

Interplay between Normal Forms and Center Manifold Reduction for Homoclinic Predictors near Bogdanov–Takens Bifurcation*

Maikel M. Bosschaert[†] and Yuri A. Kuznetsov[‡]

Abstract. This paper provides for the first time correct third-order homoclinic predictors in n -dimensional ODEs near a generic Bogdanov–Takens bifurcation point, which can be used to start the numerical continuation of the appearing homoclinic orbits. To achieve this, higher-order time approximations to the nonlinear time transformation in the Lindstedt–Poincaré method are essential. Moreover, a correct transform between approximations to solutions in the normal form and approximations to solutions on the parameter-dependent center manifold is derived rigorously. A detailed comparison is done between applying different normal forms (smooth and orbital), different phase conditions, and different perturbation methods (regular and Lindstedt–Poincaré) to approximate the homoclinic solution near Bogdanov–Takens points. Examples demonstrating the correctness of the predictors are given. The new homoclinic predictors are implemented in the open-source MATLAB/GNU Octave continuation package `MatCont`.

Key words. Bogdanov–Takens bifurcation, homoclinic asymptotics, center manifold reduction

MSC codes. 37M20, 65P30, 34C37, 34B08, 34B15, 34B40, 34E10

DOI. 10.1137/22M151354X

1. Introduction. Let $f: \mathbb{R}^n \times \mathbb{R}^2 \rightarrow \mathbb{R}^n$ with $n \geq 2$ be smooth, and suppose that the autonomous ordinary differential equation (ODE)

$$(1.1) \quad \dot{x}(t) = f(x(t), \alpha)$$

has equilibrium $x_0 = 0$ that undergoes a codimension two local bifurcation at the critical parameter value $\alpha_0 = 0$. Here, the dot means the derivative with respect to the independent variable $t \in \mathbb{R}$. To understand the dynamics near the bifurcation point (x_0, α_0) for nearby parameter values, one typically first restricts the ODE to the center manifold. By projecting the solutions on the center manifold onto the center subspace, one then obtains an n_c -dimensional ODE that locally governs the restricted dynamics. Using the normal form theory, one further tries to transform the restricted ODE into a simpler form, called the *critical normal form*.

If the canonical unfolding of the critical normal form is known and only qualitative behavior near the equilibrium is of interest, one can stop here. However, if one is interested in relating solutions of the unfolding to those of the original system (1.1) near the bifurcation point, one needs a relation between the *parameter-dependent normal form* and the restricted ODE, and also a relation between this restricted ODE on the parameter-dependent center

*Received by the editors August 3, 2022; accepted for publication (in revised form) by S. Wiczorek September 18, 2023; published electronically January 25, 2024.

<https://doi.org/10.1137/22M151354X>

[†]Department of Mathematics, Hasselt University, 3590 Diepenbeek, Belgium (maikel.bosschaert@uhasselt.be).

[‡]Department of Mathematics, Utrecht University, 3508 TA Utrecht, The Netherlands, and Department of Applied Mathematics, University of Twente, 7500AE Enschede, The Netherlands (i.a.kouznetsov@uu.nl).

manifold and the original system (1.1). These two relations can be found simultaneously utilizing the *homological equation* approach; see [5].

The solutions of interest here are the codimension one bifurcation curves emanating from the codimension two point and the corresponding orbits in phase space. In general, the bifurcation curves in the parameter-dependent normal form are not known exactly, but only by an approximation up to a certain order. Similarly, the transformation from the normal form to the (parameter-dependent) center manifold is generally also only known up to a certain order. Then, by combining these two transformations, an approximation to the codimension one bifurcation curve and the corresponding phase orbits is obtained for the original system (1.1).

These approximations are particularly useful in numerical continuation software to start the continuation of the codimension one bifurcation curves emanating from the codimension two bifurcation points, where the defining systems for the orbits of interest become degenerate. A codimension two bifurcation that has attracted much attention is the *Bogdanov–Takens bifurcation*, at which the critical equilibrium has a double zero eigenvalue. It is well known that under certain nondegeneracy and transversality conditions, three codimension one bifurcation curves emanate from the Bogdanov–Takens point: a saddle-node, an (Andronov–)Hopf, and a saddle-homoclinic bifurcation curve. Since the standard defining systems for the equilibrium bifurcations are nondegenerate at the Bogdanov–Takens point, one does not need an approximation to start continuation there. On the contrary, the standard defining system for the homoclinic solution (see section SM9 of the supplementary materials, linked from the main article webpage) does become degenerate, which is easily seen since the homoclinic orbit shrinks to the equilibrium point when we approach the Bogdanov–Takens bifurcation.

Starting continuation of the homoclinic orbits from a Bogdanov–Takens point in ODEs attracted much attention. In planar systems, Melnikov’s method was first applied to solve this problem in [29]. A first attempt to provide asymptotic approximations to the homoclinic bifurcation curve near a generic codimension two Bogdanov–Takens bifurcation point in general n -dimensional systems was made in [4]. By applying a singular rescaling to the (one of the equivalent) parameter-dependent normal form on the center manifold, a perturbed planar Hamiltonian system is obtained. The unperturbed Hamiltonian system contains an explicit homoclinic solution. A first-order correction in parameter-space can subsequently be obtained by reformulating the problem as a branching problem in a suitable Banach space; see [4]. Then, by using the regular perturbation method, higher-order approximations to the homoclinic bifurcation curve can be obtained. Unfortunately, in [4], even the first-order correction in the phase space was not derived. Nonetheless, it was proven that the obtained homoclinic predictor converges to the true solution under the Newton iterations in the perturbed Hamiltonian systems.

In [26], the work continued by obtaining a second-order correction in parameter *and* phase space to the homoclinic bifurcation curve for the perturbed Hamiltonian system. However, a new problem was overlooked. The normal form used in [26] is a normal form for C^∞ -equivalence (also called *smooth orbital equivalence*); i.e., besides a C^∞ -coordinate change, also a time-reparametrization must be taken into account, which was not the case in [26]. In the subsequent paper [27], this problem was resolved by considering a smooth normal form for the Bogdanov–Takens bifurcation point, which is a normal form for C^∞ -conjugacy (*smooth equivalence*).

In the follow-up paper [1], progress was made in obtaining a uniform approximation in time of the homoclinic solution, using a generalization of the Lindstedt–Poincaré method. This removes the so-called parasitic turns near the saddle point, as observed in [26]. Although, as pointed out by [2], there were mistakes in the third-order approximation with the Lindstedt–Poincaré method, the asymptotic approximation for the homoclinic predictor from [26] for the smooth normal form improved significantly in the phase space. The relations between the original system, the parameter-dependent center manifold, the normal forms, and the perturbation method used (regular or Lindstedt–Poincaré) are depicted in Figure 1.

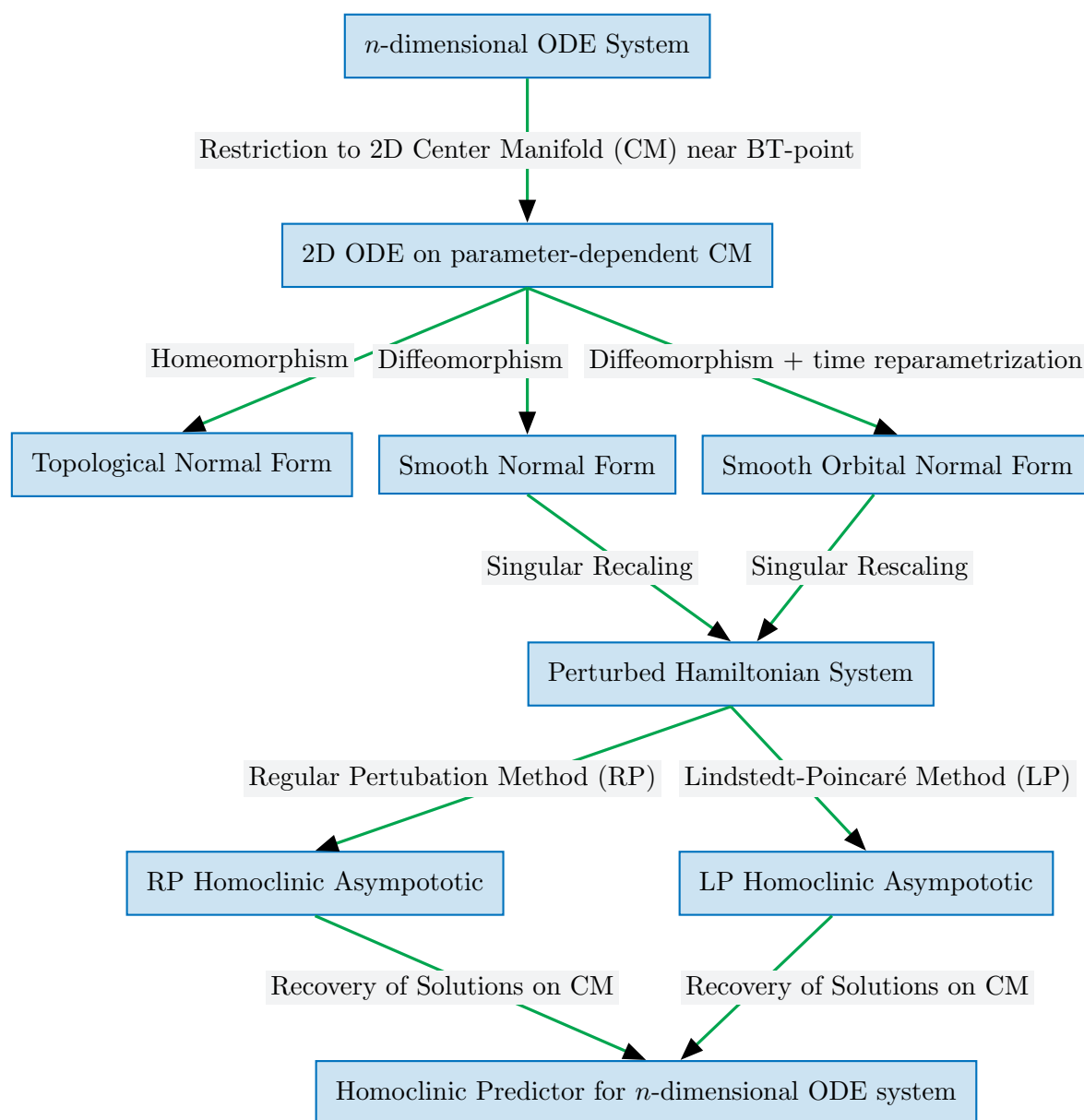


Figure 1. Schematic overview relating the concepts discussed in the introduction. Notice that different paths in the diagram lead to different predictors.

Nonetheless, the task of correctly lifting the asymptotic approximation in the normal form to the parameter-dependent center manifold was not accomplished. Effectively, only the zeroth-order approximation to the homoclinic solutions in the phase space, i.e., a transformed homoclinic solution of the *unperturbed* Hamiltonian system, was available for a general n -dimensional system.

In this paper, we will provide for the first time the third-order homoclinic predictor, with respect to a perturbation parameter to be precisely defined in the text, for the homoclinic solutions emanating from a generic Bogdanov–Takens point for a general n -dimensional system. For this, we first consider a general lifting of an asymptotic approximation of a codimension one bifurcation curve in parameter and phase space derived in the normal form of a codimension two bifurcation point to the parameter-dependent center manifold for this bifurcation. In section 2, we will show how to systematically determine which coefficients to include in the parameter-dependent center manifold parameterization and parameter transformation in order to maintain the approximation order of the available asymptotic approximation; see, in particular, Recipe 1.

Then we conclude that for the specific case of lifting the asymptotic approximation of the homoclinic bifurcation curve obtained either in the smooth or smooth orbital normal form of the Bogdanov–Takens bifurcation, we need to consider several additional systems to be solved in the homological equation method that were previously not taken into account. During the derivation of the coefficients of the normal form and the transformations, we will show that there is no need to solve certain systems simultaneously (cf. the so-called big system in [26, 27, 1]), making the expressions also suitable for infinite-dimensional ODEs generated by partial and delay differential equations to which the (parameter-dependent) center manifold theorem applies.

Furthermore, by allowing a transformation of time between the normal form and the original system, we can use the parameter-dependent *smooth orbital* normal form of the codimension two Bogdanov–Takens bifurcation point when approximating the homoclinic solution up to order three. This normal form is considerably simpler than previously employed smooth normal forms. The derivation of the coefficients will be the subject of the remainder of section 2.

Having derived the parameter-dependent center manifold transformation suitable for lifting the third-order homoclinic asymptotic approximation for the smooth orbital normal form, we turn our attention to obtain asymptotic approximations of the homoclinic solution for the smooth orbital normal form. In section 3, the generalized Lindstedt–Poincaré method for the approximation of homoclinic orbits is improved by introducing an additional transformation of time after applying the usual nonlinear time transformation. The resulting novel algorithm solely relies on polynomial division and does not involve any hyperbolic or trigonometric functions as in [2, 1]. We show that for the quadratic Bogdanov–Takens normal form, we can represent the homoclinic solution in phase space with only one single parameter.

In subsection 3.2, we provide an explicit third-order homoclinic approximation in the perturbed Hamiltonian system using the algorithm described in subsection 3.1. Here we also provide a third-order approximation to the reparametrization of time. The profiles of the homoclinic solution will only then be approximated accurately, resulting in a robust initial predictor for starting continuation of the branch of homoclinic orbits. In [1], the importance of the time-reparametrization was not recognized, and the zeroth-order approximation was used. We will demonstrate in detail that it is essential to use the higher

time-reparametrization by comparing the Lindstedt–Poincaré method with and without the higher-order time-reparametrization. Effectively, using the Lindstedt–Poincaré method without the higher-order time-reparametrization is equivalent to the zeroth-order regular perturbation method as illustrated in Figure SM3.

