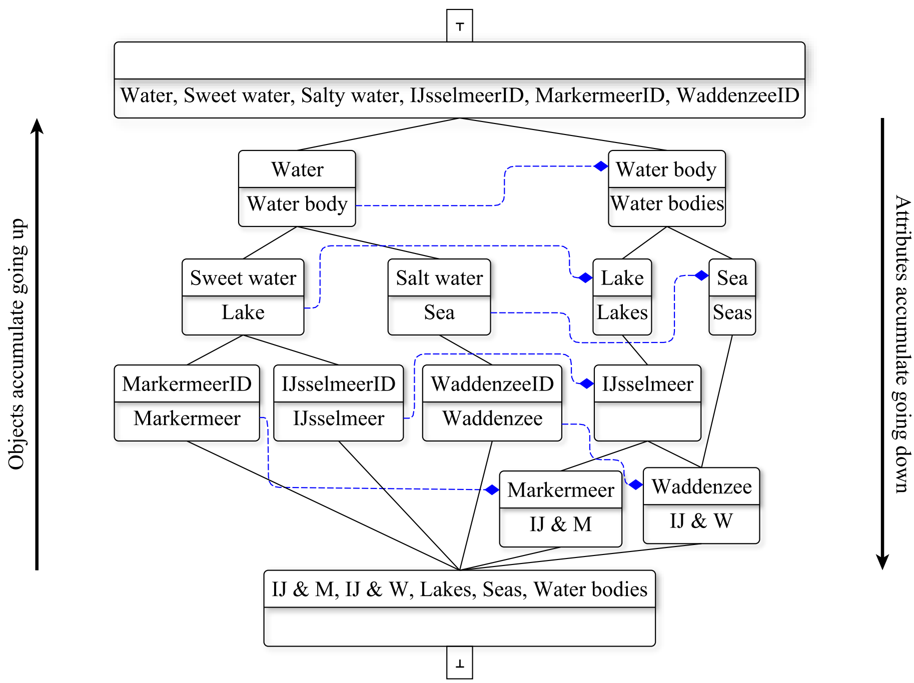


Figure 2.: Concept lattice derived from incidences x in Table 1. Attributes are shown in the upper half and objects in the lower half of the nodes. Black lines denote the $\leq$-order according to the principles of [3]. Sub-concepts inherit attributes from super-concepts and super-concepts inherit objects from sub-concepts. Blue arrows denote the $\preceq$-relation, with the arrowhead landing at the most general preceded concept.

C_a*(IJsselmeer)* introduces no new attributes. The reason this concept emerges is because otherwise the principles of the lattice structure would be violated; C_o*(IJ & M)* and C_o*(IJ & W)* would have had two candidates for their join, namely C_o*(Lakes)* and C_o*(Seas)*.

From these concepts another oddity emerges, namely that, e.g., C_o*(IJ & W)* is a sub-concept of C_o*(Lakes)*. However, de Waddenzee (W) is not a lake, but a sea, which begs the question why C_o*(IJ & W)* would be considered lakes. The technical answer is that the attributes of C_o*(IJ & W)* are a superset of those of C_o*(Lakes)*. It seems that in this case the object *Lakes* is misleading. In actuality, the concept represents things that are at least in part lakes. In other words, the concept represents some lakes, but perhaps also some other things.

3.3. Concepts from an irreflexive, transitive context

The concept lattice of the context over the irreflexive, transitive incidence is shown in figure 3. We now see that the $\preceq$-relation is embedded in the $\leq$-order. This also suggests there may be cases of homeomerosity, even though irreflexivity makes it so that there are no cases where $c_x \preceq c_x$.

