Homotopical commutative rings and bispans

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Abstract

We prove that commutative semirings in a cartesian closed presentable ∞ -category, as defined by Groth, Gepner, and Nikolaus, are equivalent to product-preserving functors from the (2, 1)-category of bispans of finite sets. In other words, we identify the latter as the Lawvere theory for commutative semirings in the ∞ -categorical context. This implies that connective commutative ring spectra can be described as grouplike product-preserving functors from bispans of finite sets to spaces. A key part of the proof is a localization result for ∞ -categories of spans, and more generally for ∞ -categories with factorization systems, that may be of independent interest.

Contents

Ι	Introduction	2
2	Background	8
	2.1 Spans	8
	2.2 Bispans	II
	2.3 Symmetric monoidal ∞ -categories	12
	2.4 Cartesian and cocartesian symmetric monoidal structures	14
3	Commutative semirings and spans	ıs
	3.1 (Co)cartesian symmetric monoidal structures via spans	16
	3.2 Symmetric monoidal structures on spans	17
	3.3 Commutative monoids and Day convolution	20
4	Comparison with bispans	23
	4.1 Localizations of span ∞-categories	23
	4.2 Localization of $\hat{\text{Span}}(\mathbb{F})^{\otimes}$	28
	4.3 Proof of the main theorem	30

1 Introduction

The goal of this paper is to give an explicit description of commutative semirings in the ∞ -categorical setting in terms of a certain (2,I)-category Bispan(\mathbb{F}) of *bispans of finite sets*:

Theorem A. Let C be a cartesian closed presentable ∞ -category. Then there is a natural equivalence

 $\mathsf{CRig}(\mathscr{C}) \simeq \mathsf{Fun}^{\times}(\mathsf{Bispan}(\mathbb{F}), \mathscr{C})$

between the ∞ -category of commutative semirings in \mathbb{C} and the ∞ -category of productpreserving functors $Bispan(\mathbb{F}) \to \mathbb{C}$.

Before properly introducing both sides of the equivalence, let us motivate this result by first discussing the analogous statement for *commutative monoids*.

Commutative monoids and spans

Given a category^I \mathcal{C} with finite products, there is a category $\mathsf{CMon}(\mathcal{C})$ of *commutative monoids in* \mathcal{C} , which consists of objects of \mathcal{C} equipped with a binary operation that is unital, associative and commutative. The theory of commutative monoids is algebraic in nature, in the sense that it can be described by a *Lawvere theory*: the data of a commutative monoid in \mathcal{C} is the same as that of a product-preserving functor $L \to \mathbf{Set}$, where $L = \{\mathbb{N}^n \mid n \ge 0\}^{\operatorname{op}}$ is the opposite of the category of free commutative monoids on finitely many generators.

The Lawvere theory *L* for commutative monoids admits an explicit description as the category $hSpan(\mathbb{F})$ of spans of finite sets. Recall that the objects of this category are finite sets, while the set of morphisms in $hSpan(\mathbb{F})$ from *S* to *T* is given as the set of isomorphism classes of *spans* (or *correspondences*)



in finite sets. The identity maps are given by taking S = X = T, and composition is defined via pullback: given two composable spans $S \leftarrow X \rightarrow T$ and $T \leftarrow Y \rightarrow U$, their composite is given by the outer (dashed) span in the following pullback diagram:



¹In this paper we will call categories *categories* and ∞ -categories ∞ -categories. We hope this does not cause too much confusion for younger readers.

One can show that the category $hSpan(\mathbb{F})$ admits finite products given by taking disjoint unions of sets.

The fact that *L* is equivalent to $hSpan(\mathbb{F})$ means that for every category \mathscr{C} with finite products there is a natural equivalence of categories

$$\mathsf{CMon}(\mathscr{C}) \simeq \mathsf{Fun}^{\times}(\mathsf{hSpan}(\mathbb{F}), \mathscr{C})$$

where the right-hand side denotes the category of product-preserving functors from $hSpan(\mathbb{F})$ to \mathscr{C} . Explicitly, a commutative monoid M in \mathscr{C} corresponds under this equivalence to the functor $M^{(-)}$: $hSpan(\mathbb{F}) \to \mathscr{C}$ given as follows:

- On objects, $M^{(-)}$ sends a finite set S to the S-indexed product $M^S \cong M^{\times |S|}$;
- On morphisms, $M^{(-)}$ sends a span $S \xleftarrow{f} X \xrightarrow{g} T$ to the composite

$$M^S \xrightarrow{f^*} M^X \xrightarrow{g_{\oplus}} M^T$$

where we define f^* by $(f^*\psi)_x = \psi_{f(s)}$ for $\psi \in M^S$ and $x \in X$, and we define g_{\oplus} by summing over the fibers of g: for $\phi \in M^X$ and $t \in T$ we set

$$(g_{\oplus}\phi)_t = \sum_{x \in g^{-1}(t)} \phi_x$$

It turns out that the above description of commutative monoids carries over to the ∞ -categorical setting, provided that we work with the (2, 1)-category **Span**(\mathbb{F}) in which we include isomorphisms of spans instead of taking isomorphism classes. In other words, for any ∞ -category \mathcal{C} with finite products there is a natural equivalence of ∞ -categories

$$\mathsf{CMon}(\mathscr{C}) \simeq \mathsf{Fun}^{\times}(\mathsf{Span}(\mathbb{F}), \mathscr{C}).$$

This seems to have been first proved in the thesis of Cranch [Cra11].

Commutative semirings and bispans

In this paper we are interested in a similar description for the Lawvere theory for commutative *semirings*. Recall that a commutative semiring in a category with finite products is an object *R* equipped with two unital, associative and commutative operations, called *addition* and *multiplication*, satisfying the property that multiplication distributes over addition. The Lawvere theory of commutative semirings is given by the category hBispan(\mathbb{F}) of *bispans* of finite sets: the objects are still finite sets, but now the set of morphisms from *S* to *T* is given by the set of isomorphism classes of *bispans* (or *polynomial diagrams*)



The fact that there are now *two* maps pointing to the right reflects the fact that we need to encode both the addition and the multiplication on *R*. Since these two operations interact via a distributivity relation, the composition rule in hBispan(\mathbb{F}) is necessarily somewhat non-trivial to describe. Given a bispan from *S* to *T* and bispan from *T* to *U*, as displayed at the bottom of the following diagram, their composite is given by the outer (dashed) bispan in the diagram:



Here four of the squares are pullbacks, as indicated, and q_* is the right adjoint to pullback along q, so that for $f: K \to F$, the function q_*f has fibres $(q_*f)_c \cong \prod_{x \in q^{-1}(c)} K_x$; the map ϵ is the counit map $q^*q_* \to \text{id}$ for the adjunction $q^* \dashv q_*$. The statement that hBispan(\mathbb{F}) is the Lawvere theory for commutative semirings amounts to having, for any category \mathscr{C} with finite products, a natural

equivalence

$$\mathsf{CRig}(\mathscr{C}) \simeq \mathsf{Fun}^{\mathsf{X}}(\mathsf{hBispan}(\mathbb{F}), \mathscr{C}) \tag{2}$$

between the category of commutative semirings in \mathscr{C} and product-preserving functors from hBispan(\mathbb{F}) to \mathscr{C} . We are unsure of where this statement was first proved, but it is discussed in Strickland's notes on Tambara functors [Str12, \S_5], and a proof appears in the work of Gambino and Kock on polynomial functors [GK13].

Commutative semirings in ∞ -categories

The goal of this paper is to show that the description of the I-categorical Lawvere theory of commutative semirings in terms of bispans generalizes to the ∞ -categorical setting. For the definition of commutative semirings in this setting, we recall that Gepner, Groth, and Nikolaus [GGN15] proved that if \mathscr{C} is a presentable ∞ -category equipped with a closed symmetric monoidal structure, then the ∞ -category CMon(\mathscr{C}) of commutative monoids in \mathscr{C} admits a unique closed symmetric monoidal structure such that the free commutative monoid functor from \mathscr{C} to CMon(\mathscr{C}) is symmetric monoidal. We then define commutative semirings in \mathscr{C} as commutative algebras with respect to this symmetric monoidal structure:

$$\mathsf{CRig}(\mathscr{C}) := \mathsf{CAlg}(\mathsf{CMon}(\mathscr{C}), \otimes).$$

The Lawvere theory of commutative semirings will be a higher category of bispans of finite sets, where we no longer take isomorphism classes of bispans as above. To define this as an ∞ -category, we make use of an observation due to Street [Str20]: The composition of *bispans* can be described in terms of pullbacks in the category of *spans*. This has been verified in the ∞ -categorical setting by Elmanto and the second author [EH23], so that we can define the (2, 1)-category Bispan(\mathbb{F}) as the (2,1)-category of spans in Span(\mathbb{F}), where the forward maps have no backward component.

Now that both sides have been defined, we recall the statement of our Theorem A: for every cartesian closed presentable ∞ -category \mathcal{C} , viewed as a symmetric monoidal category via the cartesian product, there is a natural equivalence

$$\operatorname{CRig}(\mathscr{C}) \simeq \operatorname{Fun}^{\times}(\operatorname{Bispan}(\mathbb{F}), \mathscr{C}).$$

In other words we exhibit $Bispan(\mathbb{F})$ as the Lawvere theory for commutative semirings in ∞ -categories. That this should be the case was suggested briefly at the end of [Ber20]; it was also proposed as a definition of commutative semirings in the thesis of Cranch [Craio], which contains the first (and rather different) construction of Bispan(\mathbb{F}) as a quasicategory.

Gepner–Groth–Nikolaus also show that the symmetric monoidal structure on CMon(\mathscr{C}) localizes to the full subcategory CGrp(\mathscr{C}) \subseteq CMon(\mathscr{C}) of *grouplike* commutative monoids, and that this is moreover compatible with the natural symmetric monoidal structure on the stabilization of \mathscr{C} . When \mathscr{C} is the ∞ category Spc of spaces, this enhances the recognition principle for infinite loop spaces of May [May72] and Boardman–Vogt [BV73] to a symmetric monoidal equivalence between CGrp(Spc) and the ∞ -category Sp^{≥ 0} of connective spectra. Combining this with our result, we get a rather concrete description of connective commutative ring spectra:

Corollary B. *There is an equivalence*

$$CAlg(Sp^{\geq 0}) \simeq Fun^{\times}(Bispan(\mathbb{F}), Spc)_{grp}$$

between connective commutative ring spectra and product-preserving functors $Bispan(\mathbb{F}) \rightarrow Spc$ whose underlying commutative monoid is grouplike.

Overview

Let us outline our strategy for the proof of Theorem A. First, we derive a more explicit description of $CRig(\mathcal{C})$:

► Recall that CMon(𝔅) is equivalent to the full subcategory of Fun(Span(𝔅), 𝔅) spanned by the product-preserving functors. We identify the symmetric monoidal structure of [GGN15] as a localization of the Day convolution structure arising from a tensor product on Span(𝔅) given by the cartesian product of finite sets. This is essentially a result of Ben-Moshe and Schlank [BMS24] (though they work in a slightly different setting).

