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Computing binary curves of genus five

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ABSTRACT

Genus 5 curves can be hyperelliptic, trigonal, or non-hyperelliptic non-trigonal, whose model is a complete intersection of three quadrics in $\mathbb{P}^4$. We present and explain algorithms we used to determine, up to isomorphism over $\mathbb{F}_2$, all genus 5 curves defined over $\mathbb{F}_2$, and we do that separately for each of the three mentioned types. We consider these curves in terms of isogeny classes over $\mathbb{F}_2$ of their Jacobians or their Newton polygons, and for each of the three types, we compute the number of curves over $\mathbb{F}_2$ weighted by the size of their $\mathbb{F}_2$-automorphism groups.

1. Introduction

A standard result is that the smooth curves of genus 5 are either hyperelliptic, trigonal, or complete intersections of three quadric hypersurfaces in $\mathbb{P}^4$; see [8, Section IV.5]. Therefore, to understand the moduli space $\mathcal{M}_5$ of smooth curves of genus 5, we should consider the subvarieties parametrizing these three kinds of smooth curves. Denote with $\mathcal{H}_5$ the subvariety of $\mathcal{M}_5$ parametrizing hyperelliptic curves of genus 5, with $\mathcal{T}_5$ the subvariety parametrizing trigonal curves of genus 5, and lastly, let $\mathcal{U}_5$ be the subvariety parametrizing non-hyperelliptic non-trigonal curves, whose canonical model in $\mathbb{P}^4$ is a complete intersection of three quadric hypersurfaces. Moreover, let us write $\mathrm{Hyp}_5(\mathbb{F}_2)$, $\mathrm{Trig}_5(\mathbb{F}_2)$, and $\mathrm{CompInt}_5(\mathbb{F}_2)$, respectively, for the sets of hyperelliptic, trigonal, and non-hyperelliptic nontrigonal (smooth) curves of genus 5 defined over $\mathbb{F}_2$, up to $\mathbb{F}_2$-isomorphisms.

This paper aims to give algorithms for computing all the $\mathbb{F}_2$-isomorphism classes of smooth curves of genus 5 defined over $\mathbb{F}_2$ and the sizes of their $\mathbb{F}_2$-automorphism groups, present the obtained results, and discuss some relevant questions, such as describing the isogeny types or Newton polygons of Jacobians of dimension 5 over $\mathbb{F}_2$; see [5, Sections 2.3, 2.4, and 3.3] for more details on computing the mentioned invariants

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over finite fields. We do that separately for curves in $\mathrm{Hyp}_5(\mathbb{F}_2)$, $\mathrm{Trig}_5(\mathbb{F}_2)$, and $\mathrm{CompInt}_5(\mathbb{F}_2)$. The sizes of the $\mathbb{F}_2$-automorphism groups that we obtain lead us to find the stack count $|\mathcal{M}_5(\mathbb{F}_2)|$ and consequently to get a piece of information about the cohomology of $\mathcal{M}_5$.

In [12], Xarles determined all curves of genus 4 defined over $\mathbb{F}_2$. His approach to computing the hyperelliptic curves is a universal one. We follow it closely in Section 2 and apply it to genus 5 hyperelliptic curves.

Our algorithm for computing the representatives of the isomorphism classes of trigonal curves is based on the explicit description of their models in $\mathbb{P}^2$ and the idea of the exhaustion of all eligible equations respecting the isomorphism; we present it in Section 3.

Lastly, for the non-hyperelliptic non-trigonal curves, a similar but more subtle idea of exhaustion of the eligible triples of quadratic polynomials in $\mathbb{F}_2[X, Y, Z, T, U]$ was used. We extensively explain the steps in our reasoning preceding the final algorithm we used for this problem and mention some intermediate steps and partial results in Section 4.

In Section 5, we discuss some of the outcomes of our computations and crosscheck them with some known results. The stack counts we get, $|\mathcal{H}_5(\mathbb{F}_2)|$ and $|\mathcal{T}_5(\mathbb{F}_2)|$, match the ones from [3] and [11], while $|\mathcal{U}_5(\mathbb{F}_2)| = 2^{12} - 2^9$ was not known before. The maximum numbers of $\mathbb{F}_2$-points for curves of genus 5 over $\mathbb{F}_2$ of the three considered types match the corresponding ones from [6] and [7]. The values of certain sums over the curves in $\mathrm{Hyp}_5(\mathbb{F}_2)$ that we get match the expected ones from [2]. Finally, among the other examples, we show that each eligible Newton polygon of height $2 \cdot 5 = 10$ occurs for a Jacobian of a smooth curve of genus 5 over $\mathbb{F}_2$.

We implement all the algorithms in SageMath, [10]. Our implementations and the obtained data of the curves and the sizes of their $\mathbb{F}_2$-automorphism groups are available on

https://github.com/DusanDragutinovic/MT_Curves.

The data has been added to the L-functions and modular forms database [9] in the section https://www.lmfdb.org/Variety/Abelian/Fq/.

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2. Hyperelliptic curves

Any hyperelliptic curve of genus g over $\mathbb{F}_2$ can be given by a standard (affine) equation

$$y^2 + q(x)y = p(x), \quad \text{for } p(x), q(x) \in \mathbb{F}_2[x], \tag{1}$$

with $2g + 1 \leq \max\{2\deg(q(x)), \deg(p(x))\} \leq 2g + 2$.

In [12], Xarles gave the approach to compute all (smooth) curves of genus 4 over $\mathbb{F}_2$ up to isomorphism. The presented algorithm for determining the hyperelliptic curves over $\mathbb{F}_2$ can be generalized to higher genera. Here, we use it to obtain the set $\mathrm{Hyp}_5(\mathbb{F}_2)$. We can directly use some of the claims made in [12, Section 1], and we mention the analogs of the others in the genus 5 case.

Let $\mathbb{F}_2[x]_n = \{h(x) \in \mathbb{F}_2[x] : \deg(h(x)) \leq n\}$ for $n \in \mathbb{Z}_{\geq 0}$, and for $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PGL}_2(\mathbb{F}_2)$ and $q(x) \in \mathbb{F}_2[x]_n$, define an action of $\mathrm{PGL}_2(\mathbb{F}_2)$ on $\mathbb{F}_2[x]_n$ by

$$\psi_n(A)(q(x)) = (cx+d)^n q\left(\frac{ax+b}{cx+d}\right);$$

we will also use the notation $A.q(x)$ for this. Further, denote the quotient set of $\mathbb{F}_2[x]_n$ under this action by $\overline{\mathbb{F}_2[x]_n} = \mathbb{F}_2[x]_n/\mathrm{PGL}_2(\mathbb{F}_2)$.

Let $H_1 : y^2 + q_1(x)y = p_1(x)$ and $H_2 : y^2 + q_2(x)y = p_2(x)$ be two hyperelliptic curves over $\mathbb{F}_2$ as in (1). Using that any isomorphism of such H_1 and H_2 has to be of the form

$$(x, y) \mapsto \left(\frac{ax+b}{cx+d}, \frac{r(x)+y}{(cx+d)^{g+1}}\right)$$

for some $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PGL}_2(\mathbb{F}_2)$ and $r(x) \in \mathbb{F}_2[x]_{g+1}$, Xarles showed the following lemma.

Lemma 2.1 *([12, Lemma 1]). Let H_1 and H_2 be as above and suppose $H_1 \cong H_2$. Then there exists $A \in \mathrm{PGL}_2(\mathbb{F}_2)$ such that $q_2(x) = \psi_{g+1}(A)(q_1(x))$.*

For any $q(x) \in \overline{\mathbb{F}_2[x]_{g+1}}$, let $\mathrm{Stab}(q(x))$ be the stabilizer of $q(x)$ under the $\mathrm{PGL}_2(\mathbb{F}_2)$-action. We cite two more results from [12].

Lemma 2.2 *([12, Lemma 4]). Let H_1 and H_2 be two hyperelliptic curves of genus g over $\mathbb{F}_2$ given by standard equations (1): $y^2 + q(x)y = p_i(x)$, $i \in \{1, 2\}$. If H_1 and H_2 are isomorphic over $\mathbb{F}_2$, then there are $A \in \mathrm{Stab}(q(x))$ and $r(x) \in \mathbb{F}_2[x], \deg(r(x)) \leq g+1$ such that*

$$p_2(x) = \psi_{2g+1}(A)(p_1(x) + r(x)^2 + q(x)r(x)).$$

Lemma 2.3 *([12, Lemma 5]). Let $g \in \mathbb{Z}_{\geq 2}$. Given a nonzero polynomial $q(x) \in \mathbb{F}_2[x]$ and a polynomial $p(x) \in \mathbb{F}_2[x]$ with $2g+1 \leq \max\{2\deg(q(x)), \deg(p(x))\} \leq 2g+2$, the equation $y^2 + q(x)y = p(x)$ defines a hyperelliptic curve of genus g if and only if*

$$\gcd(q(x), p'(x)^2 + q'(x)^2 p(x)) = 1,$$

and either $\deg(q(x)) = g+1$ or $a_{2g+1}^2 \neq a_{2g+2}b_g^2$, where $p(x) = \sum_{i=0}^{2g+2} a_i x^i$ and $q(x) = \sum_{i=0}^{g+1} b_i x^i$.

Using the lemmas above, to compute $\mathrm{Hyp}_5(\mathbb{F}_2)$, we should first find $Q_5(\mathbb{F}_2)$, a complete set of representatives of $\overline{\mathbb{F}_2[x]_6}$. We do that below using the ideas from [12, Lemma 2].

Lemma 2.4. *For $q(x) \in \mathbb{F}_2[x]_6$ and $\mathcal{Z}'(q(x)) = \{P \in \overline{\mathbb{F}}_2 : q(P) = 0\}$, let*

$$D_{q(x)} = \mathcal{Z}'(q(x)) + (6 - \deg(q(x))) \cdot \infty$$

be the zero divisor of $q(x)$ in $\mathbb{P}^1$. Then the action of $\mathrm{PGL}_2(\mathbb{F}_2)$ on $\mathbb{F}_2[x]_6$ naturally translates to the (standard) action of $\mathrm{PGL}_2(\mathbb{F}_2)$ on $\mathrm{Div}_6(\mathbb{F}_2)$, and these actions are compatible, i.e., $D_{A.q(x)} = A.D_{q(x)}$.