By combining the homoclinic asymptotic approximation derived in section 3 with the parameter-dependent center manifold transformation obtained in section 2, we get a correct homoclinic predictor for a general n -dimensional system. It will be shown in section 4 how to incorporate the time translation into the homoclinic predictor.

All the above methods are implemented in the open-source bifurcation and continuation software **MatCont**. The effectiveness of the new predictor is demonstrated on a four-dimensional model from quantum field theory in section 5. A comparison between the new homoclinic predictor near a generic codimension two Bogdanov–Takens bifurcation and the predictor from [1] is given. It will be shown that the order of the higher-order approximations to the homoclinic solutions in the normal form is preserved under the parameter-dependent center manifold transformation.

To separate the theoretical results from practical considerations, only the newly derived third-order homoclinic predictor based on the orbital normal form is implemented in the standard distribution of **MatCont** [17]. All the other predictors derived in this paper are implemented in a separate version of **MatCont** [10]. Accompanying this version of **MatCont** is an online Jupyter Book [9] in which 10 different models are considered using the new homoclinic predictor and comparing different approximation methods in detail.

To improve the readability of the main text, supplementary materials in which a more in-depth analysis of certain topics are provided. A complete overview of what is covered in the supplementary materials can be found in section SM1. In particular, we will revisit and correct both the regular perturbation method and the generalized Lindstedt–Poincaré method in section SM7 considered previously in [26, 27, 1]. This will allow us to compare our new asymptotic homoclinic approximation derived in the main text with previous work.

The nonuniqueness of the homoclinic solution due to a time shift results in the nonuniqueness for the systems to be solved in the regular perturbation method. To obtain uniqueness, a so-called phase condition needs to be satisfied. The phase condition used in [26] originates from a theoretical setting in [4]. In section SM3, we use another geometrically motivated phase condition which slightly improves the regular perturbation solution. It is shown in section SM11 that by considering this *norm minimizing* phase condition the homoclinic asymptotic approximation is improved. It turns out, however, that the equations to be solved become much more complicated; in particular, see (SM3.14). Furthermore, by modifying Proposition 4.3 from [4], we use symmetry arguments to simplify the calculations. The resulting proposition (Proposition SM3.1) is used in Corollary 3.3 to show that, under certain conditions, the asymptotic approximation to the homoclinic solution has a particular structure.

We will describe in section SM9 the new implementation of the homoclinic predictor in the latest version of **MatCont**. We show how to use the obtained predictors to construct an initial prediction for the defining system of the homoclinic solutions. In addition to an initial prediction, an initial tangent vector is also necessary to start continuation. Our implementation prevents the issue of possible continuation in the wrong direction, i.e., towards the

Bogdanov–Takens point. This section also provides details on installing and using the separate version of `MatCont` with the Jupyter Notebooks.

Last, to gain some insight on which interval the homoclinic asymptotic expansion might converge and the speed of the algorithm, we implement the algorithm proposed in subsection 3.1 in the programming language Julia [6] for the quadratic normal form for the codimension two Bogdanov–Takens bifurcation in section SM13. This will additionally help us to provide an initial guess to the perturbation parameter in order to start the homoclinic continuation in `MatCont`.

2. Parameter-dependent center manifold reduction combined with normalization and time-reparametrization. Without loss of generality, we may assume that the system (1.1) has a codimension two bifurcation of the equilibrium $x_0 = 0$ at $\alpha_0 = 0$. Let the normal form on the n_c -dimensional center manifold be given by

$$(2.1) \quad \frac{d}{d\eta} w(\eta) = G(w(\eta), \beta), \quad G: \mathbb{R}^{n_c} \times \mathbb{R}^2 \rightarrow \mathbb{R}^{n_c}.$$

Here G is assumed to be one of the (known) equivalent normal forms; i.e., it is known which terms are in G and which are not there. Thus, G admits a series expansion of the form

$$(2.2) \quad G(w(\eta), \beta) = \sum_{|\nu|+|\mu|\geq 1} \frac{1}{\nu! \mu!} g_{\nu\mu} w^\nu \beta^\mu,$$

where the multi-indices ν and μ have lengths n_0 and 2, respectively, and the coefficients $g_{\nu\mu} \in \mathbb{R}^{n_c}$.

Similarly, f admits the series expansion given by

$$(2.3) \quad f(x, \alpha) = Ax + J_1 \alpha + \sum_{k,l \geq 0, k+l > 1} \frac{1}{k! l!} D_x^k D_\alpha^l f_{(0,0)}(x^{(k)}, \alpha^{(l)}).$$

Here $A := D_x f_{(0,0)}$ is the Jacobian matrix of (1.1) evaluated at $(x, \alpha) = (0, 0)$. Likewise, $J_1 := D_\alpha f_{(0,0)}$ and $D_x^k D_\alpha^l f_{(0,0)}: (\mathbb{R}^n)^k \times (\mathbb{R}^2)^l \rightarrow \mathbb{R}^n$ are the mixed Fréchet derivatives of order $k + l$. Last, $x^{(k)} := (x, \dots, x) \in (\mathbb{R}^n)^k$, with $x \in \mathbb{R}^n$ repeated k times and similarly for $\alpha^{(l)}$.

Now suppose that a parameter-dependent approximation to an emanating codimension one bifurcation curve in the normal form (2.1) is known and given by

$$(2.4) \quad \epsilon \mapsto (w_\epsilon(\eta), \beta_\epsilon).$$

We assume here that $w_\epsilon(\eta)$ and β_ϵ have been derived up to a certain (not necessarily the same) order in ϵ . Let Φ be a basis for the center subspace of A and Ψ a basis for the center subspace of the adjoint A^T , normalized such that $\Psi^T \Phi$ is the identity matrix on \mathbb{R}^{n_c} . Then the solutions on the parameter-dependent center manifold can be parametrized by

$$(2.5) \quad x(t(\eta)) = \Phi w(\eta) + H(w(\eta), \beta), \quad H: \mathbb{R}^{n_c} \times \mathbb{R}^2 \rightarrow \mathbb{R}^n,$$

$$(2.6) \quad \alpha = K(\beta), \quad K: \mathbb{R}^2 \rightarrow \mathbb{R}^2,$$

where H describes a correction to the critical center subspace and K provides a correspondence between the original parameters in (1.1) and those of the normal form (2.1). The regularity of the map K at zero is referred to as the *transversality* condition for the considered bifurcation. Next, let the time t in the ODE (1.1) and the time η in the normal form (2.1) be related through the parameter-dependent non linear time rescaling

$$(2.7) \quad \frac{dt}{d\eta} = \theta(w, \beta), \quad \theta: \mathbb{R}^{n_c} \times \mathbb{R}^2 \rightarrow \mathbb{R}.$$

Then the invariance of the center manifold implies the *homological equation*

$$(HOM) \quad f(\Phi w + H(w, \beta), K(\beta))\theta(w, \beta) = (\Phi + D_w H(w, \beta))G(w, \beta).$$

The mappings H , K , and θ can be expanded as

$$(2.8) \quad H(w, \beta) = \sum_{\substack{|\nu|+|\mu|\geq 1, \\ \neg(|\nu|\neq 1 \wedge |\mu|=0)}} \frac{1}{\nu! \mu!} H_{\nu\mu} w^\nu \beta^\mu, \quad K(\beta) = \sum_{|\mu|\geq 1} \frac{1}{\mu!} K_\mu \beta^\mu,$$

and

$$(2.9) \quad \theta(w, \beta) = \sum_{|\nu|+|\mu|\geq 0} \frac{1}{\nu! \mu!} \theta_{\nu\mu} w^\nu \beta^\mu.$$

Substituting (2.2), (2.3), (2.8), and (2.9) into (HOM) and collecting terms of equal power in (w, β) , we obtain linear systems which can be solved at each order recursively. However, these solutions may be nonunique.

The question now is which coefficients need to be included in the expansion of (2.3), (2.8), and (2.9) in order to translate the approximation (2.4) obtained up to a certain order to the parameter-dependent center manifold of the codimension two point in such a way that the approximation order of (2.4) is maintained.

The first task here is to determine which terms in the expansion of the system restricted to the center manifold of (1.1) at the bifurcation point alter the approximation (2.4). It is important here to not only take into account the terms that alter the approximation that are present in the normal form G , but also terms that are *not present in the normal form*, as long as the approximation (2.4) is affected by those terms.

The coefficients $g_{\nu\mu}$ corresponding to these terms can be computed by reconstructing the normal form from (2.5) and (2.6), or equivalently from (HOM). Indeed, by multiplying the homological equation (HOM) by Ψ , using (2.2), and rearranging terms we obtain

$$\begin{aligned} \frac{d}{d\eta} w(\eta) &= \sum_{|\nu|+|\mu|\geq 1} \frac{1}{\nu! \mu!} g_{\nu\mu} w^\nu \beta^\mu \\ &= \Psi^T (f(\Phi w + H(w, \beta), K(\beta))\theta(w, \beta) - D_w H(w, \beta)G(w, \beta)). \end{aligned}$$

The next observation is that the coefficient $g_{\nu\mu}$ first appears in the system obtained from the homological equation (HOM) by considering the coefficient corresponding to the term $w^\nu \beta^\mu$. Using the expansion (2.3), we see that this system has the following form:

$$(2.10) \quad \theta_{00}(AH_{\nu\mu} + J_1 K_\mu) = \Phi g_{\nu\mu} + \Pi_{\nu\mu}.$$

Here $\Pi_{\nu\mu}$ contains the remaining terms. The exact representation of $\Pi_{\nu\mu}$ is not important here. What is important, however, is that it only depends on coefficients g_{rs} , H_{rs} , K_μ , and θ_{rs} such that $|r| \leq |\nu|$ and $|s| \leq |\mu|$, except for the coefficient $g_{\nu\mu}$ itself. Note that, depending on the normal form under consideration, the coefficients $H_{\nu\mu}$ and K_μ may also be present in $\Pi_{\nu\mu}$.

Thus, in order to compute the coefficient $g_{\nu\mu}$, we need to solve the system (2.10). Only then are we able to reconstruct the normal form on the center manifold at the bifurcation point such that the terms $g_{\nu\mu}$, which alter the predictor, are either present or transformed away into higher-order terms. Therefore, we conclude with the following statement.

Recipe 1. Consider the normal form (2.1) for a codimension two bifurcation point at $(x, \alpha) \equiv (0, 0)$ of (1.1). Let the asymptotic approximation (2.4) for a codimension one bifurcation curve emanating from the bifurcation point be determined up to a certain order. Suppose that the approximation is affected by terms with the coefficients $g_{\nu\mu}$ of the ODE restricted to the center manifold of the bifurcation point, which may or may not be present in the normal form.

Then, in order to lift the available asymptotics in the normal form to the full system, such that the approximation order of (2.4) is maintained, it is generically necessary and sufficient to include the coefficients H_{rs} and K_{rs} , with $|r| \leq |\nu|$ and $|s| \leq |\mu|$, into the expansion of H and K in (2.8) in such a way that the systems (2.10) are solvable. The systems to be solved reveal simultaneously which higher-order derivatives must be included in the Taylor expansion of f in (2.3).

Although the above statement may look obvious, it has not been understood correctly in all previous attempts to derive a higher-order approximation of the homoclinic bifurcation curve emanating from the generic codimension two Bogdanov–Takens bifurcation in n -dimensional systems; see [1, 26, 27].

The word *generically* in Recipe 1 refers to the exclusion of special cases in which the system (1.1) contains certain symmetry or other restrictions; see, for example, [20, 2.3.3 Symmetries and Reversibility]. In such situations, certain coefficients and higher-order derivatives are known to vanish beforehand.

In subsection 2.2, we will show how to determine which coefficients in the ODE restricted to the center manifold of a generic codimension two Bogdanov–Takens bifurcation point in (1.1) alter the homoclinic asymptotics up to order three. Based on this, and using Recipe 1, we are able to rigorously determine which coefficients to include in the parameter-dependent center manifold parametrization H and parameter transformation K .

Remark 2.1. We have not discussed yet which coefficients of the time-reparametrization θ need to be included. These terms are determined by the requirement that the systems in (2.10) are solvable for the by-now-known multi-indices ν and μ . We know that this is indeed possible by the existence of the considered normal form. Furthermore, the determined coefficients $H_{\nu\mu}$ and K_μ are left unchanged if a reparametrization of time is not allowed. Indeed, the time-reparametrization θ only simplifies the normal form but does not change which terms in and not in the normal form are affected by the asymptotic approximation.

2.1. Parameter-dependent normal form. Suppose that the ordinary differential equation (1.1) undergoes a generic codimension two Bogdanov–Takens bifurcation at $(x, \alpha) \equiv (x_0, \alpha_0)$.

That is, the linearization of (1.1) has a double, but not semisimple, zero eigenvalue, while all other eigenvalues are away from the imaginary axis. The critical smooth normal form on the two-dimensional center manifold of (x_0, α_0) is given by [3, 19]

$$\begin{cases} \dot{w}_0 = w_1, \\ \dot{w}_1 = aw_0^2 + bw_0w_1 + \mathcal{O}(\|w\|^3), \end{cases}$$

where $ab \neq 0$, w_i is a shorthand notation for $w_i(\eta)$ for $i = 0, 1$, and the dot is the derivative with respect to η .

Under these nondegeneracy and certain transversality conditions, the *topological normal form* for the codimension two Bogdanov–Takens bifurcation is given by

$$(2.11) \quad \begin{cases} \dot{w}_0 = w_1, \\ \dot{w}_1 = \beta_1 + \beta_2w_1 + aw_0^2 + bw_0w_1; \end{cases}$$

see [7, 8, 30, 19, 25]. It is well known that in system (2.11) three codimension one bifurcation curves emanate from $(\beta_1, \beta_2) = (0, 0)$: a saddle-node, a Hopf, and a saddle-homoclinic bifurcation curve.

