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Two generalizations of the semi-graphoid rule of probabilistic independence and more

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ABSTRACT

Probabilistic independence is a key concept in probability theory and statistics. For probabilistic independence a set of well known qualitative rules exists, the so-called semi-graphoid rules, which can be summarized into a single semi-graphoid rule. This rule system was conjectured to be complete, it is however incomplete and an additional five rules were formulated. The generalization of one of those rules subsequently showed that no finite rule system exists and in recent work even all five additional rules were (further) generalized. In this paper, two new generalized rules are stated, both involving n, $n \geq 1$ variable sets C_i. These rules generalize the semi-graphoid rule for n is odd and generalize one of the additional rules for n is even. Furthermore two new rules of probabilistic independence are given. The paper thereby contributes to the insights into the structural properties of probabilistic independence and provides an enhanced description of probabilistic independence by means of rules.

1. Introduction

Probabilistic independence is a key concept in probability theory and statistics and the notion plays a fundamental role in learning and reasoning in systems dealing with knowledge and uncertainty. Within the topic of probabilistic independence the so called probabilistic implication problem is of great importance. This problem asks for an answer to the question whether or not some probabilistic independence statement is enforced by a given set by probabilistic independences.

In [3], a set of sound qualitative rules of probabilistic independence is given; the well known semi-graphoid rules. These rules can be summarized into a single semi-graphoid rule which, given the symmetry property of probabilistic independence, is just as powerful [2,4]. Pearl conjectured that the semi-graphoid rules would be complete for probabilistic independence, however, in [5] a new rule of probabilistic independence was formulated. The semi-graphoid rules thus are incomplete and a set of independence statements that is closed under these rules, may lack statements that are enforced by probabilistic independence. In [7], yet another four new rules are stated and in [6,8] the authors show that there even is no finite complete system of rules of probabilistic independence by generalizing one of the new rules to an unlimited number of variable sets involved. The generalized rule then includes the original rule as a special case. In the proofs of the correctness of the rules, properties of the conditional mutual information between (sets of) variables and the relation

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between conditional mutual information and the multi-information function were used. Recent work showed that not just one, but that all five additional rules can be generalized to an unlimited number of variable sets [1].

In this paper, two new generalized rules are formulated, both these rules include generalizations of the semi-graphoid rule and, surprisingly, both rules also comprise yet another generalization of one of the rules generalized in [1]. With the generalizations stated in this paper it thus is established that the semi-graphoid rule is just a special case of, at least, two general rules. Moreover, with these generalizations the question whether more than one generalization of some of the additional rules exists is answered. To conclude, two more rules of probabilistic independence are stated. These rules show that the generalized rules stated thus far for do not capture yet all rules of probabilistic independence that can be found by the proof method used. All in all, the paper gives an enhanced description of probabilistic independence by means of rules. The paper thereby provides for better tools to solve the probabilistic independence problem and further contributes to the insights into the structural properties of probabilistic independence.

This paper is an extended version of the paper 'Two generalizations of the semi-graphoid rule of probabilistic independence' presented at the WUPES 2022 workshop.

2. Preliminaries

2.1. Probabilistic and semi-graphoid independence relations

Throughout the paper, a set V of discrete random variables with subsets A, B, C, ... is considered. Set union is noted by concatenation of the sets, for example, $A \cup B$ is written as AB. A triplet $\langle A, B \mid C \rangle$, with A, B, C pairwise disjoint subsets of V and A and B nonempty, states that the sets of variables A and B are probabilistic independent given observations for the variables in set C. An elementary triplet is a triplet with A and B singletons. The set of all disjoint triplets is noted as $\mathcal{T}(V)$. A subset $\mathcal{I}$ of $\mathcal{T}(V)$ is called a probabilistic independence relation if there exists a probability distribution $\Pr(V)$ for which $\langle A, B \mid C \rangle$ is true for all $\langle A, B \mid C \rangle \in \mathcal{I}$ and $\langle A, B \mid C \rangle$ is false for all $\langle A, B \mid C \rangle \notin \mathcal{I}$.

In [3] Pearl proposed four rules of probabilistic independence. These rules sum up in the following two rules [4]:

$A1$: $\langle A, B \mid C \rangle \ \leftrightarrow \ \langle B, A \mid C \rangle$
$A2$: $\langle AB, C \mid D \rangle \ \leftrightarrow \ \langle A, C \mid BD \rangle \wedge \langle B, C \mid D \rangle$

A set of valid independence statements closed under these two rules is called a semi-graphoid independence relation.

In [2] Matúš argued that any semi-graphoid independence relation is fully captured by its elementary triplets. He moreover considered the first two positions of a triplet as unordered and alternatively defined a semi-graphoid independence relation as a set of elementary triplets that is closed under the rule:

$A2'$: $\langle A, C \mid D \rangle \wedge \langle B, C \mid AD \rangle \ \leftrightarrow \ \langle A, C \mid BD \rangle \wedge \langle B, C \mid D \rangle$

In this paper rule $A2'$ will also be called the semi-graphoid rule.

The semi-graphoid rules are sound with respect to probabilistic independence relations. The system of rules, however, is not complete as shown by Studený in [5]. In this paper he stated a new rule of probabilistic independence, which he numbered $A3$. The correctness of this new rule was proved by using the relation between condition mutual information and the so-called multiinformation function. This is further discussed in the next section. In [7] another four new rules, rules $A4$ to $A7$, were proposed.

In [6,8] the authors established that there is no finite complete system of rules of probabilistic independence by generalizing rule $A6$ to an unlimited number of variables and in [1] it is shown that all rules $A3$ to $A7$ can be (further) generalized resulting in rules $G3$ to $G7$. An overview of all rules $A3$ to $A7$ and $G3$ to $G7$ is given in Appendix B. Rule $A2'$ plus the rules $G3$ to $G7$ will be referred to as rule system $\mathcal{R}$.

2.2. Mutual information, multiinformation and probabilistic independence

In proving the correctness of the new rules of probabilistic independence, the relation between the mutual conditional information and the so-called multiinformation function was used, as discussed below [5,8].

Given a probability distribution Pr over V, the mutual information of two sets of random variables A and B in the context of a third set C, noted $I(A; B|C)$, is a measure of the mutual dependence between A and B in the context of C (see for example [9]). The conditional mutual information given a discrete probability distribution Pr is defined as:

$$I(A; B \mid C) \ = \ \sum_{abc} \Pr(abc) \cdot \log \frac{\Pr(ab \mid c)}{\Pr(a \mid c) \cdot \Pr(b \mid c)}$$

where abc ranges over all possible value combinations for the variables in ABC with $\Pr(a|c)$, $\Pr(b|c) \neq 0$. The conditional mutual information has as properties that for any A, B, C

- $I(A; B|C) \geq 0$;
- $I(A; B|C) = 0$ iff $\langle A, B \mid C\rangle$ is true.

The multiinformation function $M(A||\Pr)$ induced by a probability distribution Pr over V is a real function $M : 2^V \to [0, \infty)$ on the elements A of the power set of V defined by:

$$M(A||\Pr) = \mathcal{H}(\Pr(A)|\prod_{i\in A}\Pr(i)),\ M(\emptyset||\Pr) = 0$$

Where $\mathcal{H}$ is the relative entropy between the distributions $\Pr(A)$ and $\prod_{i\in A}\Pr(i)$. Note that $M(A||\Pr) \geq 0$ and that $M(A||\Pr) = 0$ iff $\Pr(A) = \Pr(\prod_{i\in A}\Pr(i))$.

The mutual conditional information is related to the multiinformation function by:

- $I(A; B \mid C) = M(ABC) + M(C) - M(AC) - M(BC)$

We thus have that:

- $M(ABC) + M(C) - M(AC) - M(BC) \geq 0$;
- $M(ABC) + M(C) - M(AC) - M(BC) = 0$ iff $\langle A, B \mid C\rangle$.

This relation enables straightforward proofs for rules of probabilistic independence: a rule is sound if the multiinformation terms of its set of premise triplets can be converted into the multiinformation terms of its set of consequent triplets. Below as an example a proof of rule $A2$.

Example 1. The probabilistic soundness of $\langle AB, C \mid D\rangle \leftrightarrow \langle A, C \mid BD\rangle \wedge \langle B, C \mid D\rangle$ is proved as follows:

We have that $\langle AB, C \mid D\rangle$ is a valid independence statement if and only if

$$\begin{aligned}
&0 = M(ABCD) + M(D) - M(ABD) - M(CD) \Leftrightarrow \\
&0 = M(ABCD) + M(D) - M(ABD) - M(CD) \\
&\qquad + M(BD) - M(BD) + M(BCD) - M(BCD) \Leftrightarrow \\
&0 = M(ABCD) + M(BD) - M(ABD) - M(BCD) \\
&\qquad + M(BCD) + M(D) - M(BD) - M(CD)
\end{aligned}$$

which is true if and only if $\langle A, C \mid BD\rangle$ and $\langle B, C \mid D\rangle$ are valid independence statements.