Proof. For an arbitrary polynomial $q(x) = b_6x^6 + b_5x^5 + \ldots + b_1x + b_0 \in \mathbb{F}_2[x]_6$ and a matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PGL}_2(\mathbb{F}_2)$ we compute

$$q_{new}(x) = A.q(x) = b_6(ax+b)^6 + b_5(ax+b)^5(cx+d) + \ldots + b_1(ax+b)(cx+d)^5 + b_0(cx+d)^6.$$

For $P = d/c = -d/c$, with $c \neq 0$, we see that $P \in \mathcal{Z}'(A.q(x))$ if and only if $\deg(q(x)) < 6$, and moreover, its multiplicity as a zero of $A.q(x)$ is precisely $6 - \deg(q(x))$; this means that the multiplicity of $P = d/c$ in

$D_{A.q(x)}$ is the same as the multiplicity of ∞ in $D_{q(x)}$. Using A^{-1} and changing the roles of $q(x)$ and $q_{new}(x)$ we can similarly get the conclusion on the degree of $q_{new}(x)$ when inspecting $P = \infty$. For other $P \in \overline{\mathbb{F}}_2$, we see $P \in \mathcal{Z}'(A.q(x))$ if and only if $\frac{aP+b}{cP+d} \in \mathcal{Z}'(q(x))$ and the corresponding multiplicities match. The result follows. $\square$

The preceding lemma implies that determining the set of representatives of $\overline{\mathbb{F}_2[x]_6}$ is equivalent to finding the one for $\mathrm{Div}_6(\mathbb{F}_2)/\mathrm{PGL}_2(\mathbb{F}_2)$. We use that in the following theorem.

Theorem 2.5. *The set $Q_5(\mathbb{F}_2)$ which consists of the elements $q(x) \in \mathbb{F}_2[x]$ listed below is a complete set of representatives of $\overline{\mathbb{F}_2[x]_6}$:*

$\deg(q(x)) \leq 2$: $1, x, x^2, x(x+1), x^2+x+1$
$\deg(q(x)) = 3$: $x^3, x^2(x+1), (x^2+x+1)x, x^3+x+1$
$\deg(q(x)) = 4$: $x^2(x+1)^2, (x^2+x+1)^2, (x^2+x+1)x^2, (x^2+x+1)x(x+1), (x^3+x+1)x, (x^3+x^2+1)x, x^4+x+1, x^4+x^3+1$
$\deg(q(x)) = 5$: $(x^2+x+1)^2x, (x^3+x+1)(x^2+x+1), (x^3+x+1)x(x+1), (x^4+x+1)x, (x^4+x^3+x^2+x+1)x, x^5+x^2+1, x^5+x^3+1, x^5+x^3+x^2+x+1$
$\deg(q(x)) = 6$: $(x^2+x+1)^3, (x^3+x+1)^2, (x^3+x+1)(x^3+x^2+1), (x^4+x+1)(x^2+x+1), x^6+x+1, x^6+x^3+1$.

Proof. For a polynomial $q(x) \in \mathbb{F}_2[x]_6$, let $D_{q(x)}$ be as in Lemma 2.4. With ζ_n, we denote any element $\zeta_n \in \overline{\mathbb{F}}_2$ of degree n over $\mathbb{F}_2$. We use the well-known fact that given any three $\mathbb{F}_2$-points p_∞, p_0, p_1 there is a (unique) projective automorphism $A \in \mathrm{PGL}_2(\mathbb{F}_2)$ that maps $p_\infty \mapsto \infty, p_0 \mapsto 0,$ and $p_1 \mapsto 1$.

Firstly, any $D_{q(x)}$ that consists only of $\mathbb{F}_2$-points in $\mathrm{Div}_6(\mathbb{F}_2)/\mathrm{PGL}_2(\mathbb{F}_2)$ is equal to the unique one $n_\infty \cdot \infty + n_0 \cdot 0 + n_1 \cdot 1$ with $n_1 \leq n_0 \leq n_\infty$. Since $\deg(D_{q(x)}) = 6$, we get that all the possible triples (n_∞, n_0, n_1) are $\{(6,0,0), (5,1,0), (4,2,0), (4,1,1), (3,3,0), (3,2,1), (2,2,2)\}$. Using the correspondence from Lemma 2.4, this gives us the subset of polynomials $q(x)$ in $Q_5(\mathbb{F}_2)$,

$$\{1, x, x^2, x(x+1), x^3, x^2(x+1), x^2(x+1)^2\}.$$

If $D_{q(x)}$ contains only one point of degree 2 and no other points of degree ≥ 2 in its support, similarly as above, we get that $D_{q(x)}$ is equal to one of

$$3\zeta_2,\ 2\zeta_2 + 2\infty,\ 2\zeta_2 + \infty + 0,\ \zeta_2 + 4\infty,\ \zeta_2 + 3\infty + 0,\ \zeta_2 + 2\infty + 2 \cdot 0,\ \zeta_2 + 2\infty + 0 + 1.$$

This induces the set of polynomials in $Q_5(\mathbb{F}_2)$:

$$\{(x^2+x+1)^3, (x^2+x+1)^2, (x^2+x+1)^2x, x^2+x+1, (x^2+x+1)x, (x^2+x+1)x^2, (x^2+x+1)x(x+1)\}.$$

If $D_{q(x)}$ contains a point of degree 3 in its support, then

$$D_{q(x)} \in \{2\zeta_3, \zeta_3 + \zeta_3', \zeta_3 + \zeta_2 + \infty, \zeta_3 + 3\infty, \zeta_3 + 2\infty + 0, \zeta_3 + \infty + 0 + 1\},$$

where ζ_3, ζ_3' are of degree 3. The mapping $x \mapsto x+1$ (induced by the action of $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$), translates $q_1(x) = x^3+x+1$ into $q_2(x) = x^3+x^2+1$. Therefore, in $\mathrm{Div}_6(\mathbb{F}_2)/\mathrm{PGL}_2(\mathbb{F}_2)$, we have that $D_{q_1(x)}$ equals $D_{q_2(x)}$, and moreover, that $D_{(q_1(x))^2}, D_{(q_1(x))(x^2+x+1)}$, and $D_{(q_1(x))x(x+1)}$ are, respectively, equal to $D_{(q_2(x))^2}, D_{(q_2(x))(x^2+x+1)}$, and $D_{(q_2(x))x(x+1)}$. The set of possible polynomials $q(x)$ we get from this case is: $\{(x^3+x+1)^2, (x^3+x+1)(x^3+x^2+1), (x^3+x+1)(x^2+x+1), x^3+x+1, (x^3+x+1)x, (x^3+x^2+1)x, (x^3+x+1)x(x+1)\}$.

If there is a point of degree 4 in the support of $D_{q(x)}$, $D_{q(x)}$ is of form $\zeta_4 + \zeta_2, \zeta_4 + 2\infty$, or $\zeta_4 + \infty + 0$. There are three irreducible polynomials over $\mathbb{F}_2$ of degree 4, so out of all possible combinations, discussing the $\mathrm{PGL}_2(\mathbb{F}_2)$ action on $\mathbb{F}_2[x]_6$ as above, we extract the following list of representatives for $q(x)$:

$$\{(x^4+x+1)(x^2+x+1), x^4+x+1, x^4+x^3+1, (x^4+x+1)x, (x^4+x^3+x^2+x+1)x\}.$$

If the support of $D_{q(x)}$ contains a point of degree 5, there is only one possibility for the form of $D_{q(x)}$, namely, $D_{q(x)} = \zeta_5 + \infty$. Among the six irreducible polynomials of degree 5, we found that, for example, the following three are representatives of $q(x)$ for the considered action:

$$\{x^5+x+1, x^5+x^3+1, x^5+x^3+x^2+x+1\}.$$

Lastly, among the nine irreducible polynomials of degree 6, we found two:

$$x^6+x+1 \text{ and } x^6+x^3+1,$$

such that by acting via $\mathrm{PGL}_2(\mathbb{F}_2)$ on them, we can get all the others. This corresponds to a choice of the divisor $D_{q(x)} = \zeta_6$, with ζ_6 (a point of degree 6, which is either) a zero of x^6+x+1 or a zero of x^6+x^3+1 in $\overline{\mathbb{F}}_2$. $\square$

The previously described reasoning leads to an algorithm for computing the set $\mathrm{Hyp}_5(\mathbb{F}_2)$, which is practically the same as the algorithm from [12, p. 6] for computing $\mathrm{Hyp}_4(\mathbb{F}_2)$, the set of all (smooth) hyperelliptic curves of genus 4 over $\mathbb{F}_2$, up to $\mathbb{F}_2$-isomorphism.

Algorithm 1. Determine $\mathrm{Hyp}_5(\mathbb{F}_2)$.

Step 0 From the preceding theorem, we get **list_of_qs** $= Q_5(\mathbb{F}_2)$, the list of all possible representatives for a polynomial $q(x)$.

Step 1 For each $q(x)$ in **list_of_qs**, compute the stabilizer $\mathrm{Stab}(q(x)) \subseteq \mathrm{PGL}_2(\mathbb{F}_2)$ of $q(x)$ under the action defined by $\psi_6(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right))(q(x)) = (cx+d)^6 q(\frac{ax+b}{cx+d})$ for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PGL}_2(\mathbb{F}_2)$.