By using either the regular perturbation or the Lindstedt–Poincaré method, an approximation to the homoclinic bifurcation curve and the corresponding solution can, theoretically, be obtained up to any order in the singular-rescaling parameter ϵ ; see [26, 27, 1, 2].

To obtain the second-order approximation to the homoclinic solutions on the center manifold in (1.1), it is, in general, insufficient to only consider the topological normal form (2.11); see [11]. One way to deal with this problem is to consider the parameter-dependent *smooth normal form*

$$(2.12) \quad \begin{cases} \dot{w}_0 = w_1, \\ \dot{w}_1 = \beta_1 + \beta_2w_1 + (a + a_1\beta_2)w_0^2 + (b + b_1\beta_2)w_0w_1 + ew_0^2w_1 + dw_0^3 + g(w, \beta), \end{cases}$$

with

$$g(w, \beta) = \mathcal{O}(|\beta_1|\|w\|^2 + |\beta_2|w_1^2 + \|\beta\|^2\|w\|^2 + \|\beta\|\|w\|^3 + \|w\|^4)$$

as in [27, 1]. Here $w_i = w_i(t)$ ($i = 0, 1$) now depends explicitly on t as in the original ODE (1.1).

However, in this paper, we will allow for a time-reparametrization and use the parameter-dependent *smooth orbital normal form* (sometimes called the C^∞ -equivalent normal form)

$$(2.13) \quad \begin{cases} \dot{w}_0 = w_1, \\ \dot{w}_1 = \beta_1 + \beta_2w_1 + aw_0^2 + bw_0w_1 + w_0^2w_1h(w_0, \beta) + w_1^2Q(w_0, w_1, \beta), \end{cases}$$

where h is C^∞ and Q is N -flat for an a priori given N ; see [11]. Here the dot represents the derivative with respect to the new time η of $w_i(\eta)$ ($i = 0, 1$). Furthermore, we will show that we can assume $h(0, 0) = 0$. Note that we do not impose the coefficients to be $a = 1$ and $b = \pm 1$ as in [11]. This simplifies the systems to be solved in the next section without complicating the solutions for the homoclinic corrections. Indeed, we can scale away the coefficients a and

b by modifying the singular rescaling used in [26, 27, 1, 2]. Also note that the normal form (2.13) was used in [11] to study degenerate (codimension 3) Bogdanov–Takens bifurcations, while we found it to be essential for constructing homoclinic predictors in the case of generic codimension two Bogdanov–Takens bifurcations.

To approximate the homoclinic solutions emanating from the Bogdanov–Takens point, we apply the modified singular rescaling

$$(2.14) \quad w_0 = \frac{a}{b^2} u \epsilon^2, \quad w_1 = \frac{a^2}{b^3} v \epsilon^3, \quad \beta_1 = -4 \frac{a^3}{b^4} \epsilon^4, \quad \beta_2 = \frac{a}{b} \tau \epsilon^2, \quad s = \frac{a}{b} \epsilon \eta, \quad (\epsilon \neq 0)$$

to (2.13) with $h(0,0) = 0$ to obtain the second-order nonlinear oscillator

$$(2.15) \quad \ddot{u} = -4 + u^2 + \dot{u}(u + \tau)\epsilon + \mathcal{O}(\epsilon^4),$$

which does not depend on (a,b) . Here the dot represents the derivative with respect to s .

2.2. Center manifold reduction for smooth orbital normal form. We want to relate the third-order homoclinic approximation in the smooth orbital normal form (2.13) to the homoclinic solutions of (1.1) near (x_0, α_0) . The third-order ϵ -expansion of the homoclinic solution depends on the coefficients in ϵ up to order three in the perturbed Hamiltonian system (2.15); see section 3. By inspecting the blowup transformation (2.14), we can determine exactly which terms must be included in the expansion of H and K and which multilinear forms to include in the expansion of f . Indeed, we search for those terms

$$(2.16) \quad \beta_1^i \beta_2^j w_0^k w_1^l, \quad \text{with } i, j, k, l \in \mathbb{N}_0,$$

in the expansion of the system restricted to the center manifold of (x_0, α_0) in (1.1) that *alter* the coefficients in ϵ up to order three in the nonlinear oscillator (2.15). Thus, by substituting (2.14) into (2.16), inspecting the resulting exponents, requiring that the resulting system implies (2.15), we obtain that these terms are determined by solving the linear Diophantine equation

$$(2.17) \quad 4i + 2j + 2k + 3l - 4 = m, \quad m \in \{-2, -1, 0, 1, 2, 3\},$$

for $i, j, k, l \in \mathbb{N}_0$. From (2.17) we obtain the solutions listed in Table 1.

Table 1

Terms in the reduced system restricted to the parameter-dependent center manifold of the codimension two Bogdanov–Takens point in (1.1) that alter the third-order homoclinic predictor.

Order in ϵ	Affected terms
ϵ^{-2}	w_0, β_2
ϵ^{-1}	w_1
ϵ^0	$w_0^2, w_0 \beta_2, \beta_2^2, \beta_1$
ϵ^1	$w_0 w_1, w_1 \beta_2$
ϵ^2	$w_1^2, w_0^3, w_0^2 \beta_2, w_0 \beta_2^2, \beta_2^3, w_0 \beta_1, \beta_1 \beta_2$
ϵ^3	$w_0^2 w_1, w_0 w_1 \beta_2, w_1 \beta_2^2, w_1 \beta_1$

Thus, we expand the mappings H , K , and θ , including precisely those terms which are needed to transfer the homoclinic predictor in the normal form to the center manifold maintaining the approximation order. Applying Recipe 1 and Table 1, we can write

$$(2.18) \quad f(x, \alpha) = Ax + J_1\alpha + \frac{1}{2}B(x, x) + A_1(x, \alpha) + \frac{1}{2}J_2(\alpha, \alpha) + \frac{1}{6}C(x, x, x) \\ + \frac{1}{2}B_1(x, x, \alpha) + \frac{1}{2}A_2(x, \alpha, \alpha) + \frac{1}{6}J_3(\alpha, \alpha, \alpha) + \mathcal{O}(\|(x, \alpha)\|^4),$$

$$(2.19) \quad H(w, \beta) = H_{0010}\beta_1 + H_{0001}\beta_2 + \frac{1}{2}H_{2000}w_0^2 + H_{1100}w_0w_1 + \frac{1}{2}H_{0200}w_1^2 \\ + H_{1010}w_0\beta_1 + H_{1001}w_0\beta_2 + H_{0110}w_1\beta_1 + H_{0101}w_1\beta_2 + \frac{1}{2}H_{0002}\beta_2^2 \\ + H_{0011}\beta_1\beta_2 + \frac{1}{6}H_{3000}w_0^3 + \frac{1}{2}H_{2100}w_0^2w_1 + H_{1101}w_0w_1\beta_2 + \frac{1}{2}H_{2001}w_0^2\beta_2 \\ + \frac{1}{6}H_{0003}\beta_2^3 + \frac{1}{2}H_{1002}w_0\beta_2^2 + \frac{1}{2}H_{0102}w_1\beta_2^2 \\ + \mathcal{O}(|w_1|^3 + |w_0w_1^2| + |\beta_2w_1^2| + |\beta_1||w|^2 + |\beta_1^2| + \|(w, \beta)\|^4),$$

$$(2.20) \quad K(\beta) = K_{10}\beta_1 + K_{01}\beta_2 + \frac{1}{2}K_{02}\beta_2^2 + K_{11}\beta_1\beta_2 + \frac{1}{6}K_{03}\beta_2^3 \\ + \mathcal{O}(|\beta_1|^2 + |\beta_1||\beta_2|^2 + |\beta_1|^2|\beta_2| + \|\beta\|^4),$$

$$(2.21) \quad \theta(w, \beta) = 1 + \theta_{1000}w_0 + \theta_{0001}\beta_2 + \mathcal{O}(|w_1| + |\beta_2| + \|(w, \beta)\|^2),$$

where $A = D_x f(x_0, \alpha_0)$, $J_1 = D_\alpha f(x_0, \alpha_0)$, and B, J_2, J_3, C, A_1, A_2 , and B_1 are the standard multilinear forms, introduced for readability.

In subsections 2.2.1 to 2.2.8, we will solve the unknowns in (2.19)–(2.21). For this, it might be useful to consult Figure 2. There a schematic overview is given in which order the coefficients are solved. Notice, in particular, that of course the critical normal form coefficients can be solved independently of the parameter-dependent normal form coefficients. Also, due to the interdependence of the normal form coefficients, it is convenient not to solve order by order.

Remark 2.2. Notice that compared with [1] there are four additional terms in the expansion of H , i.e., with coefficients $H_{0011}, H_{1002}, H_{0102}$, and H_{0003} , and two additional terms in the expansion of K , with coefficients K_{11} and K_{03} . Also, there are two additional higher-order derivatives, namely A_2 and J_3 , included in the Taylor expansion of f previously not taken into account. By Recipe 1, we know that all displayed terms in (2.19)–(2.21) and the higher-order derivatives used in (2.18) are both necessary and sufficient to consider. The coefficients in the time-reparametrization (2.21) are determined by the requirement that the systems obtained from the homological equation are solvable.

2.2.1. (Generalized) eigenvectors. We assume that the equilibrium (x_0, α_0) has a double (but not semisimple) zero eigenvalue, while all other eigenvalues are away from the imaginary axis. Thus, there exist two real linearly independent (generalized) eigenvectors, $q_0, q_1 \in \mathbb{R}^n$, of A , such that

$$(2.22) \quad Aq_0 = 0, \quad Aq_1 = q_0,$$

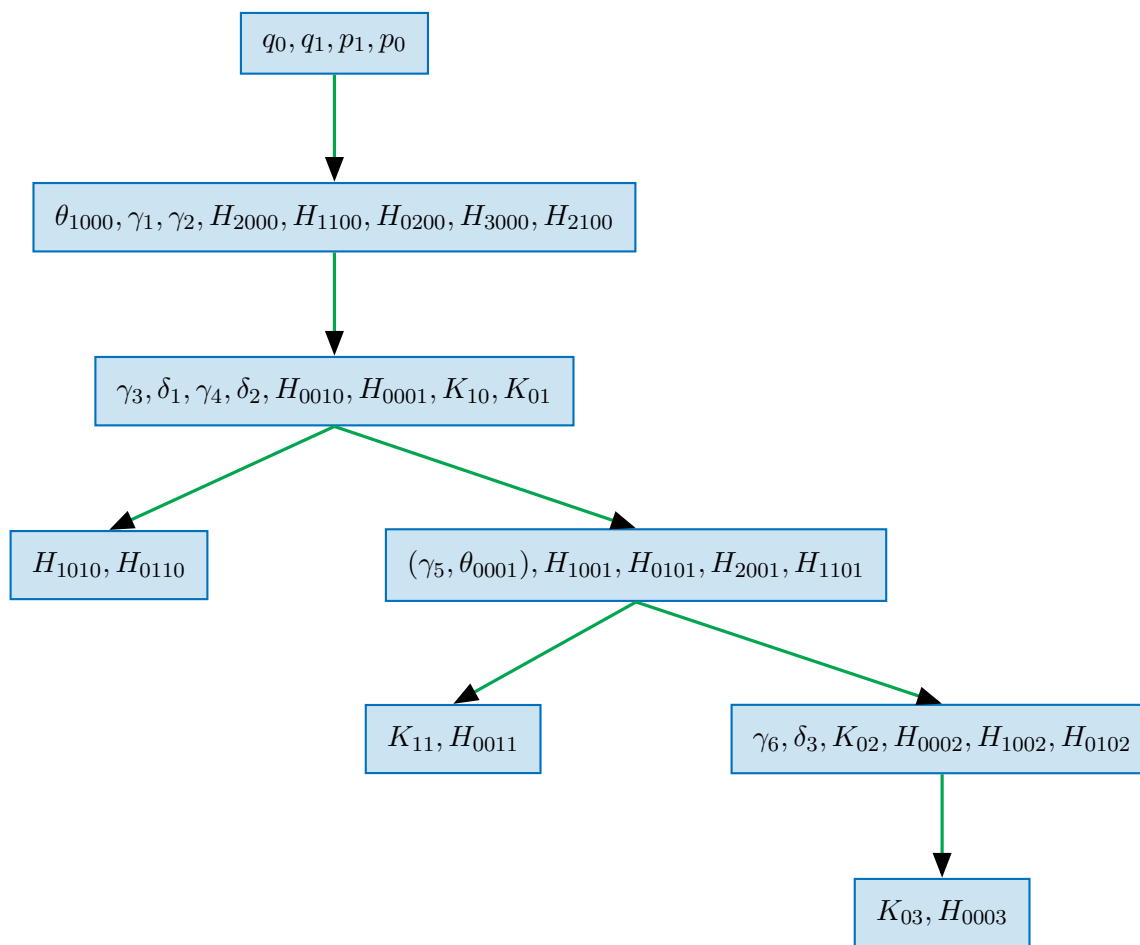


Figure 2. Schematic overview in which order the coefficients in the expansion of H , K , and θ are derived.

and two left (generalized) eigenvectors $p_1^T, p_0^T \in \mathbb{R}^n$, of A , such that

$$p_1 A = 0, \quad p_0 A = p_1.$$

These vectors can be normalized to satisfy

$$p_i q_j = \delta_{ij}, \quad i = 0, 1, j = 0, 1.$$

As in [24], we impose the condition

$$(2.23) \quad q_0^T q_0 = 1, \quad q_1^T q_0 = 0$$

to uniquely define the vectors $\{q_0, q_1, p_1, p_0\}$ up to a plus or minus sign. Using these vectors, we can explicitly write the basis for the center subspace as $\Phi = (q_0 \ q_1)$ and the adjoint subspace as $\Psi = (p_1^T \ p_0^T)$.