The last step in the proof above is based on the fact that the conditional mutual information for any three sets of variables is larger than or equal to 0. Note that the order of the first two positions of the triplets indeed doesn't affect the proof.

3. Two generalizations of rule $A2'$

In this section two generalizations of rule $A2'$ are stated. These generalizations include an unlimited number of rules of independence, defined by the choice of n. Both generalizations involve the variable sets A and B and a set $\mathbf{C}$ of $n, n \geq 1$ variable sets C_i. For each C_i two triplets are found both in the premise and in the consequent of the rules. One triplet with A as first argument and C_i as second argument and one triplet with B as first argument and C_i as second argument. The sets $\mathbf{C} \setminus C_i$ are distributed over the third arguments of those two triplets; in one of those triplets supplemented with the set A or B. The two rules differ in the specific composition of the third arguments of their triplets. Proofs of the propositions are constructed with the method described in the previous section and are provided in the appendix. Note that the proofs of both proposition are given for n is odd and n is even separately.

Remark furthermore that the triplets of rule $A1$, the triplets of rule $A2$ and the triplets of rule $A2'$ share a set of conditioning variables. (The set C in rule $A1$ and the set D in rules $A2$ and $A2'$.) In the generalizations given in this paragraph and in the two new rules in the next paragraph such a shared set is omitted for clarity of exposition. A shared condition set can be added to the triplets of a rule without affecting its validity. The proof of a rule's validity with or without such a set is fully analogous.

Proposition 1. *Let $A, B, C_1, \ldots, C_n$ with $n \geq 1$, be non-empty, mutually disjoint sets of variables. Then (taking $C_i \cdots C_{i-1} := \emptyset$)*

G2a:

$$\bigwedge_{\substack{i \in \{1,\ldots,n\},\\ i \text{ odd}}} \left[\, \langle A, C_i \mid C_1 \cdots C_{i-1}\rangle \;\wedge\; \langle B, C_i \mid A\, C_{i+1} \cdots C_n\rangle \,\right] \;\wedge$$

$$\bigwedge_{\substack{i \in \{1,\ldots,n\},\\ i \text{ even}}} \left[\langle A, C_i \mid B\, C_1 \cdots C_{i-1} \rangle \wedge \langle B, C_i \mid C_{i+1} \cdots C_n \rangle \right] \leftrightarrow$$

$$\bigwedge_{\substack{i \in \{1,\ldots,n\},\\ i \text{ odd}}} \left[\langle A, C_i \mid B\, C_{i+1} \cdots C_n \rangle \wedge \langle B, C_i \mid C_1 \cdots C_{i-1} \rangle \right] \wedge$$

$$\bigwedge_{\substack{i \in \{1,\ldots,n\},\\ i \text{ even}}} \left[\langle A, C_i \mid C_{i+1} \cdots C_n \rangle \wedge \langle B, C_i \mid A\, C_1 \cdots C_{i-1} \rangle \right]$$

is a sound rules of probabilistic independence.

In the sequel specific instances of generalize rules will be indicated by a subscript, for example, rule $G2$a for $n = 1$ will be noted by $G2\text{a}_1$.
Rule $G2\text{a}_1$ equals (with an additional conditioning set D) rule $A2'$ and rule $G2\text{a}_2$ equals rule $A7$ (taking $A = A$, $B = D$, $C_1 = C$ and $C_2 = B$). Rule $G2\text{a}_3$, for example, states that

$$\begin{aligned}
&\langle A, C_1 \mid \emptyset \rangle \wedge \langle A, C_3 \mid C_1 C_2 \rangle \wedge \langle B, C_1 \mid A\, C_2 C_3 \rangle \wedge \langle B, C_3 \mid A \rangle \wedge \\
&\langle A, C_2 \mid B\, C_1 \rangle \wedge \langle B, C_2 \mid C_3 \rangle \leftrightarrow \\
&\langle A, C_1 \mid B\, C_2 C_3 \rangle \wedge \langle A, C_3 \mid B \rangle \wedge \langle B, C_1 \mid \emptyset \rangle \wedge \langle B, C_3 \mid C_1 C_2 \rangle \wedge \\
&\langle A, C_2 \mid C_3 \rangle \wedge \langle B, C_2 \mid A\, C_1 \rangle
\end{aligned}$$

In Fig. 1 the structure of $G2$a is clarified by a graphical representation of rules $G2\text{a}_1$ to $G2\text{a}_5$. Each $X \overset{Z}{\text{———}} Y$ in this figure represents a triplet $\langle X, Y \mid Z \rangle$. This representation shows clearly the way the rule develops with increasing n.

Proposition 2. *Let $A, B, C_1, \ldots, C_n$ with $n \geq 1$, be non-empty, mutually disjoint sets of variables. Then (taking $C_i \cdots C_{i-1} := \emptyset$)*

G2b:

$$\bigwedge_{\substack{i \in \{1,\ldots,n\},\\ i \text{ odd}}} \left[\langle A, C_i \mid C_1 \cdots C_{i-1} \rangle \wedge \langle B, C_i \mid A\, C_{i+1} \cdots C_n \rangle \right] \wedge$$

$$\bigwedge_{\substack{i \in \{1,\ldots,n\},\\ i \text{ even}}} \left[\langle A, C_i \mid B\, C_{i+1} \cdots C_n \rangle \wedge \langle B, C_i \mid C_1 \cdots C_{i-1} \rangle \right] \leftrightarrow$$

$$\bigwedge_{\substack{i \in \{1,\ldots,n\},\\ i \text{ odd}}} \left[\langle A, C_i \mid B\, C_1 \cdots C_{i-1} \rangle \wedge \langle B, C_i \mid C_{i+1} \cdots C_n \rangle \right] \wedge$$

$$\bigwedge_{\substack{i \in \{1,\ldots,n\},\\ i \text{ even}}} \left[\langle A, C_i \mid C_{i+1} \cdots C_n \rangle \wedge \langle B, C_i \mid A\, C_1 \cdots C_{i-1} \rangle \right]$$

is a sound rules of probabilistic independence.

Rule $G2\text{b}_1$ equals (with an additional conditioning set D) rule $A2'$ and rule $G2\text{b}_2$ equals rule $A4$ (taking $A = C$, $B = A$, $C_1 = B$ and $C_2 = D$). Rule $G2\text{b}_3$, for example, states that

$$\begin{aligned}
&\langle A, C_1 \mid \emptyset \rangle \wedge \langle A, C_3 \mid C_1 C_2 \rangle \wedge \langle B, C_1 \mid A\, C_2 C_3 \rangle \wedge \langle B, C_3 \mid A \rangle \wedge \\
&\langle A, C_2 \mid B\, C_3 \rangle \wedge \langle B, C_2 \mid C_1 \rangle \leftrightarrow \\
&\langle A, C_1 \mid B \rangle \wedge \langle A, C_3 \mid B C_1 C_2 \rangle \wedge \langle B, C_1 \mid C_2 C_3 \rangle \wedge \langle B, C_3 \mid \emptyset \rangle \wedge \\
&\langle A, C_2 \mid C_3 \rangle \wedge \langle B, C_2 \mid A\, C_1 \rangle
\end{aligned}$$

In Fig. 2 the structure of $G2\text{b}_n$ is clarified by a graphical representation of rules $G2\text{b}_1$ to $G2\text{b}_5$.

All rules in $\mathcal{R} \cup \{G2\text{a}\} \cup \{G2\text{b}\}$ (and also the rules given in Section 4) are bi-implications so the premise triplets may be either the triplets at the left hand side or at the right hand site of the rule. Note however, that for each rule, it holds