Step 2 For a fixed $q(x)$ in **list_of_qs**, check whether a polynomial $p(x) \in \mathbb{F}_2[x]$, which satisfies $11 \leq \max\{2\deg(q(x)), \deg(p(x))\} \leq 12$, is such that $C : y^2 + q(x)y = p(x)$ is a (nonsingular) curve; collect all such $p(x)$'s in the list **q_list_of_ps** of potential $p(x)$'s for $q(x)$. The smoothness condition can be checked using Lemma 2.3, saying that C is a (nonsingular) curve of genus 5 if and only if $\gcd(q(x), p'(x)^2 + q'(x)^2 p(x)) = 1$ and either $\deg(q(x)) = 6$ or $a_{11}^2 \neq a_{12} b_5^2$, where $p(x) = \sum_{i=0}^{12} a_i x^i$ and $q(x) = \sum_{i=0}^{6} b_i x^i$.

Step 3 Fix a $q(x)$ in **list_of_qs** and consider **q_list_of_ps**, the list of potential $p(x)$'s associated with it. We write $p_1(x) \sim p_2(x)$ if the curves $C_1 : y^2 + q(x)y = p_1(x)$ and $C_2 : y^2 + q(x)y = p_2(x)$ are isomorphic over $\mathbb{F}_2$. Using Lemma 2.2, we find that the relation $\sim$ is defined as: $p_1(x) \sim p_2(x)$ if and only if $(cx+d)^{12} p_2(\frac{ax+b}{cx+d}) = p_1(x) + r(x)^2 + r(x)q(x)$ for some $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Stab}(q(x))$ and some $r(x) \in \mathbb{F}_2[x]$ of degree $\deg(q(x)) \leq 6$. We iterate over all such $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $r(x)$ to refine **q_list_of_ps** by taking only the representatives $p(x)$ for this relation $\sim$.

In such a manner, using the mathematical software SageMath, we computed the list of all non-isomorphic hyperelliptic curves of genus 5 defined over $\mathbb{F}_2$. There are in total 1070 such curves, i.e., $|\mathrm{Hyp}_5(\mathbb{F}_2)| = 1070$, and we confirmed [3, Proposition 7.1] that

$$|\mathcal{H}_5(\mathbb{F}_2)| = \sum_{C \in \mathrm{Hyp}_5(\mathbb{F}_2)} \frac{1}{|\mathrm{Aut}_{\mathbb{F}_2}(C)|} = 512 = 2^{2\cdot 5 - 1}.$$

In addition, for the curves in $\mathrm{Hyp}_5(\mathbb{F}_2)$, we computed their numbers of points over finite fields $\mathbb{F}_{2^N}$, for $N \in \{1,2,3,4,5\}$, and then we found their Newton polygons. We mention some of the observations we got in Section 5.

3. Trigonal curves

We say that a singularity of a plane (projective) curve C is of *delta invariant one* if it is either a node (an ordinary double point), where a curve is locally of the form $xy = 0$, or an ordinary cusp, where C is locally of the form $y^2 = x^3$.

Let C be a trigonal curve of genus 5, and let $D \in \mathfrak{g}^1_3$ on C. Then, $K_C - D$ belongs to $\mathfrak{g}^2_5$, the dual linear system of that $\mathfrak{g}^1_3$ on C, and we can use it to get a morphism $C \to \mathbb{P}^2$ such that the image of C is a plane quintic. Moreover, the computations based on the Riemann-Roch theorem and the genus-degree formula, together with some further reasoning, give us the following well-known fact, which we use to compute $\mathrm{Trig}_5(\mathbb{F}_2)$.

Theorem 3.1 *([1, p. 207]). A curve C of genus 5 is trigonal if and only if it can be represented as a plane quintic with one singularity of delta invariant one and no other singularities.*

Any isomorphism of such curves C_1 and C_2 extends to an automorphism of $\mathbb{P}^2$. (Note that the $\mathfrak{g}^1_3$ on C_i is unique, and so is its dual $\mathfrak{g}^2_5$; see [8, Exercise IV.5.5] or [1, p. 207].)

For a matrix $M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$ in $\mathrm{PGL}_3(\mathbb{F}_2)$ and $q(X,Y,Z)$ a homogeneous polynomial in $\mathbb{F}_2[X,Y,Z]$, the formula

$$M.q(X,Y,Z) = q(m_{11}X + m_{12}Y + m_{13}Z, m_{21}X + m_{22}Y + m_{23}Z, m_{31}X + m_{32}Y + m_{33}Z) \tag{2}$$

defines an action of $\mathrm{PGL}_3(\mathbb{F}_2)$ on the set of all homogeneous polynomials in $\mathbb{F}_2[X,Y,Z]$. Alternatively, we can define $M.q(X,Y,Z) = q(M.(X,Y,Z))$ with

$$M.(X,Y,Z) = M \cdot (X,Y,Z)^t.$$

Therefore, to determine the list of all trigonal curves of genus 5 defined over $\mathbb{F}_2$, it is sufficient to find the $\mathrm{PGL}_3(\mathbb{F}_2)$-representatives among all the quintic homogeneous polynomials in X,Y,Z that define projective plane curves with one singularity of delta invariant one and no other singularities.

To compute all trigonal curves of genus 5 over $\mathbb{F}_2$, we have implemented the following algorithm in SageMath.

Algorithm 2. Determine $\mathrm{Trig}_5(\mathbb{F}_2)$.

Step 1 Make a list of all the monomials in X,Y,Z of degree 5 and fix the order of these, e.g. the lexicographic order $X^5 > X^4Y > X^4Z > X^3Y^2 > X^3YZ > X^3Z^2 > X^2Y^3 > X^2Y^2Z > X^2YZ^2 > X^2Z^3 > XY^4 > XY^3Z > XY^2Z^2 > XYZ^3 > XZ^4 > Y^5 > Y^4Z > Y^3Z^2 > Y^2Z^3 > YZ^4 > Z^5$. Since there are 21 monomials, we can represent all homogeneous polynomials of degree 5 using the 21-tuples in $(\mathbb{F}_2)^{21} - \{0\}$. Call **quintics** the lexicographically sorted list of all 21-tuples. The action of $\mathrm{PGL}_3(\mathbb{F}_2)$ on **quintics** is induced by acting on X,Y,Z and using the mentioned correspondence. (*For example,* $X^5 + YZ^4 \longleftrightarrow (1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0)$ *under this correspondence.*)

Step 2 Get list **quintics_repr** of the representatives of the quotient set **quintics**/$\mathrm{PGL}_3(\mathbb{F}_2)$ formed as follows. Start from an empty list **quintics_repr**. Take the first element from **quintics**, put it in **quintics_repr**, delete from **quintics** all of its $\mathrm{PGL}_3(\mathbb{F}_2)$-conjugates, and repeat this until **quintics** is empty.

Step 3 Deduce whether a plane quintic corresponding to an element of **quintics_repr** has exactly one singularity of order 2 (and no other singularities) to reduce the preceding list and get the list **good_quintics**.

Step 4 For each quintic with exactly one singularity P of order 2, represented by an element of **good_quintics**, find a $\mathrm{PGL}_3(\mathbb{F}_2)$-isomorphic quintic $q_{(0:0:1)}$ with a singularity at the point $(0:0:1)$ that, locally at $(0:0:1)$ up to order 3, equals to either xy or x^2+xy+y^2 (nodal case), or y^2 (potentially cuspidal case). We call **good_quintics_001** the list of all the 21-tuples corresponding to such quintics $q_{(0:0:1)}$.

Step 5 For each element of **good_quintics_001** with the corresponding quintic from the potentially cuspidal case, decide whether there is a $\mathrm{PGL}_3(\mathbb{F}_2)$-isomorphic quintic to it, with lowest terms y^2+x^3 (locally at $(0:0:1)$) - if there is none, delete that element from **good_quintics_001**. Collect all the curves defined by the quintics corresponding to the remaining elements of **good_quintics_001** into the resulting set $\mathrm{Trig}_5(\mathbb{F}_2)$.

We found 2854 non-isomorphic trigonal curves over $\mathbb{F}_2$, i.e., $|\mathrm{Trig}_5(\mathbb{F}_2)| = 2854$, and we computed the sizes of their automorphism groups over $\mathbb{F}_2$. In particular, we have obtained that

$$|\mathcal{T}_5(\mathbb{F}_2)| = \sum_{C\in\mathrm{Trig}_5(\mathbb{F}_2)} \frac{1}{|\mathrm{Aut}_{\mathbb{F}_2}(C)|},$$

the number of (non-isomorphic) smooth trigonal curves of genus 5 defined over the finite field with two elements weighted by the size of their automorphism group, i.e. the stack count for trigonal curves of genus 5 over $\mathbb{F}_2$, precisely equals

$$|\mathcal{T}_5(\mathbb{F}_2)| = 2817 = 2^{11} + 2^{10} - 2^8 + 1.$$

This matches Wennink's results from [11, Theorem 1], where he, using a partial sieve method for plane curves, computed these weighted numbers for any finite field with q elements $\mathbb{F}_q$, and obtained $|\mathcal{T}_5(\mathbb{F}_q)| = q^{11} + q^{10} - q^8 + 1$. We comment further on the obtained results in Section 5.

4. Complete intersections of three quadrics in $\mathbb{P}^4$

The remaining set we need to compute is $\mathrm{CompInt}_5(\mathbb{F}_2)$, the set of curves of genus 5 over $\mathbb{F}_2$ are the ones whose canonical embedding in $\mathbb{P}^4$ is a complete intersection of three quadric hypersurfaces; see [8, Example IV.5.5.3]. In other words, its elements are of the form

$$C = Z(q_P, q_Q, q_R),$$

for $q_P, q_Q, q_R \in \mathbb{F}_2[X,Y,Z,T,U]$, three homogeneous geometrically irreducible polynomials of degree 2 with no non-trivial $\mathbb{F}_2$-linear relation between them.