Take the example of C_o*(IJ & M)*, which is a sub-concept of C_o*(Markermeer)*, which in turn is a sub-concept of C_o*(Lake)*. Just as with the case we discussed in the last sub-section, sc represents only that the concept has some *Markermeer*, not that all attributes are cases of *Markermeer*. In fact, C_o*(IJsselmeer)* is not a sub-concept of C_o*(Markermeer)*, which means C_o*(IJ & M)* is not homeomerous with respect to C_o*(Markermeer)* However, we can observe that all attributes of C_o*(IJ & M)* are objects of sub-concepts of C_o*(Lake)*. That means that the concept of the merge of IJsselmeer and Markermeer is homeomerous with respect to the concept of lake, i.e., $H(C_o\textit{(IJ \& M)}, C_o\textit{(Lake)})$.

The concept of the object lakes is also homeomerous with respect to the concept of the object lake. However, while C_o*(IJ & M)* is a sub-concept of the concept of lakes, it is not homeomerous with respect to the concept of lakes. That is because het IJsselmeer and het Markermeer are each a lake, but they do not 'have' lakes.

C_o*(IJ & W)* is only homeomerous with respect to C_o*(Water body)*, since one of its attributes is a sea and another is a lake. Also, C_o*(IJ & M)* and C_o*(IJ & W)* are both homeomerous with respect to C_o*(Water body)*. This suggests that if there were an aggregation over all three water bodies, then this would also be a water body.

3.4. Concepts from a reflexive, transitive context

The concept lattice of the context over the reflexive, transitive incidence is shown in figure 4. At first glance, two characteristics stand out. Firstly, all concepts depend on themselves and are the most general concept that they depend on. This seems to be due to the reflexivity of the incidence relation. Secondly, the $\leq$-order is often mediated by blank concepts. These concepts emerged to preserve the lattice structure. Since they suggest new concepts that were not considered during the construction of the context, such concepts are useful for concept mining purposes, but for our goals here they can mostly be ignored.

Just like before, examples of homeomerosity can be found. For example, C_o*(IJ & M)* is homeomerous with respect to C_o*(Water)*. However, due to reflexivity, many of the in Figure 3 observable cases of homeomerosity do not re-occur here. For example, C_o*(IJ & M)* is not homeomerous with respect to C_o*(Lake)*, because the concepts of het IJsselmeer

