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# Polynomial Tau-Functions of the $n$ -th Sawada–Kotera Hierarchy

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**Abstract:** We give a review of the B-type Kadomtsev–Petviashvili (BKP) hierarchy and find all polynomial tau-functions of the  $n$ -th reduced BKP hierarchy (=  $n$ -th Sawada–Kotera hierarchy). The name comes from the fact that, for  $n = 3$ , the simplest equation of the hierarchy is the famous Sawada–Kotera equation.

**Keywords:** soliton equations; affine Lie algebras; tau-functions

**MSC:** 17B67; 17B80; 22E65

## 1. Introduction

The three most famous hierarchies of Lax equations on one function  $u$  are the Korteweg–de Vries (KdV) hierarchy, the Kaup–Kupershmidt hierarchy, and the Sawada–Kotera hierarchy. The Lax operators are, respectively,

$$\mathcal{L} = \partial^2 + u, \quad (1)$$

$$\mathcal{L} = \partial^3 + u\partial + \frac{1}{2}u', \quad (2)$$

$$\mathcal{L} = \partial^3 + u\partial. \quad (3)$$

Let  $t = (t_1, t_2, t_3, \dots)$  and  $\tilde{t} = (t_1, t_3, t_5, \dots)$ . Recall that the Kadomtsev–Petviashvili (KP) hierarchy is the following hierarchy of Lax equations on the pseudodifferential operator  $L(t, \partial) = \partial + u_1(t)\partial^{-1} + u_2(t)\partial^{-2} + \dots$  in  $\partial = \frac{\partial}{\partial t_1}$  [1]:

$$\frac{\partial L(t, \partial)}{\partial t_k} = \left[ \left( L(t, \partial)^k \right)_{\geq 0}, L(t, \partial) \right], \quad k = 1, 2, \dots \quad (4)$$

The KdV hierarchy is the second reduced KP hierarchy, meaning that one imposes the following constraint on  $L(t, \partial)$ :

$$\mathcal{L}(t, \partial) = L(t, \partial)^2 \quad \text{is a differential operator.} \quad (5)$$

In this case, the operator  $\mathcal{L}$  is defined by (1), with  $u(t) = 2u_1(t)$ , and the KP hierarchy (4) reduces to the KdV hierarchy

$$\frac{\partial \mathcal{L}(t, \partial)}{\partial t_k} = \left[ \left( \mathcal{L}(t, \partial)^{\frac{k}{2}} \right)_{\geq 0}, \mathcal{L}(t, \partial) \right], \quad k = 1, 3, 5, \dots \quad (6)$$

For even  $k$ , this equation is trivial; for  $k = 3$ , Equation (6) is the KdV equation [2].

Recall that in order to construct solutions of the KP hierarchy and the reduced KP hierarchies, one introduces the tau-function  $\tau(t)$ , defined by [1,2]:

$$L(t, \partial) = P(t, \partial) \circ \partial \circ P(t, \partial)^{-1}, \quad (7)$$



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where  $P(t, \partial)$  is a pseudodifferential operator, with the symbol

$$P(t, z) = \frac{1}{\tau(t)} \exp\left(-\sum_{k=1}^{\infty} \frac{z^{-k}}{k} \frac{\partial}{\partial t_k}\right) (\tau(t)). \tag{8}$$

The tau-function has a geometric meaning as a point on an infinite-dimensional Grassmannian, and in [1], Sato showed that all Schur polynomials are tau-functions of the KP hierarchy. Recently, all polynomial tau-functions of the KP hierarchy and its  $n$ -reductions have been constructed in [3] (see also [4]).

The CKP hierarchy (KP hierarchy of type C) can be constructed by making use of the KP hierarchy, and assuming the additional constraint  $L(\tilde{t}, \partial)^* = -L(\tilde{t}, \partial)$  (see, e.g., [5] for details). Its 3-reduction is defined by the constraint that  $\mathcal{L}(\tilde{t}, \partial) = L(\tilde{t}, \partial)^3$  is a differential operator, and the corresponding hierarchy is

$$\frac{\partial \mathcal{L}(\tilde{t}, \partial)}{\partial t_k} = \left[ \left( \mathcal{L}(\tilde{t}, \partial)^{\frac{k}{3}} \right)_{\geq 0}, \mathcal{L}(\tilde{t}, \partial) \right], \quad k = 1, 3, 5, \dots, k \notin 3\mathbb{Z}, \tag{9}$$

where  $\mathcal{L}$  is given by (2). For  $k = 5$ , we obtain the Kaup–Kupershmidt equation, the simplest non-trivial equation in this hierarchy. All polynomial tau-functions of (9) (and all  $n$  reductions of the CKP hierarchy) have been constructed in [5].

In the present paper, we construct all polynomial tau-functions of the  $n$ -reduced BKP hierarchies (KP hierarchy of type B). These are hierarchies of Lax equations on the differential operator

$$\mathcal{L}(\tilde{t}, \partial) = \partial^n + u_{n-2}(\tilde{t})\partial^{n-2} + \dots + u_1(\tilde{t})\partial, \tag{10}$$

satisfying the constraint

$$\mathcal{L}(\tilde{t}, \partial)^* = (-1)^n \partial^{-1} \mathcal{L}(\tilde{t}, \partial) \partial. \tag{11}$$

The  $n$ -th reduced BKP hierarchy is

$$\frac{\partial \mathcal{L}(\tilde{t}, \partial)}{\partial t_k} = \left[ \left( \mathcal{L}(\tilde{t}, \partial)^{\frac{k}{n}} \right)_{\geq 0}, \mathcal{L}(\tilde{t}, \partial) \right], \quad k = 1, 3, 5, \dots, k \notin n\mathbb{Z}. \tag{12}$$

We call it the  $n$ -th Sawada–Kotera hierarchy, since, for  $n = 3$ ,  $\mathcal{L}$  is given by (3), and for  $k = 5$ , Equation (12) is the Sawada–Kotera equation [6] (see Equation (33)).

In the present paper, using the description of polynomial tau-functions of the BKP hierarchy [4,7] (see Theorem 1), we find all polynomial tau-functions for the  $n$ -th Sawada–Kotera hierarchies (see Theorem 2), and, in particular, for the Sawada–Kotera hierarchy (see Corollary 1).

### 2. The BKP Hierarchy and Its Polynomial Tau-Functions

In this section, we recall the construction of the BKP hierarchy [8] and description of its polynomial tau-functions from [4,7].