The idea behind computing curves C in $\mathrm{CompInt}_5(\mathbb{F}_2)$ is as follows. First, we find a suitable set Σ of triples (q_P, q_Q, q_R) of homogeneous quadratic polynomials in $\mathbb{F}_2[X,Y,Z,T,U]$, such that, for any C in $\mathrm{CompInt}_5(\mathbb{F}_2)$, there is an element of Σ defining a curve isomorphic to C. We want the first quadrics q_P of the triples in Σ to be the representatives under the projective automorphisms of $\mathbb{P}^4$ and that Σ is of a reasonable size. Then, we filter the set Σ to get $\mathrm{CompInt}_5(\mathbb{F}_2)$, in which no two elements define curves isomorphic over $\mathbb{F}_2$.

Recall that the curves with canonical embedding into $\mathbb{P}^4$ are isomorphic over $\mathbb{F}_2$ if and only if their canonical models in $\mathbb{P}^4$ are isomorphic via some projective automorphism $M \in \mathrm{PGL}_5(\mathbb{F}_2)$.

The group $\mathrm{PGL}_5(\mathbb{F}_2)$ acts on the subset of homogeneous quadratic polynomials in the polynomial ring $\mathbb{F}_2[X, Y, Z, T, U]$ by acting on variables X, Y, Z, T, U via

$$M.(X, Y, Z, T, U) = M \cdot (X, Y, Z, T, U)^t \tag{3}$$

for $M \in \mathrm{PGL}_5(\mathbb{F}_2)$. This induces the right action on the set of homogeneous quadratic polynomials.

For the practical reasons of working in the mathematical software SageMath, we represent the quadrics

$$\begin{aligned} q_P = p_0X^2 + p_1XY + p_2XZ + p_3XT + p_4XU + p_5Y^2 + p_6YZ + p_7YT+ \\ + p_8YU + p_9Z^2 + p_{10}ZT + p_{11}ZU + p_{12}T^2 + p_{13}TU + p_{14}U^2 \end{aligned}$$

using the 15-tuples $P \in (\mathbb{F}_2)^{15} - \{0\}$ of the coefficients in q_P,

$$P = (p_0, p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}, p_{11}, p_{12}, p_{13}, p_{14}). \tag{4}$$

As usual, with $\mathrm{Stab}(q)$ and $\mathrm{Orbit}(q)$ we denote the stabilizer and the orbit of a homogeneous quadratic polynomial q with respect to the action (3). Using the correspondence above, for a 15-tuple $P \in (\mathbb{F}_2)^{15} - \{0\}$ we set

$$\mathrm{Stab}(P) := \mathrm{Stab}(q_P) \quad \text{and} \quad \mathrm{Orbit}(P) := \mathrm{Orbit}(q_P).$$

To justify this notation, we can also define the action of $\mathrm{PGL}_5(\mathbb{F}_2)$ on $(\mathbb{F}_2)^{15} - \{0\}$, where we define $M.P = Q$, for $M \in \mathrm{PGL}_5(\mathbb{F}_2)$ and $P, Q \in (\mathbb{F}_2)^{15} - \{0\}$ if it holds that $M.q_P = q_Q$.

First, we determine the set Σ.

Algorithm 3. Determine Σ.
<u>*Description.*</u> *Form a set Σ of triples of quadrics (q_P, q_Q, q_R) such that for any non-hyperelliptic non-trigonal curve C of genus five over $\mathbb{F}_2$, there is a triple (q'_P, q'_Q, q'_R) in Σ such that C is isomorphic to $Z(q'_P, q'_Q, q'_R)$ over $\mathbb{F}_2$.*

Step 1 Among all (nonzero) 15-tuples representing the quadratic polynomials, find the representatives for the $\mathrm{PGL}_5(\mathbb{F}_2)$-action irreducible over $\overline{\mathbb{F}}_2$. There are seven representatives for the $\mathrm{PGL}_5(\mathbb{F}_2)$-action. Two of them are reducible over $\mathbb{F}_2$, while one of them is irreducible over $\mathbb{F}_2$, but not over $\mathbb{F}_4$. The remaining four are irreducible over $\overline{\mathbb{F}}_2$. *Output:* **list_of_Ps**; *call its elements P_1, P_2, P_3, and P_4.* (*We can always find an isomorphism of $C = Z(q_1, q_2, q_3)$ and $C' = Z(q_P, q_Q, q_R)$ induced by a matrix from* $\mathrm{PGL}_5(\mathbb{F}_2)$*, such that P in* **list_of_Ps**.)

Step 2 For $P = P_i$ and $i \in \{1, 2, 3, 4\}$, let **P_potential_QRs** be the list consisting of all elements contained in the union $\mathrm{Orbit}(P_1) \cup \mathrm{Orbit}(P_2) \cup \ldots \cup \mathrm{Orbit}(P_i)$.
Fix P in **list_of_Ps**, and find the representatives for the second quadric: take an element Q from **P_potential_QRs**, put it in **P_list_of_Qs**, and remove all the $\mathrm{Stab}(P)$-conjugates of Q and $P+Q$ from **P_potential_QRs**; repeat this. *Output:* ***P_list_of_Qs*** *for $P = P_1, P_2, P_3, P_4$.*
(*As above, using matrices from* $\mathrm{Stab}(q_P)$*, we can find an isomorphism over* $\mathbb{F}_2$

$$C = Z(q_1, q_2, q_3) \cong C' = Z(q_P, q_Q, q_R)$$

with P in **list_of_Ps** *and Q in* **P_list_of_Qs**.)

Step 3 Find the representatives for the third quadric. Fix P in **list_of_Ps** and fix Q in **P_list_of_Qs**. Use the same reasoning as in Step 2: take R from the list **P_potential_QRs**, put it in **PQ_list_of_Rs_apriori**, and erase all the $(\mathrm{Stab}(P) \cap \mathrm{Stab}(Q))$-conjugates of $R, P+R, Q+R$, and $P+Q+R$ from the list **P_potential_QRs**; repeat this.
For fixed P and Q as above and R in **PQ_list_of_Rs_apriori**, check whether all the elements from $(\mathbb{F}_2 \cdot P + \mathbb{F}_2 \cdot Q + \mathbb{F}_2 \cdot R) - \{0\}$ are in **P_potential_QRs** and whether the triple (q_P, q_Q, q_R) defines a non-singular curve over $\overline{\mathbb{F}}_2$. If those are satisfied, put (P, Q, R) into **P_list**.

The union of all these lists **P_list** is the desired set Σ.

Remark. As we indicated, working with polynomial rings is technically demanding. Therefore, in all three Steps, we implemented that structure and found the $\mathrm{PGL}_5(\mathbb{F}_2)$-, $\mathrm{Stab}(P)$-, or $\mathrm{Stab}(P) \cap \mathrm{Stab}(Q)$-orbits using (3) and (4) as directly as we could, to make the computations take only a reasonable amount of time.

Furthermore, for a fixed P in Step 3, we use the idea of multithread computations by parallelizing the processes for different Q's, relying on the fact that those processes do not depend on each other.

Theorem 4.1. *For any non-hyperelliptic non-trigonal curve C of genus five over $\mathbb{F}_2$, there is an element P in* **list_of_Ps** *and a triple (P, Q, R) in* **P_list** *such that $C \cong Z(q_P, q_Q, q_R)$ over $\mathbb{F}_2$.*

Proof. Take $C = Z(q_1, q_2, q_3)$ and let $i \in \{1, 2, 3, 4\}$ be the largest index such that there is an element q in the intersection of $\mathrm{Orbit}(q_{P_i})$ and $(\mathbb{F}_2 \cdot q_1 + \mathbb{F}_2 \cdot q_2 + \mathbb{F}_2 \cdot q_3) - \{0\}$. We may without of loss of generality assume that $q_1 = q$ and set $q_2 = q_2', q_3 = q_3'$ such that $\langle q_1, q_2, q_3 \rangle = \langle q, q_2', q_3' \rangle$. Write $P = P_i$.

By the construction, we can find an isomorphism $C \overset{\cong}{\to} C' = Z(q_P, q_R', q_Q')$ induced by a matrix $M \in \mathrm{PGL}_5(\mathbb{F}_2)$, such that $M.q_1 = q_P$, $M.q_2 = q_Q'$ and $M.q_3 = q_R'$. Then, we can find $M' \in \mathrm{Stab}(P)$, which induces an isomorphism of C' and $C'' = Z(q_P, q_Q, q_R'') = Z(q_P, q_Q + q_P, q_R'')$ such that q_Q' goes to either q_Q or $q_Q + q_P$ for some Q in **P_list_of_Qs**. Lastly, there is some $M'' \in \mathrm{Stab}(P) \cap \mathrm{Stab}(Q)$, which induces an isomorphism of C'' and $C''' = Z(q_P, q_Q, q_R)$ such that q_R''' goes to an element from $\{q_R, q_R + q_P, q_R + q_Q, q_R + q_P + q_Q\}$ for R in **PQ_list_of_Rs**. In particular,

$$C \cong Z(q_P, q_Q, q_R),$$

for (P, Q, R) in $\mathbf{P_i}$**_list**.

Therefore, our construction of the lists **P_list** for $P = P_1, P_2, P_3$, and P_4 (and thus of Σ) is satisfactory.

Checking the condition of whether the curves are smooth over a field $\overline{\mathbb{F}}_2$ was done by using the function `is_smooth()`. Furthermore, checking whether the considered varieties are curves indeed, i.e., one-dimensional, was done using the function `dimension()`. Both mentioned functions were already implemented in SageMath, [10]. □

The second part of computing the final list $\mathbf{CompInt_5F2}$ consists of reducing the lists **P_list** for $P = P_1, P_2, P_3, P_4$. For each curve $C \in \mathrm{CompInt}_5(\mathbb{F}_2)$, we want to have precisely one element of $\mathbf{CompInt_5F2}$ representing the $\mathbb{F}_2$-isomorphism class of C.