- Using the universal property of Day convolution, this means that we can identify CRig(𝔅) with a full subcategory of the ∞-category of lax symmetric monoidal functors (Span(𝑘), ⊗) → (𝔅, ×).
- Using the universal property of the cartesian symmetric monoidal structure, we can identify this in turn as a full subcategory of the functors

 $\operatorname{Span}(\mathbb{F})^{\otimes} \longrightarrow \mathscr{C},$

where $\text{Span}(\mathbb{F})^{\otimes}$ is the total space of the cocartesian fibration that encodes the symmetric monoidal structure on $\text{Span}(\mathbb{F})$.

We will prove this in §3 (culminating with Corollary 3.3.7), where we also give an explicit identification of the ∞ -category Span(\mathbb{F})^{\otimes}. Surprisingly, it ends up being another ∞ -category of bispans, now in Ar(\mathbb{F}) (Corollary 3.2.5).

From this description of Span(\mathbb{F})^{\otimes} we obtain an evident functor to Bispan(\mathbb{F}). We are left with proving that this functor is a localization, as well as showing that under this localization the ∞ -category of product-preserving functors from Bispan(\mathbb{F}) corresponds precisely to that of commutative semirings. This is the content of §4. We deduce the first statement as a special case of a general recognition theorem for localizations which we believe may be of independent interest:

Theorem C. Let $f: \mathcal{C} \to \mathcal{C}'$ be a functor, and assume we have equipped \mathcal{C} and \mathcal{C}' with factorization systems (E, M) and (E', M'), respectively, such that

(1) f restricts to a localization $E \to E'$ at some class $W \subset E$, and

(2) f restricts to a right fibration $M \to M'$.

Then $f: \mathcal{C} \to \mathcal{C}'$ is also a localization at W.

Future work

In follow-up work, we intend to generalize the main theorem of this paper to the more general setting of *parametrized higher algebra*. For simplicity, let us merely explain the statement in the *G*-equivariant setting for a finite group *G*. Instead of working with ∞ -categories, one should now work with G- ∞ *categories*, defined as contravariant functors $\operatorname{Orb}_{G}^{\operatorname{op}} \to \operatorname{Cat}_{\infty}$ from the *orbit category* of *G*. The statement and proof strategy for Theorem A can be generalized to this setting:

- ► For a G-∞-category D admitting so-called *finite G-products*, there is a G-∞-category CMon(D) of G-commutative monoids in D defined by [Nar16].
- ► If D is presentably G-symmetric monoidal, this G-∞-category inherits a G-symmetric monoidal structure, and so we can consider the ∞-category

$$\mathsf{CRig}_G(\mathfrak{D}) := \mathsf{CAlg}_G(\underline{\mathsf{CMon}}_G(\mathfrak{D}))$$

of G-commutative semirings in \mathfrak{D} .

► Every cartesian closed presentable ∞ -category \mathscr{C} gives rise to a presentably *G*-symmetric monoidal *G*- ∞ -category $\underline{\mathscr{C}}(G/H) := \operatorname{Fun}(\operatorname{Orb}_{H}^{\operatorname{op}}, \mathscr{C})$, and the same strategy as for Theorem A will show that there exists an equivalence

$$\operatorname{CRig}_{G}(\underline{\mathscr{C}}) \simeq \operatorname{Fun}^{\times}(\operatorname{Bispan}(\mathbb{F}_{G}), \mathscr{C}),$$

where $Bispan(\mathbb{F}_G)$ denotes the (2, 1)-category of bispans of finite *G*-sets.

Product-preserving functors from **Bispan**(\mathbb{F}_G) to sets that are also grouplike in an appropriate sense are one definition of *Tambara functors*, see e.g. [Str12, Definition 6.2]. Adding the same group-like condition, we can therefore interpret this result as saying that the ∞ -category of *G*-commutative rings in $\underline{\mathscr{C}}$ is equivalent to the ∞ -category of \mathscr{C} -valued Tambara functors. In particular, we expect to obtain a description of connective *G*-commutative ring spectra as Tambara functors valued in spaces,

$$\mathsf{CAlg}_G(\underline{\mathsf{Sp}}_G^{\geq 0}) \simeq \mathsf{Fun}^{\times}(\mathsf{Bispan}(\mathbb{F}_G), \mathsf{Spc})_{\mathrm{grp}}.$$

As some of the required foundational material on parametrized higher category theory still needs to be developed, we postpone a precise treatment of these results to a future article.

Notation

• We write \mathbb{F} for the category of finite sets, and

$$\mathbf{n} := \{1, ..., n\}$$

for a set with *n* elements.

- ► We write Cat_∞ for the ∞-category of ∞-categories and Spc for the ∞category of spaces or ∞-groupoids.
- If C is an ∞-category, then we usually denote its underlying ∞-groupoid by C[~]; in a few instances it will instead by notationally convenient to denote it by C_{eq}, however.
- ▶ We denote generic ∞-categories as A, B, C,
- ▶ We denote the arrow ∞-category of \mathscr{C} as $Ar(\mathscr{C}) := Fun([1], \mathscr{C})$.
- ▶ A subcategory \mathscr{C}_0 of an ∞-category \mathscr{C} is a functor $i: \mathscr{C}_0 \to \mathscr{C}$ such that $\mathscr{C}_0^{\simeq} \to \mathscr{C}^{\simeq}$ and $\mathsf{Map}_{\mathscr{C}_0}(x, y) \to \mathsf{Map}_{\mathscr{C}}(i(x), i(y))$ for all x, y are all monomorphisms of ∞-groupoids. In other words, subcategories are by definition always "replete", meaning that they must contain all equivalences among their objects. A subcategory is *wide* if it contains all objects, or equivalently if $\mathscr{C}_0^{\simeq} \to \mathscr{C}^{\simeq}$ is an equivalence.

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2 Background

In this section we recall some background material: We review spans in §2.1 and bispans in §2.2, and then look briefly at symmetric monoidal ∞ -categories in general in §2.3 and the special cases of (co)cartesian symmetric monoidal structures in §2.4.

2.1 Spans

In this subsection we briefly recall some definitions and results relating to ∞ categories of spans. These were originally constructed in [Bar17] using quasicategories, though our primary reference will be the model-independent reworking in [HHLN23].

Definition 2.1.1. A span pair $(\mathcal{C}, \mathcal{C}_F)$ consists of an ∞ -category \mathcal{C} together with a wide subcategory \mathcal{C}_F , whose morphisms we call the *forward* morphisms, such that:

(I) for any forward morphism $f: x \to y$ and any morphism $g: z \to y$ there exists a pullback square

$$\begin{array}{c} w \xrightarrow{f'} z \\ g' \downarrow & \downarrow g \\ x \xrightarrow{f} y, \end{array}$$

in C,

(2) and in this pullback square the morphism f' again lies in \mathscr{C}_F .

We write SpanPair for the ∞ -category of span pairs; a morphism $(\mathcal{C}, \mathcal{C}_F) \rightarrow (\mathfrak{D}, \mathfrak{D}_F)$ here is a functor $\mathcal{C} \rightarrow \mathfrak{D}$ that preserves the forward maps as well as pullbacks along forward maps.

Remark 2.1.2. The setup in [Bar17] and [HHLN23] is a bit more general than this, and uses instead *adequate triples* ($\mathcal{C}_F, \mathcal{C}_B$) where we specify subcategories of both forward and backward maps. Here we will only consider span pairs, as this simplifies the notation and encompasses all the examples we will encounter in this paper.

Observation 2.1.3. The ∞ -category SpanPair has limits and filtered colimits, which are both computed in Cat_{∞}, by [HHLN23, 2.4].

Example 2.1.4. For any ∞ -category \mathcal{C} , we always have the span pair $(\mathcal{C}, \mathcal{C}^{\simeq})$.

Example 2.1.5. If \mathcal{C} is an ∞ -category with pullbacks, then $(\mathcal{C}, \mathcal{C})$ is a span pair.

Notation 2.1.6. We write Cat_{∞}^{pb} for the subcategory of Cat_{∞} consisting of ∞ -categories with pullbacks and functors that preserve these; the functor $Cat_{\infty}^{pb} \rightarrow$ SpanPair, $\mathscr{C} \mapsto (\mathscr{C}, \mathscr{C})$ identifies it as a full subcategory of SpanPair that is closed under limits and filtered colimits.

Given a span pair $(\mathcal{C}, \mathcal{C}_F)$, we can construct an ∞ -category

$$\operatorname{Span}_{F}(\mathscr{C}) = \operatorname{Span}(\mathscr{C}, \mathscr{C}_{F}),$$

which informally has the same objects as C, with a morphism from x to y given by a *span* (or *correspondence*)



where f is in \mathcal{C}_F and b is arbitrary; composition is given by taking pullbacks in \mathcal{C} . See [HHLN23, 2.12] for a definition of Span_{*F*}(\mathcal{C}) as a complete Segal space, which gives a functor

Span: SpanPair
$$\longrightarrow$$
 Cat _{∞} .

Given $\mathscr{C} \in \mathsf{Cat}^{\mathsf{pb}}_{\infty}$, for the span pair $(\mathscr{C}, \mathscr{C}_F) = (\mathscr{C}, \mathscr{C})$ we will write

$$\operatorname{Span}(\mathscr{C}) := \operatorname{Span}_F(\mathscr{C}).$$

We note some important properties of this functor:

- ▶ We have $\operatorname{Span}_{eq}(\mathscr{C}) = \operatorname{Span}(\mathscr{C}, \mathscr{C}^{\sim}) \simeq \mathscr{C}^{\operatorname{op}}[\operatorname{HHLN23}, 2.15].$
- ► The functor Span preserves limits in fact, on the larger ∞-category of adequate triples it has a left adjoint, given by the twisted arrow ∞-category [HHLN23, 2.18]

Observation 2.1.7. A morphism in $\text{Span}_F(\mathscr{C})$, that is a span



is invertible if and only if the maps f and g are invertible in \mathcal{C} ; see for instance [Hau18, 8.2] for a proof.

Next, we recall two useful results relating spans and (co)cartesian fibrations:

Theorem 2.1.8 (Barwick). Suppose \mathcal{C} is an ∞ -category with pullbacks and

$$\Phi \colon \mathscr{C}^{\mathrm{op}} \longrightarrow \mathsf{Cat}_{\infty}$$

is a functor such that

- for every morphism $f: x \to y$ in \mathcal{C} , the functor $f^* := \Phi(f)$ has a left adjoint f_i ,
- ► for every pullback square

$$\begin{array}{c} w \xrightarrow{f'} z \\ \downarrow^{g'} & \downarrow^{g} \\ x \xrightarrow{f} y, \end{array}$$

in *C*, the Beck–Chevalley transformation

$$f'_{!}g'^{*} \longrightarrow g^{*}f_{!}$$

is an equivalence.