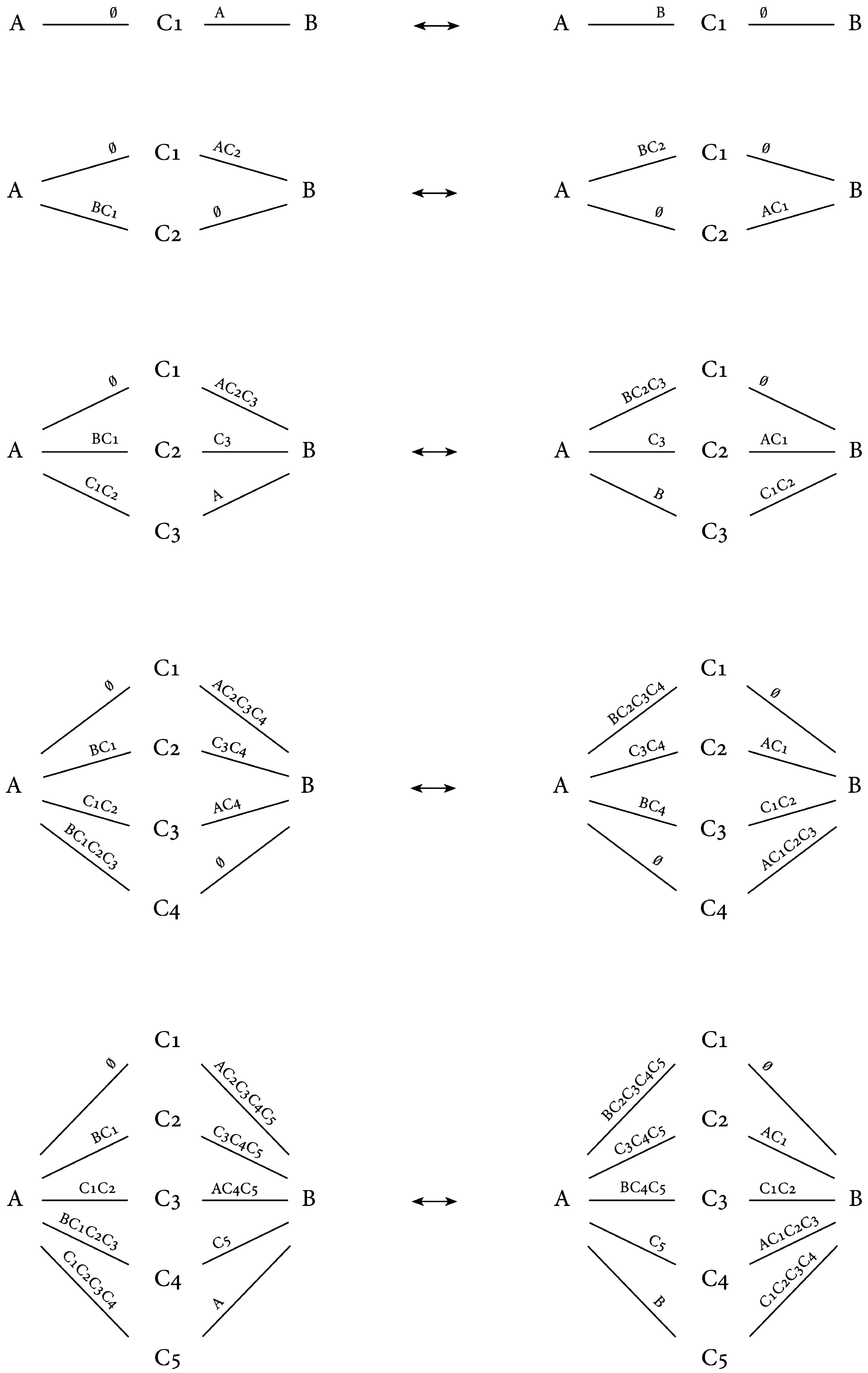

Fig. 1. The structure of rule $G2a$ for $n = 1$ to $n = 5$.

that set of triplets at left and right hand side only differ in the names of variable sets. The set premise triplets of each rule therefore can be represented by a single set of triplets. For rule $G2a_3$, for example this set is $\{\langle V_1, V_3 \mid \emptyset\rangle, \langle V_2, V_3 \mid V_1V_4V_5\rangle, \langle V_1, V_4 \mid V_2V_3\rangle, \langle V_2, V_4 \mid V_5\rangle, \langle V_1, V_5 \mid V_3V_4\rangle, \langle V_2, V_5 \mid V_1\rangle\}$.

Rules $G2a$ and $G2b$ both strengthen (for $n > 2$) rule system $\mathcal{R}$, as stated in Proposition 3. This proposition is based on Lemmas 1 to 6.

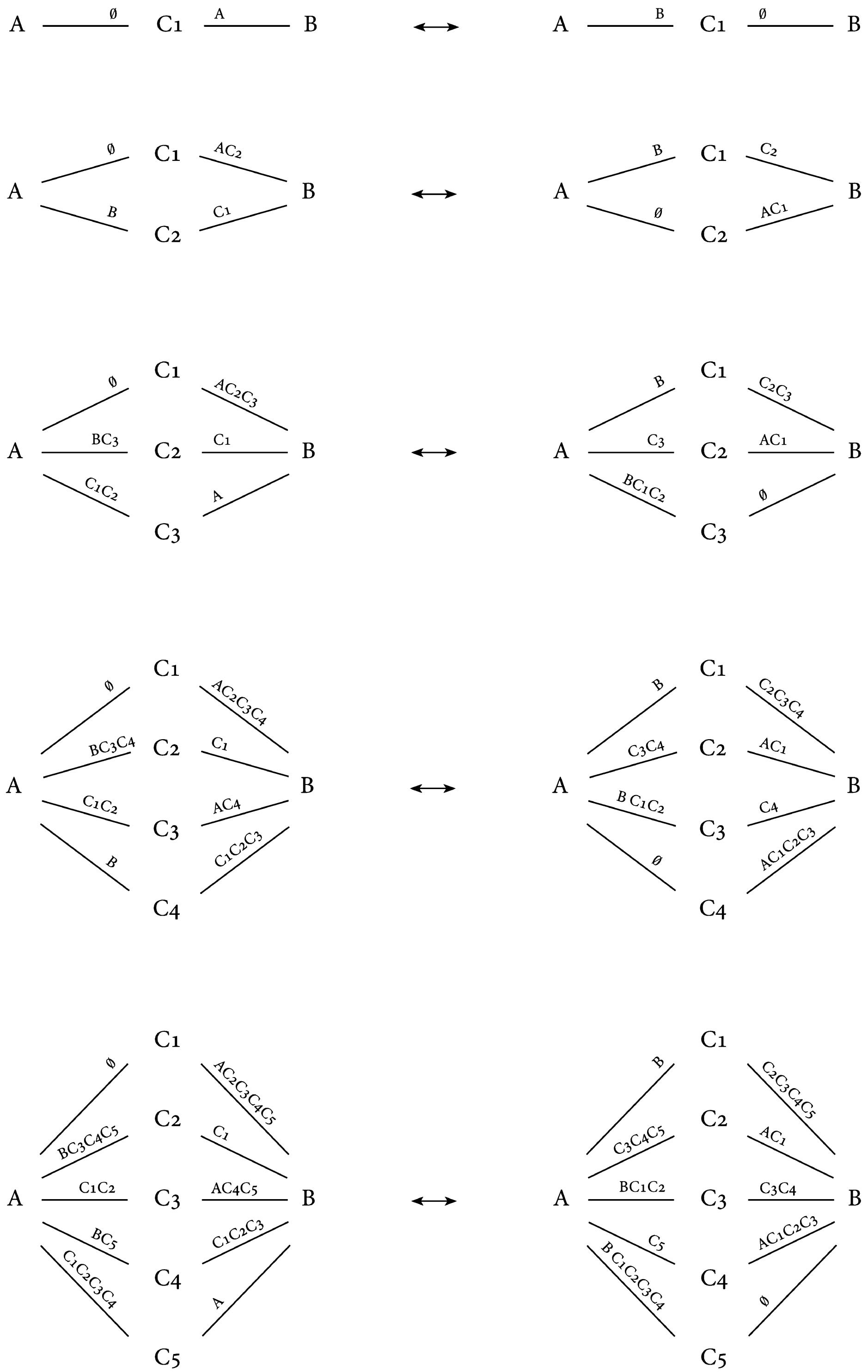

Fig. 2. The structure of rule $G2$b for $n = 1$ to $n = 5$.

Lemma 1. *Let $\mathcal{T}_{G2a}$ be the set of premise triplets of rule G2a and let $\mathcal{T}_{G2b}$ be the set of premise triplets of rule G2b. Then*

$$\forall_{k,l;k\neq l} \quad \mathcal{T}_{G2a_l} \nsubseteq \mathcal{T}_{G2a_k} \quad \textit{and}$$
$$\forall_{k,l;k\neq l} \quad \mathcal{T}_{G2b_l} \nsubseteq \mathcal{T}_{G2b_k}$$

Proof. By definition of rule $G2$a, for all l it holds that

$$\{\langle V_1, V_3 \mid \emptyset\rangle, \langle V_2, V_3 \mid V_1 V_4, \dots V_{l+2}\rangle\} \subseteq \mathcal{T}_{G2a_l}$$

however for all k; $k \neq l$ it holds that

$$\{\langle V_1, V_3 \mid \emptyset\rangle, \langle V_2, V_3 \mid V_1 V_4, \dots V_{l+2}\rangle\} \nsubseteq \mathcal{T}_{G2a_k}$$

Therefore $\mathcal{T}_{G2a_l} \nsubseteq \mathcal{T}_{G2a_k}$ for all k, l with $k \neq l$. The proof of $\mathcal{T}_{G2b_l} \nsubseteq \mathcal{T}_{G2b_k}$ is similar. □