First, note that by our construction, for $i, j \in \{1, 2, 3, 4\}$ and $i \neq j$, no two triples

$$(P_i, Q, R) \in \mathbf{P_i\_list}, \quad \text{and } (P_j, Q', R') \in \mathbf{P_j\_list}$$

can define isomorphic curves. Without loss of generality, assume $i < j$. Then, inside the set $(\mathbb{F}_2 \cdot q_{P_i} + \mathbb{F}_2 \cdot q_Q + \mathbb{F}_2 \cdot q_R) - \{0\}$ there cannot be elements in $\mathrm{Orbit}(P_j)$ by our construction, so we cannot find a projective transformation establishing the desired isomorphism.

Then, for a fixed list $\mathbf{P_i\_list}$, with $i \in \{1, 2, 3, 4\}$ consider two of its elements (P, Q, R) and (P, Q', R'). Note that the triples (P, Q, R) and (P, Q', R') define the same curve if and only if $\langle P, Q, R\rangle_{\mathbb{F}_2} = \langle P, Q', R'\rangle_{\mathbb{F}_2}$. Moreover, if (P, Q, R) and (P, Q', R') define isomorphic curves, then in $\langle P, Q', R'\rangle_{\mathbb{F}_2}$, there has to be an element S mapping to P, and hence inside the orbit of P.

Algorithm 4. Determine $\mathrm{CompInt}_5(\mathbb{F}_2)$.
Description. *Consider the triples in Σ one after the other. For a chosen triple (P, Q, R), which defines a curve C, find all the triples in $\Sigma - \{(P, Q, R)\}$ defining a curve that is isomorphic over $\mathbb{F}_2$ to C and remove them from Σ.*

Step 0 Let $\mathbf{CompInt_5F2}$ be an empty list at the beginning.
Step 1 For a fixed $P = P_i$ with $i \in \{1, 2, 3, 4\}$, take a triple (P, Q, R) in **P_list**. Add it to the final list $\mathbf{CompInt_5F2}$. Let $C = Z(q_P, q_Q, q_R)$.
Step 2 For a triple (P, Q, R) from Step 1, consider the vector space $V = \mathbb{F}_2 \cdot P + \mathbb{F}_2 \cdot Q + \mathbb{F}_2 \cdot R$ and let $D = V \cap \mathrm{Orbit}(P_i)$. For each element in D, find all matrices $M \in \mathrm{PGL}_5(\mathbb{F}_2)$ mapping it to $P = P_i$.
Step 3 Use the matrices M, obtained in Step 2, to act on (P, Q, R). If $(M.P, M.Q, M.R)$ defines the same curve as some triple from **P_list**, remove such a triple from **P_list**.

Remark. To find all matrices $M \in \mathrm{PGL}_5(\mathbb{F}_2)$ that map an element $S \in \mathrm{Orbit}(P_i)$ to P_i, it is enough to have only one such matrix M_0 and to know all matrices in $\mathrm{Stab}(P_i)$. Namely, it is not hard to see for any such S and M that there is a matrix $N \in \mathrm{Stab}(P_i)$ such that $M.S = N.(M_0.S)$. We use that in our implementation.

Remark. In every iteration of Step 3, we will remove at least one element from **P_list**, namely (P, Q, R), so the process terminates. As discussed in the last paragraph before the presentation of *Algorithm 4*, in this way, we will indeed remove from **P_list** all the triples (P', Q', R') defining a curve isomorphic over $\mathbb{F}_2$ to $C = Z(q_P, q_Q, q_R)$. Lastly, in Step 3, we check whether $(M.P, M.Q, M.R)$ defines the same curve as some triple from **P_list** using the criterion occurring in the paragraph just mentioned by checking the equality of the corresponding vector spaces.

4.1. Computing the automorphisms over $\mathbb{F}_2$

An $\mathbb{F}_2$-automorphism of a curve C in $\mathrm{CompInt}_5(\mathbb{F}_2)$ is induced by a matrix in $\mathrm{PGL}_5(\mathbb{F}_2)$. If a matrix $M \in \mathrm{PGL}_5(\mathbb{F}_2)$ is such that $M : C \overset{\cong}{\to} C$ and $C = Z(q_P, q_Q, q_R)$, then

$$\langle Mq_P, Mq_Q, Mq_R\rangle_{\mathbb{F}_2} = \langle q_P, q_Q, q_R\rangle_{\mathbb{F}_2}.$$

Therefore, for triples (P, Q, R) in $\mathbf{CompInt_5F2}$ with $P = P_i$ and $i \in \{1, 2, 3, 4\}$, either $M \in \mathrm{Stab}(P)$ or M maps an element of $W = (\mathbb{F}_2 \cdot P + \mathbb{F}_2 \cdot Q + \mathbb{F}_2 \cdot R) - \{0, P\}$ to P. In the latter case, we see that M needs to belong to the set of matrices mapping elements from $W \cap \mathrm{Orbit}(P)$ to P; call that set $D = D_{(P,Q,R)}$. Therefore, for a fixed (P, Q, R) as above, we can only check whether

$$\mathbb{F}_2 \cdot M.P + \mathbb{F}_2 \cdot M.Q + \mathbb{F}_2 \cdot M.R = \mathbb{F}_2 \cdot P + \mathbb{F}_2 \cdot Q + \mathbb{F}_2 \cdot R$$

for M in $\mathrm{Stab}(P) \cup D_{(P,Q,R)}$.

From the preceding discussion, we can easily get the precise steps of an algorithm for computing $\mathrm{Aut}_{\mathbb{F}_2}(C)$ as a set, for each $C \in \mathrm{CompInt}_5(\mathbb{F}_2)$.

We implemented the algorithms from this section in SageMath, [10] and computed the sets Σ and $\mathrm{CompInt}_5(\mathbb{F}_2)$, as well as $|\mathrm{Aut}_{\mathbb{F}_2}(C)|$ for each curve $C \in \mathrm{CompInt}_5(\mathbb{F}_2)$. For example, the set $\mathrm{CompInt}_5(\mathbb{F}_2)$ consists of 3905 elements.

5. Obtained results

As we already indicated, using the algorithms from Section 2 for hyperelliptic, Section 3 for trigonal, and Section 4 for non-hyperelliptic non-trigonal curves of genus 5 over $\mathbb{F}_2$, we computed $\mathrm{Hyp}_5(\mathbb{F}_2), \mathrm{Trig}_5(\mathbb{F}_2)$, and $\mathrm{CompInt}_5(\mathbb{F}_2)$, the sets of all the isomorphism representatives. For all the obtained curves, we computed the number of their points over $\mathbb{F}_{2^N}$ for $N \in \{1,2,3,4,5\}$ and determined the sizes of their $\mathbb{F}_2$-automorphism groups.

There are 1070 hyperelliptic, 2854 trigonal, and 3905 non-hyperelliptic non-trigonal curves of genus 5 over $\mathbb{F}_2$, so in total, there are 7829 pairwise non-isomorphic curves of genus 5 over $\mathbb{F}_2$.

We already mentioned the stack counts

$$|\mathcal{H}_5(\mathbb{F}_2)| = 512 \quad \text{and} \quad |\mathcal{T}_5(\mathbb{F}_2)| = 2817,$$

and we got the stack count that was not known before

$$|\mathcal{U}_5(\mathbb{F}_2)| = \sum_{C \in \mathrm{CompInt}_5(\mathbb{F}_2)} \frac{1}{|\mathrm{Aut}_{\mathbb{F}_2}(C)|} = 3584 = 2^{12} - 2^9.$$

Therefore, we have

$$|\mathcal{M}_5(\mathbb{F}_2)| = 6913 = 2^{12} + 2^{11} + 2^{10} - 2^8 + 1.$$

The Honda-Tate theorem gives us that there are 4339 isogeny classes over $\mathbb{F}_2$ that contain a Jacobian of a genus 5 curve defined over $\mathbb{F}_2$.

Furthermore, Jacobian varieties of dimension $g = 5$ over $\mathbb{F}_2$ realize all eligible Newton polygons of height $2g = 10$. In other words, for any eligible Newton polygon $\mathcal{N}$ of height 10, there is a curve of genus 5 defined over $\mathbb{F}_2$, which has $\mathcal{N}$ as its Newton polygon.

In Table 1, for each of the three discussed classes of genus 5 curves, we mention the number of such curves occurring for indicated Newton polygons of height 10.

Table 1
Numbers of curves for indicated Newton polygon.

Newton polygon slopes	$\mathrm{Hyp}_5(\mathbb{F}_2)$	$\mathrm{Trig}_5(\mathbb{F}_2)$	$\mathrm{CompInt}_5(\mathbb{F}_2)$	Total
$[0,0,0,0,0,1,1,1,1,1]$	550	1417	1617	3584
$[0,0,0,0,\frac{1}{2},\frac{1}{2},1,1,1,1]$	156	623	868	1647
$[0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},1,1,1]$	108	404	672	1184
$[0,0,\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{2}{3},\frac{2}{3},\frac{2}{3},1,1]$	32	122	206	360
$[0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},1,1]$	88	80	176	344
$[0,\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},1]$	0	64	88	152
$[0,\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{2},\frac{1}{2},\frac{2}{3},\frac{2}{3},\frac{2}{3},1]$	48	24	40	112
$[0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},1]$	56	28	108	192
$[\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{4}{5},\frac{4}{5},\frac{4}{5},\frac{4}{5},\frac{4}{5}]$	0	48	48	96
$[\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4}]$	0	8	24	32
$[\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{2}{3},\frac{2}{3},\frac{2}{3}]$	16	18	26	60
$[\frac{2}{5},\frac{2}{5},\frac{2}{5},\frac{2}{5},\frac{2}{5},\frac{3}{5},\frac{3}{5},\frac{3}{5},\frac{3}{5},\frac{3}{5}]$	8	4	4	16
$[\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}]$	8	14	28	50

In Table 2, we collect the stack counts for all three types of genus 5 curves over $\mathbb{F}_2$ possessing specified Newton polygon.

Table 2
Stack counts for indicated Newton polygon.