Following Date, Jimbo, Kashiwara, and Miwa [8] (see also [7] for details), we introduce the BKP hierarchy in terms of the so-called twisted neutral fermions  $\phi_i, i \in \mathbb{Z}$ , which are generators of a Clifford algebra over  $\mathbb{C}$ , satisfying the following anti-commutation relation:

$$\phi_i \phi_j + \phi_j \phi_i = (-1)^i \delta_{i,-j}. \tag{13}$$

Consider the right (resp., left) irreducible module  $F = F_r$  (resp.,  $F_l$ ) over this algebra by the following action on the vacuum vector  $|0\rangle$  (resp.,  $\langle 0|$ ):

$$\phi_0|0\rangle = \frac{1}{\sqrt{2}}|0\rangle, \quad \phi_j|0\rangle = 0 \quad \left( \text{resp., } \langle 0|\phi_0 = \frac{1}{\sqrt{2}}\langle 0|, \quad \langle 0|\phi_{-j} = 0 \right), \quad \text{for } j > 0. \tag{14}$$

The quadratic elements

$$\phi_j \phi_k - \phi_k \phi_j \text{ for } j, k \in \mathbb{Z}, j > k,$$

form a basis of the infinite-dimensional Lie algebra  $so_{\infty, odd}$  over  $\mathbb{C}$ . Let  $SO_{\infty, odd}$  be the corresponding Lie group. We proved in ([9], Theorem 1.2a) that a non-zero element  $\tau \in F$  lies in this Lie group orbit of the vacuum vector  $|0\rangle$  if and only if it satisfies the BKP hierarchy in the fermionic picture, i.e., the following equation in  $F \otimes F$ :

$$\sum_{j \in \mathbb{Z}} (-1)^j \phi_j \tau \otimes \phi_{-j} \tau = \frac{1}{2} \tau \otimes \tau. \tag{15}$$

Non-zero elements of  $F$ , satisfying (15), are called tau-functions of the BKP hierarchy in the fermionic picture.

The group  $SO_{\infty, odd}$  consists of elements  $G$  leaving the symmetric bilinear form

$$(\phi_j, \phi_k) = (-1)^j \delta_{j, -k} \tag{16}$$

on  $F$  invariant, i.e.,

$$(G\phi_j, G\phi_k) = (\phi_j, \phi_k). \tag{17}$$

Stated differently,

$$G\phi_k G^{-1} = \sum_{j \in \mathbb{Z}} a_{jk} \phi_k \text{ (finite sum) with } \sum_{j \in \mathbb{Z}} (-1)^j a_{jk} a_{-j\ell} = (-1)^k \delta_{k, -\ell}. \tag{18}$$

The group orbit of the vacuum vector is the disjoint union of Schubert cells (see Section 3 of [7] for details). These cells are parametrized by the strict partitions  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ , with  $\lambda_1 > \lambda_2 > \dots > \lambda_k > 0$ . Namely, the cell, attached to the partition  $\lambda$  is

$$C_\lambda = \{v_1 v_2 \dots v_{k-1} v_k |0\rangle | v_j = \sum_{i \geq -\lambda_j} a_{ij} \phi_i \text{ (finite sum) with } a_{-\lambda_j, j} \neq 0\}. \tag{19}$$

An element  $\tau \in C_\lambda$  corresponds to the following point in the maximal isotropic Grassmannian (i.e., a maximal isotropic subspace of  $V = \bigoplus_{j \in \mathbb{Z}} \mathbb{C} \phi_j$ ):

$$\text{Ann } \tau = \{v \in V | v v_1 v_2 \dots v_{k-1} v_k |0\rangle = 0\}. \tag{20}$$

For instance,  $\text{Ann } |0\rangle = \text{span}\{\phi_1, \phi_2, \dots\}$ .

Using the bosonization of Equation (15), one obtains a hierarchy of differential equations on  $\tau$  ([4,7,8,10], Section 3). This bosonization is an isomorphism  $\sigma$  between the spin module  $F$  and the polynomial algebra  $B = \mathbb{C}[\tilde{t}] = \mathbb{C}[t_1, t_3, t_5, \dots]$ . Explicitly, we introduce the twisted neutral fermionic field

$$\phi(z) = \sum_{j \in \mathbb{Z}} \phi_j z^{-j},$$

and the bosonic field

$$\alpha(z) = \sum_{j \in \mathbb{Z}} \alpha_{2j+1} z^{-2j-1} = \frac{1}{2} : \phi(z) \phi(-z) :,$$

where the normal ordering  $:$  is defined by

$$: \phi_j \phi_k := \phi_j \phi_k - \langle 0 | \phi_j \phi_k | 0 \rangle;$$

equivalently:  $\phi_j \phi_k := \phi_j \phi_k$  if  $j \leq k$  and  $-\phi_k \phi_j$  if  $j > k$ , except when  $j = k = 0$ , then  $\phi_0 \phi_0 := 0$ . The operators  $\alpha_j$  satisfy the commutation relations of the Heisenberg Lie algebra

$$[\alpha_j, \alpha_k] = \frac{j}{2} \delta_{j-k}, \quad \alpha_i |0\rangle = \langle 0 | \alpha_{-i} = 0, \quad \text{for } i > 0, \tag{21}$$

and its representation on  $F$  is irreducible ([9], Theorem 3.2). Using this, we obtain a vector space isomorphism  $\sigma : F \rightarrow B$ , uniquely defined by the following relations:

$$\sigma(|0\rangle) = 1, \quad \sigma \alpha_j \sigma^{-1} = \frac{\partial}{\partial t_j}, \quad \sigma \alpha_{-j} \sigma^{-1} = \frac{j}{2} t_j, \quad \text{for } j > 0 \text{ odd.} \tag{22}$$

Explicitly ([9], Section 3.2):

$$\sigma \phi(z) \sigma^{-1} = \frac{1}{\sqrt{2}} \exp \sum_{j=1}^{\infty} t_{2j-1} z^{2j-1} \exp \sum_{j=1}^{\infty} -2 \frac{\partial}{\partial t_{2j-1}} \frac{z^{-2j+1}}{2j-1}. \tag{23}$$