Corollary 1. *Let $\mathcal{T}_{G2a}$ and $\mathcal{T}_{G2b}$ as before and let $\mathcal{T}_{A2}$ be the set of premise triplets of rule A2. Since $\mathcal{T}_{A2} = \mathcal{T}_{G2a_1} = \mathcal{T}_{G2b_1}$, by Lemma 1*

$$\begin{aligned} &\forall_{k>1} \quad \mathcal{T}_{A2} \nsubseteq \mathcal{T}_{G2a_k} \quad \textit{and} \\ &\forall_{k>1} \quad \mathcal{T}_{A2} \nsubseteq \mathcal{T}_{G2b_k} \quad \square \end{aligned}$$

Lemma 2. *Let $\mathcal{T}_{G2a}$ and $\mathcal{T}_{G2b}$ as before and let $\mathcal{T}_{G3}$ be the set of premise triplets of rule G3. Then*

$$\begin{aligned} &\forall_{k,l} \quad \mathcal{T}_{G3_l} \nsubseteq \mathcal{T}_{G2a_k} \quad \textit{and} \\ &\forall_{k,l} \quad \mathcal{T}_{G3_l} \nsubseteq \mathcal{T}_{G2b_k} \end{aligned}$$

Proof. By definition of rule $G3$, for all l it holds that

$$\{\langle V_1, V_2 \mid \emptyset\rangle, \langle V_1, V_2 \mid V_3, \dots V_l\rangle\} \subseteq \mathcal{T}_{G3_l}$$

however, by definition of rule $G2$a, for all k it holds that

$$\{\langle V_1, V_2 \mid \emptyset\rangle, \langle V_1, V_2 \mid V_3, \dots V_l\rangle\} \nsubseteq \mathcal{T}_{G2a_k}$$

Therefore, $\mathcal{T}_{G3_l} \nsubseteq \mathcal{T}_{G2a_k}$ for all k, l. The proof of $\mathcal{T}_{G3_l} \nsubseteq \mathcal{T}_{G2b_k}$ is similar. □

Lemma 3. *Let $\mathcal{T}_{G2a}$ and $\mathcal{T}_{G2b}$ as before and let $\mathcal{T}_{G4}$ be the set of premise triplets of rule G4. Then*

$$\begin{aligned} &\forall_{k,l} \quad \mathcal{T}_{G4_l} \nsubseteq \mathcal{T}_{G2a_k} \quad \textit{and} \\ &\forall_{k>2,l} \quad \mathcal{T}_{G4_l} \nsubseteq \mathcal{T}_{G2b_k} \end{aligned}$$

Proof. By definition of rule $G4$, for all l it holds that

$$\{\langle V_1, V_2 \mid V_3\rangle, \langle V_2, V_3 \mid V_1, V_4 \dots V_{l+2}\rangle\} \subseteq \mathcal{T}_{G4_l}$$

however, by definition of rule $G2$a, for all k it holds that

$$\{\langle V_1, V_2 \mid V_3\rangle, \langle V_2, V_3 \mid V_1, V_4 \dots V_{l+2}\rangle\} \nsubseteq \mathcal{T}_{G2a_k}$$

Therefore, $\mathcal{T}_{G4_l} \nsubseteq \mathcal{T}_{G2a_k}$ for all k, l. The proof of $\mathcal{T}_{G4_l} \nsubseteq \mathcal{T}_{G2b_k}$ for all $k > 2, l$ is similar. Note that $\mathcal{T}_{G4_2} = \mathcal{T}_{G2b_2}$, therefor now the condition $k > 2$ has to be made. □

Lemma 4. *Let $\mathcal{T}_{G2a}$ and $\mathcal{T}_{G2b}$ as before and let $\mathcal{T}_{G5}$ be the set of premise triplets of rule G5. Then*

$$\begin{aligned} &\forall_{k,l} \quad \mathcal{T}_{G5_l} \nsubseteq \mathcal{T}_{G2a_k} \quad \textit{and} \\ &\forall_{k,l} \quad \mathcal{T}_{G5_l} \nsubseteq \mathcal{T}_{G2b_k} \end{aligned}$$

Proof. By definition of rule $G5$, it holds that for all l, the set $\mathcal{T}_{G5_l}$ includes at least four triplets with a single set at the third position. By definition of rules $G2$a and $G2$b, it holds that for all k, the sets $\mathcal{T}_{G2a_k}$ and $\mathcal{T}_{G2b_k}$ include at most two triplets with a single set at the third position. Therefore $\mathcal{T}_{G5_l} \nsubseteq \mathcal{T}_{G2a_k}$ and $\mathcal{T}_{G5_l} \nsubseteq \mathcal{T}_{G2b_k}$ for all k, l. □

Lemma 5. *Let $\mathcal{T}_{G2a}$ and $\mathcal{T}_{G2b}$ as before and let $\mathcal{T}_{G6}$ be the set of premise triplets of rule G6. Then*

$$\begin{aligned} &\forall_{k,l} \quad \mathcal{T}_{G6_l} \nsubseteq \mathcal{T}_{G2a_k} \quad \textit{and} \\ &\forall_{k,l} \quad \mathcal{T}_{G6_l} \nsubseteq \mathcal{T}_{G2b_k} \end{aligned}$$

Proof. By definition of rule $G6$, for all l it holds that the set $\mathcal{T}_{G6_l}$ includes at least three triplets with the same set at one of the first two positions plus an equal number of sets at the third position. By definition of rules $G2$a and $G2$b, for all k it holds that the sets $\mathcal{T}_{G2a_k}$ and $\mathcal{T}_{G2b_k}$ do not fulfill this condition. Therefore $\mathcal{T}_{G6_l} \nsubseteq \mathcal{T}_{G2a_k}$ and $\mathcal{T}_{G6_l} \nsubseteq \mathcal{T}_{G2b_k}$ for all k, l. $\square$

Lemma 6. *Let $\mathcal{T}_{G2a}$ and $\mathcal{T}_{G2b}$ as before and let $\mathcal{T}_{G7}$ be the set of premise triplets of rule G7. Then*

$$\forall_{k>2,l} \quad \mathcal{T}_{G7_l} \nsubseteq \mathcal{T}_{G2a_k} \text{ and}$$
$$\forall_{k,l} \quad \mathcal{T}_{G7_l} \nsubseteq \mathcal{T}_{G2b_k}$$

Proof. By definition of rule $G7$, for all l it holds that

$$\{\langle V_1, V_2 \mid \emptyset\rangle, \langle V_3, V_4 \mid \emptyset\rangle\} \subseteq \mathcal{T}_{G7_l}$$

however, by definition of rule $G2$b, for all k it holds that

$$\{\langle V_1, V_2 \mid \emptyset\rangle, \langle V_3, V_4 \mid \emptyset\rangle\} \nsubseteq \mathcal{T}_{G2b_k}$$

Therefore, $\mathcal{T}_{G7_l} \nsubseteq \mathcal{T}_{G2b_k}$ for all k, l. Further, $\mathcal{T}_{G7_1} = \mathcal{T}_{G2a_2}$. By Lemma 1, therefore $\mathcal{T}_{G7_1} \nsubseteq \mathcal{T}_{G2a_k}$ for $k > 2$. By definition of rule $G7$, for $l > 1$ it holds that

$$\{\langle V_1, V_2 \mid \emptyset\rangle, \langle V_3, V_4 \mid \emptyset\rangle, \langle V_5, V_6 \mid \emptyset\rangle\} \subseteq \mathcal{T}_{G7_l}$$

however, by definition of $G2$a, for all k it holds that

$$\{\langle V_1, V_2 \mid \emptyset\rangle, \langle V_3, V_4 \mid \emptyset\rangle, \langle V_5, V_6 \mid \emptyset\rangle\} \nsubseteq \mathcal{T}_{G2a_k}$$

Therefore, $\mathcal{T}_{G7_l} \nsubseteq \mathcal{T}_{G2a_k}$ for all $k > 2, l$. $\square$

Lemma 7. *Let $\mathcal{T}_{G2a}$ and $\mathcal{T}_{G2b}$ as before then*

$$\forall_{k>1,l} \quad \mathcal{T}_{G2b_l} \nsubseteq \mathcal{T}_{G2a_k} \text{ and}$$
$$\forall_{k>1,l} \quad \mathcal{T}_{G2a_l} \nsubseteq \mathcal{T}_{G2b_k}$$

Proof. $\mathcal{T}_{G2a_1} = \mathcal{T}_{G2b_1}$ therefore, by Lemma 1, $\forall_{k>1}\ \mathcal{T}_{G2b_1} \nsubseteq \mathcal{T}_{G2a_k}$. Further, by definition of $G2$b, for all $l > 1$ it holds that

$$\{\langle V_1, V_2 \mid \emptyset\rangle, \langle V_2, V_3 \mid V_1, V_4 \ldots V_{l+2}\emptyset\rangle, \langle V_3, V_4 \mid V_2\rangle\} \subseteq \mathcal{T}_{G2b_l}$$

however, by definition of $G2$a, for all k it holds that

$$\{\langle V_1, V_2 \mid \emptyset\rangle, \langle V_2, V_3 \mid V_1, V_4 \ldots V_{l+2}\emptyset\rangle, \langle V_3, V_4 \mid V_2\rangle\} \nsubseteq \mathcal{T}_{G2a_k}$$

Therefore, $\mathcal{T}_{G2b_l} \nsubseteq \mathcal{T}_{G2a_k}$ for all $k > 1, l$.