Newton polygon slopes	$\mathcal{H}_5(\mathbb{F}_2)$	$\mathcal{T}_5(\mathbb{F}_2)$	$\mathcal{U}_5(\mathbb{F}_2)$	$\mathcal{M}_5(\mathbb{F}_2)$
$[0,0,0,0,0,1,1,1,1,1]$	264	1405	1524	3193
$[0,0,0,0,\frac{1}{2},\frac{1}{2},1,1,1,1]$	76	610	838	1524
$[0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},1,1,1]$	52	402	574	1028
$[0,0,\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{2}{3},\frac{2}{3},\frac{2}{3},1,1]$	16	122	198	336
$[0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},1,1]$	40	78	154	272
$[0,\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},1]$	0	64	88	152
$[0,\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{2},\frac{1}{2},\frac{2}{3},\frac{2}{3},\frac{2}{3},1]$	24	24	32	80
$[0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},1]$	24	24	64	112
$[\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{4}{5},\frac{4}{5},\frac{4}{5},\frac{4}{5},\frac{4}{5}]$	0	48	48	96
$[\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4}]$	0	8	24	32
$[\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{2}{3},\frac{2}{3},\frac{2}{3}]$	8	14	18	40
$[\frac{2}{5},\frac{2}{5},\frac{2}{5},\frac{2}{5},\frac{2}{5},\frac{3}{5},\frac{3}{5},\frac{3}{5},\frac{3}{5},\frac{3}{5}]$	4	4	4	12
$[\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}]$	4	14	18	36

Example 5.1. There are 161 isogeny classes over $\mathbb{F}_2$ such that in each of them, we can find (pairwise non-isomorphic) Jacobians defined by all three types of genus 5 curves over $\mathbb{F}_2$. For example, we can consider the hyperelliptic curve defined by the standard affine equation

$$y^2+y+x^{11}+x^{10}+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x=0,$$

the trigonal curve defined by

$$\begin{aligned}X^4Y+&X^3Y^2+XY^4+X^3YZ+X^2Y^2Z+XY^3Z+\\&+X^3Z^2+X^2YZ^2+Y^3Z^2+XYZ^3+Y^2Z^3=0 \text{ in } \mathbb{P}^2,\end{aligned}$$

and the non-hyperelliptic non-trigonal curve defined by

$$\begin{cases}Y^2+YZ+Z^2+XT+ZT=0\\ XT+XU+YU+ZU=0\\ X^2+XY+Y^2+XZ+YZ+XU+YU+ZU+TU+U^2=0\end{cases} \quad \text{in } \mathbb{P}^4.$$

Since for each curve C of them we have that

$$|C(\mathbb{F}_2)|=5,\ |C(\mathbb{F}_{2^2})|=9,\ |C(\mathbb{F}_{2^3})|=11,\ |C(\mathbb{F}_{2^4})|=33, \text{ and } |C(\mathbb{F}_{2^5})|=25,$$

it follows that their Jacobians are isogenous over $\mathbb{F}_2$.

Example 5.2. The non-hyperelliptic non-trigonal curve C given in $\mathbb{P}^4$ by

$$\begin{cases}Y^2+XZ+YZ=0\\ XY+XZ+YT+ZT+XU+ZU+U^2=0\\ XY+XZ+YZ+Z^2+XT+ZT+T^2+YU+ZU=0\end{cases}$$

is the unique curve of genus 5 over $\mathbb{F}_2$ with 9 $\mathbb{F}_2$-points, which is the maximal number of $\mathbb{F}_2$-points among all genus 5 curves over $\mathbb{F}_2$; this agrees with [7] for $q=2$ and $g=5$ in their notation. The maximum number of $\mathbb{F}_2$-points among hyperelliptic curves (of genus 5 over $\mathbb{F}_2$) is 6, and there are 44 such curves with this property. Among trigonal curves (of genus 5 over $\mathbb{F}_2$), the maximum number of $\mathbb{F}_2$-points is 8, and there are 6 trigonal curves attaining this number. The obtained values in these three cases agree with [6, Table 2]. There are 308 (out of which 44 hyperelliptic, 23 trigonal, and 241 non-hyperelliptic non-trigonal) curves of genus 5 over $\mathbb{F}_2$ without $\mathbb{F}_2$-points.

Example 5.3. In Table 3, we mention the sizes of $\mathbb{F}_2$-automorphism groups occurring for the curves in $\mathrm{Hyp}_5(\mathbb{F}_2)$ and the number of such curves C with the indicated size $|\mathrm{Aut}_{\mathbb{F}_2}(C)|$.

Table 3
Sizes of $\mathbb{F}_2$-automorphism groups of curves in $\mathrm{Hyp}_5(\mathbb{F}_2)$.

$\lvert\mathrm{Aut}_{\mathbb{F}_2}(C)\rvert$	2	4	6	12
$\#C$ in $\mathrm{Hyp}_5(\mathbb{F}_2)$	983	76	7	4

The four curves given in our list $\mathrm{Hyp}_5(\mathbb{F}_2)$ by the equations

$$\begin{aligned}
&y^2+(x+1)^2x^2y+x^{11}+x^9+x^8+x^5+x^3+x^2+x+1=0,\\
&y^2+(x+1)^2x^2y+x^{11}+x^{10}+x^3+x=0,\\
&y^2+(x^3+x^2+1)(x^3+x+1)y+x^{10}+x^6+x^4+x^3=0,\\
&y^2+(x^3+x^2+1)(x^3+x+1)y+x^{12}+x^{10}+x^8+x^7+x^5+x^3+1=0
\end{aligned}$$

are the ones with $\mathbb{F}_2$-automorphism group of size 12, the maximal one. Using the description from Section 2, we can take a closer look at the elements of these groups, compute their orders, and using [4, Table 2], find that the first two $\mathbb{F}_2$-automorphism groups are isomorphic to $\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$, while the last two are isomorphic to the dihedral group D_6.

Note that, a priori, we only counted the number of elements of each $\mathrm{Aut}_{\mathbb{F}_2}(C)$ for C in one of the lists $\mathrm{Hyp}_5(\mathbb{F}_2)$, $\mathrm{Trig}_5(\mathbb{F}_2)$, or $\mathrm{CompInt}_5(\mathbb{F}_2)$, and we do not know what the structure of the group $\mathrm{Aut}_{\mathbb{F}_2}(C)$ is.

Example 5.4. Let $\mathbb{F}_q$ be a finite field of cardinality $q = p^r$ for $r \in \mathbb{Z}_{>0}$ and a prime number $p \in \mathbb{Z}_{>0}$, let $g \in \mathbb{Z}_{\geq 2}$, and let $\lambda = [1^{\lambda_1}, \dots, \nu^{\lambda_\nu}]$ be a partition of an integer $m \in \mathbb{Z}_{\geq 1}$ with non-negative integers $\lambda_1, \dots, \lambda_\nu$, such that $|\lambda| := \sum_{i=1}^{\nu} i\lambda_i = m$. In [2], Bergström made $\mathbb{S}_n$-equivariant counts of points defined over finite fields of the moduli space $\mathcal{H}_{g,n}$ of n-pointed smooth hyperelliptic curves of genus g over $\mathbb{F}_q$. The counts depend on the numbers

$$a_\lambda|_g := \sum_{C \in \mathrm{Hyp}_g(\mathbb{F}_q)} \frac{1}{|\mathrm{Aut}_{\mathbb{F}_q}(C)|} \prod_{i=1}^{\nu} a_i(C)^{\lambda_i},$$

where $a_i(C) = q^i + 1 - |C(\mathbb{F}_{q^i})|$, for $i \in \mathbb{Z}_{\geq 1}$. In that paper, for arbitrary q and g as above, some explicit formulas were mentioned: $a_{[2]}|_g = (-1)^g - q^{2g}$, $a_{[1^2]}|_g = -1 + q^{2g}$,

$$a_{[1^2,2]}|_g = -\frac{q^{2g+2}-1}{q+1} - q^{2g} + \frac{1}{2}g(q^3+q-2) + \frac{1}{2}\begin{cases} 2q & \text{if } g \equiv 0 \mod 2 \\ q^3-q-2 & \text{if } g \equiv 1 \mod 2 \end{cases},$$

and $a_\lambda|_g = 0$, if $|\lambda|$ is odd. Using our data, for $q = 2$ and $g = 5$, we computed the sums from the definition of $a_\lambda|_g$ for some λ, and the outcomes agree with the mentioned formulas:

$$a_{[2]}|_5 = -1025, \quad a_{[1^2]}|_5 = 1023, \quad a_{[1^2,2]}|_5 = -2367, \quad a_{[1^2,3]}|_5 = a_{[3,4^2]}|_5 = a_{[1,2,5^2]}|_5 = 0.$$

Example 5.5. In Table 4, we mention the sizes of $\mathbb{F}_2$-automorphism groups occurring for the curves in $\mathrm{Trig}_5(\mathbb{F}_2)$ and the number of such curves C with the indicated size $|\mathrm{Aut}_{\mathbb{F}_2}(C)|$.

Table 4
Sizes of $\mathbb{F}_2$-automorphism groups of curves in $\mathrm{Trig}_5(\mathbb{F}_2)$.

$\lvert\mathrm{Aut}_{\mathbb{F}_2}(C)\rvert$	1	2	3	6
$\#C$ in $\mathrm{Trig}_5(\mathbb{F}_2)$	2783	63	7	1

The curve C given in $\mathbb{P}^2$ by the equation

$$\begin{aligned}X^5+&Y^5+X^4Z+X^3YZ+XY^3Z+Y^4Z+X^3Z^2+\\&+X^2YZ^2+XY^2Z^2+Y^3Z^2+X^2Z^3+XYZ^3+Y^2Z^3=0,\end{aligned}$$

is the one with $|\mathrm{Aut}_{\mathbb{F}_2}(C)|=6$.

Example 5.6. In Table 5, we mention the sizes of $\mathbb{F}_2$-automorphism groups occurring for the curves in $\mathrm{CompInt}_5(\mathbb{F}_2)$ and the number of such curves C with the indicated size $|\mathrm{Aut}_{\mathbb{F}_2}(C)|$.