Since (15) can be rewritten as

$$\text{Res}_z \phi(z) \tau \otimes \phi(-z) \tau \frac{dz}{z} = \frac{1}{2} \tau \otimes \tau,$$

under the isomorphism  $\sigma$ , Equation (15) turns into:

$$\text{Res}_z e^{\sum_{j=1}^{\infty} (t_{2j-1} - t'_{2j-1}) z^{2j-1}} e^{\sum_{j=1}^{\infty} 2 \left( \frac{\partial}{\partial t'_{2j-1}} - \frac{\partial}{\partial t_{2j-1}} \right) \frac{z^{-2j+1}}{2j-1}} \tau(\tilde{t}) \tau(\tilde{t}') \frac{dz}{z} = \tau(\tilde{t}) \tau(\tilde{t}'), \tag{24}$$

where  $\tilde{t} = (t_1, t_3, t_5, \dots)$  and  $\tilde{t}' = (t'_1, t'_3, t'_5, \dots)$ . Therefore,  $\tau(\tilde{t})$  is the vacuum expectation value

$$\tau(\tilde{t}) = \sigma \tau \sigma^{-1} = \langle 0 | e^{\sum_{j=1}^{\infty} t_{2j-1} \alpha_{2j-1}} \tau. \tag{25}$$

Furthermore, by making a change of variables, as in [10] (p. 972), viz.  $t_{2k-1} = x_{2k-1} - y_{2k-1}$  and  $t'_{2k-1} = x'_{2k-1} - y'_{2k-1}$ , and using the elementary Schur polynomials  $s_j(r)$ , which are defined by

$$\exp \sum_{k=1}^{\infty} r_k z^k = \sum_{j=0}^{\infty} s_j(r) z^j, \tag{26}$$

we can rewrite (24), where we assume  $x_{2k} = y_{2k} = 0$ :

$$\sum_{j=1}^{\infty} s_j(-2\tilde{y}) s_j(2\tilde{\delta}_y) \tau(\tilde{x} - \tilde{y}) \tau(\tilde{x} + \tilde{y}) = 0, \tag{27}$$

where  $\tilde{y} = (y_1, 0, y_3, 0, \dots)$  and  $\tilde{\delta}_y = (\frac{\partial}{\partial y_1}, 0, \frac{1}{3} \frac{\partial}{\partial y_3}, 0, \frac{1}{5} \frac{\partial}{\partial y_5}, \dots)$ . Using Taylor's formula, we thus obtain the BKP hierarchy of Hirota bilinear equations [10] (p. 972):

$$\sum_{j=1}^{\infty} s_j(-2\tilde{y}) s_j(2\tilde{\delta}_u) \exp \sum_{j=1}^{\infty} y_{2j-1} \frac{\partial}{\partial u_{2j-1}} \tau(\tilde{x} - \tilde{u}) \tau(\tilde{x} + \tilde{u}) \Big|_{\tilde{u}=0} = 0. \tag{28}$$

Using the notation  $p(D)f \cdot g = p(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \dots) f(\tilde{x} + \tilde{u}) g(\tilde{x} - \tilde{u}) \Big|_{\tilde{u}=0}$ , this turns into

$$\sum_{j=1}^{\infty} s_j(-2\tilde{y}) s_j(2\tilde{D}) e^{\sum_{j=1}^{\infty} y_{2j-1} D_{2j-1}} \tau \cdot \tau = 0. \tag{29}$$

The simplest equation in this hierarchy is ([10], Appendix 3):

$$(D_1^6 - 5D_1^3 D_3 - 5D_3^2 + 9D_1 D_5) \tau \cdot \tau = 0. \tag{30}$$

If we assume that our tau-function does not depend on  $t_3$ , then this gives

$$(D_1^6 + 9D_1D_5)\tau \cdot \tau = 0. \tag{31}$$

Letting  $x = t_1, t = \frac{1}{9}t_5$ , and

$$u(x, t) = 2 \frac{\partial^2 \log \tau(x, t)}{\partial x^2}, \tag{32}$$

and viewing the remaining  $t_j$  as parameters, Equation (31) turns into the famous Sawada–Kotera equation [6]:

$$u_t + 15(uu_{xxx} + u_xu_{xx} + 3u^2u_x) + u_{xxxxx} = 0. \tag{33}$$

Another approach is by using the wave function; see [8] (p. 345),

$$\begin{aligned} w(\tilde{t}, z) &= \frac{1}{\tau(\tilde{t})} \exp \sum_{j=1}^{\infty} t_{2j-1} z^{2j-1} \exp - \sum_{j=1}^{\infty} 2 \frac{\partial}{\partial t_{2j-1}} \frac{z^{-2j+1}}{2j-1} \tau(\tilde{t}) \\ &= P(\tilde{t}, z) e^{\sum_{j=1}^{\infty} (t_{2j-1}) z^{2j-1}}, \end{aligned} \tag{34}$$

where  $P(\tilde{t}, z) = 1 + \sum_{j=1}^{\infty} p_j(\tilde{t}) z^{-j}$ , and, in particular,

$$p_1(\tilde{t}) = -2 \frac{\partial \log \tau(\tilde{t})}{\partial t_1}. \tag{35}$$

Letting  $P(\tilde{t}, \partial)$  be the pseudodifferential operator in  $\partial = \frac{\partial}{\partial t_1}$  with the symbol  $P(\tilde{t}, z)$ , Equation (24) turns into

$$\text{Res}_z P(\tilde{t}, \partial) P(\tilde{t}', \partial') e^{\sum_{j=1}^{\infty} (t_{2j-1} - t'_{2j-1}) z^{2j-1}} \frac{dz}{z} = 1. \tag{36}$$

Now, using the fundamental lemma, Lemma 1.1 of [2] or Lemma 4.1 of [11], we deduce from (36):

$$\begin{aligned} P(\tilde{t}, \partial) \partial^{-1} P(\tilde{t}, \partial)^* \partial &= 1, \\ \frac{\partial P(\tilde{t}, \partial)}{\partial t_{2j-1}} &= -(P(\tilde{t}, \partial) \partial^{2j-1} P(\tilde{t}, \partial)^{-1} \partial^{-1})_{<0} \partial P(\tilde{t}, \partial), \quad j = 1, 2, \dots \end{aligned} \tag{37}$$

Next, introducing the Lax operator

$$L(\tilde{t}, \partial) = P(\tilde{t}, \partial) \partial P(\tilde{t}, \partial)^{-1} = \partial + u_1(\tilde{t}) \partial^{-1} + u_2(\tilde{t}) \partial^{-2} + \dots,$$

we deduce from (37) that  $L$  satisfies [8]

$$\begin{aligned} L(\tilde{t}, \partial)^* &= -\partial^{-1} L(\tilde{t}, \partial) \partial, \\ \frac{\partial L(\tilde{t}, \partial)}{\partial t_{2j-1}} &= \left[ \left( L(\tilde{t}, \partial) \right)^{2j-1} \right]_{\geq 0} L(\tilde{t}, \partial), \quad j = 1, 2, \dots \end{aligned} \tag{38}$$

Note that, since  $u_1(\tilde{t}) = -\frac{\partial p_1(\tilde{t})}{\partial t_1}$  and the fact that  $p_1(t)$  is given by (35), we find that

$$u_1(\tilde{t}) = 2 \frac{\partial^2 \log \tau(\tilde{t})}{\partial t_1^2}, \tag{39}$$

which explains the choice (32) of  $u(x, t)$  to obtain the Sawada–Kotera equation from the Hirota bilinear Equation (31).