Let $p: \mathcal{C} \to \mathcal{C}$ be the cartesian fibration for Φ and write \mathcal{C}_{cart} for the subcategory of \mathcal{C} spanned by the *p*-cartesian edges. Then $(\mathcal{C}, \mathcal{C}_{cart})$ is a span pair, and moreover the functor

$$\operatorname{Span}(p)^{\operatorname{op}}$$
: $\operatorname{Span}_{\operatorname{cart}}(\mathscr{C})^{\operatorname{op}} \longrightarrow \operatorname{Span}(\mathscr{C})^{\operatorname{op}} \simeq \operatorname{Span}(\mathscr{C})$

is a cocartesian fibration for a functor $\text{Span}(\mathcal{C}) \to \text{Cat}_{\infty}$ that restricts to Φ on \mathcal{C}^{op} and is given on forward maps by taking left adjoints.

Proof. This is a special case of [Bar17, 11.6]; see also [HHLN23, 3.2 and 3.4] for further discussion.

Remark 2.1.9. In the situation of Theorem 2.1.8, the universal property of the $(\infty, 2)$ -category SPAN(\mathscr{C}) of spans in \mathscr{C} says that there is a unique functor of $(\infty, 2)$ -categories SPAN(\mathscr{C}) \rightarrow CAT $_{\infty}$ that extends Φ . We expect that its underlying functor of ∞ -categories corresponds to the cocartesian fibration Span $(p)^{\text{op}}$.

Theorem 2.1.10. Suppose $p: \mathcal{C} \to \mathcal{C}$ is the cartesian fibration for a functor $F: \mathcal{C}^{op} \to \operatorname{Cat}_{\infty}^{pb}$. Then the cocartesian fibration for the composite functor $\operatorname{Span} \circ F: \mathcal{C}^{op} \to \operatorname{Cat}$ is

 $\operatorname{Span}(p)$: $\operatorname{Span}_{\mathrm{fw}}(\mathscr{C}) \longrightarrow \operatorname{Span}_{\mathrm{eq}}(\mathscr{C}) \simeq \mathscr{C}^{\mathrm{op}}$,

where \mathcal{E}_{fw} contains the ("fibrewise") maps in \mathcal{E} that map to equivalences in \mathcal{C} under p.

Proof. This is a special case of [HHLN23, 3.9].

2.2 Bispans

We now recall the definition of ∞ -categories of *bispans*, following [EH23].

Definition 2.2.1 ([EH23, 2.4.3 and 2.4.6]). A *bispan triple* (\mathcal{C} , \mathcal{C}_F , \mathcal{C}_L) consists of an ∞ -category \mathcal{C} together with two wide subcategories \mathcal{C}_F and \mathcal{C}_L such that the following conditions hold:

- (I) Both $(\mathcal{C}, \mathcal{C}_F)$ and $(\mathcal{C}, \mathcal{C}_L)$ are span pairs.
- (2) Let $\mathscr{C}_{/x}^{L} \subseteq \mathscr{C}_{/x}$ be the full subcategory spanned by the maps to x that lie in \mathscr{C}_{L} ; for $f: x \to y$ in \mathscr{C}_{F} , the functor $f^{*}: \mathscr{C}_{/y}^{L} \to \mathscr{C}_{/x}^{L}$ given by pullback along f has a right adjoint f_{*} .
- (3) For every pullback square

$$\begin{array}{ccc} x' & \xrightarrow{f'} & y' \\ \xi & & & \downarrow^{\eta} \\ x & \xrightarrow{f} & y \end{array}$$

with f in \mathcal{C}_F , the commutative square

$$\begin{array}{ccc} \mathscr{C}_{/y}^{L} & \xrightarrow{f^{*}} & \mathscr{C}_{/x}^{L} \\ & & & \downarrow^{x} \\ & & & \downarrow^{x^{*}} \\ & & & & \downarrow^{x^{*}} \\ & & & & & \downarrow^{x^{*}} \\ & & & & & & \downarrow^{x^{*}} \end{array}$$

is right adjointable, i.e. the mate transformation $\eta^* f_* \to f'_* \xi^*$ is an equivalence.

Remark 2.2.2. If $C_L = C$, then condition (2) says precisely that C is locally cartesian closed; in this case, condition (3) is automatic.

Theorem 2.2.3 ([EH23, 2.5.2(I)]). Suppose $(\mathcal{C}, \mathcal{C}_F)$ is a span pair and suppose \mathcal{C}_L is a wide subcategory of \mathcal{C} . Then $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ is a bispan triple if and only if $(\operatorname{Span}_F(\mathcal{C})^{\operatorname{op}}, \mathcal{C}_L)$ is a span pair, where we regard \mathcal{C}_L as contained in the subcategory $\mathcal{C} \simeq \operatorname{Span}_{eq}(\mathcal{C})^{\operatorname{op}}$ inside $\operatorname{Span}_F(\mathcal{C})^{\operatorname{op}}$.

Definition 2.2.4. Suppose $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ is a bispan triple. Then we define

$$\mathsf{Bispan}_{FL}(\mathscr{C}) := \mathsf{Span}_{L}(\mathsf{Span}_{F}(\mathscr{C})^{\mathrm{op}}).$$

If $\mathcal{C}_L = \mathcal{C}$ we abbreviate this to $\text{Bispan}_F(\mathcal{C})$, and if also $\mathcal{C}_F = \mathcal{C}$ we just write $\text{Bispan}(\mathcal{C})$.

Remark 2.2.5. The fact that we get the correct composition of bispans by viewing them as "spans in spans" was first observed by Street [Str20]. The first (and rather different) construction of $Bispan(\mathbb{F})$ as an ∞ -category is in the thesis of Cranch [Cra10].

Definition 2.2.6. If $(\mathscr{C}, \mathscr{C}_F, \mathscr{C}_L)$ and $(\mathfrak{D}, \mathfrak{D}_F, \mathfrak{D}_L)$ are bispan triples, then a *morphism of bispan triples* is a functor $\Phi \colon \mathscr{C} \to \mathfrak{D}$ that induces morphisms of span pairs $(\mathscr{C}, \mathscr{C}_F) \to (\mathfrak{D}, \mathfrak{D}_F), (\mathscr{C}, \mathscr{C}_L) \to (\mathfrak{D}, \mathfrak{D}_L)$ and

$$(\operatorname{\mathsf{Span}}_F(\mathscr{C})^{\operatorname{op}}, \mathscr{C}_L) \longrightarrow (\operatorname{\mathsf{Span}}_F(\mathfrak{D})^{\operatorname{op}}, \mathfrak{D}_L).$$

Assuming the previous conditions, the final condition is equivalent to the square

$$\begin{array}{c} \mathscr{C}_{/y}^{L} \xrightarrow{f^{*}} \mathscr{C}_{/x}^{L} \\ \downarrow^{\Phi} & \downarrow^{\downarrow} \\ \mathfrak{D}_{/\Phi(y)}^{L} \xrightarrow{\Phi(f)^{*}} \mathfrak{D}_{/\Phi(x)}^{L} \end{array}$$

being right adjointable for every morphism $f: x \to y$ in \mathcal{C}_F , i.e. the Beck-Chevalley transformation

$$\Phi \circ f_* \longrightarrow \Phi(f)_* \circ \Phi$$

is an equivalence; this follows from the identification of pullbacks in $\text{Span}_F(\mathscr{C})^{\text{op}}$ of maps $\mathscr{C} \subset \text{Span}_F(\mathscr{C})^{\text{op}}$ along forwards and backwards morphisms in $\text{Span}_F(\mathscr{C})^{\text{op}}$ with pullbacks in \mathscr{C} and *distributivity diagrams* in \mathscr{C} respectively, see [EH23, 2.5.10, 2.5.12].

2.3 Symmetric monoidal ∞-categories

In this subsection we recall some definitions related to commutative monoids and symmetric monoidal structures on ∞ -categories.

Definition 2.3.1. Let \mathcal{C} be an ∞ -category with finite products. A *commutative monoid* in \mathcal{C} is a functor

 $M: \operatorname{Span}(\mathbb{F}) \longrightarrow \mathscr{C}$

that preserves finite products. We write $CMon(\mathcal{C})$ for the full subcategory of $Fun(Span(\mathbb{F}), \mathcal{C})$ spanned by the commutative monoids.

Remark 2.3.2. This definition of commutative monoids fits into the framework for algebraic structures defined by Segal conditions from [CH21]: We can endow $\text{Span}(\mathbb{F})$ with the structure of an *algebraic pattern* where the inertactive factorization system is that given by the backwards and forwards maps, and the point is the only elementary object. Then a Segal Span(\mathbb{F})-object in \mathcal{C} is a functor M such that

$$M(\mathbf{n}) \xrightarrow{\sim} \lim_{(\{1\}_{/\mathbf{n}})^{\mathrm{op}}} M(1) \simeq \prod_{i=1}^{n} M(1)$$

for every n.

Remark 2.3.3. This definition of commutative monoids is equivalent to that used in [Lur17] in terms of finite pointed sets; see for instance [BH21, C.1].

Definition 2.3.4. A symmetric monoidal ∞ -category is a commutative monoid in Cat_{∞}; its *underlying* ∞ -category is the value at 1. Given a symmetric monoidal structure on an ∞ -category \mathcal{C} , we will denote the corresponding cocartesian and cartesian fibrations by²

$$\mathscr{C}^{\otimes} \longrightarrow \operatorname{Span}(\mathbb{F}), \quad \mathscr{C}_{\otimes} \longrightarrow \operatorname{Span}(\mathbb{F})^{\operatorname{op}}.$$

We say that a morphism in \mathscr{C}^{\otimes} is *inert* if it is cocartesian over a backwards morphism in Span(\mathbb{F}); similarly, a morphism in \mathscr{C}_{\otimes} is *inert* if it is cartesian over a (reversed) backwards morphism in Span(\mathbb{F})^{op}.

Definition 2.3.5. Suppose $\mathscr{C}^{\otimes}, \mathfrak{D}^{\otimes} \to \text{Span}(\mathbb{F})$ are symmetric monoidal ∞ -categories. A symmetric monoidal functor from \mathscr{C} to \mathfrak{D} is a commutative triangle



where *F* preserves cocartesian morphisms. We say that *F* is *lax symmetric monoidal* if it instead only preserves inert morphisms. We write $\operatorname{Fun}_{/\operatorname{Span}(\mathbb{F})}^{\operatorname{LSM}}(\mathscr{C}^{\otimes}, \mathfrak{D}^{\otimes})$ for the full subcategory of $\operatorname{Fun}_{/\operatorname{Span}(\mathbb{F})}(\mathscr{C}^{\otimes}, \mathfrak{D}^{\otimes})$ spanned by the lax symmetric monoidal functors.

Remark 2.3.6. It follows from [BHS22, 5.1.15] that this definition of lax symmetric monoidal functors agrees with the more standard one, with \mathbb{F}_* in place of Span(\mathbb{F}). By [Hau23, 2.2.7 and 2.2.10], we can also equivalently define a lax symmetric monoidal functor to be a commutative triangle as above where *F* preserves finite products.