Table 5
Sizes of $\mathbb{F}_2$-automorphism groups of curves in $\mathrm{CompInt}_5(\mathbb{F}_2)$.

$\lvert\mathrm{Aut}_{\mathbb{F}_2}(C)\rvert$	1	2	3	4	6	8	12	16	24
$\#C$ in $\mathrm{CompInt}_5(\mathbb{F}_2)$	3319	490	3	60	4	24	2	2	1

The curve C given in $\mathbb{P}^4$ by

$$\begin{cases}Y^2+XZ+YZ=0\\Y^2+XZ+YZ+Z^2+YT+T^2+XU+YU+ZU=0\\X^2+Y^2+XZ+YT+T^2+YU+U^2=0\end{cases}$$

is the (only) one with $\mathbb{F}_2$-automorphism group of size 24; this is the largest possible size among all $\mathbb{F}_2$-automorphism groups of genus 5 curves defined over $\mathbb{F}_2$.

Example 5.7. In Appendix A, we present the tables of the equations defining all the supersingular curves C of genus 5 over $\mathbb{F}_2$. Also, we present there the number of their $\mathbb{F}_{2^N}$-points for $N\in\{1,2,3,4,5\}$ in the form $[|C(\mathbb{F}_2)|,|C(\mathbb{F}_4)|,|C(\mathbb{F}_8)|,|C(\mathbb{F}_{16})|,|C(\mathbb{F}_{32})|]$, which one can use to see that their Jacobians occur in 19 distinct $\mathbb{F}_2$-isogeny classes.

Example 5.8. In the first step of Algorithm 3 for determining the set $\mathrm{CompInt}_5(\mathbb{F}_2)$ of non-hyperelliptic non-trigonal curves of genus 5 over $\mathbb{F}_2$, we mentioned that we found models for curves C in $\mathrm{CompInt}_5(\mathbb{F}_2)$, such that one quadric on which such a curve lies is always one of the quadrics corresponding to the 15-tuples P_1,P_2,P_3, or P_4. Explicitly, we have

$$\begin{aligned}q_{P_1}&=X^2+XZ+YZ+XT+ZT+TU,\\q_{P_2}&=XY+Y^2+Z^2+YT+ZT,\\q_{P_3}&=Y^2+YZ+Z^2+XT+ZT,\\q_{P_4}&=Y^2+XZ+YZ.\end{aligned}$$

In the union $\bigcup_{i\in\{1,2,3,4\}} \mathrm{Orbit}(q_{P_i})$ there are 32116 elements, so in the set difference between all quadratic polynomials and this union, there are 651 elements. It can be checked that all of them are reducible over $\overline{\mathbb{F}}_2$ and that they split into three orbits as was mentioned in the description of Algorithm 3.

For completeness, let us mention the remaining three (reducible over $\overline{\mathbb{F}}_2$) quadratic polynomials we obtained in the first step of Algorithm 3. They are

$$
\begin{aligned}
q_{P_5} &= Y^2,\\
q_{P_6} &= X^2 + XY,\\
q_{P_7} &= X^2 + XY + XT + Y^2 + YZ + YT + Z^2 + ZT + T^2\\
&= (X + \zeta_2 Y + Z + (\zeta_2+1)T)(X + (\zeta_2+1)Y + Z + \zeta_2 T),
\end{aligned}
$$

where $\zeta_2 \in \overline{\mathbb{F}}_2$, such that $\zeta_2^2 + \zeta_2 + 1 = 0$.

All the relevant data that we used, including the sets $\mathrm{Orbit}(P_i)$ and $\mathrm{Stab}(P_i)$ for $i \in \{1,2,3,4\}$ and the matrices M appearing in the second step of Algorithm 4, can be found in the codes available on the mentioned GitHub page. Finally, we note that the choice of the representatives P_i for $i \in \{1,2,3,4,5,6,7\}$ is not canonical and depends on our implementation of the first step of Algorithm 3. However, one can use the mentioned additional data to find some other defining equations of the isomorphism classes of non-hyperelliptic non-trigonal curves of genus 5 over $\mathbb{F}_2$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Tables of supersingular binary curves of genus five

In this section, we present the defining equations of all the supersingular curves C of genus 5 over $\mathbb{F}_2$ and their number of $\mathbb{F}_{2^N}$-points for $N \in \{1,2,3,4,5\}$ as mentioned in Example 5.7, and for each of them we comment on the size of $\mathrm{Aut}_{\mathbb{F}_2}(C)$.

There are 8 hyperelliptic, 14 trigonal, and 28 non-hyperelliptic non-trigonal supersingular curves of genus 5 over $\mathbb{F}_2$, up to an isomorphism over $\mathbb{F}_2$. All the hyperelliptic ones have an $\mathbb{F}_2$-automorphism group of size 2 and we present their equations in Table 6. All the trigonal ones have a trivial $\mathbb{F}_2$-automorphism group and we present their equations in Table 7. In Table 8, we present the equations of supersingular curves in $\mathrm{CompInt}_5(\mathbb{F}_2)$ with a trivial $\mathbb{F}_2$-automorphism group, and in Table 9, we present the ones with a non-trivial $\mathbb{F}_2$-automorphism group - the top four ones have an $\mathbb{F}_2$-automorphism group of size 2, the next six of size 4, and the bottom four of size 8.

Table 6
Supersingular curves C in $\mathrm{Hyp}_5(\mathbb{F}_2)$.

Defining equation of C	List of $\lvert C(\mathbb{F}_{2^N})\rvert$ for $N = 1, 2, 3, 4, 5$
$y^2 + y + x^{12} + x^{11} + x^{10} + x^5 + x^2 + x + 1 = 0$	$[1, 5, 13, 25, 41]$
$y^2 + y + x^{11} + x^{10} + x^9 + x^4 + x^3 + x + 1 = 0$	$[1, 9, 1, 17, 41]$
$y^2 + y + x^{11} + x^8 + x^2 + 1 = 0$	$[3, 5, 9, 17, 33]$
$y^2 + y + x^{11} + x^{10} + x^6 + x^5 + x^3 + x^2 + x = 0$	$[3, 5, 9, 17, 33]$
$y^2 + y + x^{12} + x^{11} + x^{10} + x^5 + x^2 = 0$	$[3, 9, 9, 25, 33]$
$y^2 + y + x^{12} + x^{11} + x^{10} + x^9 + x^8 + x^3 + x = 0$	$[3, 9, 9, 25, 33]$
$y^2 + y + x^{12} + x^{11} = 0$	$[5, 5, 5, 25, 25]$
$y^2 + y + x^{11} + x^{10} + x^9 + x^8 + x^6 + x^4 + x^2 + x = 0$	$[5, 9, 17, 17, 25]$

Table 7
Supersingular curves C in $\mathrm{Trig}_5(\mathbb{F}_2)$.

Defining equation of C	List of $\lvert C(\mathbb{F}_{2^N})\rvert$ for $N = 1, 2, 3, 4, 5$
$X^5 + X^3Y^2 + X^2Y^3 + X^4Z + XY^3Z + Y^4Z+$ $+X^2YZ^2 + Y^3Z^2 + X^2Z^3 + XYZ^3 + Y^2Z^3 = 0$	$[1, 5, 13, 41, 41]$
$X^5 + X^4Y + X^3Y^2 + X^2Y^3 + XY^4 + Y^4Z + X^3Z^2+$ $+X^2YZ^2 + XY^2Z^2 + Y^3Z^2 + X^2Z^3 + XYZ^3 + Y^2Z^3 = 0$	$[1, 9, 13, 33, 41]$
$X^5 + X^4Y + X^3Y^2 + X^2Y^3 + Y^5 + X^3YZ+$ $+X^2Y^2Z + X^2YZ^2 + XY^2Z^2 + XYZ^3 + Y^2Z^3 = 0$	$[3, 5, 9, 17, 33]$
$X^5 + X^4Z + XY^3Z + Y^4Z + XY^2Z^2 + X^2Z^3 + XYZ^3 = 0$	$[3, 5, 9, 17, 33]$
$X^5 + X^4Y + X^3Y^2 + X^2Y^3 + Y^5 + X^3YZ+$ $+XY^3Z + Y^4Z + X^2YZ^2 + XY^2Z^2 + XYZ^3 = 0$	$[3, 5, 9, 17, 33]$
$X^5 + X^4Y + X^2Y^3 + X^4Z + X^3YZ + X^2Y^2Z+$ $+Y^4Z + X^3Z^2 + X^2YZ^2 + XY^2Z^2 + XYZ^3 = 0$	$[3, 5, 9, 17, 73]$
$X^4Y + X^3Y^2 + X^2Y^3 + XY^4 + Y^5+$ $+X^3Z^2 + X^2Z^3 + XYZ^3 + Y^2Z^3 = 0$	$[3, 9, 9, 25, 33]$
$X^5 + X^2Y^3 + XY^4 + Y^5 + X^4Z + X^3YZ + X^2Y^2Z+$ $+X^3Z^2 + X^2YZ^2 + XY^2Z^2 + X^2Z^3 + XYZ^3 + Y^2Z^3 = 0$	$[3, 9, 9, 25, 33]$
$X^5 + X^4Y + X^4Z + Y^4Z + X^2YZ^2+$ $+XY^2Z^2 + Y^3Z^2 + X^2Z^3 + XYZ^3 + Y^2Z^3 = 0$	$[3, 9, 9, 25, 33]$
$X^4Y + Y^5 + X^4Z + X^3Z^2 + X^2YZ^2+$ $+XY^2Z^2 + X^2Z^3 + XYZ^3 + Y^2Z^3 = 0$	$[3, 13, 9, 17, 33]$
$X^4Y + X^3Y^2 + X^2Y^3 + Y^5 + X^3YZ + X^2Y^2Z+$ $+X^3Z^2 + X^2YZ^2 + XY^2Z^2 + XYZ^3 + Y^2Z^3 = 0$	$[5, 9, 17, 17, 25]$
$X^3Y^2 + Y^5 + X^4Z + X^2Y^2Z + XY^3Z + X^3Z^2 + X^2Z^3 + XYZ^3 = 0$	$[5, 9, 17, 17, 25]$
$X^5 + X^2Y^3 + XY^4 + Y^5 + X^2Y^2Z + XY^3Z+$ $+Y^4Z + X^2YZ^2 + XY^2Z^2 + X^2Z^3 + XYZ^3 = 0$	$[5, 9, 17, 17, 25]$
$X^4Y + X^3Y^2 + X^2Y^2Z + XY^3Z+$ $+Y^4Z + X^3Z^2 + X^2Z^3 + XYZ^3 = 0$	$[7, 9, 13, 25, 17]$

Table 8
Supersingular curves C in $\mathrm{CompInt}_5(\mathbb{F}_2)$ with $|\mathrm{Aut}_{\mathbb{F}_2}(C)| = 1$.