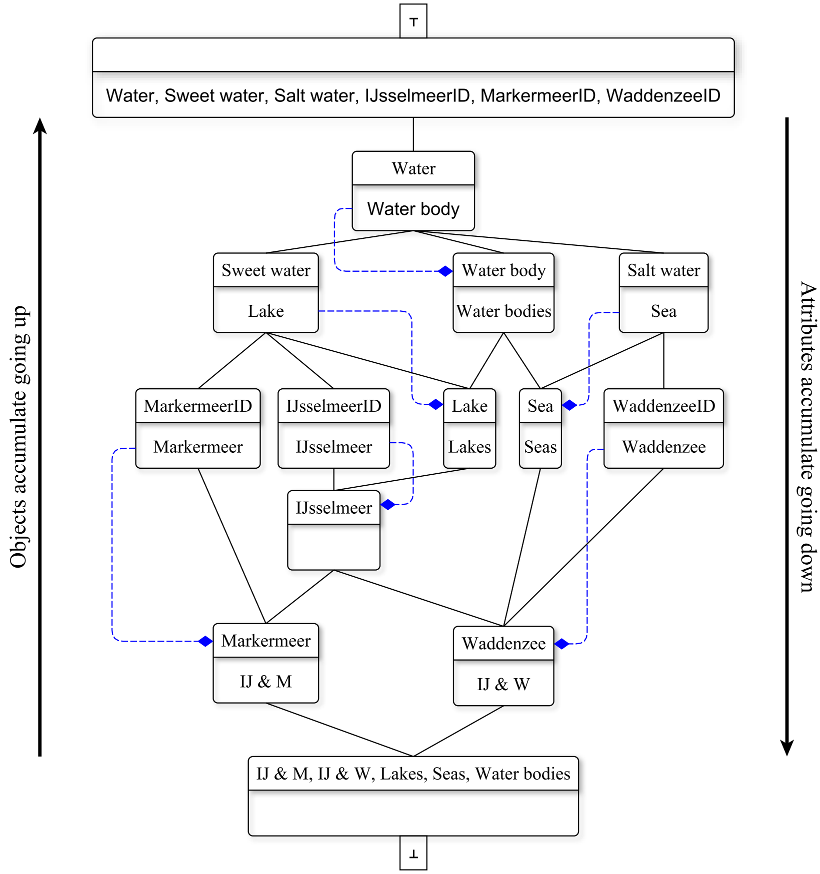


Figure 3.: Concept lattice derived from incidences $x \cup T$ in Table 1. Attributes are shown in the upper half and objects in the lower half of the nodes. Black lines denote the $\leq$-order according to the principles of [3]. Sub-concepts inherit attributes from super-concepts and super-concepts inherit objects from sub-concepts. Blue arrows denote the $\preceq$-relation, with the arrowhead landing at the most general preceded concept.

and het Markermeer are no longer sub-concepts of the concept of lake. C_o*(IJsselmeer)* also has an attribute *IJsselmeer*, which C_o*(Lake)* does not and C_o*(Lake)* has an attribute *Lake*, which C_o*(IJsselmeer)* does not.

There are also examples of continuous homeomerosity. For instance, C_o*(Lakes)* is continuously homeomerous with respect to C_o*(Water)*, since all its attributes are also continuously homeomerous with respect to C_o*(Water)*. Due to reflexivity C_o*(Water)* is also continuously reflexive with respect to itself. Note that, however, C_o*(Lakes)* is not continuously homeomerous with respect to, e.g., C_o*(Lake)*, since not all attributes of the latter are themselves lakes.

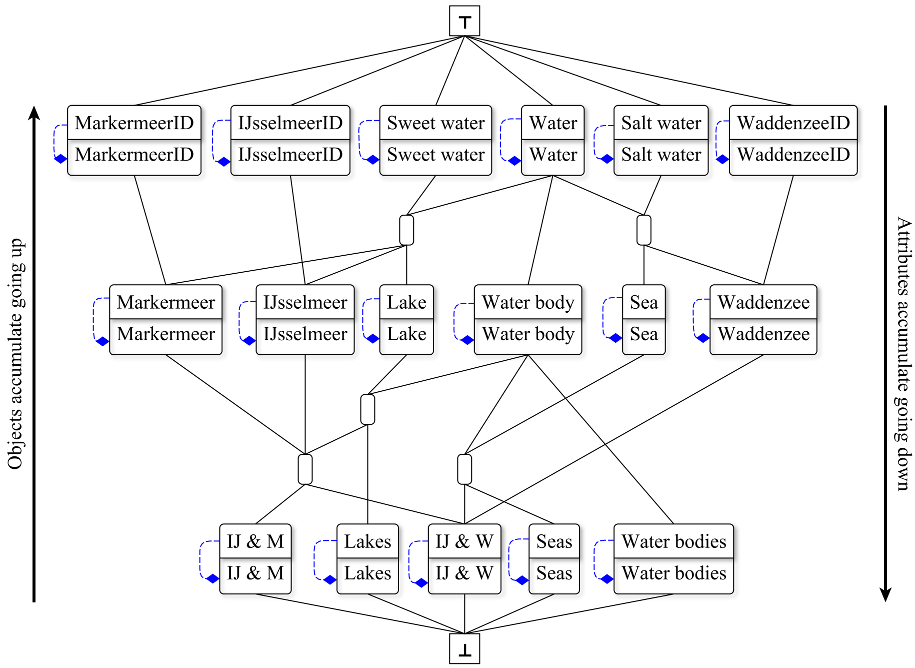


Figure 4.: Concept lattice derived from incidences $x \cup T \cup R$ in Table 1. Attributes are shown in the upper half and objects in the lower half of the nodes. Black lines denote the $\leq$-order according to the principles of [3]. Sub-concepts inherit attributes from super-concepts and super-concepts inherit objects from sub-concepts. Blue arrows denote the $\preceq$-relation, with the arrowhead landing at the most general preceded concept.