Definition 2.3.7. A *commutative algebra* in a symmetric monoidal ∞ -category \mathscr{C}^{\otimes} is a lax symmetric monoidal functor from $*^{\otimes} = \text{Span}(\mathbb{F})$; we write

$$\mathsf{CAlg}(\mathscr{C}) := \mathsf{Fun}^{\mathrm{LSM}}_{/\mathsf{Span}(\mathbb{F})}(\mathsf{Span}(\mathbb{F}), \mathscr{C}^{\otimes})$$

for the ∞ -category of commutative algebras in C.

²Here $\text{Span}(\mathbb{F})^{\text{op}} \simeq \text{Span}(\mathbb{F})$, but we write op as a reminder that this is a cartesian fibration.

2.4 Cartesian and cocartesian symmetric monoidal structures

In this subsection we review some results from [Lur17] on cartesian and cocartesian symmetric monoidal structures, meaning those arising from products and coproducts in an ∞ -category. Let us first recall a precise definition of such structures:

Definition 2.4.1 ([Lur17, 2.4.0.1]). Let \mathscr{C}^{\otimes} be a symmetric monoidal ∞ -category. We say that this is *cocartesian* if

- ▶ the unit 1 in *C* is initial,
- ▶ for all objects $X, Y \in \mathcal{C}$, the maps

 $X\simeq X\otimes \mathbb{1} \longrightarrow X\otimes Y \longleftarrow \mathbb{1}\otimes Y\simeq Y$

exhibit $X \otimes Y$ as the coproduct of X and Y.

Dually, \mathcal{C}^{\otimes} is cartesian if

- the unit $\mathbb{1}$ in \mathcal{C} is terminal,
- ▶ for all objects $X, Y \in \mathcal{C}$, the maps

 $X\simeq X\otimes \mathbb{1} \longleftrightarrow X\otimes Y \longrightarrow \mathbb{1}\otimes Y\simeq Y$

exhibit $X \otimes Y$ as the product of X and Y.

Remark 2.4.2. As the unit $\mathbb{1}_{\mathscr{C}}$ is simply the image of the essentially unique object of $\mathscr{C}(\mathbf{0}) \simeq *$ under the functor $\mathscr{C}(\mathbf{0} = \mathbf{0} \to \mathbf{1})$, in the cocartesian case the first condition is equivalent to demanding that $\mathscr{C}(\mathbf{0} = \mathbf{0} \to \mathbf{1})$ is a left adjoint. If in this case also $\mathscr{C}(\mathbf{2} = \mathbf{2} \to \mathbf{1})$ is a left adjoint, then the second condition follows, because $(X, \mathbb{1}) \to (X, Y) \leftarrow (\mathbb{1}, Y)$ is a coproduct diagram in $\mathscr{C}(\mathbf{2})$. Conversely, it will follow from Proposition 3.1.1 that all forward maps in a cocartesian symmetric monoidal structure are left adjoint to the corresponding backwards map.

Theorem 2.4.3 (Lurie). Suppose C is an ∞ -category with finite coproducts. Then there is a unique cocartesian symmetric monoidal ∞ -category C^{II} such that $C_1^{II} \simeq C$. Dually, if C is an ∞ -category with finite products, then there is a unique cartesian symmetric monoidal ∞ -category C^{\times} such that $C_1^{\times} \simeq C$.

Proof. This is part of [Lur17, 2.4.1.8 and 2.4.3.12]. □

In the cartesian case, we can further describe lax symmetric monoidal functors to \mathcal{C}^{\times} in terms of certain functors to \mathcal{C} : **Definition 2.4.4.** Suppose $\mathfrak{D}^{\otimes} \to \mathsf{Span}(\mathbb{F})$ is a symmetric monoidal ∞ -category and \mathscr{C} is an ∞ -category with finite products. A \mathfrak{D}^{\otimes} -monoid in \mathscr{C} is a functor

$$M\colon \mathfrak{D}^{\otimes} \longrightarrow \mathscr{C}$$

such that for an object $X \in \mathfrak{D}^{\otimes}$ over $\mathbf{n} \in \text{Span}(\mathbb{F})$, if $\rho_i \colon \mathbf{1} \to \mathbf{n}$ for i = 1, ..., n are the summand inclusions, viewed as backwards maps in $\text{Span}(\mathbb{F})$, then

$$M(X) \xrightarrow{\sim} \prod_{i=1}^{n} M(\rho_{i,!}X)$$

We write $Mon_{\mathfrak{D}^{\otimes}}(\mathfrak{C})$ for the full subcategory of $Fun(\mathfrak{D}^{\otimes},\mathfrak{C})$ spanned by the monoids.

Remark 2.4.5. By [Hau23, 2.3.3], we can equivalently characterize \mathfrak{D}^{\otimes} -monoids in \mathfrak{C} as functors $\mathfrak{D}^{\otimes} \to \mathfrak{C}$ that preserve finite products.

Theorem 2.4.6 (Lurie). Suppose \mathcal{C} is an ∞ -category with finite products. Then there is a functor $\mathcal{C}^{\times} \to \mathcal{C}$ that induces an equivalence

$$\mathsf{Fun}^{\mathrm{LSM}}_{/\mathsf{Span}(\mathbb{F})}(\mathfrak{D}^{\otimes}, \mathfrak{C}^{\times}) \xrightarrow{\sim} \mathsf{Mon}_{\mathfrak{D}^{\otimes}}(\mathfrak{C})$$

between lax symmetric monoidal functors and monoids.

Proof. This is the content of [Lur17, $\S2.4.1$], translated through the equivalence between ∞ -operads over \mathbb{F}_* and Span(\mathbb{F}) from [BHS22].

Applied to $\mathfrak{D}^{\otimes} = *^{\otimes}$, the previous theorem gives the following corollary:

Corollary 2.4.7. Let \mathcal{C} be an ∞ -category with finite products. Then the functor $\mathcal{C}^{\times} \to \mathcal{C}$ from Theorem 2.4.6 induces an equivalence $\mathsf{CAlg}(\mathcal{C}^{\times}) \simeq \mathsf{CMon}(\mathcal{C})$.

3 Commutative semirings and spans

Our goal in this section is to give an explicit description of commutative semirings, defined as commutative algebras in the symmetric monoidal structure on commutative monoids constructed by Gepner–Groth–Nikolaus. We first obtain a concrete construction of the fibration $\text{Span}(\mathbb{F})^{\otimes} \rightarrow \text{Span}(\mathbb{F})$ for the symmetric monoidal structure on spans induced by the cartesian product of finite sets, by first describing (co)cartesian symmetric monoidal structures in terms of spans in §3.1 and then describing symmetric monoidal structures on ∞ -categories of spans in §3.2. In §3.3 we then prove, following Ben–Moshe and Schlank, that the symmetric monoidal structure on commutative monoids is a localization of the Day convolution for this symmetric monoidal structure on $\text{Span}(\mathbb{F})$; from this we then get the desired description of commutative semirings in \mathscr{C} as lax symmetric monoidal functors from $\text{Span}(\mathbb{F})$.

3.1 (Co)cartesian symmetric monoidal structures via spans

In this subsection we will see that the cocartesian fibration for a cocartesian symmetric monoidal structure can be described in terms of spans. Moreover, so can the cartesian fibration for a cartesian symmetric monoidal structure, in a way that is *not* simply dual to the first description.³

Proposition 3.1.1. Suppose \mathcal{C} is an ∞ -category with finite coproducts, and let

$$p: \mathscr{C}_{\mathbb{F}} \longrightarrow \mathbb{F}$$

denote the cartesian fibration for the functor

$$\mathscr{C}^{(-)} \colon \mathbb{F}^{\mathrm{op}} \to \mathsf{Cat}_{\infty}, \quad S \mapsto \mathsf{Fun}(S, \mathscr{C}).$$

Then $\text{Span}_{cart}(\mathfrak{C}_{\mathbb{F}})^{op} \to \text{Span}(\mathbb{F})^{op} \simeq \text{Span}(\mathbb{F})$ *is the cocartesian fibration for the cocartesian symmetric monoidal structure on* \mathfrak{C} *.*

Proof. Because \mathscr{C} admits coproducts, the functor $\mathscr{C}^{(-)}$: $\mathbb{F}^{\text{op}} \to \text{Cat}$ satisfies the assumptions of Theorem 2.1.8, showing that $(\mathscr{C}_{\mathbb{F}}, \mathscr{C}_{\mathbb{F},\text{cart}})$ is a span pair and that the functor

$$\operatorname{Span}_{\operatorname{cart}}(\mathscr{C}_{\mathbb{F}})^{\operatorname{op}} \longrightarrow \operatorname{Span}(\mathbb{F})$$

is a cocartesian fibration.

Furthermore, as explained there, the restriction of the corresponding functor $\text{Span}(\mathbb{F}) \rightarrow \text{Cat}$ to $\mathbb{F}^{\text{op}} \subset \text{Span}(\mathbb{F})$ agrees with the original functor $\mathscr{C}^{(-)}$, and so $\text{Span}_{\text{cart}}(\mathscr{C}_{\mathbb{F}})^{\text{op}}$ is a symmetric monoidal ∞ -category, while the restriction to $\mathbb{F} \subset \text{Span}(\mathbb{F})$ agrees with the functor obtained from $\mathscr{C}^{(-)}$ by passing to left adjoints. In particular, it is cocartesian monoidal by Remark 2.4.2.

Corollary 3.1.2. Let $Ar(\mathbb{F})_{pb}$ denote the subcategory of $Ar(\mathbb{F})$ whose morphisms are the commutative squares that are pullbacks. Then

$$ev_1: \operatorname{Span}_{\operatorname{pb}}(\operatorname{Ar}(\mathbb{F}))^{\operatorname{op}} \longrightarrow \operatorname{Span}(\mathbb{F})$$

is the cocartesian fibration for the cocartesian symmetric monoidal structure on \mathbb{F} .

Proof. By Proposition 3.1.1 it suffices to show that $ev_1 : \operatorname{Ar}(\mathbb{F}) \to \mathbb{F}$ is the cartesian fibration for the functor $\mathbb{F}^{(-)} : \mathbb{F}^{\operatorname{op}} \to \operatorname{Cat}_{\infty}$. But ev_1 is the cartesian fibration for $S \mapsto \mathbb{F}_{/S}$, with functoriality given by pullbacks, so this follows from the natural straightening equivalence $\mathbb{F}_{/S} \simeq \operatorname{Fun}(S, \mathbb{F})$, under which pullbacks correspond to compositions.

We now give an explicit description of the cartesian fibration for a cartesian symmetric monoidal structure:

³Such a dual description does also exist, but this is in terms of *cospans* rather than spans.