Defining equations of C	List of $\lvert C(\mathbb{F}_{2^N})\rvert$ for $N = 1, 2, 3, 4, 5$
$\begin{cases} TX + Y^2 + TZ + YZ + Z^2 = 0 \\ TU + TX + UX + X^2 + UY + Y^2 + Z^2 = 0 \\ T^2 + TU + U^2 + TX + UY + TZ + UZ + Z^2 = 0 \end{cases}$	$[1, 5, 13, 25, 41]$
$\begin{cases} Y^2 + XZ + YZ = 0 \\ T^2 + TX + UX + TY + UY + XY + TZ + UZ = 0 \\ U^2 + UX + X^2 + UY + XY + Y^2 + TZ + XZ + YZ + Z^2 = 0 \end{cases}$	$[1, 5, 13, 25, 41]$
$\begin{cases} TX + Y^2 + TZ + YZ + Z^2 = 0 \\ TU + X^2 + TY + UY + TZ + UZ + XZ = 0 \\ TU + U^2 + TX + UX + TY + UY + Y^2 + TZ + XZ = 0 \end{cases}$	$[1, 5, 13, 41, 41]$
$\begin{cases} TX + Y^2 + TZ + YZ + Z^2 = 0 \\ TU + TX + UX + X^2 + UY + Y^2 + Z^2 = 0 \\ U^2 + TX + UY + XY + Y^2 + XZ = 0 \end{cases}$	$[1, 9, 1, 17, 41]$
$\begin{cases} Y^2 + XZ + YZ = 0 \\ T^2 + TX + UX + TY + UY + XY + TZ + UZ = 0 \\ U^2 + TX + X^2 + TY + UZ + XZ + YZ + Z^2 = 0 \end{cases}$	$[3, 5, 9, 17, 33]$
$\begin{cases} TX + Y^2 + TZ + YZ + Z^2 = 0 \\ TU + X^2 + TY + UY + TZ + UZ + XZ = 0 \\ U^2 + TY + UY + XY + TZ + YZ = 0 \end{cases}$	$[3, 5, 9, 17, 33]$
$\begin{cases} TX + Y^2 + TZ + YZ + Z^2 = 0 \\ TU + X^2 + TY + UY + TZ + UZ + XZ = 0 \\ T^2 + U^2 + UX + XY + TZ + UZ + YZ + Z^2 = 0 \end{cases}$	$[3, 5, 9, 17, 33]$
$\begin{cases} TU + TX + X^2 + TZ + XZ + YZ = 0 \\ TU + TX + UX + X^2 + UY + Y^2 + Z^2 = 0 \\ T^2 + TX + UX + X^2 + XY + TZ + UZ + XZ + Z^2 = 0 \end{cases}$	$[3, 5, 9, 17, 73]$
$\begin{cases} TX + Y^2 + TZ + YZ + Z^2 = 0 \\ TU + X^2 + TY + UY + TZ + UZ + XZ = 0 \\ T^2 + U^2 + TY + UY + XZ + YZ + Z^2 = 0 \end{cases}$	$[3, 5, 9, 17, 73]$
$\begin{cases} TX + Y^2 + TZ + YZ + Z^2 = 0 \\ TU + X^2 + TY + UY + TZ + UZ + XZ = 0 \\ T^2 + TU + U^2 + TX + UX + X^2 + UY + Y^2 + YZ + Z^2 = 0 \end{cases}$	$[3, 9, 9, 25, 33]$
$\begin{cases} Y^2 + XZ + YZ = 0 \\ T^2 + TX + UX + TY + UY + XY + TZ + UZ = 0 \\ U^2 + TX + TY + UZ + XZ + YZ = 0 \end{cases}$	$[3, 9, 9, 25, 33]$
$\begin{cases} TX + Y^2 + TZ + YZ + Z^2 = 0 \\ T^2 + TU + TX + TY + XY + TZ + UZ + XZ + YZ = 0 \\ TU + U^2 + UX + TY + XY + Y^2 + UZ + XZ + YZ = 0 \end{cases}$	$[5, 9, 17, 17, 25]$
$\begin{cases} Y^2 + XZ + YZ = 0 \\ T^2 + TX + UX + TY + UY + XY + TZ + UZ = 0 \\ U^2 + TX + TY + XY + Y^2 + UZ + YZ = 0 \end{cases}$	$[5, 9, 17, 17, 25]$
$\begin{cases} TY + XY + Y^2 + TZ + Z^2 = 0 \\ TU + UX + TY + UY + YZ + Z^2 = 0 \\ TU + U^2 + TX + UX + TY + UY + XY + UZ + XZ = 0 \end{cases}$	$[5, 9, 17, 17, 25]$

Table 9
Supersingular curves C in $\mathrm{CompInt}_5(\mathbb{F}_2)$ with $|\mathrm{Aut}_{\mathbb{F}_2}(C)| > 1$.

Defining equations of C	List of $\lvert C(\mathbb{F}_{2^N})\rvert$ for $N = 1, 2, 3, 4, 5$
$\begin{cases} Y^2+XZ+YZ=0 \\ U^2+TX+UX+X^2+UY+Y^2+TZ+UZ+XZ=0 \\ T^2+U^2+UX+TY+UY+Y^2+UZ+XZ+YZ+Z^2=0 \end{cases}$	$[1, 1, 13, 33, 41]$
$\begin{cases} Y^2+XZ+YZ=0 \\ T^2+U^2+TX+TY+UY+Y^2+TZ+XZ=0 \\ T^2+TX+UX+TY+Y^2+TZ+UZ+Z^2=0 \end{cases}$	$[3, 5, 9, 17, 33]$
$\begin{cases} Y^2+XZ+YZ=0 \\ U^2+TX+UX+X^2+UY+Y^2+TZ+UZ+XZ=0 \\ T^2+U^2+UX+X^2+TY+UY+Y^2+UZ+Z^2=0 \end{cases}$	$[3, 13, 9, 17, 33]$
$\begin{cases} Y^2+XZ+YZ=0 \\ U^2+TX+UX+X^2+UY+Y^2+TZ+UZ+XZ=0 \\ T^2+TX+TY+TZ+XZ=0 \end{cases}$	$[5, 9, 5, 33, 25]$
$\begin{cases} Y^2+XZ+YZ=0 \\ U^2+TX+UX+X^2+XY+UZ+XZ+Z^2=0 \\ T^2+TX+UX+TZ+XZ+YZ=0 \end{cases}$	$[1, 9, 1, 17, 41]$
$\begin{cases} Y^2+XZ+YZ=0 \\ U^2+TX+UX+X^2+XY+UZ+XZ+Z^2=0 \\ T^2+U^2+X^2+XY+TZ+UZ+Z^2=0 \end{cases}$	$[1, 9, 13, 49, 41]$
$\begin{cases} Y^2+XZ+YZ=0 \\ U^2+TX+UX+X^2+XY+UZ+XZ+Z^2=0 \\ T^2+U^2+X^2+XY+Y^2+TZ+UZ+Z^2=0 \end{cases}$	$[1, 17, 13, 17, 41]$
$\begin{cases} Y^2+XZ+YZ=0 \\ T^2+U^2+UX+X^2+Y^2+TZ+UZ+XZ+Z^2=0 \\ U^2+X^2+XY+Y^2+UZ+YZ=0 \end{cases}$	$[5, 9, 5, 49, 25]$
$\begin{cases} Y^2+XZ+YZ=0 \\ T^2+U^2+UX+X^2+Y^2+TZ+UZ+XZ+Z^2=0 \\ T^2+UX+X^2+XY+TZ+XZ=0 \end{cases}$	$[5, 9, 17, 17, 25]$
$\begin{cases} Y^2+XZ+YZ=0 \\ T^2+U^2+UX+X^2+Y^2+TZ+UZ+XZ+Z^2=0 \\ T^2+UX+X^2+XY+TZ=0 \end{cases}$	$[5, 17, 5, 17, 25]$
$\begin{cases} Y^2+XZ+YZ=0 \\ U^2+TX+UX+X^2+XY+UZ+XZ+Z^2=0 \\ T^2+TX+UX+X^2+Y^2+TZ+XZ+YZ+Z^2=0 \end{cases}$	$[1, 9, 1, 17, 41]$
$\begin{cases} Y^2+XZ+YZ=0 \\ T^2+U^2+UX+X^2+Y^2+TZ+UZ+XZ+Z^2=0 \\ U^2+XY+Y^2+UZ+YZ+Z^2=0 \end{cases}$	$[1, 9, 1, 17, 41]$
$\begin{cases} Y^2+XZ+YZ=0 \\ T^2+U^2+UX+X^2+Y^2+TZ+UZ+XZ+Z^2=0 \\ U^2+X^2+XY+Y^2+UZ+YZ+Z^2=0 \end{cases}$	$[1, 9, 25, 17, 41]$
$\begin{cases} Y^2+XZ+YZ=0 \\ U^2+UX+TY+XY+TZ+UZ+XZ=0 \\ T^2+TX+UY+XY+TZ+UZ+XZ+YZ+Z^2=0 \end{cases}$	$[9, 9, 9, 17, 9]$

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