Note that collecting the coefficients of the linear terms in w in the homological (HOM) equation provides precisely the systems defining the (generalized) eigenvectors (2.22).

2.2.2. Critical coefficients. Collecting the quadratic coefficients in w from the homological (HOM) yields the systems

$$(2.24) \quad -AH_{2000} = B(q_0, q_0) - 2aq_1,$$

$$(2.25) \quad -AH_{1100} = B(q_0, q_1) - bq_1 + \theta_{1000}q_0 - H_{2000},$$

$$(2.26) \quad -AH_{0200} = B(q_1, q_1) - 2H_{1100}.$$

The Fredholm solvability condition for the first two systems yields the well-known expressions

$$a = \frac{1}{2}p_1B(q_0, q_0),$$

$$b = p_1B(q_0, q_1) + p_0B(q_0, q_0)$$

for the critical coefficients; see, for example, [23]. By the nondegeneracy conditions, we have that $ab \neq 0$.

Remark 2.3. Since we assume that $p_1B(q_0, q_0) \neq 0$, we can use the freedom in the eigenvectors to normalize the critical coefficient a to one. Alternatively, the freedom could have been used to set $b = 1$. To have the situation $a = 1$ and $b = \pm 1$ as in [11], the coefficient in front of the constant term in the expansion of θ , i.e., θ_{0000} , should be used. For convenience, we fixed the constant θ_{0000} to 1.

Now that (2.24) and (2.25) are solvable, we can define

$$\hat{H}_{2000} = -A^{\text{INV}}(B(q_0, q_0) - 2aq_1),$$

$$\hat{H}_{1100} = -A^{\text{INV}}(B(q_0, q_1) - bq_1 - \hat{H}_{2000}).$$

The expression $x = A^{\text{INV}}y$ is defined by solving the nonsingular bordered system

$$\begin{pmatrix} A & p_1^T \\ q_0^T & 0 \end{pmatrix} \begin{pmatrix} x \\ s \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix}$$

for the unknown $(x, s) \in \mathbb{R}^{n+1}$ that necessarily satisfies $s = 0$. While this uniquely defines x , the solutions to system $Ax = y$, are of course still nonunique. The properties of bordered linear systems and their role in numerical bifurcation analysis are discussed in [22] and [18, Chapter 3].

It follows that the general solutions of the systems (2.24) and (2.25) are given by

$$H_{2000} = \hat{H}_{2000} + \gamma_1 q_0,$$

$$H_{1100} = \hat{H}_{1100} + \gamma_1 q_1 - \theta_{1000}q_1 + \gamma_2 q_0.$$

The constant γ_1 is determined by the solvability condition from (2.26), which gives

$$\gamma_1 = -p_1\hat{H}_{1100} + \frac{1}{2}p_1B(q_1, q_1) + \theta_{1000}.$$

To determine the constant γ_2 and the coefficient θ_{1000} , we consider the w_0^3 and $w_0^2w_1$ terms in the homological equation. After some simplification, we obtain the systems

$$(2.27) \quad -AH_{3000} = 3B(H_{2000}, q_0) + C(q_0, q_0, q_0) + 6a\theta_{1000}q_1 - 6aH_{1100},$$

$$(2.28) \quad -AH_{2100} = 2B(H_{1100}, q_0) + B(H_{2000}, q_1) + C(q_0, q_0, q_1) - 2aH_{0200} - 2bH_{1100} \\ - H_{3000} + 2\theta_{1000}(bq_1 - \theta_{1000}q_0 + H_{2000}).$$

The solvability condition of the first equation determines θ_{1000} as

$$(2.29) \quad \theta_{1000} = -\frac{1}{12a}p_1 \left\{ 3B(\hat{H}_{2000}, q_0) + C(q_0, q_0, q_0) \right\} + \frac{1}{2}p_1 \hat{H}_{1100}.$$

The solvability condition for the system in (2.28) yields, after a rather lengthy calculation, that γ_2 is determined by

$$(2.30) \quad \gamma_2 = \frac{1}{6a} \left[p_1 \left\{ 2B(\hat{H}_{1100}, q_0) + B(\hat{H}_{2000}, q_1) + C(q_0, q_0, q_1) \right\} \right. \\ \left. + 2ap_0B(q_1, q_1) + 2bp_0 \left(B(q_0, q_1) - \hat{H}_{2000} \right) \right. \\ \left. + p_0 \left(3B(\hat{H}_{2000}, q_0) + C(q_0, q_0, q_0) \right) \right. \\ \left. + \gamma_1b - 10ap_0\hat{H}_{1100} + 2b\theta_{1000} \right].$$

The solvability of (2.28) implies that we can assume that $h(0,0) = 0$ in the smooth orbital normal form (2.13). Since the systems in (2.26)–(2.28) are now all consistent, we are allowed to take the bordered inverses to obtain

$$(2.31) \quad H_{0200} = -A^{\text{INV}} [B(q_1, q_1) - 2H_{1100}],$$

$$(2.32) \quad H_{3000} = -A^{\text{INV}} [3B(H_{2000}, q_0) + C(q_0, q_0, q_0) + 6a\theta_{1000}q_1 - 6aH_{1100}],$$

$$(2.32) \quad H_{2100} = -A^{\text{INV}} [-2aH_{0200} - 2bH_{1100} - H_{3000} + 2B(H_{1100}, q_0) \\ + B(H_{2000}, q_1) + 2\theta_{1000}(bq_1 - \theta_{1000}q_0 + H_{2000}) + C(q_0, q_0, q_1)].$$

2.2.3. Parameter-dependent linear coefficients. The coefficients of the linear terms in β give the systems

$$(2.33) \quad -AH_{0001} = J_1K_{01}, \\ -AH_{0010} = J_1K_{10} - q_1.$$

Since p_1 and J_1 are known, we can calculate

$$\nu := (p_1J_1)^T.$$

Given that $ab \neq 0$, the transversality condition indicates that the vector ν is nonzero; see [26]. It then follows from the Fredholm alternative that

$$K_{01} = \delta_1 \hat{K}_{01}, \\ H_{0001} = \delta_1 \left(\hat{H}_{0001} + \gamma_3 q_0 \right), \\ K_{10} = \hat{K}_{10} + \delta_2 K_{01}, \\ H_{0010} = \hat{H}_{0010} + \delta_2 H_{0001} + \gamma_4 q_0,$$

where

$$\begin{aligned}\hat{K}_{10} &= \frac{1}{\|\nu\|^2} \nu, \\ \hat{H}_{0010} &= -A^{\text{INV}} \left(J_1 \hat{K}_{10} - q_1 \right), \\ \hat{K}_{01} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \hat{K}_{10}, \\ \hat{H}_{0001} &= -A^{\text{INV}} J_1 \hat{K}_{01},\end{aligned}$$

and $\delta_{1,2}$, $\gamma_{3,4}$ are real constants determined by the solvability condition of the $w\beta$ terms in the homological equation. Collecting the corresponding systems in the homological equation yields

$$\begin{aligned}-AH_{1001} &= B(H_{0001}, q_0) + A_1(q_0, K_{01}), \\ -AH_{0101} &= B(H_{0001}, q_1) + A_1(q_1, K_{01}) - H_{1001} - q_1 + \theta_{0001}q_0, \\ -AH_{1010} &= B(H_{0010}, q_0) + A_1(q_0, K_{10}) - H_{1100} + \theta_{1000}q_1, \\ -AH_{0110} &= B(H_{0010}, q_1) + A_1(q_1, K_{10}) - H_{0200} - H_{1010}.\end{aligned}$$

The solvability condition for the first two systems yields

$$\begin{aligned}\gamma_3 &= -\frac{1}{2a} p_1 \left(B(\hat{H}_{0001}, q_0) + A_1(q_0, \hat{K}_{01}) \right), \\ \delta_1 &= \left[p_1 \left(B(\hat{H}_{0001}, q_1) + A_1(q_1, \hat{K}_{01}) \right) + p_0 \left(B(\hat{H}_{0001}, q_0) + A_1(q_0, \hat{K}_{01}) \right) + \gamma_3 b \right]^{-1},\end{aligned}$$

while the solvability condition for the latter two systems yields

$$\begin{aligned}\gamma_4 &= \frac{1}{2a} p_1 H_{1100} - \theta_{1000} - p_1 \left(B(\hat{H}_{0010}, q_0) + A_1(q_0, \hat{K}_{10}) \right), \\ \delta_2 &= -p_1 \left(B(\hat{H}_{0010}, q_1) + A_1(q_1, \hat{K}_{10}) \right) - \gamma_4 b + p_1 H_{0200} \\ &\quad - p_0 \left(B(\hat{H}_{0010}, q_0) + A_1(q_0, \hat{K}_{10}) - H_{1100} \right).\end{aligned}$$

Note that the denominator in δ_1 is nonzero by the transversality condition if $ab \neq 0$.

2.2.4. Coefficients H_{1010} and H_{0110} . Since we do not need to use the nonuniqueness in the systems for the coefficients H_{1010} and H_{0110} to simplify higher-order systems we let

$$\begin{aligned}H_{1010} &= -A^{\text{INV}} [B(H_{0010}, q_0) + A_1(q_0, K_{10}) - H_{1100} + \theta_{1000}q_1], \\ H_{0110} &= -A^{\text{INV}} [B(H_{0010}, q_1) + A_1(q_1, K_{10}) - H_{0200} - H_{1010}].\end{aligned}$$

2.2.5. Coefficients $(\theta_{0001}, \gamma_5), H_{1001}, H_{0101}, H_{2001}, H_{1101}$. Define

$$\begin{aligned}\hat{H}_{1001} &= -A^{\text{INV}} [B(H_{0001}, q_0) + A_1(q_0, K_{01})], \\ \hat{H}_{0101} &= -A^{\text{INV}} [B(H_{0001}, q_1) + A_1(q_1, K_{01}) - \hat{H}_{1001} - q_1],\end{aligned}$$

so that

$$\begin{aligned} H_{1001} &= \hat{H}_{1001} + \gamma_5 q_0, \\ H_{0101} &= \hat{H}_{0101} + \gamma_5 q_1 - \theta_{0001} q_1. \end{aligned}$$

In order to determine γ_5 and θ_{0001} , we consider the systems corresponding to the $w_0^2\beta_2$ and $w_0w_1\beta_2$ terms in the homological equation. These are given by

$$\begin{aligned} -AH_{2001} &= -2aH_{0101} + A_1(H_{2000}, K_{01}) + B(H_{0001}, H_{2000}) + 2B(H_{1001}, q_0) \\ &\quad + 2a\theta_{0001}q_1 + B_1(q_0, q_0, K_{01}) + C(H_{0001}, q_0, q_0), \\ -AH_{1101} &= -bH_{0101} - H_{1100} - H_{2001} + A_1(H_{1100}, K_{01}) \\ &\quad + \theta_{1000}(H_{1001} + q_1 - \theta_{0001}q_0) + B(H_{0001}, H_{1100})B(H_{0101}, q_0) \\ &\quad + B(H_{1001}, q_1) + \theta_{0001}(H_{2000} + bq_1 - \theta_{1000}q_0) + B_1(q_0, q_1, K_{01}) \\ &\quad + C(H_{0001}, q_0, q_1). \end{aligned} \quad (2.34)$$

The Fredholm solvability condition leads to the following system to be solved:

$$\begin{pmatrix} 2a & 4a \\ b & b \end{pmatrix} \begin{pmatrix} \gamma_5 \\ \theta_{0001} \end{pmatrix} = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}, \quad (2.35)$$

where

$$\begin{aligned} \zeta_1 &= -p_1 \left[-2a\hat{h}_{0101} + A_1(h_{2000}, K_{01}) + B(h_{0001}, h_{2000}) \right. \\ &\quad \left. + 2B(\hat{h}_{1001}, q_0) + B_1(q_0, q_0, K_{01}) + C(h_{0001}, q_0, q_0) \right], \\ \zeta_2 &= -p_1 \left[-b\hat{h}_{0101} - h_{1100} + A_1(h_{1100}, K_{01}) \right. \\ &\quad \left. + \theta_{1000}(\hat{h}_{1001} + q_1) + B(h_{0001}, h_{1100}) + B(\hat{h}_{0101}, q_0) \right. \\ &\quad \left. + B(\hat{h}_{1001}, q_1) + B_1(q_0, q_1, K_{01}) + C(h_{0001}, q_0, q_1) \right] \\ &\quad - p_0 \left[-2a\hat{h}_{0101} + A_1(h_{2000}, K_{01}) + B(h_{0001}, h_{2000}) \right. \\ &\quad \left. + 2B(\hat{h}_{1001}, q_0) + B_1(q_0, q_0, K_{01}) + C(h_{0001}, q_0, q_0) \right]. \end{aligned} \quad (2.36)$$

Notice that the matrix is invertible by the nondegeneracy condition. Now that the systems in (2.34) are solvable, we obtain

$$\begin{aligned} H_{2001} &= -A^{\text{INV}} \left[-2aH_{0101} + A_1(H_{2000}, K_{01}) + B(H_{0001}, H_{2000}) \right. \\ &\quad \left. + 2B(H_{1001}, q_0) + 2a\theta_{0001}q_1 + B_1(q_0, q_0, K_{01}) + C(H_{0001}, q_0, q_0) \right], \\ H_{1101} &= -A^{\text{INV}} \left[-bH_{0101} - H_{1100} - H_{2001} + A_1(H_{1100}, K_{01}) \right. \\ &\quad \left. + \theta_{1000}(H_{1001} + q_1 - \theta_{0001}q_0) + B(H_{0001}, H_{1100}) + B(H_{0101}, q_0) \right. \\ &\quad \left. + B(H_{1001}, q_1) + \theta_{0001}(H_{2000} + bq_1 - \theta_{1000}q_0) + B_1(q_0, q_1, K_{01}) \right. \\ &\quad \left. + C(H_{0001}, q_0, q_1) \right]. \end{aligned} \quad (2.37)$$