Proposition 3.1.3. Suppose C is an ∞ -category with finite products. Then

$$\mathsf{Span}_{\mathsf{cart}}(\mathscr{C}_{\mathbb{F}})^{\mathrm{op}} \longrightarrow \mathsf{Span}(\mathbb{F})^{\mathrm{op}}$$

is a cartesian fibration, which classifies the cartesian symmetric monoidal structure on ℃. That is, **Span_{cart}(℃**_F)^{op} ≃ ℃×.

Proof. Assume first that \mathscr{C} in addition has finite coproducts, so we can apply Proposition 3.1.1 to describe the cocartesian unstraightening of the *co*cartesian symmetric monoidal structure on \mathscr{C} as $\text{Span}_{cart}(\mathscr{C}_{\mathbb{F}})^{op} \to \text{Span}(\mathbb{F})$.

Because \mathscr{C} also admits finite products, the corresponding functor $\text{Span}(\mathbb{F}) \rightarrow \text{Cat}$ sends every map to a left adjoint, and so $\text{Span}_{cart}(\mathscr{C}_{\mathbb{F}}) \rightarrow \text{Span}(\mathbb{F})$ is also a cartesian fibration, and as a cartesian fibration it unstraightens to the diagram $\text{Span}(\mathbb{F})^{\text{op}} \rightarrow \text{Cat}$ obtained by passing to right adjoints. On the backward maps this functor thus recovers $\mathscr{C}^{(-)} \colon \mathbb{F}^{\text{op}} \rightarrow \text{Cat}_{\infty}$, since we have taken left adjoints and then right adjoints for this diagram. The new functor therefore also preserves products, as this condition only depends on the backwards maps, and so $\text{Span}_{cart}(\mathscr{C}_{\mathbb{F}}) \rightarrow \text{Span}(\mathbb{F})$ is the cartesian unstraightening of a symmetric monoidal ∞ -category. By the dual of Remark 2.4.2 it is moreover cartesian. This completes the proof under the additional assumption that \mathscr{C} also has finite coproducts.

For general \mathcal{C} , let \mathfrak{D} denote the finite-coproduct-completion of \mathcal{C} , i.e. the full subcategory of the ∞ -category of presheaves on \mathcal{C} spanned by finite coproducts of representables. As \mathcal{C} has finite products, and since coproducts and products distribute in spaces, \mathfrak{D} also has finite products, and the Yoneda embedding $\mathcal{C} \hookrightarrow \mathfrak{D}$ preserves finite products.

By the above special case, $\operatorname{Span}_{\operatorname{cart}}(\mathfrak{D}_{\mathbb{F}})^{\operatorname{op}} \to \operatorname{Span}(\mathbb{F})$ is then the cartesian fibration for the cartesian symmetric monoidal on on \mathfrak{D} . As the inclusion $\mathscr{C} \hookrightarrow \mathfrak{D}$ preserves products, it extends to a fully faithful functor $\mathscr{C}_{\times} \hookrightarrow \mathfrak{D}_{\times} = \operatorname{Span}_{\operatorname{cart}}(\mathfrak{D}_{\mathbb{F}})^{\operatorname{op}}$ with essential image those tuples (X_1, \ldots, X_n) such that all X_i are contained in \mathscr{C} . As the natural inclusion $\operatorname{Span}_{\operatorname{cart}}(\mathscr{C}_{\mathbb{F}})^{\operatorname{op}} \to \operatorname{Span}_{\operatorname{cart}}(\mathfrak{D}_{\mathbb{F}})^{\operatorname{op}}$ is also fully faithful with the same essential image, the claim follows.

Combining this with Corollary 3.1.2, we get the following special case:

Corollary 3.1.4. There is an equivalence of cartesian fibrations

$$\operatorname{Span}_{\operatorname{pb}}(\operatorname{Ar}(\mathbb{F}))^{\operatorname{op}} \simeq \mathbb{F}_{\times}$$

over $\text{Span}(\mathbb{F})^{\text{op}}$ (using evaluation at 1 in the arrow category).

3.2 Symmetric monoidal structures on spans

In this subsection we discuss symmetric monoidal structures on span ∞ -categories, and in particular on Span(\mathbb{F}). These symmetric monoidal structures were first constructed in [BGS20].

Construction 3.2.1. Since the functor **Span:** SpanPair \rightarrow Cat_{∞} preserves limits, and in particular finite products, it also preserves commutative monoids. Therefore, **Span** applied to any commutative monoid in **SpanPair** (or more generally in adequate triples) is a symmetric monoidal ∞ -category. Here we will only consider this construction for objects of Cat^{pb}_{∞} \subseteq SpanPair. A commutative monoid in Cat^{pb}_{∞} is a symmetric monoidal ∞ -category M: Span(\mathbb{F}) \rightarrow Cat_{∞} such that

- ▶ the underlying ∞-category $\mathscr{C} = M(1)$ has pullbacks,
- ▶ the tensor product functor $\otimes -: \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ preserves pullbacks.⁴

In this case, Span(M): $\text{Span}(\mathbb{F}) \to \text{Cat}_{\infty}$ is a commutative monoid, which endows $\text{Span}(\mathscr{C})$ with a symmetric monoidal structure whose tensor product is inherited from \mathscr{C} .

Example 3.2.2. If $M: \text{Span}(\mathbb{F}) \to \text{Cat}_{\infty}$ is the cartesian symmetric monoidal structure on an ∞ -category \mathscr{C} with finite limits, then $-\times -: \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ clearly preserves pullbacks, and so induces a symmetric monoidal structure on $\text{Span}(\mathscr{C})$.⁵ For example, we conclude that $\text{Span}(\mathbb{F})$ is endowed with a symmetric monoidal structure given by the cartesian product in \mathbb{F} .

We can describe the cocartesian fibrations for these symmetric monoidal structures rather explicitly:

Proposition 3.2.3. Suppose $p: \mathcal{C}_{\otimes} \to \text{Span}(\mathbb{F})^{\text{op}}$ is the cartesian fibration for a commutative monoid in $\text{Cat}_{\infty}^{\text{pb}}$. Then the cocartesian fibration $\text{Span}(\mathcal{C})^{\otimes} \to \text{Span}(\mathbb{F})$ for the induced symmetric monoidal structure on spans from Construction 3.2.1 is given by

$$\operatorname{Span}(\mathscr{C})^{\otimes} \simeq \operatorname{Span}_{\mathrm{fw}}(\mathscr{C}_{\otimes}),$$

where $(\mathcal{C}_{\otimes})_{\mathrm{fw}}$ denotes the subcategory of maps that go to equivalences under p.

Proof. This follows immediately from Theorem 2.1.10.

Remark 3.2.4. When the symmetric monoidal structure on \mathscr{C} is cartesian, the description of Span $(\mathscr{C})^{\otimes}$ above shows that it agrees with the symmetric monoidal structure on Span (\mathscr{C}) constructed as a special case of [BGS20, 2.14].

Combining Proposition 3.2.3 with the description of \mathbb{F}_{\times} from Corollary 3.1.4, we get the following special case:

⁴Here we mean pullbacks in $\mathscr{C} \times \mathscr{C}$, so that the tensor product of a pair of pullback squares in \mathscr{C} is again a pullback. In particular, this condition is stronger than \otimes preserving pullbacks in each variable.

⁵Note that this is *not* the cartesian monoidal structure on $Span(\mathscr{C})!$

Corollary 3.2.5. The symmetric monoidal structure on $\text{Span}(\mathbb{F})$ induced by the cartesian symmetric monoidal structure on \mathbb{F} is given by

$$\text{Span}(\mathbb{F})^{\otimes} \simeq \text{Bispan}_{pb,tgt=eq}(\text{Ar}(\mathbb{F})) = \text{Span}_{tgt=eq}(\text{Span}_{pb}(\text{Ar}(\mathbb{F}))^{op})$$

where $\operatorname{Ar}(\mathbb{F})_{pb}$ denotes the subcategory of $\operatorname{Ar}(\mathbb{F})$ where the morphisms are pullback squares, and $\operatorname{Span}(\operatorname{Ar}(\mathbb{F}))_{tgt=eq}$ denotes the subcategory of morphisms whose image under ev_1 is a span of equivalences in \mathbb{F} .

Remark 3.2.6. A morphism in $\text{Span}(\mathbb{F})^{\otimes}$ is then a diagram

which simplifies to

This amounts to specifying a family of spans

$$\prod_{t \in g^{-1}(s')} X_{f(t)} \longleftarrow X'_{s'} \longrightarrow X''_{s'}$$

indexed by $s' \in S'$, as we expect.

Remark 3.2.7. For later use, let us identify some of the cocartesian edges of the cocartesian fibration p: $Bispan_{pb,tgt=eq}(Ar(\mathbb{F})) \rightarrow Span(\mathbb{F})$. By construction, the restriction of the cartesian fibration $Span_{pb}(Ar(\mathbb{F}))^{op} \rightarrow Span(\mathbb{F})$ to \mathbb{F} is the cartesian fibration $Ar(\mathbb{F}) \rightarrow \mathbb{F}$, whose cartesian arrows are precisely the pullback squares in \mathbb{F} . Applying the characterization of cocartesian edges from [HHLN23, 3.2], we therefore see that the *p*-cocartesian lifts of a map

$$S \xleftarrow{f} T \xrightarrow{=} T$$

in $\text{Span}(\mathbb{F})$ are precisely given by the bispans of the form

$$\begin{array}{c} X \longleftarrow Y = Y = Y \\ \downarrow & \downarrow \\ S \leftarrow f \end{array} \begin{array}{c} Y = Y \\ \downarrow & \downarrow \\ T = T \end{array} \begin{array}{c} Y \\ \downarrow \\ T = T \end{array}$$

Observation 3.2.8. Evaluation at 0 gives a functor

$$\text{Span}(\mathbb{F})^{\otimes} \simeq \text{Bispan}_{pb,tgt=eq}(\text{Ar}(\mathbb{F})) \longrightarrow \text{Bispan}(\mathbb{F}).$$

We will prove in Corollary 4.2.3 below that this is a localization.

3.3 Commutative monoids and Day convolution

In this subsection we briefly recall the definition of commutative semirings from $[GGN_{15}]$ and then study its relation to the symmetric monoidal structure on $Span(\mathbb{F})$.

Definition 3.3.1. We say a symmetric monodial ∞ -category \mathscr{C} is *presentably* symmetric monoidal if \mathscr{C} is presentable and the tensor product $-\otimes -: \mathscr{C} \times \mathscr{C} \rightarrow \mathscr{C}$ preserves colimits in each variable separately.⁶ A presentably symmetric monoidal ∞ -category is equivalently a commutative monoid in Pr^L equipped with the Lurie tensor product.

Theorem 3.3.2 (Gepner–Groth–Nikolaus [GGN15]). Suppose \mathcal{C} is a presentably symmetric monoidal ∞ -category. Then CMon(\mathcal{C}) has a unique presentably symmetric monoidal structure such that the free monoid functor $\mathcal{C} \to \text{CMon}(\mathcal{C})$ is symmetric monoidal.