Since $\mathcal{T}_{G2a_1} = \mathcal{T}_{G2b_1}$ by Lemma 1 also $\forall_{k>1}$, $\mathcal{T}_{G2a_1} \nsubseteq \mathcal{T}_{G2b_k}$. Further, by definition of $G2$a, for all $l > 1$ it holds that[1]

$$\{\langle V_1, V_2 \mid \emptyset\rangle, \langle V_2, V_3 \mid V_1, V_4 \ldots V_{l+2}\rangle, \langle V_3, V_4 \mid V_5 \ldots V_{l+2}\rangle\} \subseteq \mathcal{T}_{G2a_l}$$

however, by definition of $G2$b for all k it holds that

$$\{\langle V_1, V_2 \mid \emptyset\rangle, \langle V_2, V_3 \mid V_1, V_4 \ldots V_{l+2}\rangle, \langle V_3, V_4 \mid V_5 \ldots V_{l+2}\rangle\} \nsubseteq \mathcal{T}_{G2b_k}$$

Therefore, $\mathcal{T}_{G2a_l} \nsubseteq \mathcal{T}_{G2b_k}$ for all $k > 1, l$. $\square$

Proposition 3. *For $k > 2$ all rules G2a$_k$ and G2b$_k$ individually strengthen rule system $\mathcal{R}$.*

Proof. In Lemmas 2 to 6 plus Corollary 1 it is shown that for $k > 2$ there is no rule in $\mathcal{R}$ of which the set of premises is a (sub)set of the set of premises of the rules $G2$a$_k$ or $G2$b$_k$. So, for $k > 2$ both rules $G2$a$_k$ and $G2$b$_k$ allow for derivations from a given set of triplets were no derivations can be made with the rules in $\mathcal{R}$. Moreover, it is shown in Lemma 1 that for each rule $G2$a$_k$ it holds that there is no rule$G2$a$_l$ with $l \neq k$ of which the set of premise triplets is a subset of the premise triplets of $G2$a$_k$. So each rule $G2$a$_k$ allows for a derivation where no derivation can be made with rules $G2$a$_l$, $l \neq k$. The same holds for $G2$b. To conclude, in Lemma 7 it is shown that for $k > 1, l$, there is no rule $G2$b$_l$ of which the premise triplets are a subset of the premise triplets of $G2$a$_k$ and vice versa. So all rules $G2$a$_k$, $k > 1$ allow for derivations, not allowed by any rule $G2$b$_l$ and all rules $G2$b$_k$, $k > 1$ allow for derivations, not allowed by any $G2$a$_l$. Thus for $k > 2$, all rules $G2$a$_k$ and $G2$b$_k$ individually strengthen rule system $\mathcal{R}$. $\square$

[1] taking $\{V_5 \ldots V_{l+2}\} := \emptyset$ for $l = 2$.

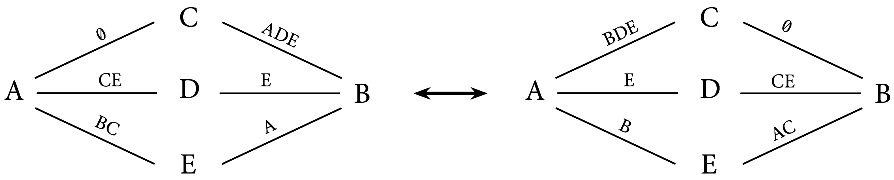

Fig. 3. Rule *A*8.

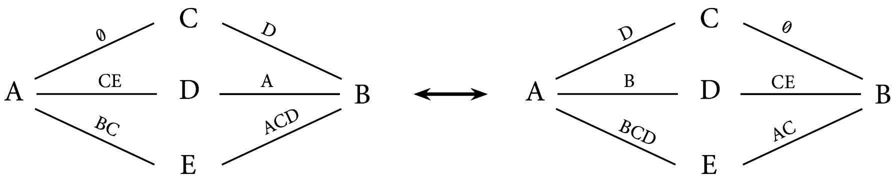

Fig. 4. Rule *A*9.

The two rules presented in this section comprise generalizations of the semi-graphoid rule. Surprisingly, the first generalization not only includes (for $n = 1$) the semi-graphoid rule, but also includes (for $n = 2$) rule *A*7, as a special case. Moreover, in the proof of its soundness a distinction has to be made between n is odd and n is even. So, in fact the first generalization can be considered as a generalization of the semi-graphoid rule for n is odd and as a generalization of rule *A*7 for n is even. Likewise, the second generalization, not only includes (for $n = 1$) the semi-graphoid rule, but also includes (for $n = 2$) rule *A*4 as a special case and, again, in the proof of its soundness a distinction has to be made between n is odd and n is even. So, in fact the second generalization can be considered as a generalization of the semi-graphoid rule for n is odd and as a generalization of *A*4 for n is even. Obviously, since the generalized rules are new, the generalizations of *A*7 and *A*4 presented here differ from their generalizations given in [1].

4. Two additional rules of probabilistic independence

Below two more rules of probabilistic independence are given.

Proposition 4. *Let A, B, C, D, E, be non-empty, mutually disjoint sets of variables. Then,*

$$\begin{aligned} A8: \quad & \langle A, C \mid \emptyset \rangle \wedge \langle A, D \mid CE \rangle \wedge \langle A, E \mid BC \rangle \wedge \\ & \langle B, C \mid ADE \rangle \wedge \langle B, D \mid E \rangle \wedge \langle B, E \mid A \rangle \quad \leftrightarrow \\ & \langle A, C \mid BDE \rangle \wedge \langle A, D \mid E \rangle \wedge \langle A, E \mid B \rangle \wedge \\ & \langle B, C \mid \emptyset \rangle \wedge \langle B, D \mid CE \rangle \wedge \langle B, E \mid AC \rangle \end{aligned}$$

is a sound rule of probabilistic independence.

Proof. The proposition can be proved straightforwardly by using the method described in Section 2.2. □

The structure of the rule is clarified in Fig. 3.

Proposition 5. *Let A, B, C, D, E, be non-empty, mutually disjoint sets of variables. Then,*

$$\begin{aligned} A9: \quad & \langle A, C \mid \emptyset \rangle \wedge \langle A, D \mid CE \rangle \wedge \langle A, E \mid BC \rangle \wedge \\ & \langle B, C \mid D \rangle \wedge \langle B, D \mid A \rangle \wedge \langle B, E \mid ACD \rangle \quad \leftrightarrow \\ & \langle A, C \mid D \rangle \wedge \langle A, D \mid B \rangle \wedge \langle A, E \mid BCD \rangle \wedge \\ & \langle B, C \mid \emptyset \rangle \wedge \langle B, D \mid CE \rangle \wedge \langle B, E \mid AC \rangle \end{aligned}$$

is a sound rule of probabilistic independence.

Proof. The proposition can be proved straightforwardly by using the method described in Section 2.2. □

The structure of the rule is clarified in Fig. 4.

None of the sets of premise triplets of the rules in $\mathcal{R} \cup \{G2a, G2b\}$ are included in the set of premise triplets in rules $A8$ and $A9$. These two rules thus show that the generalized rules stated thus far do not capture yet all rules of probabilistic independence that can be found by the method described in Section 2.2.

5. Conclusions and future research

In this paper two generalizations of the semi-graphoid rule of probabilistic independence were formulated. These generalizations include an unlimited number of rules of independence, defined by the choice of n. The rules were proved to be sound by a proof method based on the relation between conditional mutual independence and the concept of multiinformation. The first generalized rule joins a generalization of the semi-graphoid rule of probabilistic independence and a generalization of rule $A7$ from [7] and the second generalized rule joins a generalization of the semi-graphoid rule and a generalization of rule $A4$ from [7]. The generalizations of $A4$ and $A7$, moreover, differ from their generalizations given in [1]. With these two generalizations it is established that the semi-graphoid rule, which summarizes the well known semi-graphoid rule system, is just a special case of at least two general rules. These generalizations moreover show that more than one generalization of rules $A4$ and $A7$ exists. Also two new rules of probabilistic independence were given. These rules show that the generalized rules stated so far thus do not capture yet all rules of probabilistic independence that can be found by the used proof method. The paper all in all contributes to the insights in the structural properties of probabilistic independence and to the description of probabilistic independence by a qualitative rule system.