To obtain the second equation of (38), we use (37) and the first equation of (38), which is equivalent (see [8], (p. 356)) to the fact that  $L(\tilde{t}, \partial)^{2j-1}$ , for  $j = 1, 2, \dots$ , has zero constant term. Let us prove that the first equation of (38) indeed implies this fact. We have

$$L^k \partial^{-1} = (-\partial^{-1} L^* \partial)^k \partial^{-1} = (-1)^k \partial^{-1} L^{*k} = (-1)^{k+1} (L^k \partial^{-1})^*.$$

Now, using the fact that the constant term of  $L^k$  is equal to

$$\text{Res}_\partial L^k \partial^{-1} = -\text{Res}_\partial (L^k \partial^{-1})^* = \text{Res}_\partial (-1)^{k+1} (L^k \partial^{-1})^* = (-1)^k \text{Res}_\partial L^k \partial^{-1},$$

we find that the constant term of  $L^k$  is zero whenever  $k$  is odd.

**Remark 1.** Note that this also means that we can replace the second equation of (38) by, cf. [12],

$$\frac{\partial L(\tilde{t}, \partial)}{\partial t_{2j-1}} = \left[ (L(\tilde{t}, \partial))^{2j-1} \right]_{\geq 1}, \quad j = 1, 2, \dots$$

In the formulation of Kupershmidt [12], this means that  $L$  satisfies not only the KP equation for the odd times, but also his formulation of the modified KP hierarchy (only for the odd times).

Next, we describe polynomial tau-functions  $\tau(t_1, t_3, \dots)$  of the BKP hierarchy obtained in ([7], Theorem 6) (see also [4]). For that, given integers  $a$  and  $b$ ,  $a > b \geq 0$ , let

$$\begin{aligned} \chi_{a,b}(t, t') &= \frac{1}{2} s_a(t') s_b(t) + \sum_{j=1}^b (-1)^j s_{a+j}(t') s_{b-j}(t), \\ \chi_{b,a}(t, t') &= -\chi_{a,b}(t, t'), \quad \chi_{a,a}(t, t') = 0, \end{aligned} \tag{40}$$

and let  $\chi_{a,b}(t, t') = 0$  if  $b < 0$ . Then

**Theorem 1** ([7], Theorem 6). All polynomial tau-functions of the BKP hierarchy (24), up to a scalar multiple, are equal to

$$\tau_\lambda(\tilde{t}) = \text{Pf} \left( \chi_{\lambda_i, \lambda_j}(\tilde{t} + c_i, \tilde{t} + c_j) \right)_{1 \leq i, j \leq 2n'} \tag{41}$$

where  $\text{Pf}$  is the Pfaffian of a skew-symmetric matrix,  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{2n})$  is an extended strict partition, i.e.,  $\lambda_1 > \lambda_2 > \dots > \lambda_{2n} \geq 0$ ,  $\tilde{t} = (t_1, 0, t_3, 0, \dots)$ ,  $c_i = (c_{1i}, c_{2i}, c_{3i}, \dots)$ ,  $c_{ij} \in \mathbb{C}$ .

**Remark 2.** The connection between the set of strict partitions and the extended strict partitions is as follows. If  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$  is a strict partition and  $k$  is even, then this partition is equal to the extended strict partition  $\lambda$ . However, if  $k$  is odd, the Pfaffian of a  $k \times k$  anti-symmetric matrix is equal to 0, hence, in that case, we extend  $\lambda$  by the element 0, i.e., the corresponding extended strict partition is then  $(\lambda_1, \lambda_2, \dots, \lambda_k, 0)$ .

### 3. The $n$ -th Sawada–Kotera Hierarchy and Its Polynomial Tau-Functions

As we have seen in Section 2, a necessary condition for a tau-function to give a solution of the Sawada–Kotera equation is that  $\frac{\partial \tau(\tilde{t})}{\partial t_3} = 0$ . This means that the tau-function lies in a smaller group orbit of the vacuum vector  $|0\rangle$ . Instead of the  $SO_{\infty, odd}$  orbit of the vacuum vector  $|0\rangle$ , we consider the twisted loop group  $G_3^{(2)}$ , corresponding to the affine Lie algebra  $sl_3^{(2)}$ , to obtain the 3-reduced BKP hierarchy [8]. More generally (see also [8]), when  $n = 2k + 1 > 1$  is odd, the  $2k + 1$ -reduced hierarchy is related to the twisted loop group  $G_{2k+1}^{(2)}$  corresponding to the twisted affine Lie algebra  $sl_{2k+1}^{(2)}$ . When  $n = 2k > 2$  is even, one has the twisted loop group  $G_{2k}^{(2)}$  corresponding to the affine Lie algebra  $so_{2k}^{(2)}$  [9,13] (see [14] (Chapter 7) for the construction of these Lie algebras). Elements  $G$  in this twisted loop

group not only satisfy (18), which implies  $\sum_{j \in \mathbb{Z}} (-1)^j a_{kj} a_{\ell, -j} = (-1)^k \delta_{k, -\ell}$ , but also the  $n$ -periodicity condition  $a_{i+n, j+n} = a_{ij}$ . This means that these group elements also commute with the operator

$$\sum_{i \in \mathbb{Z}} (-1)^{pn-i} \phi_i \otimes \phi_{pn-i}, \quad \text{for } p = 1, 2, 3, \dots,$$

namely

$$\begin{aligned} (G \otimes G) \sum_{i \in \mathbb{Z}} (-1)^{pn-i} \phi_i \otimes \phi_{pn-i} &= \sum_{i \in \mathbb{Z}} (-1)^{pn-i} G \phi_i G^{-1} \otimes G \phi_{pn-i} G^{-1} G \\ &= \sum_{i, j, k \in \mathbb{Z}} (-1)^{pn-i} a_{ji} a_{k, pn-i} \phi_j G \otimes \phi_k G \\ &= \sum_{i, j, k \in \mathbb{Z}} (-1)^{pn-i} a_{ji} a_{k-pn, -i} \phi_j G \otimes \phi_k G \\ &= \sum_{j \in \mathbb{Z}} (-1)^{pn-j} \phi_j G \otimes \phi_{pn-j} G \\ &= \sum_{j \in \mathbb{Z}} (-1)^{pn-j} \phi_j \otimes \phi_{pn-j} (G \otimes G). \end{aligned}$$