Definition 3.3.3. A *commutative semiring* in a presentably symmetric monoidal ∞ -category \mathscr{C} is a commutative algebra in CMon(\mathscr{C}). We write

 $CRig(\mathscr{C}) := CAlg(CMon(\mathscr{C}))$

for the ∞ -category of commutative semirings in $C.^7$

Recall that $\mathsf{CMon}(\mathscr{C})$ is by definition a full subcategory of $\mathsf{Fun}(\mathsf{Span}(\mathbb{F}), \mathscr{C})$. We will now prove that the symmetric monoidal structure of Theorem 3.3.2 is a localization of a Day convolution structure on this functor ∞ -category. To see this we first need to know that this Day convolution indeed localizes to the product-preserving functors. This is a special case of the following proposition, which is really just [BMS24, 4.24] stated in a more general form, but we include a proof for completeness.

Proposition 3.3.4 (Ben-Moshe–Schlank). Fix a set $\mathcal{R} \subset \operatorname{Cat}_{\infty}$ of ∞ -categories. Suppose \mathcal{I} is an ∞ -category with \mathcal{R} -shaped limits that is equipped with a symmetric monoidal structure such that for every $X \in \mathcal{I}$, the functor $X \otimes -$ preserves \mathcal{R} -shaped limits. Then for every presentably symmetric monoidal ∞ -category \mathcal{C} , the Day convolution structure on $\operatorname{Fun}(\mathcal{I}, \mathcal{C})$ localizes to a symmetric monoidal structure on the full subcategory $\operatorname{Fun}^{\mathcal{R}}(\mathcal{I}, \mathcal{C})$ of \mathcal{R} -limit-preserving functors.

Proof. We first consider the case where \mathscr{C} is the ∞ -category Spc of spaces, with the cartesian product. Let L: Fun $(\mathscr{I}, Spc) \to Fun^{\mathscr{R}}(\mathscr{I}, Spc)$ denote the localization functor. We must show that for any $\Phi \in Fun(\mathscr{I}, Spc)$, the functor $\Phi \otimes$ - preserves L-equivalences. Since L-equivalences are closed under colimits in the arrow ∞ -category, it suffices to show this when Φ is of the form y(X) for $X \in \mathscr{I}^{op}$,

 $^{^{6}}$ By the adjoint functor theorem for presentable ∞ -categories, this is equivalent to the symmetric monoidal structure being closed.

⁷Note that this notation is slightly abusive, as this ∞ -category really depends not just on \mathcal{C} , but on its symmetric monoidal structure.

where y is the Yoneda embedding, since these generate $\operatorname{Fun}(\mathcal{J}, \operatorname{Spc})$ under colimits. Also, for any Φ the statement is equivalent to: for $M \in \operatorname{Fun}^{\mathcal{R}}(\mathcal{J}, \operatorname{Spc})$, the internal hom M^{Φ} in $\operatorname{Fun}(\mathcal{J}, \operatorname{Spc})$ also preserves \mathcal{R} -shaped limits, as is immediate from the natural equivalence

$$Map(\Phi \otimes -, M) \simeq Map(-, M^{\Phi})$$

and the relation between *L*-equivalences and *L*-local objects.

Now we claim that we can describe $M^{y(X)}$ as $M(-\otimes X)$. Indeed, we know that $y: \mathcal{F}^{\mathrm{op}} \to \operatorname{Fun}(\mathcal{F}, \operatorname{Spc})$ is symmetric monoidal, so that we have natural equivalences

$$M^{y(X)}(-) \simeq \mathsf{Map}(y(-), M^{y(X)})$$
$$\simeq \mathsf{Map}(y(-) \otimes y(X), M)$$
$$\simeq \mathsf{Map}(y(- \otimes X), M)$$
$$\simeq M(- \otimes X).$$

Now by assumption $- \otimes X \colon \mathcal{F} \to \mathcal{F}$ preserves \mathcal{R} -shaped limits, hence so does the composite $M(- \otimes X)$.

For a general *C*, we have a commutative square

where the top equivalence is symmetric monoidal for the Day convolution by [BMS24, 3.10] and the bottom equivalence will be shown below. The result then follows from [BMS24, 4.21], which shows that the left map is again a symmetric monoidal localization.

It remains to show that the map

$$\operatorname{Fun}^{\mathscr{R}}(\mathscr{F},\operatorname{Spc})\otimes \mathscr{C} \to \operatorname{Fun}^{\mathscr{R}}(\mathscr{F},\mathscr{C})$$

is an equivalence. For this we may compute

$$\mathsf{Fun}^{\mathscr{R}}(\mathscr{I},\mathsf{Spc})\otimes \mathscr{C} \simeq \mathsf{Fun}^{\mathsf{R}}(\mathscr{C}^{\mathrm{op}},\mathsf{Fun}^{\mathscr{R}}(\mathscr{I},\mathsf{Spc}))$$
$$\simeq \mathsf{Fun}^{\mathscr{R}}(\mathscr{I},\mathsf{Fun}^{\mathsf{R}}(\mathscr{C}^{\mathrm{op}},\mathsf{Spc}))$$
$$\simeq \mathsf{Fun}^{\mathscr{R}}(\mathscr{I},\mathscr{C}),$$

where we have used that for any two presentable ∞ -categories \mathscr{C} and \mathfrak{D} the tensor product $\mathscr{C} \otimes \mathfrak{D}$ is equivalent to the ∞ -category Fun^R($\mathscr{C}^{op}, \mathfrak{D}$) of (small) limit-preserving functors from \mathscr{C}^{op} to \mathfrak{D} ; see [Lur17, 4.8.1.17].

Applying this to $\text{Span}(\mathbb{F})$, we get the following variant of [BMS24, 4.26]:

Corollary 3.3.5. Let \mathcal{C} be a presentably symmetric monoidal ∞ -category. The symmetric monoidal structure on CMon(\mathcal{C}) from [GGN15] is a localization of the Day convolution on Fun(Span(\mathbb{F}), \mathcal{C}) using the symmetric monoidal structure on Span(\mathbb{F}) induced by the cartesian product in \mathbb{F} .

Proof. The cartesian product in Span(\mathbb{F}) is given by the coproduct in \mathbb{F} , and the tensor product on Span(\mathbb{F}), given by the cartesian product in \mathbb{F} , indeed preserves this in each variable. Thus the assumptions of Proposition 3.3.4 are satisfied, so that the Day convolution on Fun(Span(\mathbb{F}), \mathcal{C}) localizes to CMon(\mathcal{C}), making this a presentably symmetric monoidal ∞ -category. It remains to show that this satisfies the condition that unquely characterizes the symmetric monoidal structure of [GGN15], namely that the free commutative monoid functor

$$F: \mathscr{C} \longrightarrow \mathsf{CMon}(\mathscr{C})$$

is symmetric monoidal. The functor F is by definition left adjoint to the forgetful functor $CMon(\mathcal{C}) \rightarrow \mathcal{C}$, which factors as

$$\mathsf{CMon}(\mathscr{C}) \hookrightarrow \mathsf{Fun}(\mathsf{Span}(\mathbb{F}), \mathscr{C}) \xrightarrow{u} \mathscr{C},$$

where u^* is evaluation at 1, i.e. restriction along $u: \{1\} \rightarrow \text{Span}(\mathbb{F})$. Hence *F* factors as

$$\mathscr{C} \xrightarrow{u_!} \operatorname{Fun}(\operatorname{Span}(\mathbb{F}), \mathscr{C}) \xrightarrow{L} \operatorname{CMon}(\mathscr{C}).$$

Here the localization *L* is symmetric monoidal for the localized Day convolution, and u_1 is symmetric monoidal by [BMS24, 3.6] since $u: * \rightarrow \text{Span}(\mathbb{F})$ is symmetric monoidal.

Using the universal property of Day convolution, we then immediately get the following description of commutative semirings:

Corollary 3.3.6. For any presentably symmetric monoidal ∞ -category \mathcal{C} , the ∞ -category $\mathsf{CRig}(\mathcal{C})$ of commutative semirings in \mathcal{C} is equivalent to the full subcategory of $\mathsf{Fun}^{\mathsf{LSM}}_{/\mathsf{Span}(\mathbb{F})}(\mathsf{Span}(\mathbb{F})^{\otimes}, \mathbb{C}^{\otimes})$ spanned by those lax symmetric monoidal functors whose underlying functors $\mathsf{Span}(\mathbb{F}) \to \mathcal{C}$ are commutative monoids.

Proof. Since $\mathsf{CMon}(\mathscr{C})$ is a symmetric monoidal localization of $\mathsf{Fun}(\mathsf{Span}(\mathbb{F}), \mathscr{C})$, we have a full subcategory inclusion

$$\mathsf{CRig}(\mathscr{C}) = \mathsf{CAlg}(\mathsf{CMon}(\mathscr{C})) \subseteq \mathsf{CAlg}(\mathsf{Fun}(\mathsf{Span}(\mathbb{F}), \mathscr{C}))$$

whose image is spanned by the commutative algebras whose underlying object in Fun(Span(\mathbb{F}), \mathcal{C}) is a commutative monoid. Moreover, the universal property of Day convolution from [Gla16] (or [Lur17, §2.2.6]) identifies commutative algebras in Fun(Span(\mathbb{F}), \mathcal{C}) with lax symmetric monoidal functors Span(\mathbb{F})^{\otimes} $\rightarrow \mathcal{C}^{\otimes}$.

In the case where the symmetric monoidal structure on \mathcal{C} is cartesian, we can simplify this further:

Corollary 3.3.7. For any cartesian closed presentable ∞ -category \mathcal{C} , the ∞ -category **CRig**(\mathcal{C}) (using the cartesian product on \mathcal{C}) is equivalent to the full subcategory of Fun(Span(\mathbb{F})^{\otimes}, \mathcal{C}) spanned by the functors

 $F: \operatorname{Span}(\mathbb{F})^{\otimes} \longrightarrow \mathscr{C}$

such that:

- (1) *F* is a Span(\mathbb{F})^{\otimes}-monoid in the sense of Definition 2.4.4.
- (2) The restriction of F to $\text{Span}(\mathbb{F})$ is a commutative monoid in \mathcal{C} .

Proof. Combine Corollary 3.3.6 with the description of lax symmetric monoidal functors to C^{\times} as monoids from Theorem 2.4.6.

4 Comparison with bispans

In this section, we first prove a general localization result for ∞ -categories of spans in §4.1. We then apply this to our functor $\text{Span}(\mathbb{F})^{\otimes} \rightarrow \text{Bispan}(\mathbb{F})$ in §4.2 before we complete the proof of our comparison result for commutative semirings in §4.3.

4.1 Localizations of span ∞-categories

In this subsection we will prove the following general result about localizations of ∞ -categories of spans:

Theorem 4.1.1. Suppose ϕ : $(\mathcal{C}, \mathcal{C}_F) \to (\mathfrak{D}, \mathfrak{D}_F)$ is a functor of span pairs such that

(i) ϕ is a localization at some class of maps W,

(ii) $\mathcal{C}_F \to \mathfrak{D}_F$ is a right fibration.