2.2.6. Coefficients K_{11} and H_{0011} . Collecting the systems corresponding to the $\beta_1\beta_2$ term in the homological equation yields

$$(2.38) \quad \begin{aligned} -AH_{0011} = & J_1K_{11} + A_1(H_{0001}, K_{10}) + A_1(H_{0010}, K_{01}) \\ & + B(H_{0001}, H_{0010}) + J_2(K_{01}, K_{10}) + \theta_{0001}q_1 - H_{0101}. \end{aligned}$$

Using the identity

$$p_1J_1K_{10} = 1$$

from the second system in (2.33) combined with the solvability condition yields

$$\begin{aligned} K_{11} = & -p_1[A_1(H_{0001}, K_{10}) + A_1(H_{0010}, K_{01}) \\ & + B(H_{0010}, H_{0001}) + J_2(K_{10}, K_{01}) + \theta_{0001}q_1 - H_{0101}]K_{10}. \end{aligned}$$

It follows that

$$(2.39) \quad \begin{aligned} H_{0011} = & -A^{\text{INV}}[J_1K_{11} + A_1(H_{0001}, K_{10}) + A_1(H_{0010}, K_{01}) \\ & + B(H_{0001}, H_{0010}) + J_2(K_{01}, K_{10}) + \theta_{0001}q_1 - H_{0101}]. \end{aligned}$$

2.2.7. Coefficients K_{02} , H_{0002} , H_{1002} , H_{0102} . The systems corresponding to the β_2^2 , $w_0\beta_2^2$, and $w_1\beta_2^2$, terms in the homological equation yield

$$(2.40) \quad \begin{aligned} -AH_{0002} = & J_1K_{02} + 2A_1(H_{0001}, K_{01}) + B(H_{0001}, H_{0001}) + J_2(K_{01}, K_{01}), \\ -AH_{1002} = & 2A_1(H_{1001}, K_{01}) + A_1(q_0, K_{02}) + A_2(q_0, K_{01}, K_{01}) \\ & + B(q_0, H_{0002}) + 2B(H_{0001}, H_{1001}) + 2B_1(q_0, H_{0001}, K_{01}) \\ & + C(q_0, H_{0001}, H_{0001}), \\ -AH_{0102} = & 2A_1(H_{0101}, K_{01}) + A_1(q_1, K_{02}) + A_2(q_1, K_{01}, K_{01}) \\ & + B(q_1, H_{0002}) + 2B(H_{0001}, H_{0101}) + 2B_1(q_1, H_{0001}, K_{01}) \\ & + C(q_1, H_{0001}, H_{0001}) + 2\theta_{0001}(H_{1001} + q_1 - \theta_{0001}q_0) \\ & - 2H_{0101} - H_{1002}. \end{aligned}$$

The first system is solved similarly to (2.38). In order to make the second and third systems consistent, define

$$\begin{aligned} \hat{K}_{02} = & -p_1[2A_1(H_{0001}, K_{01}) + B(H_{0001}, H_{0001}) + J_2(K_{01}, K_{01})]K_{10}, \\ \hat{H}_{0002} = & -A^{\text{INV}}\left[J_1\hat{K}_{02} + 2A_1(H_{0001}, K_{01}) + B(H_{0001}, H_{0001}) + J_2(K_{01}, K_{01})\right]. \end{aligned}$$

Then the general solution to the first system in (2.40) can be written as

$$\begin{aligned} K_{02} = & \hat{K}_{02} + \delta_3K_{01}, \\ H_{0002} = & \hat{H}_{0002} + \delta_3H_{0001} + \gamma_6q_0. \end{aligned}$$

Substituting these two expressions into the last two system of (2.40) and using the solvability condition yields

$$\begin{aligned}
\gamma_6 = & -\frac{1}{2a}p_1[2A_1(H_{1001}, K_{01}) + A_1(q_0, \hat{K}_{02}) + A_2(q_0, K_{01}, K_{01}) \\
& + B(q_0, \hat{H}_{0002}) + 2B(H_{0001}, H_{1001}) + 2B_1(q_0, H_{0001}, K_{01}) \\
& + C(q_0, H_{0001}, H_{0001})], \\
\delta_3 = & -p_1[2A_1(H_{0101}, K_{01}) + A_1(q_1, \hat{K}_{02}) + A_2(q_1, K_{01}, K_{01}) \\
& + B(q_1, \hat{H}_{0002}) + 2B(H_{0001}, H_{0101}) + 2B_1(q_1, H_{0001}, K_{01}) \\
& + C(q_1, H_{0001}, H_{0001}) + 2\theta_{0001}(H_{1001} + q_1) - 2H_{0101}] \\
& - p_0[2A_1(H_{1001}, K_{01}) + A_1(q_0, \hat{K}_{02}) + A_2(q_0, K_{01}, K_{01}) \\
& + B(q_0, \hat{H}_{0002}) + 2B(H_{0001}, H_{1001}) + 2B_1(q_0, H_{0001}, K_{01}) \\
& + C(q_0, H_{0001}, H_{0001})] - \gamma_6 b.
\end{aligned}$$

Now that the last two systems in (2.40) are consistent, we obtain

$$\begin{aligned}
H_{1002} = & -A^{\text{INV}}[2A_1(H_{1001}, K_{01}) + A_1(q_0, K_{02}) + A_2(q_0, K_{01}, K_{01}) \\
& + B(q_0, H_{0002}) + 2B(H_{0001}, H_{1001}) + 2B_1(q_0, H_{0001}, K_{01}) \\
& + C(q_0, H_{0001}, H_{0001})], \\
H_{0102} = & -A^{\text{INV}}[2A_1(H_{0101}, K_{01}) + A_1(q_1, K_{02}) + A_2(q_1, K_{01}, K_{01}) \\
& + B(q_1, H_{0002}) + 2B(H_{0001}, H_{0101}) + 2B_1(q_1, H_{0001}, K_{01}) \\
& + C(q_1, H_{0001}, H_{0001}) + 2\theta_{0001}(H_{1001} + q_1 - \theta_{0001}q_0) \\
& - 2H_{0101} - H_{1002}].
\end{aligned}$$

2.2.8. Coefficients K_{03} and H_{0003} . Collecting the systems corresponding to the β_2^3 term in the homological equation yields

$$\begin{aligned}
-AH_{0003} = & J_1K_{03} + A_1(H_{0001}, K_{02}) + A_1(H_{0002}, K_{01}) + 2(A_1(H_{0001}, K_{02}) \\
& + A_1(H_{0002}, K_{01}) + 3B(H_{0001}, H_{0002}) + 3J_2(K_{01}, K_{02}) \\
& + 3A_2(H_{0001}, K_{01}, K_{01}) + 3B_1(H_{0001}, H_{0001}, K_{01}) \\
& + C(H_{0001}, H_{0001}, H_{0001}) + J_3(K_{01}, K_{01}, K_{01})).
\end{aligned}$$

This equation is solved similarly to (2.38). We obtain

$$\begin{aligned}
K_{03} = & -p_1[A_1(H_{0001}, K_{02}) + A_1(H_{0002}, K_{01}) + 2A_1(H_{0001}, K_{02}) \\
& + 2A_1(H_{0002}, K_{01}) + 3B(H_{0001}, H_{0002}) + 3J_2(K_{01}, K_{02}) \\
& + 3A_2(H_{0001}, K_{01}, K_{01}) + 3B_1(H_{0001}, H_{0001}, K_{01}) \\
& + C(H_{0001}, H_{0001}, H_{0001}) + J_3(K_{01}, K_{01}, K_{01})]K_{10}, \\
H_{0003} = & -A^{\text{INV}}[J_1K_{03} + A_1(H_{0001}, K_{02}) + A_1(H_{0002}, K_{01}) + 2A_1(H_{0001}, K_{02}) \\
& + 2A_1(H_{0002}, K_{01}) + 3B(H_{0001}, H_{0002}) + 3J_2(K_{01}, K_{02}) \\
& + 3A_2(H_{0001}, K_{01}, K_{01}) + 3B_1(H_{0001}, H_{0001}, K_{01}) \\
& + C(H_{0001}, H_{0001}, H_{0001}) + J_3(K_{01}, K_{01}, K_{01})].
\end{aligned}$$

Remark 2.4. In section SM2, we will illustrate the necessary modifications to the above calculation needed to derive the parameter-dependent center manifold reduction for the smooth normal form given in (2.12).

3. Homoclinic approximation in the normal form. The Lindstedt–Poincaré method considers a nonlinear time transformation defined implicitly through the relation

$$(3.1) \quad \frac{d\xi}{ds} = \omega(\xi),$$

which can be used to remove the so-called secular terms, i.e., terms growing without bound, appearing in the process of approximating periodic orbits in weakly nonlinear oscillators using the regular perturbation approach.

The Lindstedt–Poincaré method is also used to approximate homoclinic solutions in nonlinear oscillators, referred to as the generalized Lindstedt–Poincaré method; see [12, 14, 15, 13, 16]. In this case, there are no terms growing without bound when applying the regular perturbation approach. Instead, there are so-called parasitic turns; see [26, Figure 1]. The nonlinear transformation (3.1) can then be used to remove the parasitic turns. In fact, using the nonlinear transformation, one can obtain a very simple form for the solution of the homoclinic orbit in phase space; see [12, equation 35] and [2].

In both cases, i.e., when approximating periodic orbits or homoclinic orbits, we do the same: a nonlinear time transformation is used to obtain a uniform approximation of the orbit in time.

3.1. Polynomial Lindstedt–Poincaré method. Substituting the parametrization of time ω (3.1) into (2.15) yields

$$(3.2) \quad \omega \frac{d}{d\xi} (\omega \hat{u}') - \hat{u}^2 + 4 = \epsilon \omega \hat{u}' (\hat{u} + \tau) + \mathcal{O}(\epsilon^4),$$

where $\hat{u}(\xi(s)) = u(s)$.

We now introduce a new time ζ via the transformation

$$(3.3) \quad \frac{d\zeta}{d\xi} = 1 - \zeta^2$$

to simplify the solutions obtained below. Note that this transformation implies that $\zeta = \tanh(\xi + c_1)$, where c_1 is some constant. Without loss of generality, we can assume that $c_1 = 0$ since c_1 just shifts the homoclinic solution in time. Substituting (3.3) into (3.2) yields

$$(3.4) \quad (1 - \zeta^2) \tilde{\omega} \frac{d}{d\zeta} ((1 - \zeta^2) \tilde{\omega} \tilde{u}') - \tilde{u}^2 + 4 = \epsilon \tilde{\omega} (1 - \zeta^2) \tilde{u}' (\tilde{u} + \tau) + \mathcal{O}(\epsilon^4),$$

where the prime $'$ now represents the derivative with respect to the variable ζ , $\tilde{u}(\zeta(\xi(s))) = u(s)$, and $\tilde{\omega}(\zeta) = \omega(\xi(\zeta))$.

Expanding \tilde{u} , $\tilde{\omega}$, and τ in ϵ ,

$$(3.5) \quad \tilde{u}(\zeta) = \sum_{i=0} \tilde{u}_i(\zeta) \epsilon^i, \quad \tilde{\omega}(\zeta) = \sum_{i=0} \tilde{\omega}_i(\zeta) \epsilon^i, \quad \tau = \sum_{i=0} \tau_i \epsilon^i,$$

substituting into (3.2), and collecting terms of equal power in ϵ , we obtain the following systems to be solved:

$$(3.6) \quad (1 - \zeta^2) \tilde{\omega}_0 \left((1 - \zeta^2) \tilde{\omega}_0 \tilde{u}'_0 \right)' - \tilde{u}_0^2 + 4 = 0,$$

$$(3.7) \quad (1 - \zeta^2) \left((1 - \zeta^2) \tilde{u}'_i \right)' - 2\tilde{u}_0 \tilde{u}_i + 2(1 - \zeta^2) \tilde{\omega}_i \left((1 - \zeta^2) \tilde{u}'_0 \right)' + (1 - \zeta^2)^2 \tilde{u}'_0 \tilde{\omega}'_i \\ = \tau_{i-1} (1 - \zeta^2) \tilde{u}'_0 + z_i, \quad i \in \mathbb{N}.$$

Here z_i contains the sums and products of terms in $\tilde{u}_j, \tilde{\omega}_j$, and τ_{j-1} , with $0 \leq j \leq i-1$, where τ_{-1} is defined to be zero.

Theorem 3.1. *Equations (3.6) and (3.7) are solvable for every $i \in \mathbb{N}_0$, with*

$$(3.8) \quad \tilde{u}_i(\zeta) = \sigma_i \zeta^2 + \delta_i,$$

where σ_i and δ_i are constants to be determined.

Proof. It is easy to see that (3.6) is solvable with $\sigma_0 = 6$, $\delta_0 = -4$, and $\tilde{\omega}_0(\zeta) = 1$.

Assume that for $i = 1, \dots, n-1$, the systems given by (3.7) are solvable for $\tilde{\omega}_i$ and \tilde{u}_i . Furthermore, also assume that for $i = 1, \dots, n$, \tilde{u}_i is of the form (3.8). We will show that the system (3.7) with $i = n$ is solvable for $\tilde{\omega}_n$.