Since  $\sum_{i \in \mathbb{Z}} (-1)^{pn-i} \phi_i |0\rangle \otimes \phi_{pn-i} |0\rangle = 0$ , we find that the elements  $\tau$  in the orbit of the vacuum vector of this twisted loop not only satisfy (15), but also satisfy the conditions

$$\sum_{j \in \mathbb{Z}} (-1)^{pn-j} \phi_j \tau \otimes \phi_{pn-j} \tau = 0, \quad p = 1, 2, \dots \tag{42}$$

This means that  $\tau(\tilde{t}) = \sigma(\tau)$  not only satisfies (24), but also the conditions

$$\text{Res}_{z \neq 0} z^{pn-1} e^{\sum_{j=1}^{\infty} (t_{2j-1} - t'_{2j-1}) z^{2j-1}} e^{\sum_{j=1}^{\infty} 2 \left( \frac{\partial}{\partial t'_{2j-1}} - \frac{\partial}{\partial t_{2j-1}} \right) \frac{z^{-2j+1}}{2j-1}} \tau(\tilde{t}) \tau(\tilde{t}') dz = 0, \quad p = 1, 2, \dots \tag{43}$$

From (43), one deduces, using the fundamental Lemma ([11], Lemma 4.1) and the first equation of (37), that

$$(P(\tilde{t}, \partial) \partial^{pn-1} P(\tilde{t}, \partial)^*)_{<0} = (P(\tilde{t}, \partial) \partial^{pn} P(\tilde{t}, \partial)^{-1} \partial^{-1})_{<0} = 0.$$

Thus, the Lax operator  $L(\tilde{t}, \partial)$  satisfies

$$(L(\tilde{t}, \partial)^{pn})_{<0} = 0, \quad p = 1, 2, \dots \tag{44}$$

Hence,  $\mathcal{L}(\tilde{t}, \partial) = L(\tilde{t}, \partial)^n$  is a monic differential operator with zero constant term. Moreover,  $\mathcal{L}(\tilde{t}, \partial)$  is equal to (10), and, by the first formula of (37), we have the relation (11).

Now, if  $n$  is odd, one can use the the Sato–Wilson equation, i.e., the second equation of (37), to find that

$$\frac{\partial P(\tilde{t}, \partial)}{\partial t_{(2j-1)n}} = 0, \quad j = 1, 2, \dots$$

From this we find that the tau-function satisfies

$$\frac{\partial \tau(\tilde{t})}{\partial t_{(2j-1)n}} = \lambda_j \tau(\tilde{t}), \quad \lambda_j \in \mathbb{C}, \quad \text{for } j = 1, 2, \dots \tag{45}$$

Since we consider only polynomial tau-functions, we find that for odd  $n$ :

$$\frac{\partial \tau(\tilde{t})}{\partial t_{(2j-1)n}} = 0, \quad j = 1, 2, \dots \tag{46}$$

If  $n$  is even, there is no such restriction, because the Sato–Wilson Equation (37) is only defined for odd flows. However, the additional Equation (43) still holds and gives additional constraints on the tau-function.

**Proposition 1.** For  $n$  odd, Equation (46) for  $j = 1$  and the BKP hierarchy (24) on  $\tau(\tilde{t})$  are equivalent to (24) and (43).

**Proof.** We only have to show that (46) for  $j = 1$  and (24) imply (43). For this, differentiate (24) by  $t_n$  and use (46); this gives Equation (43) for  $p = 1$ . Next, differentiate (43) for  $p = 1$  again by  $t_n$  and use again (46); this gives (43) for  $p = 2$ , etc.  $\square$

**Remark 3.** If  $n$  is odd, Proposition 1 gives that a polynomial BKP tau-function is  $n$ -th Sawada–Kotera tau-function if and only if  $\tau$  satisfies  $\frac{\partial \tau}{\partial t_n} = 0$ .

Since  $L$  satisfies the BKP hierarchy,  $\mathcal{L} = L^n$  also satisfies the BKP hierarchy. For  $n = 3$ , assuming the constraint that  $\mathcal{L}$  is a differential operator,  $\mathcal{L}$  is given by (3) and  $\frac{\partial \mathcal{L}}{\partial \tilde{t}_3} = [(\mathcal{L}^{\frac{5}{3}})_{\geq 0}, \mathcal{L}]$  is the Sawada–Kotera equation (33). This leads to the following definition.

**Definition 1.** Let  $\mathcal{L} = \partial^n + u_{n-2}\partial^{n-2} + \dots + u_1\partial$  be a differential operator, satisfying (11). The system of Lax equations

$$\frac{\partial \mathcal{L}(\tilde{t}, \partial)}{\partial t_{2j-1}} = \left[ \left( \mathcal{L}(\tilde{t}, \partial)^{\frac{2j-1}{n}} \right)_{\geq 0}, \mathcal{L}(\tilde{t}, \partial) \right], \quad j = 1, 2, \dots, \tag{47}$$

is called the  $n$ -th reduced BKP hierarchy or the  $n$ -th Sawada–Kotera hierarchy. For  $n = 3$ , it is called the Sawada–Kotera hierarchy.