4. Discussion and conclusion

In this study we investigated how to implement the notion of homeomerosity in formal concept analysis (FCA). We considered a thing homeomerous if it has what it is and is what it has. We used the specialization operator ($\leq$) as an analogy to the verb ‘being’. Additionally, we formalized a new relation between FCA-concepts, namely dependence (the $\preceq$-relation), to formalize a notion of ‘having’. From the synergy between the two relations we defined a notion of homeomerosity in FCA. We tested our definition on a simple example of an analytical operation commonly-applied in geographic information science. The results show that a theory of homeomerosity in FCA is tenable for both parthood and proper parthood relations in mereology. We also demonstrate that letting the incidence relation have varying properties, such as reflexivity and transitivity, has profound effects on the conceptual outcomes.

We should note that the results are still somewhat preliminary, since our definitions are not fully axiomatized. Specifically, we have not provided a formal mereology and instead relied on parthood examples. We leave this formalization for future work. Should our findings hold, they would have important implications for the field of mereology. It is said that ”*nothing substantive follows from* [choosing proper parthood as a primitive instead of parthood]” [23, section 2.1], but our results show that whether or not something is considered part of itself leads to different conceptualizations. Precisely, the $\leq$-order and $\preceq$-relation arising from an irreflexive incidence are not embedded in the concept lattice that arises from the incidence's reflexive extension. This problem can potentially be resolved by annotating additional incidences according to sub-concept - super-concept relations in the $\leq$-order (E.g., if sea is part of itself, then it also has a water body as part), but this begs the question of whether this knowledge can be considered available *a priori*, i.e., before the $\leq$-order is established.

Our definition of homeomerosity is not affected by this problem. As we have shown, it generalizes over both cases. We considered how merges of water bodies would stand to the individual water bodies in the same FCA-context. Depending on the assumed properties of the incidence relations, they are either $\leq$-unordered or the merges are sub-concepts of the individual water bodies. In no case is a specific merge homeomerous with respect to one of its individuals, but in generality, they have been found to both be water bodies.

We discovered the FCA-specific notion of dependence during our search for a notion of 'having' for the definition of homeomerosity. From our initial experience, it seems thedependence relation is a powerful tool for reasoning in FCA, on par with the specification order. We have not found earlier work (e.g., not in [18]) where such a relation is formalized.

Our tests of continuous homeomerosity were limited and questions remain on this part. It seems infinitely-continuous homeomerosity only applies to concepts with respect to properly-reflexive concepts. What we mean by the latter are concepts where $A = B$. This could be problematic because this gives too little potential for defining meaningful concepts. For example, things can only be continuously-homeomerous with respect to water if water is not defined in terms of its parts (e.g., three atoms), but only in terms of it being incidental with itself. There are two possible approaches for alleviating this problem. First, the definition itself could be revised, and second, the theory could be extended to host maneuvering multiple contexts. In the latter case, something could be continuously-homeomerous in one context and not in another. We believe practice shows evidence of the latter. In geographic information science, a geometric shape representing water can be partitioned infinitely many times without losing its representative quality of being water. At no point, the water is split into atoms.

We also did not consider how merges of merges stand to merges and the merged individuals. There seem to be two possibilities. Firstly, a merge of merges could be specified as a merge of the individuals. This would make it a sub-concept of the first merges. It could also be specified with the first merges as attributes. In either case, we expect homeomerosity to be preserved. To show this requires overcoming challenges regarding the specification of the concepts themselves. Strictly speaking, this was beyond our goals in this paper. However, for a notion of homeomerosity to become workable, this seems like a problem that should be addressed in future research.

If so, we believe homeomerosity and FCA can be applied for the development of a metadata structure that automatically tracks class parthood through mereological procedures. We showed merges as examples, but we believe this could also be applied during geometric intersections and maybe even some data transformations.

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