Then $\text{Span}(\phi)$: $\text{Span}_F(\mathscr{C}) \to \text{Span}_F(\mathfrak{D})$ *is a localization, at the class of maps of the form* $X \xleftarrow{w} Y \xrightarrow{=} Y$ *where* w *is in* W.

In fact, this theorem is a special case of a more general result about localizations of factorization systems, in the following sense:

Definition 4.1.2. A *factorization system* on an ∞ -category C consists of two wide subcategories $E, M \subset C$ satisfying the following conditions:

- (I) Every morphism in *E* is *left orthogonal* to every morphism in *M*.
- (2) Every morphism f of \mathcal{C} admits a factorization $f \simeq me$ with e in E and m in M.

If (\mathcal{C}, E, M) and (\mathcal{C}', E', M') are two factorization systems, then a *map of fac*torization systems $f: (\mathcal{C}, E, M) \to (\mathcal{C}', E', M')$ is a functor $f: \mathcal{C} \to \mathcal{C}'$ of their underlying ∞ -categories satisfying $f(E) \subset E'$ and $f(M) \subset M'$.

Remark 4.1.3. The above definition follows [ABFJ22, 3.1.6]. As a consequence of Lemma 3.1.9 of the same paper, it is equivalent to [Luro9, 5.2.8.8]; see also [CLL23, 3.32].

Proposition 4.1.4. Let (\mathcal{C}, E, M) be a factorization system and consider the commutative square of inclusions

$$\begin{array}{ccc} \mathscr{C}^{\simeq} & \stackrel{\alpha}{\longrightarrow} & E \\ i & & & & \downarrow^{j} \\ M & \stackrel{\beta}{\longrightarrow} & \mathscr{C}. \end{array}$$

$$(3)$$

Then the induced Beck–Chevalley transformation

$$i_!\alpha^* \longrightarrow \beta^* j_!$$

on functors to any cocomplete ∞ -category is an equivalence.

Proof. Using the colimit formula for left Kan extensions, we see that it suffices to show that for every $X \in \mathcal{C}$, the induced functor $\mathcal{C}^{\approx} \times_{M} M_{/X} \to E \times_{\mathcal{C}} \mathcal{C}_{/X}$ is (co)final, i.e. that restriction along it induces an equivalence on all colimits. We will show that the functor above is a right adjoint, from which the result follows by [Cis19, 6.1.13].

For this, we note that by [CLL23, Proof of 3.33], the inclusion $\kappa: M_{/X} \rightarrow \mathcal{C}_{/X}$ is fully faithful and admits a left adjoint λ such that for every $(Y \rightarrow X) \in \mathcal{C}_{/X}$ the unit map $Y \rightarrow \kappa \lambda(Y)$ belongs to *E*; moreover, in the same proof it is also shown that λ is a localization at *E*. It follows directly that the left adjoint λ restricts to a functor between the non-full subcategories

$$E \times_{\mathscr{C}} \mathscr{C}_{/X} \longrightarrow \mathscr{C}^{\simeq} \times_M M_{/X},$$

and that also the unit restricts accordingly. As the counit is an equivalence by the full faithfulness of $M_{/X} \rightarrow \mathcal{C}_{/X}$, it similarly restricts to these subcategories, so that the restricted functor is a right adjoint, as claimed.

Using this, we will now prove the following general base change criterion:

Theorem 4.1.5. Let

$$\begin{array}{cccc} (\mathscr{C}_{00}, E_{00}, M_{00}) & \xrightarrow{g_0} & (\mathscr{C}_{10}, E_{10}, M_{10}) \\ & & & & & \\ f_0 & & & & & \\ f_0 & & & & & \\ (\mathscr{C}_{01}, E_{01}, M_{01}) & \xrightarrow{g_1} & (\mathscr{C}_{11}, E_{11}, M_{11}) \end{array}$$

$$(4)$$

be a commutative diagram of small factorization systems and assume that the restricted maps $f_0: M_{00} \rightarrow M_{01}$ and $f_1: M_{10} \rightarrow M_{11}$ are right fibrations.

Let \mathcal{T} be a complete ∞ -category and let $X: \mathfrak{C}_{10} \to \mathcal{T}$ be arbitrary. Then the Beck–Chevalley map $g_1^*f_{1*}X \to f_{0*}g_0^*X$ associated to (4) is an equivalence if and only if the analogous Beck–Chevalley map $g_1^*f_{1*}(X|_{E_{10}}) \to f_{0*}g_0^*(X|_{E_{10}})$ associated to

$$\begin{array}{ccc} E_{00} & \xrightarrow{g_0} & E_{10} \\ f_0 & & & \downarrow f_1 \\ E_{01} & \xrightarrow{g_1} & E_{11} \end{array}$$

is an equivalence.

We begin with the following key special case:

Lemma 4.1.6. Let $f: (\mathcal{C}, E, M) \to (\mathcal{C}', E', M')$ be a map of small factorization systems such that $f: M \to M'$ is a right fibration. Then the Beck–Chevalley map $j^*f_* \to f_*i^*$ associated to

$$E \xrightarrow{i} \mathscr{C}$$

$$f \downarrow \qquad \qquad \downarrow f$$

$$E' \xrightarrow{j} \mathscr{C}'$$

is an equivalence for functors to any complete ∞ -category.

Proof. Let \mathcal{T} be a complete ∞ -category, which by changing universe we may assume is small. Then we may embed \mathcal{T} continuously into a complete and cocomplete ∞ -category \mathcal{T}' via the Yoneda embedding. Proving the statement for \mathcal{T}' implies the statement for \mathcal{T} , and so it suffices to prove the statement for (co)complete \mathcal{T} . Having made this reduction, we may pass to total mates, and instead show that the Beck–Chevalley map $i_! f^* \to f^* j_!$ is an equivalence. For this, let us consider the commutative cube

Taking functors into \mathcal{T} and passing to left adjoints horizontally we obtain a

commutative diagram



in the homotopy 2-category of ∞ -categories, where the left and right face are filled by the naturality equivalences and the remaining squares are filled by the respective Beck–Chevalley maps, which a priori might or might not be invertible; our goal is to prove that the natural transformation filling the back square is an equivalence.

For this we observe that the Beck–Chevalley maps in the top and bottom face are invertible by Proposition 4.1.4. On the other hand, as $f: M \to M'$ is conservative, the front face of (5) is a pullback square, and as this restriction of f is moreover assumed to be a right fibration, we see that the associated Beck–Chevalley map is an equivalence by the dual of [Cis19, 6.4.13].

Altogether, we have shown that all squares except possibly the back square are filled with invertible transformations. By coherence, we conclude that $i_!f^* \rightarrow f^*j_!$ becomes an equivalence after restricting to $M \subset \mathcal{C}$. However, M is a wide subcategory, so restriction is conservative and the claim follows.

Proof of Theorem 4.1.5. Consider the commutative cube

Mapping into \mathcal{T} and passing to right adjoints vertically, we again obtain a coherent cube, where the top and bottom face are filled by the naturality equivalences and all other faces are filled via the appropriate Beck–Chevalley maps.

By Lemma 4.1.6, the Beck–Chevalley maps filling the left and right face are invertible. Fixing now $X: \mathscr{C}_{10} \to \mathcal{T}$, we conclude from coherence that the

diagram of Beck-Chevalley maps

$$\begin{array}{ccc} (g_1^*f_{1*}X)|_{E_{01}} & \longrightarrow & (f_{0*}g_0^*X)|_{E_{01}} \\ & & \swarrow & & & \downarrow^{\sim} \\ g_1^*f_{1*}(X|_{E_{10}}) & \longrightarrow & f_{0*}g_0^*(X|_{E_{10}}) \end{array}$$

commutes up to homotopy. The claim follows via 2-out-of-3, using again that we can test equivalences on a wide subcategory.

Using this we can now prove the following localization criterion for maps of factorization systems:

Theorem 4.1.7 ('Separation of variables'). Let $f: (\mathcal{C}, E, M) \to (\mathcal{C}', E', M')$ be a map of factorization systems. Assume the following:

- (1) $f: E \to E'$ is a localization at some class $W \subset E$.
- (2) $f: M \to M'$ is a right fibration.
- Then $f: \mathcal{C} \to \mathcal{C}'$ is also a localization at W.

Proof. Passing to a larger universe, we may assume without loss of generality that all participating ∞ -categories are small.

As $f: E \to E'$ is a localization, it is in particular essentially surjective, whence so is $f: \mathcal{C} \to \mathcal{C}'$. It is moreover clear that the latter inverts all maps in W. By [Cis19, 7.1.11] it will therefore be enough to show that for every presentable \mathcal{T} the restriction $f^*: \operatorname{Fun}(\mathcal{C}', \mathcal{T}) \to \operatorname{Fun}(\mathcal{C}, \mathcal{T})$ is fully faithful and that its essential image contains all functors inverting W.

For this we consider the two commutative diagrams

Full faithfulness of f^* amounts to saying that the unit id $\rightarrow f_*f^*$ is an equivalence. However, this is precisely the Beck–Chevalley map associated to the left-hand square, so the claim follows from full faithfulness of $(f|_E)^*$ via Theorem 4.1.5. Similarly, if $X: \mathscr{C}' \rightarrow \mathcal{T}$ is arbitrary, then Theorem 4.1.5 applied to the right hand square shows that the counit $f^*f_*X \rightarrow X$ is an equivalence if and only if the counit of $X|_E$ is so. The latter holds if (and only if) $X|_E$ inverts W, proving that any such functor is contained in the essential image of f^* , as claimed.

Proof of Theorem **4.1.1**. Given a span pair (\mathscr{C} , \mathscr{C}_F), the ∞ -category $\operatorname{Span}_F(\mathscr{C})$ admits a canonical factorization system by [HHLN23, 4.9] such that $E = \mathscr{C}^{\operatorname{op}}$ and $M = \mathscr{C}_F$. Moreover given any map of span pairs $\phi : (\mathscr{C}, \mathscr{C}_F) \to (\mathfrak{D}, \mathfrak{D}_F)$, the induced map $\operatorname{Span}(\phi) : \operatorname{Span}_F(\mathscr{C}) \to \operatorname{Span}_F(\mathfrak{D})$ is a map of factorization systems, which agrees with ϕ^{op} and ϕ_F when restricted to the backward and forward morphisms, respectively. In particular we may immediately apply the previous criterion.