First notice that (3.7) is just a first-order ODE in $\tilde{\omega}_i$:

$$(3.9) \quad \tilde{\omega}'_i + \frac{2 \left((1 - \zeta^2) \tilde{u}'_0 \right)'}{(1 - \zeta^2) \tilde{u}'_0} \tilde{\omega}_i = \frac{2\tilde{u}_0 \tilde{u}_i - (1 - \zeta^2) \left((1 - \zeta^2) \tilde{u}'_i \right)' + \tau_{i-1} (1 - \zeta^2) \tilde{u}'_0 + z_i}{(1 - \zeta^2)^2 \tilde{u}'_0}.$$

Multiplying by the integrating factor

$$(3.10) \quad (1 - \zeta^2)^2 (\tilde{u}'_0)^2$$

and subsequently integrating with respect to ζ yields the identity

$$(3.11) \quad \tilde{\omega}_i = \frac{(1 - \zeta^2) \left((1 - \zeta^2) \tilde{u}'_0 \right)' \tilde{u}_i - (1 - \zeta^2)^2 \tilde{u}'_0 \tilde{u}'_i + (g_i(\zeta) - g_i(1))}{((1 - \zeta^2) \tilde{u}'_0)^2} \\ = -\frac{\sigma_i}{12} - \frac{(1 - \zeta^2) \left((1 - \zeta^2) \tilde{u}'_0 \right)' \tilde{u}_i + (g_i(\zeta) - g_i(1))}{((1 - \zeta^2) \tilde{u}'_0)^2},$$

where

$$g_i(\zeta) = \tau_{i-1} \int (1 - \zeta^2) (\tilde{u}'_0)^2 d\zeta + \int \tilde{u}'_0 z_i d\zeta.$$

Here we used the identity

$$\left((1 - \zeta^2) \left((1 - \zeta^2) \tilde{u}'_0 \right)' \right)' = 2\tilde{u}_0 \tilde{u}'_0,$$

obtained from differentiating (3.6) and then using integration by parts. Furthermore, we have chosen the integration constant $g_i(1)$ such that numerator in (3.11) vanishes for $\zeta = 1$. Indeed, for $\tilde{\omega}_i$ to be well-defined, the numerator in (3.11) must have roots of at least multiplicity two at $\zeta = 0$ and $\zeta = \pm 1$. By setting $\zeta = -1, 0$ in the numerator of (3.11), we obtain the equations

$$(3.12) \quad 0 = g_i(-1) - g_i(1),$$

$$(3.13) \quad 0 = 12\delta_i - (g_i(0) - g_i(1)),$$

respectively. The first equation can be solved explicitly for τ_{i-1} . Since $g_i(0) = 0$, it follows that $\delta_i = \frac{g_i(1)}{12}$. To show that the roots $\zeta = 0$ and $\zeta = \pm 1$ have multiplicity two, we notice that differentiation of the numerator in (3.11) with respect to ζ is equal to multiplying the right-hand side of (3.9) with the integrating factor (3.10), i.e.,

$$(3.14) \quad \tilde{u}'_0 \left(2\tilde{u}_0\tilde{u}_i - (1 - \zeta^2) \left((1 - \zeta^2)\tilde{u}'_i \right)' + \tau_{i-1} (1 - \zeta^2) \tilde{u}'_0 - z_i \right).$$

Since $\tilde{u}'_0 = 12\zeta$, we can factor out $\zeta = 0$. Then substituting $\zeta = \pm 1$ into (3.14), the following equation needs to be satisfied:

$$2\tilde{u}_0(\pm 1)\tilde{u}_i(\pm 1) + z_i(\pm 1) = 0.$$

Notice that this condition is equivalent to the condition obtained by substituting $\zeta = \pm 1$ into (3.7). Therefore, solving the above equation for either ± 1 yields

$$(3.15) \quad \sigma_i = -\delta_i - \frac{z_i(1)}{4}.$$

Last, notice that for $i = 1$ we have the solution

$$\tau_0 = \frac{10}{7}, \quad \sigma_1 = 0, \quad \delta_1 = 0, \quad \tilde{\omega}_1(\zeta) = \frac{6}{7}\zeta. \quad \blacksquare$$

Corollary 3.2. *For $i \in \mathbb{N}_0$, the polynomials $\tilde{\omega}_i$ (3.11) have rational coefficients. Also, the τ_i, σ_i , and δ_i are rational.*

Proof. The proof follows from a simple induction argument taking into account to structure of z_i , $i \in \mathbb{N}$, in (3.7). ■

Corollary 3.3. *The following relation holds:*

$$\sigma_i = \delta_i = \tau_i = 0 \quad \text{for } i \text{ odd.}$$

Proof. From Proposition SM3.1 we have that the branch of nontrivial homoclinic orbits has the following symmetry:

$$u(-s, \epsilon) = u(s, -\epsilon) + \gamma \dot{u}_0(s), \quad \tau(\epsilon) = \tau(-\epsilon)$$

for $s \in \mathbb{R}$ and some open neighborhood of $\epsilon = 0$. Since $u(s, \epsilon) = \tilde{u}(\zeta(\xi(s)), \epsilon)$, we have by Theorem 3.1 that

$$\tilde{u}(\zeta(\xi(s)), \epsilon) = \sigma(\epsilon)\zeta^2(\xi(s)) + \delta(\epsilon),$$

where $\sigma(\epsilon) = \sum_i \sigma_i \epsilon^i$ and $\delta(\epsilon) = \sum_i \delta_i \epsilon^i$. Furthermore, by the specific structure of the solutions \tilde{u}_i in Theorem 3.1 it follows that $\gamma_i = 0$, for all i ; see also section SM6. Thus, we obtain that

$$\sigma(\epsilon)\zeta^2(\xi(-s)) + \delta(\epsilon) = u(-s, \epsilon) = u(s, -\epsilon) = \sigma(-\epsilon)\zeta^2(\xi(s)) + \delta(-\epsilon).$$

By substituting $s = 0$ in the last equation and comparing coefficients in $\zeta(\xi(0))$, we obtain that σ and δ are indeed even functions in ϵ . The assertion follows. ■

Corollary 3.4. *For the quadratic Bogdanov–Takens normal form (2.11), we have the relation that*

$$(3.16) \quad \sigma_i = -\delta_i \quad \text{for } i \geq 1.$$

Proof. Applying the singular rescaling (2.14) to the normal form (2.11), and consecutively applying the nonlinear time transformations (3.1) and (3.3), we obtain (3.4) without the higher-order terms in ϵ . After some calculations, we obtain the explicit expression for z_i with $i \geq 1$ in (3.7), namely

$$\begin{aligned} z_i(\zeta) = & \sum_{k=1}^{i-1} u_k u_{i-k} + (1 - \zeta^2) \left\{ \sum_{l=1}^{i-1} u'_l \tau_{i-1-l} + \sum_{k=1}^{i-1} \sum_{l=0}^{i-1-k} \omega_k u'_l \tau_{i-1-l-k} \right. \\ & \left. + \sum_{k=0}^{i-1} \sum_{l=0}^{i-1-k} \omega_k u'_l u_{i-1-l-k} - \sum_{l=1}^{i-1} \omega_l ((1 - \zeta^2) u'_{i-l})' - \sum_{k=1}^{i-1} \sum_{l=0}^{i-k} \omega_l ((1 - \zeta^2) \omega_k u'_{i-l-k})' \right\}. \end{aligned}$$

From Corollary 3.3, we have that $z_1(1) = 0$. By assuming that the relation (3.16) holds for $i = 1, 2, \dots, n-1$, $n \in \mathbb{N}$, we see directly that $z_i(1) = 0$. The assertion now follows by (3.15), with $i = n$. ■

Remark 3.5. From Corollary 3.3 it follows that the solution \tilde{u} for the quadratic Bogdanov–Takens normal form (2.11) can be represented by the single parameter $\sigma(\epsilon)$:

$$\tilde{u}(\zeta) = 2 - (1 - \zeta^2) \sum_{i \geq 0} \sigma_i \epsilon^i.$$

Consequently, \hat{u} becomes

$$\hat{u}(\xi) = 2 - \operatorname{sech}^2(\xi) \sum_{i \geq 0} \sigma_i \epsilon^i.$$

3.2. Third-order orbital homoclinic approximation. For the third-order homoclinic predictor, we obtain

$$\begin{aligned} (3.17) \quad \sigma &= 6 + \frac{18}{49} \epsilon^2 + \mathcal{O}(\epsilon^4), \\ \delta &= -4 - \frac{18}{49} \epsilon^2 + \mathcal{O}(\epsilon^4), \\ \tau &= \frac{10}{7} + \frac{288}{2401} \epsilon^2 + \mathcal{O}(\epsilon^4), \\ \tilde{\omega}(\zeta) &= 1 - \frac{6}{7} \zeta \epsilon + \left(\frac{9}{98} + \frac{27}{98} \zeta^2 \right) \epsilon^2 + \left(-\frac{198}{2401} \zeta + \frac{18}{343} \zeta^3 \right) \epsilon^3 + \mathcal{O}(\epsilon^4), \end{aligned}$$

from which it follows that

$$(3.18) \quad \tilde{u}(\zeta) = 2 - (1 - \zeta^2) \left(6 + \frac{18}{49} \epsilon^2 \right) + \mathcal{O}(\epsilon^4),$$

$$\begin{aligned} (3.19) \quad \tilde{v}(\zeta) &= -2\tilde{\omega}(\zeta)\sigma(1 - \zeta^2)\zeta = - \left[-12 + \frac{72}{7} \zeta \epsilon - \left(\frac{90}{49} + \frac{162}{49} \zeta^2 \right) \epsilon^2 \right. \\ &\quad \left. + \left(\frac{3888}{2401} \zeta - \frac{216}{343} \zeta^3 \right) \epsilon^3 \right] (1 - \zeta^2) \zeta + \mathcal{O}(\epsilon^4). \end{aligned}$$

The function $\xi(s)$ is obtained by solving the ODE

$$(3.20) \quad \frac{d\xi}{ds}(s) = \tilde{\omega}(\tanh(\xi(s))).$$

Thus, we substitute

$$\xi(s) = s + \xi_1(s)\epsilon + \xi_2(s)\epsilon^2 + \xi_3(s)\epsilon^3 + \mathcal{O}(\epsilon^4)$$

into (3.20) and expand the resulting equation in ϵ to obtain

$$(3.21) \quad \frac{d\xi_1}{ds}(s) = -\frac{6 \tanh(s)}{7},$$

$$(3.22) \quad \frac{d\xi_2}{ds}(s) = \frac{18 + 54 \tanh^2(s) - 168\xi_1(s) + 168 \tanh^2(s)\xi_1(s)}{196},$$

$$\begin{aligned} \frac{d\xi_3}{ds}(s) = & -\frac{198 \tanh(s)}{2401} + \frac{18 \tanh^3(s)}{343} - \frac{27}{49}(-\tanh(s)\xi_1(s) + \tanh^3(s)\xi_1(s)) \\ & - \frac{6}{7}(-\tanh(s)\xi_1^2(s) + \tanh^3(s)\xi_1^2(s) + \xi_2(s) - \tanh^2(s)\xi_2(s)). \end{aligned}$$

Here we directly used that $\xi_0(s) = s$. By solving these equations recursively, we obtain

$$\begin{aligned} \xi_1(s) &= c_1 - \frac{6}{7} \log(\cosh(s)), \\ \xi_2(s) &= c_2 - \frac{18s}{49} + \frac{45 \tanh(s)}{98} - \frac{6}{7} c_1 \tanh(s) + \frac{36}{49} \tanh(s) \log(\cosh(s)), \\ \xi_3(s) &= c_3 + \frac{1}{4802} (3 \operatorname{sech}^2(s) [-504 \log^2(\cosh(s)) - 276 \cosh(2s) \log(\cosh(s)) \\ &\quad + 102 \log(\cosh(s)) + 14(18s - 49c_2) \sinh(2s) + 1176c_1 \log(\cosh(s)) \\ &\quad + 546 - 686c_1^2 - 441c_1]). \end{aligned}$$

The constants c_i ($i = 1, 2, 3$) lead to different phase conditions. A computationally simple phase condition is given by

$$(3.23) \quad \xi_i(0) = 0 \quad \text{for } i = 1, 2, 3.$$

These result in the constraint $v(0) = 0$, i.e., the phase condition used in [26]. Solving (3.23) leads to the solution

$$(3.24) \quad c_1 = 0, \quad c_2 = 0, \quad c_3 = -\frac{117}{343}.$$

Substituting the above expression for ξ into (2.14), we obtain the third-order predictor

$$(3.25) \quad \begin{cases} w_0(\eta) = \frac{a}{b^2} \tilde{u} \left(\tanh \left(\xi \left(\frac{a}{b} \epsilon \eta \right) \right) \right) \epsilon^2, \\ w_1(\eta) = \frac{a^2}{b^3} \tilde{v} \left(\tanh \left(\xi \left(\frac{a}{b} \epsilon \eta \right) \right) \right) \epsilon^3, \\ \beta_1 = -4 \frac{a^3}{b^4} \epsilon^4, \\ \beta_2 = \frac{a}{b} \epsilon^2 \tau, \end{cases}$$

where τ , \tilde{u} , and \tilde{v} are given by (3.17)–(3.19), respectively.

Remark 3.6. By expanding $\tilde{u}(\tanh(\xi(s)))$ in ϵ up to third order, we obtain precisely the solution obtained by using the regular perturbation method to (2.15) with phase condition $\dot{u}(0) = 0$; see section SM5. Note that for the conjecture in [1, section 7] to hold, the phase condition (3.23) must be satisfied.