The geometric meaning of Equation (42) is that the space  $\text{Ann } \tau$  is invariant under the shift  $\Lambda_n$ , where

$$\Lambda_n(\phi_i) = \phi_{i+n}. \tag{48}$$

As in the  $SO_{\infty, \text{odd}}$  case, all polynomial tau-functions in this  $n$ -reduced case lie in some Schubert cell. Such a Schubert cell has a “lowest” element  $w_\lambda$ , for  $\lambda$  a certain strict partition. This element can be obtained from the vacuum vector by the action of the Weyl group corresponding to  $G_n^{(2)}$ . The element

$$w_\lambda = \phi_{-\lambda_1} \phi_{-\lambda_2} \dots \phi_{-\lambda_k} |0\rangle \tag{49}$$

lies in the Weyl group orbit of  $|0\rangle$ , corresponding to  $SO_{\infty, \text{odd}}$ , (see [15]), however, not all such elements lie in the Weyl group orbit of  $|0\rangle$  for  $G_n^{(2)}$ . For this, consider

$$\text{Ann } w_\lambda = \text{span}\{\phi_{-\lambda_1}, \phi_{-\lambda_2}, \dots, \phi_{-\lambda_k}\} \oplus \text{span}\{\phi_i \mid i > 0, i \neq \lambda_j, j = 1, \dots, k\}. \tag{50}$$

The element  $w_\lambda$  lies in the  $G_n^{(2)}$  Weyl group orbit if and only if  $\text{Ann } w_\lambda$  is invariant under the action of  $\Lambda_n$ , which means that the  $(\lambda_1 + 1)$  shifted set

$$V_\lambda = \{\lambda_1 + \lambda_i + 1, i = 1, \dots, k\} \cup \{\lambda_1 - j + 1 \mid 0 < j < \lambda_1, j \neq \lambda_i \text{ for } i = 1, \dots, k\}, \tag{51}$$

must satisfy the  $-n$  shift condition, i.e.,

$$\text{if } \mu_j \in V_\lambda, \text{ then } \mu_j - n \in V_\lambda \text{ or } \mu_j - n \leq 0. \tag{52}$$

Only the elements  $w_\lambda$ , for which the corresponding  $V_\lambda$  satisfies condition (52), lie in the  $G_n^{(2)}$  group orbit.



**Example 1.** (a) For  $n = 2$ , the only strict partition  $\lambda$  that satisfies condition (52) is  $\lambda = \emptyset$ .  
 (b) For  $n = 3$ , the only strict partitions  $\lambda$  that satisfy condition (52) are

$$(3m + 1, 3m - 2, 3m - 5, \dots, 4, 1) \text{ and } (3m + 2, 3m - 1, 3m - 4, \dots, 5, 2), \quad m \in \mathbb{Z}_{\geq 0}.$$

**Remark 4.** Note that (52) means that  $\lambda$  is a strict partition that is the union of strict partitions  $(nm + a_i, n(m - 1) + a_i, \dots, n + a_i, a_i)$ , with  $1 \leq a_i < n$  and  $1 \leq i < n$ , such that  $a_j - n \neq -a_\ell$ . In other words,  $a_j + a_\ell \neq n$ . Hence there are at most  $\lfloor \frac{n}{2} - 1 \rfloor$  such  $a_i$ .

To a strict partition  $\lambda$  that satisfies condition (52), the corresponding Schubert cell is then obtained through the action on a  $w_\lambda$  by an upper-triangular matrix in the group  $G_n^{(2)}$ . This produces, up to a constant factor, elements

$$v_\lambda = v_1 v_2 \cdots v_k |0\rangle, \text{ where } v_j = \phi_{-\lambda_j} + \sum_{i \geq 1 - \lambda_j} a_{ij} \phi_i \text{ (finite sum)}, \tag{53}$$

and

$$(v_j, v_\ell) = 0, \text{ for } j, \ell = 1, \dots, k, \text{ and if } \lambda_i = \lambda_j - n, \text{ then } v_i = \Lambda_n(v_j). \tag{54}$$

We first express the constants  $a_{ij}$  in terms of other constants by letting

$$a_{ij} = s_{i+\lambda_j}(c_{\bar{\lambda}_j}), \text{ where the } s_i \text{ are elementary Schur polynomials.}$$

Here, we use that

$$1 + \sum_{i=1-\lambda_j}^{\infty} a_{ij} z^{i+\lambda_j} = \exp\left(\sum_{k=1}^{\infty} c_{k, \bar{\lambda}_j} z^k\right),$$

hence, for every  $\bar{\lambda}_j$ , one can recursively obtain the  $c_{k, \bar{\lambda}_j}$ . Since  $a_{ij} = 0$  for  $i \gg 0$ , one only has a finite number of  $c_{k, \bar{\lambda}_j}$ . Thus,

$$v_j = \phi_{-\lambda_j} + \sum_{i > -\lambda_j} s_{i+\lambda_j}(c_{\bar{\lambda}_j}) \phi_i, \tag{55}$$

where  $c_{\bar{\lambda}_j} = (c_{1, \bar{\lambda}_j}, c_{2, \bar{\lambda}_j}, c_{3, \bar{\lambda}_j}, \dots)$ . Here,  $\bar{\lambda}_j = \lambda_j \pmod n$ , which means that there are at most  $\lfloor \frac{n}{2} - 1 \rfloor$  of such infinite series of constants (see Remark 4) and the  $v_j$  satisfy the condition

$$\text{if } \lambda_i = \lambda_j - n, \text{ then } v_i = \Lambda_n(v_j). \tag{56}$$

We can now use the isomorphism  $\sigma$  to calculate the bosonization of elements  $v_\lambda$ . For this, we use formula (25) and apply this to  $v_\lambda$  (which is given by (53) with  $v_j$  given by (55)). Now, using (22) and (23) and the fact that

$$e^{\sum_{j=1}^{\infty} t_{2j-1} \frac{\partial}{\partial s_{2j-1}}} e^{\sum_{j=1}^{\infty} s_{2j-1} z^{2j-1}} = e^{\sum_{j=1}^{\infty} (t_{2j-1} + s_{j-1}) z^{2j-1}} e^{\sum_{j=1}^{\infty} t_{2j-1} z^{2j-1} \frac{\partial}{\partial s_{2j-1}}}$$

we find that

$$e^{\sum_{j=1}^{\infty} t_{2j-1} \alpha_{2j-1}} \phi(z) e^{-\sum_{j=1}^{\infty} t_{2j-1} \alpha_{2j-1}} = e^{\sum_{j=1}^{\infty} t_{2j-1} z^{2j-1}} \phi(z).$$