An obvious question for future research is whether the two new non-general rules can be generalized as well. Another, more fundamental, question is whether or not the number of (generalized) rules that can be found by the proof method based on the relation between conditional mutual independence and the concept of multiinformation is limited.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Appendix A. Proofs of Propositions 1 and 2

Both propositions are proved using the method described in Section 2.2.

In both proofs $\{i, \ldots, i-2\} := \emptyset$ and

$$\begin{aligned}
m_1(i) &= M(C_1 \cdots C_{i-1}) & m_9(i) &= M(BC_1 \cdots C_{i-1}) \\
m_2(i) &= M(C_1 \cdots C_i) & m_{10}(i) &= M(BC_1 \cdots C_i) \\
m_3(i) &= M(C_i \cdots C_n) & m_{11}(i) &= M(BC_i \cdots C_n) \\
m_4(i) &= M(C_{i+1} \cdots C_n) & m_{12}(i) &= M(BC_{i+1} \cdots C_n) \\
m_5(i) &= M(AC_1 \cdots C_{i-1}) & m_{13}(i) &= M(ABC_1 \cdots C_{i-1}) \\
m_6(i) &= M(AC_1 \cdots C_i) & m_{14}(i) &= M(ABC_1 \cdots C_i) \\
m_7(i) &= M(AC_i \cdots C_n) & m_{15}(i) &= M(ABC_i \cdots C_n) \\
m_8(i) &= M(AC_{i+1} \cdots C_n) & m_{16}(i) &= M(ABC_{i+1} \cdots C_n)
\end{aligned}$$

Moreover $\sum_{i \in \{1,\ldots,n\},\, i \text{ odd}}$ is abbreviated by $\sum_{i \text{ odd}}$ and $\sum_{i \in \{1,\ldots,n\},\, i \text{ even}}$ by $\sum_{i \text{ even}}$.

Proposition 1

We have that

$$\bigwedge_{\substack{i \in \{1,\ldots,n\},\\ i \text{ odd}}} \left[\langle A, C_i \mid C_1 \cdots C_{i-1} \rangle \wedge \langle B, C_i \mid A\, C_{i+1} \cdots C_n \rangle \right] \wedge$$

$$\bigwedge_{\substack{i \in \{1,\ldots,n\},\\ i \text{ even}}} \left[\langle A, C_i \mid B\, C_1 \cdots C_{i-1} \rangle \wedge \langle B, C_i \mid C_{i+1} \cdots C_n \rangle \right]$$

are valid independence statements if and only if

$$0 = \sum_{i \text{ odd}} \big[m_6(i) + m_1(i) - m_5(i) - m_2(i) + m_{15}(i) + m_8(i) - m_{16}(i) - m_7(i) \big]$$
$$+ \sum_{i \text{ even}} \big[m_{14}(i) + m_9(i) - m_{13}(i) - m_{10}(i) + m_{11}(i) + m_4(i) - m_{12}(i) - m_3(i) \big]$$

Given n is odd, we have that

$$\sum_{i \text{ odd}} \big[m_6(i) + m_8(i) \big] = \sum_{i \text{ even}} \big[m_5(i) + m_7(i) \big] + M(AC_1 \cdots C_n) + M(A)$$
$$\sum_{i \text{ odd}} \big[-m_5(i) - m_7(i) \big] = \sum_{i \text{ even}} \big[-m_6(i) - m_8(i) \big] - M(A) - M(AC_1 \cdots C_n)$$
$$\sum_{i \text{ even}} \big[m_9(i) + m_{11}(i) \big] = \sum_{i \text{ odd}} \big[m_{10}(i) + m_{12}(i) \big] - M(BC_1 \cdots C_n) - M(B)$$
$$\sum_{i \text{ even}} \big[-m_{10}(i) - m_{12}(i) \big] = \sum_{i \text{ odd}} \big[-m_9(i) - m_{11}(i) \big] + M(B) + M(BC_1 \cdots C_n)$$

and thus find that

$$0 = \sum_{i \text{ odd}} \big[m_{15}(i) + m_{12}(i) - m_{16}(i) - m_{11}(i) + m_{10}(i) + m_1(i) - m_9(i) - m_2(i) \big]$$
$$+ \sum_{i \text{ even}} \big[m_7(i) + m_4(i) - m_8(i) - m_3(i) + m_{14}(i) + m_5(i) - m_{13}(i) - m_6(i) \big]$$

which is true if and only if

$$\bigwedge_{\substack{i \in \{1,\ldots,n\},\\ i \text{ odd}}} \big[\langle A, C_i \mid B\, C_{i+1} \cdots C_n \rangle \ \wedge\ \langle B, C_i \mid C_1 \cdots C_{i-1} \rangle \big] \ \wedge$$
$$\bigwedge_{\substack{i \in \{1,\ldots,n\},\\ i \text{ even}}} \big[\langle A, C_i \mid C_{i+1} \cdots C_n \rangle \ \wedge\ \langle B, C_i \mid A\, C_1 \cdots C_{i-1} \rangle \big]$$

are valid independence statements.

Given n is even, we have that

$$\sum_{i \text{ odd}} \big[m_6(i) + m_8(i) \big] = \sum_{i \text{ even}} \big[m_5(i) + m_7(i) \big]$$
$$\sum_{i \text{ odd}} \big[-m_5(i) - m_7(i) \big] = \sum_{i \text{ even}} \big[-m_6(i) - m_8(i) \big]$$

(since $m_5(1) = m_8(n)$ $(= M(A))$ and $m_7(1) = m_6(n)$ $(= M(AC_1 \ldots C_n))$)

$$\sum_{i \text{ even}} \big[m_9(i) + m_{11}(i) \big] = \sum_{i \text{ odd}} \big[m_{10}(i) + m_{12}(i) \big]$$
$$\sum_{i \text{ even}} \big[-m_{10}(i) - m_{12}(i) \big] = \sum_{i \text{ odd}} \big[-m_9(i) - m_{11}(i) \big]$$

(since $m_{10}(n) = m_{11}(1)$ $(= M(BC_1 \cdots C_n))$ and $m_{12}(n) = m_9(1)$ $(= M(B))$)

and thus again find that

$$0 = \sum_{i \text{ odd}} \big[m_{15}(i) + m_{12}(i) - m_{16}(i) - m_{11}(i) + m_{10}(i) + m_1(i) - m_9(i) - m_2(i) \big]$$
$$+ \sum_{i \text{ even}} \big[m_7(i) + m_4(i) - m_8(i) - m_3(i) + m_{14}(i) + m_5(i) - m_{13}(i) - m_6(i) \big]$$

which concludes the proof.

Proposition 2

We have that

$$\bigwedge_{\substack{i \in \{1,\ldots,n\},\\ i \text{ odd}}} \left[\langle A, C_i \mid C_1 \cdots C_{i-1}\rangle \wedge \langle B, C_i \mid A\, C_{i+1} \cdots C_n\rangle\right] \wedge \bigwedge_{\substack{i \in \{1,\ldots,n\},\\ i \text{ even}}} \left[\langle A, C_i \mid B\, C_{i+1} \cdots C_n\rangle \wedge \langle B, C_i \mid C_1 \cdots C_{i-1}\rangle\right]$$

are valid independence statements if and only if:

$$\begin{aligned} 0 = &\sum_{i \text{ odd}} \left[m_6(i) + m_1(i) - m_5(i) - m_2(i) + m_{15}(i) + m_8(i) - m_{16}(i) - m_7(i)\right] \\ &+ \sum_{i \text{ even}} \left[m_{15}(i) + m_{12}(i) - m_{16}(i) - m_{11}(i) + m_{10}(i) + m_1(i) - m_9(i) - m_2(i)\right] \end{aligned}$$

We observe that, given n is odd,

$$\begin{aligned} \sum_{i \text{ odd}} \left[m_1(i) + m_{15}(i)\right] + \sum_{i \text{ even}} \left[-m_2(i) - m_{16}(i)\right] &= M(\emptyset) + M(ABC_1 \cdots C_n), \\ \sum_{i \text{ odd}} \left[-m_2(i) - m_{16}(i)\right] + \sum_{i \text{ even}} \left[m_1(i) + m_{15}(i)\right] &= -M(C_1 \cdots C_n) - M(AB) \end{aligned}$$

and thus find

$$\begin{aligned} 0 = &\sum_{i \text{ odd}} \left[m_6(i) - m_5(i) + m_8(i) - m_7(i)\right] \\ &+ \sum_{i \text{ even}} \left[m_{12}(i) - m_{11}(i) + m_{10}(i) - m_9(i)\right] \\ &+ M(\emptyset) - M(C_1 \cdots C_n) + M(ABC_1 \cdots C_n) - M(AB) \end{aligned}$$