4.2 Localization of $\text{Span}(\mathbb{F})^{\otimes}$

To apply Theorem 4.1.1 to $\text{Span}(\mathbb{F})^{\otimes}$, we must first show that $\text{Span}_{pb}(\text{Ar}(\mathbb{F})) \rightarrow \text{Span}(\mathbb{F})$ is a localization. This is a special case of the following more general statement:

Proposition 4.2.1. Let $(\mathcal{C}, \mathcal{C}_F)$ be a span pair. Then $(Ar(\mathcal{C}), Ar(\mathcal{C})_{F-pb})$ is also a span pair, where $Ar(\mathcal{C})_{F-pb}$ is the subcategory whose morphisms are the pullback squares

$$\begin{array}{ccc} X' & \stackrel{f'}{\longrightarrow} & Y' \\ \downarrow & & \downarrow \\ X & \stackrel{f}{\longrightarrow} & Y \end{array}$$

where f (and hence also f') lies in C_F . Moreover, ev_0 is a morphism of span pairs, and the induced functor

$$\operatorname{Span}_{F\operatorname{-bb}}(\operatorname{Ar}(\mathscr{C})) \longrightarrow \operatorname{Span}_{F}(\mathscr{C})$$

is a localization.

Proof. That $(Ar(\mathscr{C}), Ar(\mathscr{C})_{F-pb})$ is again a span pair, and that ev_0 is a map of span pairs, is clear.

Now consider the constant arrow functor $c: \mathcal{C} \to Ar(\mathcal{C})$, and note that it is also a map of span pairs. We claim that Span(c) provides an inverse to $Span(ev_0)$ after localization. First note that $Span(ev_0) \circ Span(c)$ is already the identity before localization. Therefore it suffices to show that the other composite is also the identity after localization. Consider the counit $\epsilon: c \circ ev_0 \Rightarrow id$ of the adjunction $c \dashv ev_0$. We claim that it induces a natural transformation $Span(c) \circ$ $Span(ev_0) \Rightarrow id$, which is an equivalence after localization. The first claim will follow from [BH2I, C.20], see also [HHLN23, 2.22], after we show that η is a natural transformation of span pairs. The second follows by observing that the component of $Span(\eta)$ on an object $x \to y$ is given by



and so is inverted by $Span(ev_0)$.

To complete the proof it only remains to show that η is a natural transformation of span pairs. Unwinding the definition, this amounts to the claim that for a commutative cube



such that the front face is a pullback in \mathcal{C} , the cube is a pullback in $Ar(\mathcal{C})$. Since limits in functor ∞ -categories are computed objectwise, this is true since the back face is evidently a pullback.

Proposition 4.2.2. *Let* $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ *be a bispan triple. Then*

$$(\mathsf{Ar}(\mathscr{C}), \mathsf{Ar}(\mathscr{C})_{F-\mathrm{pb}}, \mathsf{Ar}(\mathscr{C})_{\Lambda})$$

is also a bispan triple, where $Ar(\mathscr{C})_{\Lambda}$ consists of the squares

$$\begin{array}{ccc} x \xrightarrow{l} y \\ \downarrow & \downarrow \\ x' \xrightarrow{\simeq} y' \end{array}$$

such that I is in C_L and the lower horizontal map is an equivalence. Moreover, $ev_0: Ar(C) \rightarrow C$ is a morphism of bispan triples, and the induced functor

$$\operatorname{Bispan}_{F-\operatorname{pb},\Lambda}(\operatorname{Ar}(\mathscr{C})) \longrightarrow \operatorname{Bispan}_{F,L}(\mathscr{C})$$

is a localization.

Proof. To show that $(\operatorname{Ar}(\mathscr{C}), \operatorname{Ar}(\mathscr{C})_{F-\mathrm{pb}}, \operatorname{Ar}(\mathscr{C})_{\Lambda})$ is a bispan triple, we first observe that condition (1) in Definition 2.2.1 is clear, as is the fact that ev_0 gives a morphism of span pairs using both classes of morphisms in $\operatorname{Ar}(\mathscr{C})$. Now note that given a map $f: x \to x'$ in \mathscr{C} , the ∞ -category $\operatorname{Ar}(\mathscr{C})_{/f}^{\Lambda}$ consists of squares of the form

$$\begin{array}{c} y \xrightarrow{l} x \\ \downarrow & \downarrow^{f} \\ y' \xrightarrow{\sim} x', \end{array}$$

with l in \mathscr{C}_L , so that evaluation at 0 gives an equivalence $\operatorname{Ar}(\mathscr{C})_{/f}^{\Lambda} \simeq \mathscr{C}_{/x}^{L}$, which is moreover natural with respect to pullback. Therefore, conditions (2) and (3)

in the definition follow immediately from the analogous facts for C_F and C_L . The adjointability condition for ev_0 to be a morphism of bispan triples from Definition 2.2.6 also follows immediately.

Next we recall that, by definition, the functor

 $\operatorname{Bispan}_{F\operatorname{-pb},\Lambda}(\operatorname{Ar}(\mathscr{C})) \longrightarrow \operatorname{Bispan}_{F,L}(\mathscr{C})$

is given by applying Span to the morphism of span pairs

 $\mathsf{Span}(\mathrm{ev}_0)\colon (\mathsf{Span}_{F\operatorname{-bb}}(\mathsf{Ar}(\mathscr{C}))^{\mathrm{op}}, \mathsf{Ar}(\mathscr{C})_\Lambda) \longrightarrow (\mathsf{Span}_F(\mathscr{C})^{\mathrm{op}}, \mathscr{C}_L).$

To complete the proof we will show that Theorem 4.1.1 applies to this functor. We saw in Proposition 4.2.1 that the functor Span(ev_0) is a localization, so we only need to show that the functor ev_0 : $Ar(\mathscr{C})_{\Lambda} \to \mathscr{C}_L$ is a right fibration. But here $Ar(\mathscr{C})_{\Lambda}$ is precisely the subcategory of cartesian arrows for the cartesian fibration ev_0 : $Ar(\mathscr{C}_L) \to \mathscr{C}_L$, so we have a right fibration by [Luro9, 2.4.2.5]. \Box

As a special case, for the bispan triple $(\mathbb{F}, \mathbb{F}, \mathbb{F})$ we get the desired localization result for bispans in \mathbb{F} :

Corollary 4.2.3. The functor

$$\text{Span}(\mathbb{F})^{\otimes} \simeq \text{Bispan}_{pb,tgt=eq}(\text{Ar}(\mathbb{F})) \longrightarrow \text{Bispan}(\mathbb{F})$$

given by evaluation at 0 is a localization.

4.3 Proof of the main theorem

In this subsection we complete the proof of Theorem A. Given Corollary 3.3.7 and Corollary 4.2.3, it remains to prove the following:

Proposition 4.3.1. *The following are equivalent for a functor* $F: \text{Span}(\mathbb{F})^{\otimes} \to \mathcal{C}$ *:*

- (1) *F* is a Span(\mathbb{F})^{\otimes}-monoid, and its restriction to Span(\mathbb{F}) is a commutative monoid.
- (2) F factors through the localization $Bispan(\mathbb{F})$, and the induced functor $Bispan(\mathbb{F}) \rightarrow \mathcal{C}$ preserves products.

We first make condition (I) above explicit using the equivalence $\text{Span}(\mathbb{F})^{\otimes} \simeq \text{Bispan}_{\text{pb,tgt=eq}}(\text{Ar}(\mathbb{F}))$ from Corollary 3.2.5:

Lemma 4.3.2. Let \mathcal{C} be a presentable ∞ -category. For a functor

F: Bispan_{pb,tgt=eq}(Ar(
$$\mathbb{F}$$
)) $\longrightarrow \mathscr{C}$,

the following conditions are equivalent:

(1) F satisfies condition (1) of Proposition 4.3.1, after applying the equivalence of Corollary 3.2.5.

(2) For all $S \to T$ in $Ar(\mathbb{F})$, we have equivalences

$$F(S \longrightarrow T) \xrightarrow{\sim} \prod_{t \in T} F(S_t \longrightarrow *), \quad F(S \longrightarrow *) \xrightarrow{\sim} \prod_{s \in S} F(* \longrightarrow *),$$

induced by the backwards maps associated to the obvious inclusions in $Ar(\mathbb{F})$.

(3) For all $S \to T$ in $Ar(\mathbb{F})$, we have equivalences

$$F(S \longrightarrow *) \xrightarrow{\sim} F(S \longrightarrow T), \qquad F(S \longrightarrow *) \xrightarrow{\sim} \prod_{s \in S} F(* \longrightarrow *),$$

induced by the obvious maps in $Ar(\mathbb{F})$.

(4) F takes all backward maps of the form

$$\begin{array}{c} S = & S \\ \downarrow & & \downarrow \\ T \leftarrow & T' \end{array} \tag{6}$$

to equivalences, and $F(S \to *) \xrightarrow{\sim} \prod_{s \in S} F(* \to *)$.

Proof. For the equivalence of (1) and (2) observe that the first condition of (2) is by Remark 3.2.7 equivalent to *F* being a $\operatorname{Bispan}_{\operatorname{pb,tgt}=eq}(\operatorname{Ar}(\mathbb{F}))$ -monoid, while the usual description of products in $\operatorname{Span}(\mathbb{F})$ (as coproducts in \mathbb{F}) shows that the second condition is equivalent to the restriction to $\operatorname{Span}(\mathbb{F})$ being a commutative monoid.

To see that (2) and (3) are equivalent, consider the following commutative square:

Here the first top horizontal map is an equivalence if (3) holds and the second if (2) holds, while the other maps are invertible under both assumptions; the equivalence of the two conditions then follows from the 2-of-3 property.

Finally, we observe that (3) is a special case of (4), and conversely (4) follows from (3) by applying the 2-of-3 property to the value of *F* at the composition

$$S = S = S$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$* \longleftarrow T \longleftarrow T'$$

which completes the proof.

Proof of Proposition 4.3.1. Let S be the class of maps in $Bispan_{pb,tgt=eq}(Ar(\mathbb{F}))$ of the form (6) and let W be the class of maps that are inverted by ev_0 . Since a span is invertible if and only if both of its components are (Observation 2.1.7), the maps in W are those whose top row consists of equivalences, which can immediately be simplified to those of the form

Using the characterization from Lemma 4.3.2(4) and the description of products in Bispan(\mathbb{F}) from [EH23, Remark 2.6.13] we see that (1) follows from (2), since *S* is contained in *W*. For the converse, Corollary 4.2.3 implies that it suffices to show that a functor that inverts the maps in *S* must invert all maps in *W*. A map in *W* is a composite of a map in *S* and a forward map of the form

$$\begin{array}{ccc} X & & & \\ & \downarrow & & & \\ & & \downarrow & & \\ T & & & \\ & & & & \\ & & & & \\ \end{array} \xrightarrow{f} f$$

so it suffices to show that such maps are inverted. For this we consider the composition

$$\begin{array}{cccc} X & & & X & & X \\ \downarrow & & & \downarrow f & & \parallel \\ T & & & & S & \leftarrow & T \end{array} \\ \end{array}$$

where the second map lies in *S*. The left square being cartesian implies that the composite is

Χ	$\longrightarrow X = $	X
\checkmark	U	U
Т	$\leftarrow X = =$	Χ,

which lies in S. The 2-of-3 property then implies that a functor that inverts S must invert all maps in W, as required. \Box

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