Thus, using (55), we find that

$$\begin{aligned}
 v_j(\tilde{t}) &:= e^{\sum_{j=1}^{\infty} t_{2j-1} \alpha_{2j-1}} v_j e^{-\sum_{j=1}^{\infty} t_{2j-1} \alpha_{2j-1}} \\
 &= e^{\sum_{j=1}^{\infty} t_{2j-1} \alpha_{2j-1}} (\phi_{-\lambda_j} + \sum_{i \geq 1-\lambda_j} s_{i+\lambda_j}(c_{\bar{\lambda}_j}) \phi_i) e^{-\sum_{j=1}^{\infty} t_{2j-1} \alpha_{2j-1}} \\
 &= e^{\sum_{j=1}^{\infty} t_{2j-1} \alpha_{2j-1}} \operatorname{Res} \sum_{\ell=0}^{\infty} s_{\ell}(c_{\bar{\lambda}_j}) z^{\ell-\lambda_j} \phi(z) e^{-\sum_{j=1}^{\infty} t_{2j-1} \alpha_{2j-1}} \frac{dz}{z} \\
 &= \operatorname{Res} \sum_{\ell=0}^{\infty} s_{\ell}(c_{\bar{\lambda}_j}) z^{\ell-\lambda_j} \phi(z) e^{\sum_{j=1}^{\infty} t_{2j-1} \alpha_{2j-1}} \frac{dz}{z} \\
 &= \operatorname{Res} z^{-\lambda_j} e^{\sum_{i=1}^{\infty} c_{i, \bar{\lambda}_j} z^i + \sum_{j=1}^{\infty} t_{2j-1} \alpha_{2j-1}} \phi(z) \frac{dz}{z} \\
 &= \operatorname{Res} z^{-\lambda_j} \sum_{k=0}^{\infty} s_k(\tilde{t} + c_{\bar{\lambda}_j}) z^k \sum_{i \in \mathbb{Z}} \phi_i z^{-i} \frac{dz}{z} \\
 &= \phi_{-\lambda_j} + \sum_{i \geq 1-\lambda_j} s_{i+\lambda_j}(\tilde{t} + c_{\bar{\lambda}_j}) \phi_i.
 \end{aligned}$$

Since  $e^{\sum_{j=1}^{\infty} t_{2j-1} \alpha_{2j-1}} |0\rangle = 0$ , we find that the corresponding tau-function is equal to the vacuum expectation value

$$\tau(\tilde{t}) = \langle 0 | v_1(\tilde{t}) v_2(\tilde{t}) \cdots v_k(\tilde{t}) | 0 \rangle. \tag{57}$$

If  $k = 2m$ , then this is the Pfaffian of a  $2m \times 2m$  skew-symmetric matrix. If  $k = 2m - 1$ , we use the fact that

$$\langle 0 | v_1(\tilde{t}) v_2(\tilde{t}) \cdots v_k(\tilde{t}) | 0 \rangle = 2 \langle 0 | v_1(\tilde{t}) v_2(\tilde{t}) \cdots v_k(\tilde{t}) \phi_0 | 0 \rangle$$

and again we find a Pfaffian. We thus arrive at the main theorem.

**Theorem 2.** All polynomial tau-functions of the  $n$ -th Sawada–Kotera hierarchy are, up to a scalar factor, equal to the Pfaffian

$$\tau_{\lambda}(\tilde{t}) = \operatorname{Pf} \left( \chi_{\lambda_i, \lambda_j}(\tilde{t} + c_{\bar{\lambda}_i}, \tilde{t} + c_{\bar{\lambda}_j}) \right)_{1 \leq i, j \leq 2m}, \tag{58}$$

where  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{2m})$ ,  $m = 0, 1, \dots$ , is an extended strict partition, which satisfies the  $-n$  shift condition (52) for (51). The polynomials  $\chi_{a,b}$  are given by (40). Here, as before,  $\tilde{t} = (t_1, 0, t_3, 0, \dots)$ , and  $c_{\bar{\lambda}_i} = (c_{1, \bar{\lambda}_i}, c_{2, \bar{\lambda}_i}, c_{3, \bar{\lambda}_i}, \dots)$  are arbitrary constants, where we replace, recursively, for all  $j = 1, 2, \dots, 2m$  (respectively, for all  $j = 1, 2, \dots, 2m - 1$ ), when  $\lambda_{2m} \neq 0$  (resp.,  $\lambda_{2m} = 0$ ), the constants  $c_{\lambda_j + \lambda_{\ell}, \bar{\lambda}_j}$ , for  $j \leq \ell \leq 2m$  (resp.,  $j \leq \ell < 2m$ ) as follows:

(1) If  $\bar{\lambda}_j \neq \bar{\lambda}_{\ell}$ , then

$$\begin{aligned}
 c_{\lambda_j + \lambda_{\ell}, \bar{\lambda}_j} &= -(-1)^{\lambda_j + \lambda_{\ell}} \times \\
 & s_{\lambda_j + \lambda_{\ell}}(c_{1, \bar{\lambda}_{\ell}} - c_{1, \bar{\lambda}_1}, c_{2, \bar{\lambda}_{\ell}} + c_{2, \bar{\lambda}_1}, \dots, c_{\lambda_j + \lambda_{\ell} - 1, \bar{\lambda}_{\ell}} + (-1)^{\lambda_j + \lambda_{\ell} - 1} c_{\lambda_j + \lambda_{\ell} - 1, \bar{\lambda}_j}, c_{\lambda_j + \lambda_{\ell}, \bar{\lambda}_{\ell}}).
 \end{aligned} \tag{59}$$

(2) If  $\bar{\lambda}_j = \bar{\lambda}_{\ell}$  and  $\lambda_j + \lambda_{\ell}$  is even, then

$$c_{\lambda_j + \lambda_{\ell}, \bar{\lambda}_j} = -\frac{1}{2} s_{\frac{\lambda_j + \lambda_{\ell}}{2}}(2c_{2, \bar{\lambda}_j}, 2c_{4, \bar{\lambda}_j}, \dots, 2c_{\lambda_j + \lambda_{\ell} - 2, \bar{\lambda}_j}, 0). \tag{60}$$

**Proof.** We still need to use the fact that all vectors  $v_i$ , for  $i = 1, \dots, k$ , form an isotropic subspace. So, assume that  $1 \leq j \leq \ell \leq k$  and that  $v_j$  and  $v_\ell$  are given by (55). Then

$$\begin{aligned}
 0 = (v_j, v_\ell) &= (-1)^{\lambda_j} \sum_{i=0}^{\lambda_j + \lambda_\ell} (-1)^i s_i(c_{\bar{\lambda}_j}) s_{\lambda_j + \lambda_\ell - i}(c_{\bar{\lambda}_\ell}) \\
 &= (-1)^{\lambda_j} s_{\lambda_j + \lambda_\ell}(c_{i, \bar{\lambda}_\ell} + (-1)^i c_{i, \bar{\lambda}_j}).
 \end{aligned}
 \tag{61}$$

Here, we have used the fact that the coefficient of  $z^m$  of

$$e^{\sum_{i=1}^{\infty} x_i z^i + y^i (-z)^i} = e^{\sum_{i=1}^{\infty} x_i z^i} e^{\sum_{i=1}^{\infty} y^i (-z)^i}$$

is equal to  $s_m(x_i + (-1)^i y_i) = \sum_{j=0}^m (-1)^{m-j} s_j(x) s_{m-j}(y)$ .