Given n is odd we moreover have that

$$\begin{aligned} \sum_{i \text{ odd}} \left[m_6(i) + m_8(i)\right] &= \sum_{i \text{ even}} \left[m_5(i) + m_7(i)\right] + M(AC_1 \cdots C_n) + M(A) \\ \sum_{i \text{ odd}} \left[-m_5(i) - m_7(i)\right] &= \sum_{i \text{ even}} \left[-m_6(i) - m_8(i)\right] - M(A) - M(AC_1 \cdots C_n) \\ \sum_{i \text{ even}} \left[m_{12}(i) + m_{10}(i)\right] &= \sum_{i \text{ odd}} \left[m_{11}(i) + m_9(i)\right] - M(BC_1 \cdots C_n) - M(B) \\ \sum_{i \text{ even}} \left[-m_{11}(i) - m_9(i)\right] &= \sum_{i \text{ odd}} \left[-m_{12}(i) - m_{10}(i)\right] + M(B) + M(BC_1 \cdots C_n) \end{aligned}$$

and that

$$\begin{aligned} M(ABC_1 \ldots C_n) + M(\emptyset) &= \sum_{i \text{ odd}} \left[m_{14}(i) + m_4(i)\right] + \sum_{i \text{ even}} \left[-m_{13}(i) - m_3(i)\right] \\ -M(AB) - M(C_1 \ldots C_n) &= \sum_{i \text{ odd}} \left[-m_{13}(i) - m_3(i)\right] + \sum_{i \text{ even}} \left[m_{14}(i) + m_4(i)\right] \end{aligned}$$

and thus find that

$$\begin{aligned} 0 = & \sum_{i \text{ odd}} \left[m_{14}(i) + m_9(i) - m_{13}(i) - m_{10}(i) + m_{11}(i) + m_4(i) - m_{12}(i) - m_3(i) \right] \\ & + \sum_{i \text{ even}} \left[m_7(i) + m_4(i) - m_8(i) - m_3(i) + m_{14}(i) + m_5(i) - m_{13}(i) - m_6(i) \right] \end{aligned}$$

which is true if and only if

$$\bigwedge_{\substack{i \in \{1,\ldots,n\}, \\ i \text{ odd}}} \left[\langle A, C_i \mid B\, C_1 \cdots C_{i-1} \rangle \ \wedge\ \langle B, C_i \mid C_{i+1} \cdots C_n \rangle \right] \ \wedge$$

$$\bigwedge_{\substack{i \in \{1,\ldots,n\}, \\ i \text{ even}}} \left[\langle A, C_i \mid C_{i+1} \cdots C_n \rangle \ \wedge\ \langle B, C_i \mid A\, C_1 \cdots C_{i-1} \rangle \right]$$

are valid independence statements.

Given n is even we have that

$$\begin{aligned} & \sum_{i \text{ odd}} \left[m_1(i) + m_{15}(i) \right] + \sum_{i \text{ even}} \left[-m_2(i) - m_{16}(i) \right] = 0, \\ & \sum_{i \text{ odd}} \left[-m_2(i) - m_{16}(i) \right] + \sum_{i \text{ even}} \left[m_1(i) + m_{15}(i) \right] = \\ & \qquad M(\emptyset) + M(ABC_1 \cdots C_n) - M(C_1 \cdots C_n) - M(AB) \end{aligned}$$

and thus find, as in the case n is odd, that

$$\begin{aligned} 0 = & \sum_{i \text{ odd}} \left[m_6(i) - m_5(i) + m_8(i) - m_7(i) \right] \\ & + \sum_{i \text{ even}} \left[m_{12}(i) - m_{11}(i) + m_{10}(i) - m_9(i) \right] \\ & + M(\emptyset) - M(C_1 \cdots C_n) + M(ABC_1 \cdots C_n) - M(AB) \end{aligned}$$

Given n is even we moreover have that

$$\begin{aligned} & \sum_{i \text{ odd}} \left[m_6(i) + m_8(i) \right] = \sum_{i \text{ even}} \left[m_5(i) + m_7(i) \right] \\ & \sum_{i \text{ odd}} \left[-m_5(i) - m_7(i) \right] = \\ & \qquad \sum_{i \text{ even}} \left[-m_6(i) - m_8(i) \right] - M(A) - M(AC_1 \cdots C_n) + M(AC_1 \cdots C_n) + M(A) \\ & \sum_{i \text{ even}} \left[m_{12}(i) + m_{10}(i) \right] = \\ & \qquad \sum_{i \text{ odd}} \left[m_{11}(i) + m_9(i) \right] + M(B) + M(BC_1 \cdots C_n) - M(BC_1 \cdots C_n) - M(B) \\ & \sum_{i \text{ even}} \left[-m_{11}(i) - m_9(i) \right] = \sum_{i \text{ odd}} \left[-m_{12}(i) - m_{10}(i) \right] \end{aligned}$$

and that

$$\begin{aligned} & M(\emptyset) + M(ABC_1 \ldots C_n) - M(C_1 \ldots C_n) - M(AB) = \\ & \qquad \sum_{i \text{ odd}} \left[m_4(i) + m_{14}(i) \right] + \sum_{i \text{ even}} \left[-m_3(i) - m_{13}(i) \right] \\ & 0 = \sum_{i \text{ even}} \left[m_4(i) + m_{14}(i) \right] + \sum_{i \text{ odd}} \left[-m_3(i) - m_{13}(i) \right] \end{aligned}$$

and thus find again that

$$0 = \sum_{i \text{ odd}} \left[m_{14} + m_9 - m_{13} - m_{10} + m_{11} + m_4 - m_{12} - m_3 \right] + \sum_{i \text{ even}} \left[m_7 + m_4 - m_8 - m_3 + m_{14} + m_5 - m_{13} - m_6 \right]$$

which concludes the proof. □

Appendix B. Rules $A3$ to $A7$ and rules $G3$ to $G7$

Below rules $A3$ to $A7$ from [5,7] are given.

$$A3: \langle A, B \mid CD \rangle \wedge \langle A, B \mid \emptyset \rangle \wedge \langle C, D \mid A \rangle \wedge \langle C, D \mid B \rangle \leftrightarrow \langle A, B \mid C \rangle \wedge \langle A, B \mid D \rangle \wedge \langle C, D \mid AB \rangle \wedge \langle C, D \mid \emptyset \rangle$$

$$A4: \langle A, B \mid CD \rangle \wedge \langle A, D \mid B \rangle \wedge \langle C, D \mid A \rangle \wedge \langle B, C \mid \emptyset \rangle \leftrightarrow \langle A, B \mid D \rangle \wedge \langle A, D \mid BC \rangle \wedge \langle C, D \mid \emptyset \rangle \wedge \langle B, C \mid A \rangle$$

$$A5: \langle A, C \mid D \rangle \wedge \langle B, D \mid C \rangle \wedge \langle B, C \mid A \rangle \wedge \langle A, D \mid B \rangle \leftrightarrow \langle A, C \mid B \rangle \wedge \langle B, D \mid A \rangle \wedge \langle B, C \mid D \rangle \wedge \langle A, D \mid C \rangle$$

$$A6: \langle A, B \mid C \rangle \wedge \langle A, C \mid D \rangle \wedge \langle A, D \mid B \rangle \leftrightarrow \langle A, B \mid D \rangle \wedge \langle A, C \mid B \rangle \wedge \langle A, D \mid C \rangle$$

$$A7: \langle A, B \mid CD \rangle \wedge \langle C, D \mid AB \rangle \wedge \langle A, C \mid \emptyset \rangle \wedge \langle B, D \mid \emptyset \rangle \leftrightarrow \langle A, B \mid \emptyset \rangle \wedge \langle C, D \mid \emptyset \rangle \wedge \langle A, C \mid BD \rangle \wedge \langle B, D \mid AC \rangle$$

Below rules $G3$ to $G7$ plus a graphical representation (of an instance) of each rule (Fig. 5) from [1] are given.

$G3$: Let A, B, C_i, D_i, $i = 1, \ldots, n$, $n \geq 1$, be non-empty, mutually disjoint sets of variables, and let $\mathbf{C} = C_1 \cdots C_n$ and $\mathbf{D} = D_1 \cdots D_n$. Then, taking $Z_j \cdots Z_{j-1} := \emptyset$, for $\mathbf{Z} = \mathbf{C}, \mathbf{D}$

$$\bigwedge_{i \in \{1,\ldots,n\}} \left[\langle A, C_i \mid \mathbf{C_{i^-}}\, \mathbf{D_{i^{++}}} \rangle \wedge \langle A, C_i \mid B\, \mathbf{C_{i^+}}\, \mathbf{D_{i^{--}}} \rangle \wedge \langle B, D_i \mid A\, \mathbf{C_{i^{++}}}\, \mathbf{D_{i^-}} \rangle \wedge \langle B, D_i \mid \mathbf{C_{i^{--}}}\, \mathbf{D_{i^+}} \rangle \right] \leftrightarrow$$
$$\bigwedge_{i \in \{1,\ldots,n\}} \left[\langle A, C_i \mid B\, \mathbf{C_{i^-}}\, \mathbf{D_{i^{++}}} \rangle \wedge \langle A, C_i \mid \mathbf{C_{i^+}}\, \mathbf{D_{i^{--}}} \rangle \wedge \langle B, D_i \mid \mathbf{C_{i^{++}}}\, \mathbf{D_{i^-}} \rangle \wedge \langle B, D_i \mid A\, \mathbf{C_{i^{--}}}\, \mathbf{D_{i^+}} \rangle \right]$$

where $\mathbf{Z_{i^-}} = Z_1 \cdots Z_{i-1}$, $\mathbf{Z_{i^+}} = Z_{i+1} \cdots Z_n$, $\mathbf{Z_{i^{--}}} = Z_1 \cdots Z_{n-i+1}$ and $\mathbf{Z_{i^{++}}} = Z_{n-i+2} \cdots Z_n$. Note that for $n = 1$ the rule equals rule $A3$.