4. Homoclinic asymptotic expansion in n -dimensional systems. In this section, we will provide third-order approximations to the homoclinic solution for (1.1) emanating from a generic codimension two Bogdanov–Takens bifurcation assumed to be at $x_0 \equiv 0$ and $\alpha_0 = 0$. Here we consider the situation where we have obtained a homoclinic predictor for the smooth orbital normal form (2.13) in combination with the polynomial Lindstedt–Poincaré method from section 3. Predictors corresponding to other passes in Figure 1 are treated in the supplementary materials.

First, we substitute (3.25) into the parameter-dependent center manifold transformations H and K defined in (2.19) and (2.20). By truncating the higher-order terms in w and β , we obtain the following approximation $(\bar{x}^o, \bar{\alpha}^o)$ to the homoclinic solution:

$$(4.1) \quad \begin{aligned} \bar{x}^o(\eta, \epsilon) = & q_0 w_0(\eta) + q_1 w_1(\eta) + H_{0010} \beta_1 + H_{0001} \beta_2 + \frac{1}{2} H_{2000} w_0^2(\eta) + H_{1100} w_0 w_1(\eta) \\ & + \frac{1}{2} H_{0200} w_1^2(\eta) + H_{1010} w_0(\eta) \beta_1 + H_{1001} w_0(\eta) \beta_2 + H_{0110} w_1(\eta) \beta_1 \\ & + H_{0101} w_1(\eta) \beta_2 + \frac{1}{2} H_{0002} \beta_2^2 + H_{0011} \beta_1 \beta_2 + \frac{1}{6} H_{3000} w_0^3(\eta) \\ & + \frac{1}{2} H_{2100} w_0^2(\eta) w_1(\eta) + H_{1101} w_0(\eta) w_1(\eta) \beta_2 + \frac{1}{2} H_{2001} w_0^2(\eta) \beta_2 \\ & + \frac{1}{6} H_{0003} \beta_2^3 + \frac{1}{2} H_{1002} w_0(\eta) \beta_2^2 + \frac{1}{2} H_{0102} w_1(\eta) \beta_2^2, \\ (4.2) \quad \bar{\alpha}^o(\epsilon) = & K_{10} \beta_1 + K_{01} \beta_2 + \frac{1}{2} K_{02} \beta_2^2 + K_{11} \beta_1 \beta_2 + K_{03} \frac{1}{6} \beta_2^3. \end{aligned}$$

Next, we use (2.21) to approximate $\eta(t)$ from the relation

$$(4.3) \quad \frac{dt}{d\eta} = 1 + \theta_{1000} w_0(\eta) + \theta_{0001} \beta_2,$$

where w_0 and β_2 are again defined by (3.25). Integrating (4.3) with respect to η yields

$$(4.4) \quad t(\eta) = \eta \left(1 + \theta_{0001} \frac{a}{b} \epsilon^2 \left(\frac{10}{7} + \frac{288}{2401} \epsilon^2 + \mathcal{O}(\epsilon^4) \right) \right) + \theta_{1000} \frac{a}{b^2} \epsilon^2 \int \hat{u} \left(\xi \left(\frac{a}{b} \eta \epsilon \right) \right) d\eta.$$

Note that we write an indefinite integral. To select a particular primitive, we use $t(0) = 0$. To approximate the integral in the above equation uniformly in η , we make the substitution $\tilde{\xi}(\eta) = \xi \left(\frac{a}{b} \epsilon \eta \right)$. Then

$$\int \hat{u} \left(\xi \left(\frac{a}{b} \eta \epsilon \right) \right) d\eta = \frac{b}{a} \frac{1}{\epsilon} \int \hat{u}(\tilde{\xi}) / \omega(\tilde{\xi}) d\tilde{\xi}.$$

Expanding the integrand $\hat{u}(\tilde{\xi})/\omega(\tilde{\xi})$, up to order three in ϵ and integrating with respect to $\tilde{\xi}$ yields

$$\begin{aligned}
(4.5) \quad \int \frac{1}{\omega(\tilde{\xi})} \hat{u}(\tilde{\xi}) d\tilde{\xi} &= 2\tilde{\xi} - 6 \tanh(\tilde{\xi}) + \frac{1}{7} \left(18 \operatorname{sech}^2(\tilde{\xi}) + 12 \log(\cosh(\tilde{\xi})) \right) \epsilon \\
&+ \frac{9}{49} \left(4\tilde{\xi} - 9 \tanh(\tilde{\xi}) + 5 \tanh(\tilde{\xi}) \operatorname{sech}^2(\tilde{\xi}) \right) \epsilon^2 \\
&+ \frac{18}{2401} \left(-21 \operatorname{sech}^4(\tilde{\xi}) + 47 \operatorname{sech}^2(\tilde{\xi}) + 8 \log(\cosh(\tilde{\xi})) \right) \epsilon^3 + \mathcal{O}(\epsilon^4).
\end{aligned}$$

Substituting $\tilde{\xi}$ with $\xi(\frac{a}{b}\eta\epsilon)$ gives the relation $t(\eta)$ up to order three in ϵ .

Since we are interested in the inverse relation, i.e., $\eta(t)$, we numerically solve the equation

$$t(\eta) - t = 0$$

for η . This can easily be done within machine precision.

5. Example: Homoclinic renormalization group (RG) flows. In this section, we will demonstrate that the approximation order for the homoclinic approximation lifts correctly to the original system. Additionally, a comparison is made between applying different normal forms (smooth and orbital), different phase conditions, and different perturbation methods (regular and Lindstedt–Poincaré) to approximate the homoclinic solution.

Here we will not demonstrate how to actually start the continuation of the homoclinic orbits with **MatCont**. For this, we refer the reader to the [online Jupyter Book](#) [9]. In the Jupyter Book, a total of 10 different models are considered, demonstrating in detail how to start continuation from either an explicitly derived or encountered during continuation codimension two Bogdanov–Takens bifurcation point. Each model is treated in a separate Jupyter Notebook, which can be executed to reproduce the results obtained. For a detailed description of how to set up the Jupyter Notebook with a MATLAB kernel, we refer the reader to subsection [SM9.8](#).

We do like to note that in all cases, the curve of homoclinic solutions could be started with the default settings without the need to adjust any parameters; see section [SM9](#). This shows that the homoclinic asymptotic approximations obtained in this paper are very robust.

In [21], an $\mathcal{N} = 1$ supersymmetric model of interacting scalar superfields Φ_{ab}^i that is invariant under the action of an $O(N) \times O(M)$ group in $d = 3 - \epsilon$ dimensions is considered. The coupling constants g_i ($i = 1, \dots, 4$) satisfy the following differential equations:

$$(5.1) \quad \dot{g} = -\epsilon g + \beta^{(2)}(g, M, N) + \mathcal{O}(g^5), \quad g \in \mathbb{R}^4,$$

where the two-loop contributions $\beta_i^{(2)}$ ($i = 1, \dots, 4$) are cubic in the coupling and the parameter ϵ is scaled to 1. The exact expressions for $\beta_i^{(2)}$ are quite long and can be found in section [SM10](#) or in the [online Jupyter Book](#) [9].

In [21], a Bogdanov–Takens point near the parameter values $M = 0.2945$ and $N = 4.036$ is located. Using these parameter values, we locate an equilibrium at

$$\begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{pmatrix} = \begin{pmatrix} 0.0701457361241472 \\ -0.06520883770451065 \\ 0.001823543197553845 \\ 0.22874527306411319 \end{pmatrix}.$$

By continuing the equilibrium in the parameter M , we detect several limit points and two Hopf points. We continue the second Hopf point at $M \approx 0.2958$ in parameters M and N . Several Bogdanov–Takens points are detected. The first Bogdanov–Takens point is located at

$$\begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{pmatrix} = \begin{pmatrix} -0.715157316845187 \\ -0.250968103603174 \\ 0.510051114588271 \\ -0.391935453715783 \end{pmatrix},$$

with parameter values

$$(M, N) = (0.294477255737036, 4.035536108506390).$$

In Figure 3, there are two log-log convergence plots that are shown. We clearly see that the order of convergence correctly lifts from the normal form to the two-dimensional center manifold in \mathbb{R}^4 . In the left plot, we compare the regular perturbation method with the Lindstedt–Poincaré method. We see that these are almost indistinguishable. In the right plot, we compare four different third approximations to the homoclinic orbit:

- (i) the Lindstedt–Poincaré method using the smooth orbital normal form (the blue diamond),
- (ii) the Lindstedt–Poincaré method using the smooth normal form (the dashed light gray line),
- (iii) the regular perturbation method using the smooth normal form (the pink square),
- (iv) the Lindstedt–Poincaré method using the hypernormal form (the green plus).

We see that both the Lindstedt–Poincaré method and the regular perturbation method using the smooth orbital normal form are slightly more accurate than the Lindstedt–Poincaré

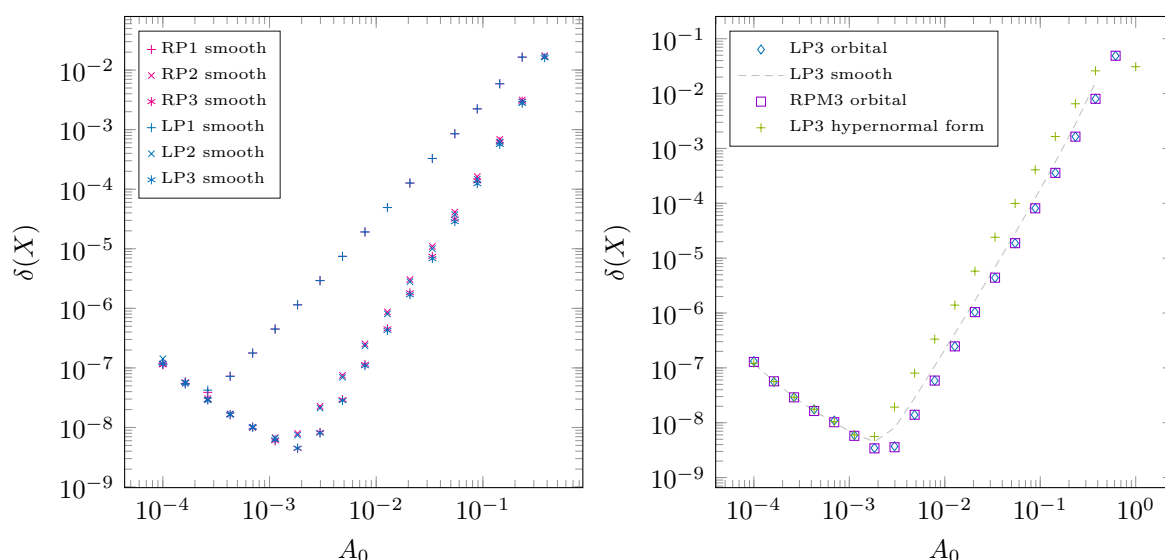


Figure 3. Convergence plot for the homoclinic predictors near one of the Bogdanov–Takens bifurcations in the homoclinic RG model (5.1). Here we show the relative error $\delta(X)$ between the predicted and corrected Newton solutions to the defining system (SM9.1) in the Euclidean distance on the mesh.

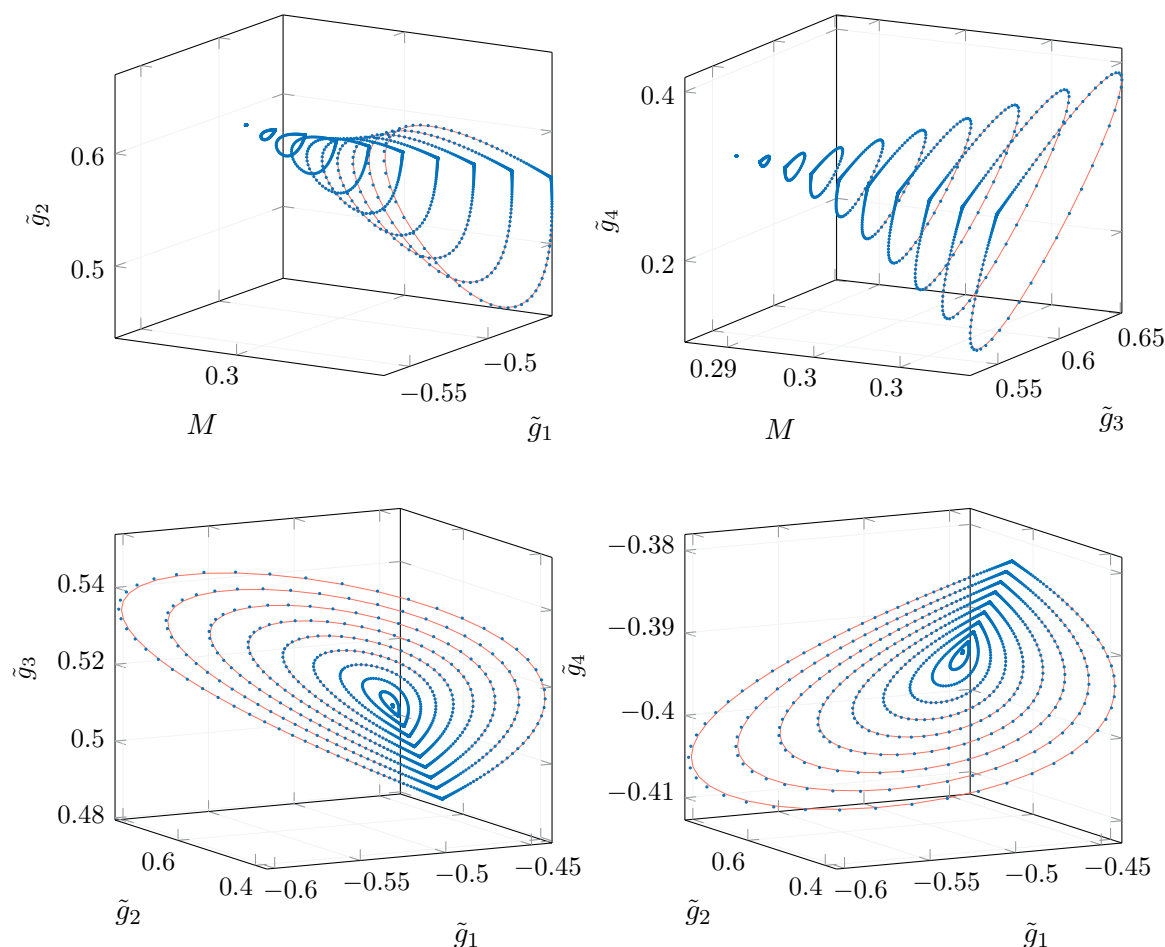


Figure 4. Comparison between the predicted (solid, red) with the corrected (dotted, blue) homoclinic orbits using the Lindstedt–Poincaré method with the smooth orbital normal form for amplitudes $A_0 = 10^{-3}$ to $A_0 = 10^{-1}$. The g_i ($i = 1, \dots, 4$) coordinates have been rotated and stretched to make the homoclinic orbits more visible.

method using the smooth normal form. The homoclinic predictor using the hypernormal form is less accurate compared to be other methods.