So, we need to investigate condition (61). Here we have two possibilities, viz.  $\bar{\lambda}_j \neq \bar{\lambda}_\ell$  and  $\bar{\lambda}_j = \bar{\lambda}_\ell$ . If  $\bar{\lambda}_j \neq \bar{\lambda}_\ell$ , then using the fact that  $s_i(x) = x_i +$  terms not containing  $x_i$ , we find (59). If  $\bar{\lambda}_j = \bar{\lambda}_\ell$ , then notice that  $s_{\lambda_j + \lambda_\ell}$  only depends on the  $c_{i, \bar{\lambda}_j}$  with  $i$  even. Thus, all elementary Schur polynomials  $s_{2i+1}$  in only the even variables are equal to zero. This means that if  $\lambda_j + \lambda_\ell$  is odd, there is no restriction on the constants, but if  $\lambda_j + \lambda_\ell$  is even, we find that

$$c_{\lambda_j + \lambda_\ell, \bar{\lambda}_j} = -\frac{1}{2} s_{\lambda_j + \lambda_\ell}(0, 2c_{2, \bar{\lambda}_j}, 0, 2c_{4, \bar{\lambda}_j}, 0, 2c_{6, \bar{\lambda}_j}, \dots, 0, 2c_{\lambda_j + \lambda_\ell - 2, \bar{\lambda}_j}, 0, 0).$$

Note that this restriction coincides with (60).  $\square$

**Remark 5.** Since  $\chi_{\lambda_j, \lambda_\ell}$  is given by (40), the constant  $c_{2\lambda_1, \bar{\lambda}_1}$  does not appear in (58) and the substitution (60) for  $c_{2\lambda_1, \bar{\lambda}_1}$  is void.

For  $n = 3$ , see Example 1(b), we only have one infinite series of constants  $c_{i, \bar{\lambda}_1}$ , which means that we only have the substitutions (60). We therefore find:

**Corollary 1.** All polynomial tau-functions of the Sawada–Kotera hierarchy are, up to a non-zero constant factor,

$$\tau_\lambda(\tilde{t}) = Pf\left(\chi_{\lambda_i, \lambda_j}(\tilde{t} + c, \tilde{t} + c)\right)_{1 \leq i, j \leq 2m'} \tag{62}$$

where  $\lambda$  is one of the following extended strict partitions:

1.  $(6m + 1, 6m - 2, 6m - 5, \dots, 4, 1)$ ;
2.  $(6m - 2, 6m - 5, 6m - 8, \dots, 4, 1, 0)$ ;
3.  $(6m + 2, 6m - 1, 6m - 4, \dots, 5, 2)$ ;
4.  $(6m - 1, 6m - 4, 6m - 7, \dots, 5, 2, 0)$ ,

where  $m = 0, 1, \dots$ , and  $c = (c_1, c_2, c_3, \dots)$  are arbitrary constants in which we substitute recursively,  $c_2 = 0$ , and  $c_8, c_{14}, c_{20}, \dots, c_{12m-4}$  in case 1;  $c_2 = 0$ , and  $c_8, c_{14}, c_{20}, \dots, c_{12m-10}$  in case 2;  $c_4, c_{10}, c_{16}, \dots, c_{12m-2}$  in case 3; and  $c_4, c_{10}, c_{16}, \dots, c_{12m-8}$  in case 4, respectively, by the following formula

$$c_{2k} = -\frac{1}{2} s_k(2c_2, 2c_4, \dots, 2c_{2k-2}, 0) \quad \text{for } k > 1. \tag{63}$$

**Remark 6.** If we choose  $c = \tilde{c} = (c_1, 0, c_3, 0, c_5, \dots)$ , then the corresponding Sawada–Kotera tau-function  $\tau_\lambda(\tilde{t})$  is equal, up to a non-zero constant factor, to a Schur Q-function  $Q_\lambda(\tilde{t} + \tilde{c})$  (cf. [15]).

**Example 2.** All the Sawada–Kotera tau-functions related to the partition  $\lambda = (5, 2)$  are given, up to a multiplicative constant, by

$$\begin{aligned} & c_1^7 + 1120c_7 + 7c_1^6t_1 - 280c_2^3t_1 - 280c_5t_1^2 + 140c_2^2t_1^3 + t_1^7 + 35c_1^3(2c_2 + t_1^2)^2 + \\ & + 7c_1t_1^5(2c_2 + 3t_1^2) + 35c_1^4(2c_2t_1 + t_1^3) - 280t_1^2t_5 - 7c_1^2(40c_5 - 60c_2^2t_1 - 20c_2t_1^3 - 3t_1^5 + 40t_5) + \\ & + 14c_2(120c_5 + t_1^5 + 120t_5) - 7c_1(40c_2^3 - 60c_2^2t_1^2 - 10c_2t_1^4 + t_1(80c_5 - t_1^5 + 80t_5)) + 1120t_7, \end{aligned}$$

with  $c_i \in \mathbb{C}$ , arbitrary. (We have eliminated  $c_4$  by the substitution  $c_4 = -c_2^2$ ; all the other constants that do not appear, disappear automatically).

#### 4. Conclusions

Three classes of solutions of soliton equations have been extensively studied in the literature: rational solutions, soliton solutions, and theta function solutions. Soliton solutions are constructed by making use of vertex operators [2,6,10,11,13,14,16,17]. Theta function solutions are obtained by Krichever’s method [18,19]. Rational solutions are obtained by constructing polynomial tau-functions using the group transformations method, introduced by Sato [1]. In the present paper, we describe all polynomial tau-functions for the  $n$ -th Sawada–Kotera hierarchy, and, in particular, for the Sawada–Kotera equation.

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