$G4$: Let $A, B, C_1, \ldots, C_n$, with $n \geq 2$ even, be non-empty, mutually disjoint sets of variables. Then,

$$\bigwedge_{i \in \{1,3,\ldots,n-1\}} [\, \langle A, C_i \mid \mathbf{C_{i^-}}\, B \rangle \wedge \langle B, C_i \mid \mathbf{C_{i^+}}\, A \rangle \,] \wedge \bigwedge_{i \in \{2,4,\ldots,n\}} [\, \langle A, C_i \mid \mathbf{C_{i^-}} \rangle \wedge \langle B, C_i \mid \mathbf{C_{i^+}} \rangle \,] \leftrightarrow$$
$$\bigwedge_{i \in \{1,3,\ldots,n-1\}} [\, \langle A, C_i \mid \mathbf{C_{i^+}}\, B \rangle \wedge \langle B, C_i \mid \mathbf{C_{i^-}}\, A \rangle \,] \wedge \bigwedge_{i \in \{2,4,\ldots,n\}} [\, \langle A, C_i \mid \mathbf{C_{i^+}} \rangle \wedge \langle B, C_i \mid \mathbf{C_{i^-}} \rangle \,]$$

where $\mathbf{C_{i^-}} = C_1 \cdots C_{i-1}$ and $\mathbf{C_{i^+}} = C_{i+1} \cdots C_n$, taking $C_j \cdots C_{j-1} := \varnothing$. Note that for $n = 2$ the rule equals $A4$.

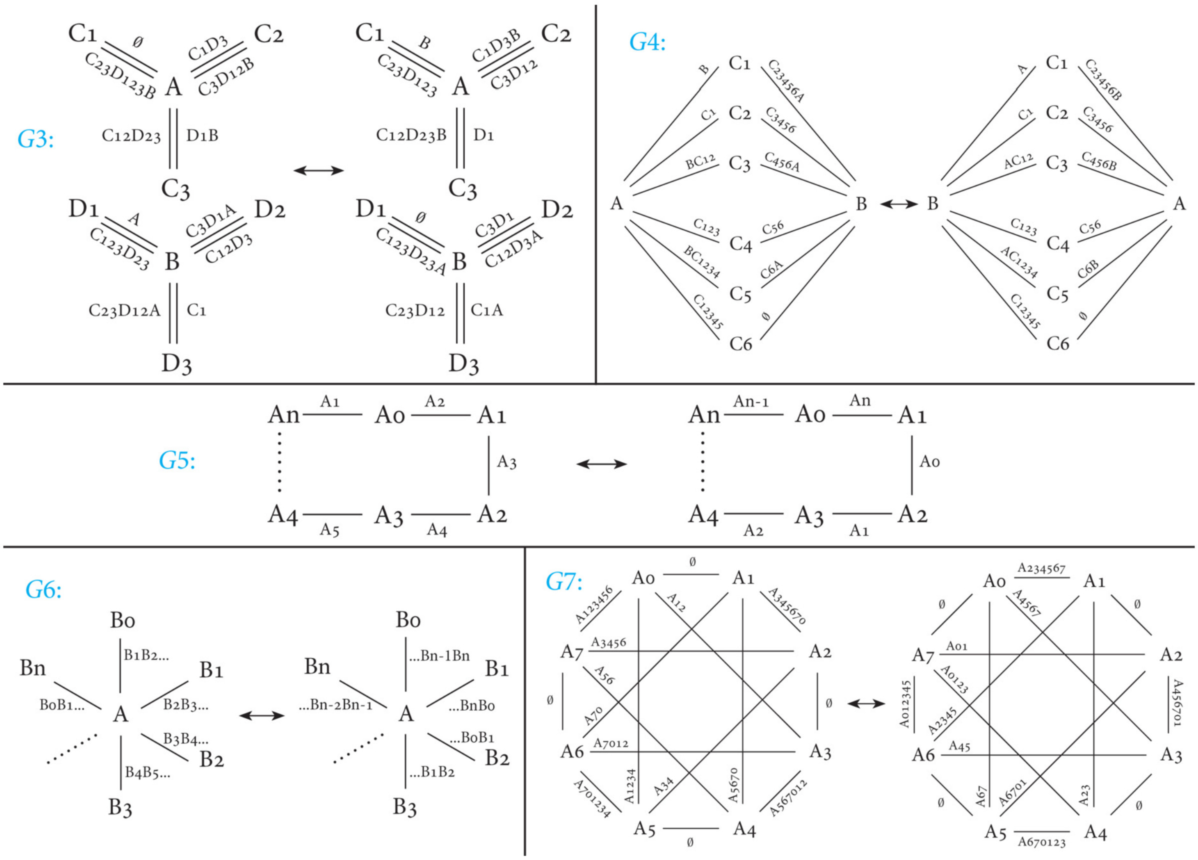

Fig. 5. A graphical representation of rule $G3$ for $n=3$, $G4$ for $n=6$, $G5$, $G6$ and $G7$ for $n=7$. Each $X \overset{Z}{\text{—}} Y$ in the figure represents a triplet $\langle X, Y \mid Z\rangle$.

$G5$: Let $A_0, \ldots, A_n, n \geq 2$, be non-empty, mutually disjoint sets of variables. Then,

$$\bigwedge_{i \in \{0,\ldots,n\}} \langle A_i, A_{\mu(i+1)} \mid A_{\mu(i+2)} \rangle \;\leftrightarrow\; \bigwedge_{i \in \{0,\ldots,n\}} \langle A_i, A_{\mu(i+1)} \mid A_{\mu(i-1)} \rangle$$

were $\mu(x) := x \bmod (n+1)$. Note that for $n=2$, $G5$ equals $A5$.

$G6$: Let $A, B_0, \ldots, B_n$, $n \geq 0$, be non-empty, mutually disjoint sets of variables. Then, for all $c \in [0, n]$,

$$\bigwedge_{i \in \{0,\ldots,n\}} \langle A, B_i \mid \mathbf{B}^c_{\mathbf{i}^+} \rangle \;\leftrightarrow\; \bigwedge_{i \in \{0,\ldots,n\}} \langle A, B_i \mid \mathbf{B}^c_{\mathbf{i}^-} \rangle$$

where $\mathbf{B}^c_{\mathbf{i}^+} = B_{\mu(i+1)} \cdots B_{\mu(i+c)}$, $\mathbf{B}^c_{\mathbf{i}^-} = B_{\mu(i-c)} \cdots B_{\mu(i-1)}$, taking $\mathbf{B}^c_{\mathbf{i}^+}, \mathbf{B}^c_{\mathbf{i}^-} := \emptyset$ for $c = 0$, and where $\mu(x) := x \bmod (n+1)$. Note that for $n = 0$, the rule equals $A6$.

$G7$: Let $A_0, \ldots, A_n$, with $n \geq 1$ an odd number, be non-empty, mutually disjoint sets of variables. Then,

$$\bigwedge_{\substack{i,k \in \{0,\ldots,n\},\\ i \text{ even}, k \text{ odd}}} \langle A_i, A_{\mu(i+k)} \mid \mathbf{A}_{\mathbf{i}^+} \rangle \;\leftrightarrow\; \bigwedge_{\substack{i,k \in \{0,\ldots,n\},\\ i \text{ even}, k \text{ odd}}} \langle A_i, A_{\mu(i-k)} \mid \mathbf{A}_{\mathbf{i}^-} \rangle$$

where $\mathbf{A}_{\mathbf{i}^+} = A_{\mu(i+1)} \cdots A_{\mu(i+k-1)}$ and $\mathbf{A}_{\mathbf{i}^-} = A_{\mu(i-k+1)} \cdots A_{\mu(i-1)}$, *taking* $\mathbf{A}_{\mathbf{i}^+}, \mathbf{A}_{\mathbf{i}^-} := \emptyset$ for $k = 1$, and where $\mu(x) := x$ mod $(n+1)$. Note that for $n = 1$, the rule equals $A7$.

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