Last, in Figure 4 we compare the predicted (solid, red) with the corrected (dotted, blue) homoclinic orbits using the Lindstedt–Poincaré method with the smooth orbital normal form for amplitudes $A_0 = 10^{-3}$ to $A_0 = 5 \times 10^{-2}$. We see that they are in excellent agreement.

6. Discussion. We have derived third-order predictors for the homoclinic curve emanating from the generic codimension two Bogdanov–Takens bifurcation in general n -dimensional autonomous ODEs. By considering the smooth orbital normal form (2.13) and incorporating the time-reparametrization in the homological equation (HOM), we were able to derive the third-order asymptotic of the homoclinic curve independent of any undetermined normal form coefficients. However, for this simplification, there is a price to pay. First, the systems to be solved to obtain the coefficients for the parameter-dependent center manifold become more complicated; see subsection 2.2. Ideally, there should be an automatic algorithm in spirit with

[28]. However, to the best of our knowledge, such algorithms do not exist yet. Second, the transformation of time in the homological equation needs to be inverted numerically. However, this can be done relatively cheaply and is very accurate, as shown by the examples.

In Recipe 1, we have shown how to obtain a correct transformation to the parameter-dependent center manifold by inspecting which terms are in, *and are not in*, the normal form that alters the homoclinic asymptotic up to a certain order. The examples in section 5, the supplementary materials, and the online Jupyter Book, but also the comparison in section SM8, confirm that we indeed have obtained the correct transformation.

The additional nonlinear transformation (3.3) greatly simplifies the computation of the coefficients in the Lindstedt–Poincaré method since all calculations become essentially polynomial, which is ideal for computers to work with. Nonetheless, the algorithmic complexity grows exponentially as the order increases linearly. Also, the radius of convergence is clearly finite, as shown in section SM13. One way to increase the convergence radius is by using transformations as in [31]. However, we didn't include any results in this direction since they would distract too much from our main objectives.

Using different phase conditions can improve the accuracy of the homoclinic approximation. However, this only holds true when applied directly to the system considered. Indeed, the phase condition isn't invariant under the parameter-dependent center manifold transformation. Thus, its applicability is very limited. Furthermore, using a different phase condition may, somewhat unexpectedly, result in complicated integrals to be solved; see section SM3.

The higher-order approximation to the nonlinear time transformation in the Lindstedt–Poincaré method turns out to be essential to obtain higher-order approximations to the homoclinic solutions. This is clearly seen by inspecting the profiles of the homoclinic solution in Figure SM3 and in the convergence plot in Figure SM2. Without the higher-order time approximation, the same convergence order as the unperturbed Hamiltonian solution, i.e., the zeroth-order solution is obtained. It should be noted that the higher-order approximation of the nonlinear time transformation is more difficult to obtain. Therefore, we conclude that there seem to be *no benefits* of the Lindstedt–Poincaré method over the regular perturbation method for starting continuation of homoclinic orbits. Indeed, the numerical comparisons in section 5 show similar accuracy of convergence at each order.

By comparing the convergence order of Lindstedt–Poincaré with the regular perturbation method, we see that, contrary to what one might expect, the regular perturbation method may result in better accuracy at the same order. A possible explanation for this might be that although the Lindstedt–Poincaré method provides a uniform approximation in time, the numerical solution is truncated to a finite interval in which the “parasitic turn” doesn't give a significant contribution. After all, it then simply depends on the higher-order nonlinear terms in the system which favor one method over the other.

Acknowledgments. The authors would like to thank Prof. Peter De Maesschalck (Hasselt University, Belgium) for multiple useful discussions during this research project, Prof. Wolf-Jürgen Beyn (Bielefeld University, Germany) for his positive comments on the preprint, and Dr. Hil Meijer (University of Twente, The Netherlands) for multiple suggestions leading to a significant improvement of the paper. Additionally, we would like to express our gratitude to the two anonymous referees for very useful remarks and suggestions.

REFERENCES

- [1] B. AL-HDAIBAT, W. GOVAERTS, YU. A. KUZNETSOV, AND H. G. E. MEIJER, *Initialization of homoclinic solutions near Bogdanov–Takens points: Lindstedt–Poincaré compared with regular perturbation method*, SIAM J. Appl. Dyn. Syst., 15 (2016), pp. 952–980, <https://doi.org/10.1137/15M1017491>.
- [2] A. ALGABA, K.-W. CHUNG, B.-W. QIN, AND A. J. RODRÍGUEZ-LUIS, *A nonlinear time transformation method to compute all the coefficients for the homoclinic bifurcation in the quadratic Takens–Bogdanov normal form*, Nonlinear Dyn., 97 (2019), pp. 979–990, <https://doi.org/10.1007/s11071-019-05025-2>.
- [3] V. I. ARNOL'D, *Geometrical Methods in the Theory of Ordinary Differential Equations*, 2nd ed., Grundlehren Math. Wiss. 250, Springer-Verlag, New York, 1988, <https://doi.org/10.1007/978-1-4612-1037-5>.
- [4] W.-J. BEYN, *Numerical analysis of homoclinic orbits emanating from a Takens–Bogdanov point*, IMA J. Numer. Anal., 14 (1994), pp. 381–410, <https://doi.org/10.1093/imanum/14.3.381>.
- [5] W.-J. BEYN, A. CHAMPNEYS, E. DOEDEL, W. GOVAERTS, YU. A. KUZNETSOV, AND B. SANDSTEDE, *Numerical continuation, and computation of normal forms*, in Handbook of Dynamical Systems, North-Holland, Amsterdam, 2002, pp. 149–219, [https://doi.org/10.1016/S1874-575X\(02\)80025-X](https://doi.org/10.1016/S1874-575X(02)80025-X).
- [6] J. BEZANSON, A. EDELMAN, S. KARPINSKI, AND V. B. SHAH, *Julia: A fresh approach to numerical computing*, SIAM Rev., 59 (2017), pp. 65–98, <https://doi.org/10.1137/141000671>.
- [7] R. BOGDANOV, *Versal deformations of a singular point of a vector field on the plane in the case of zero eigenvalues*, Funct. Anal. Appl., 9 (1975), pp. 144–145, <https://doi.org/10.1007/BF01075453>.
- [8] R. I. BOGDANOV, *The versal deformation of a singular point of a vector field on the plane in the case of zero eigenvalues*, Trudy Sem. Petrovsk., 2 (1976), pp. 37–65.
- [9] M. BOSSCHAERT, *Jupyter Book: Interplay between Normal Forms and Center Manifold Reduction for Homoclinic Predictors near Bogdanov–Takens Bifurcation*, <https://mmbosschaert.github.io/MatCont7p2NewInitBTHom-/>, 2022.
- [10] M. BOSSCHAERT, *MATCONT: Interplay between Normal Forms and Center Manifold Reduction for Homoclinic Predictors near Bogdanov–Takens Bifurcation*, <https://github.com/mmbosschaert/MatCont7p2NewInitBTHom->, 2022.
- [11] H. W. BROER, F. DUMORTIER, S. J. VAN STRIEN, AND F. TAKENS, *Structures in Dynamics: Finite-Dimensional Deterministic Studies*, Stud. Math. Phys. 2, North-Holland, Amsterdam, 1991.
- [12] S. CHEN, Y. CHEN, AND K. SZE, *A hyperbolic perturbation method for determining homoclinic solution of certain strongly nonlinear autonomous oscillators*, J. Sound Vib., 322 (2009), pp. 381–392, <https://doi.org/10.1016/j.jsv.2008.11.015>.
- [13] S. CHEN, Y. CHEN, AND K. Y. SZE, *Homoclinic and heteroclinic solutions of cubic strongly nonlinear autonomous oscillators by hyperbolic Lindstedt–Poincaré method*, Sci. China Technol. Sci., 53 (2010), pp. 692–702, <https://doi.org/10.1007/s11431-010-0069-5>.
- [14] Y. CHEN AND S. CHEN, *Homoclinic and heteroclinic solutions of cubic strongly nonlinear autonomous oscillators by the hyperbolic perturbation method*, Nonlinear Dyn., 58 (2009), pp. 417–429, <https://doi.org/10.1007/s11071-009-9489-9>.
- [15] Y. CHEN, S. CHEN, AND K. SZE, *A hyperbolic Lindstedt–Poincaré method for homoclinic motion of a kind of strongly nonlinear autonomous oscillators*, Acta Mech. Sin., 25 (2009), pp. 721–729, <https://doi.org/10.1007/s10409-009-0276-0>.
- [16] Y.-Y. CHEN, L.-W. YAN, K.-Y. SZE, AND S.-H. CHEN, *Generalized hyperbolic perturbation method for homoclinic solutions of strongly nonlinear autonomous systems*, Appl. Math. Mech. (English Ed.), 33 (2012), pp. 1137–1152, <https://doi.org/10.1007/s10483-012-1611-6>.
- [17] A. DHOOGHE, W. GOVAERTS, YU. A. KUZNETSOV, H. G. E. MEIJER, AND B. SAUTOIS, *New features of the software MatCont for bifurcation analysis of dynamical systems*, Math. Comput. Model. Dyn. Syst., 14 (2008), pp. 147–175, <https://doi.org/10.1080/13873950701742754>.
- [18] W. J. F. GOVAERTS, *Numerical Methods for Bifurcations of Dynamical Equilibria*, SIAM, Philadelphia, 2000, <https://doi.org/10.1137/1.9780898719543>.
- [19] J. GUCKENHEIMER AND P. HOLMES, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Appl. Math. Sci. 42, Springer, New York, 1983.

- [20] M. HARAGUS AND G. IOOSS, *Local Bifurcations, Center Manifolds, and Normal Forms in Infinite-dimensional Dynamical Systems*, Universitext, Springer-Verlag, London; EDP Sciences, Les Ulis, 2011, <https://doi.org/10.1007/978-0-85729-112-7>.
- [21] C. B. JEPSEN AND F. K. POPOV, *Homoclinic renormalization group flows, or when relevant operators become irrelevant*, Phys. Rev. Lett. 127 (2021), 141602.
- [22] H. B. KELLER, *Lectures on Numerical Methods in Bifurcation Problems*, Tata Inst. Fund. Res. Lectures on Math. and Phys. 79, Springer-Verlag, Berlin, 1987.
- [23] YU. A. KUZNETSOV, *Numerical normalization techniques for all codim 2 bifurcations of equilibria in ODE's*, SIAM J. Numer. Anal., 36 (1999), pp. 1104–1124, <https://doi.org/10.1137/S0036142998335005>.
- [24] YU. A. KUZNETSOV, *Practical computation of normal forms on center manifolds at degenerate Bogdanov–Takens bifurcations*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., 15 (2005), pp. 3535–3546.
- [25] YU. A. KUZNETSOV, *Elements of Applied Bifurcation Theory*, 4th ed., Appl. Math. Sci. 112, Springer, Cham, 2023, <https://doi.org/10.1007/978-3-031-22007-4>.
- [26] YU. A. KUZNETSOV, H. G. E. MEIJER, B. AL-HDAIBAT, AND W. GOVAERTS, *Improved homoclinic predictor for Bogdanov–Takens bifurcation*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., 24 (2014), 1450057, <https://doi.org/10.1142/s0218127414500576>.
- [27] YU. A. KUZNETSOV, H. G. E. MEIJER, B. AL-HDAIBAT, AND W. GOVAERTS, *Accurate approximation of homoclinic solutions in Gray–Scott kinetic model*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., 25 (2015), 1550125, <https://doi.org/10.1142/S0218127415501254>.
- [28] J. MURDOCK, *Normal Forms and Unfoldings for Local Dynamical Systems*, Springer, New York, 2003, <https://doi.org/10.1007/b97515>.
- [29] A. J. RODRÍGUEZ-LUIS, E. FREIRE, AND E. PONCE, *A method for homoclinic and heteroclinic continuation in two and three dimensions*, in Continuation and Bifurcations: Numerical Techniques and Applications, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1990, pp. 197–210, https://doi.org/10.1007/978-94-009-0659-4_13.
- [30] F. TAKENS, *Forced oscillations and bifurcations*, in Applications of Global Analysis, I, Commun. Math. Inst. Rijksuniv. Utrecht, No. 3-1974 (1974).
- [31] M. VAN DYKE, *Analysis and improvement of perturbation series*, Quart. J. Mech. Appl. Math., 27 (1974), pp. 423–450, <https://doi.org/10.1093/qjmam/27.4.423>.