Concrete-abstract-new-concrete: Freudenthal and Davydov in advancing embodied design framework

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Freudenthal and Davydov—two giants of mathematics education research—had a similar sophisticated vision on the position of mathematical models and symbols in abstract–concrete dialectics. In our vertical analysis of those approaches, we integrate them in a spiral *vision of mathematics learning. Starting from concrete enactment, students abstract previously latent aspects of their reality. Further, students fixate abstract ideas in mathematical artifacts, which in turn enable new concrete experiences. We show how local integration of two theoretical approaches can support an empirical analysis of embodied design for proportions and act as backing a design heuristic for embodied technologically supported activities.*

Introduction

The relation between abstraction and concrete experience inevitably lies in the core of mathematics education studies. Should concrete experience become a starting point for teaching abstraction? Can abstract ideas be derived from empirical observation or enactment with concrete objects, or shall abstract ideas be exposed to students directly, through 'symbolically structured environments' (Coles & Sinclair, 2019)? Traditional views on abstract and concrete—established within inductive empirical science—are questioned in contemporary investigations of mathematics learning, particularly by those who take an embodied stand.

The variety of visions on the problem of abstraction in mathematics learning creates a need to base theoretical work on a solid ground. In this paper, we coordinate the approaches of two giants in the field of mathematics education research, Davydov and Freudenthal, with respect to the role of mathematical artifacts and students' concrete experiences in teaching abstract ideas. Revealed similarities in the views of such major yet independent figures makes, in our opinion, the emerging theoretical proposal particularly strong and related design heuristics well backed up. The compatibility of views is based on Davydov's approach belonging to the Marxist tradition of cultural-historical activity theory, which can be aligned with Freudenthal's approach to mathematics as a human activity of mathematizing. Neither Davydov, nor Freudenthal used the term *artifacts*, however symbols, models, and visuals lied in the core of their ideas. We refer to these and any other instances of material culture developed within mathematical activity as *mathematical artifacts*. We interpret *cultural artifacts* as a broader category of entities developed within cultural practices.

The paper consists of two main parts. Firstly, we integrate Davydov's and Freudenthal's thinking into a joint view on concrete and abstract. For this local integration of theories (Bikner-Ahsbahs & Prediger, 2014) we conduct a *vertical analysis* (Shvarts & Bakker, 2021) as we go beyond comparing current states of the theories and dive into the history and philosophical roots of their development. Secondly, we apply the results of this theoretical analysis to guide an analysis of students' interaction with (cultural) artifacts within embodied design (Abrahamson & Sánchez-

García, 2016) and further suggest some design heuristics for technologically supported embodied activities. Two research questions guide our study: (1) What are Freudenthal's and Davydov's positions towards the role of mathematical artifacts in facilitating concrete experiences and abstract ideas? (2) How to introduce mathematical artifacts in embodied designs informed by the integrated Freudenthal-Davydov approach to mathematical abstraction?

Freudenthal and Davydov: concrete, abstraction, and mathematical artifacts

Freudenthal dedicated quite some publications to Davydov's approach (Freudenthal, 1974, 1977, 1979). Moreover, the Davydov's curriculum has been implemented in some Dutch schools, and compared to Wiskobas—a curriculum inspired by Freudenthal (Nelissen, 1987). In our view, Freudenthals' interest in Davydov is rooted in a deep agreement on the research and teaching methods. Within both research programs, intensive investigatory implementations were conducted in schools and referred to as *formative experiments* in Russia and *developmental research* in the Netherlands, presenting historical variants of what we know now as design research (Bakker, 2018). As we explain below, the similarities in the teaching methods convey insights on abstraction, the role of cultural artifacts, and the progressive development of concrete experiences.

(1) An abstraction is not based on recollection of empirical impressions

The fundamental innovation of the Davydov approach lies in an intensive critique of empirical thinking and pedagogy that treats abstraction as deriving from empirical examples (Davydov, 1990). As van Oers (2019) explains, Ernest Cassirer was apparently the first to criticize this type of abstraction because of the impossibility to limit the set of empirical observations from which to derive abstract qualities. Per Davydov (1990), new classes of objects are created within human practical activity, and theoretical thinking later describes those classes not through empirical observation but through transformative actions that reveal otherwise hidden properties. In the course of learning, students are to "develop special object-related actions by which they can disclose in the instructional material and reproduce in models the essential connection in an entity, then study its properties" (p. 174). Analyzing Davydov's approach, Freudenthal highly appreciated this perspective on abstraction and stressed that “abstraction and generality are—in many cases—not reached by abstracting and generalizing from a large number of concrete and special cases” (Freudenthal, 1974, p. 412). Later, Freudenthal tried out Davydov's approach of deriving arithmetic operations from practical actions with magnitudes—such as length and volume—with his grandson and found this approach to be effective (Freudenthal, 1977, 2002b, p. 102).

(2) Children need to reinvent mathematical models and symbols

Another point of the clear coordination between the approaches of Davydov and Freudenthal lies in addressing the role of mathematical models and symbols. Per Freudenthal, the mathematical activity consists of progressive schematization and algorithmization of solving problems that are meaningful for students. Those schematizations and algorithmizations are later preserved in the form of mathematical models and formalized rules (Freudenthal, 2002a). The rules preserve the history of problem-solving for those who came up with them in their own problem-solving. So, the only way for the learners to meaningfully extend their understanding of reality through mathematics, lies in reinventing mathematical rules and symbols. Otherwise, “having been imposed, they [rules and symbols], never had a real chance to develop into common sense of a higher order" (p. 8).

Davydov similarly assigned a primary role in scientific thinking to models, symbols, and signs. "Symbols and signs, as well as mixed forms of them, serve as the material means of idealizing and constructing scientific objectness" (Davydov, 1990, p. 121). Constructing these material means (artifacts) is exactly a process of abstraction, which fixates (reifies) the essential (for a practical activity) aspects of the object under investigation: "The construction of this new object [idealized model] functions as a certain mode of activity—as abstraction" (p. 117). In learning, children pass through a quasi-investigation, in which they uncover an essential (theoretical, mathematical) aspect of an object and reproduce it "in particular object-related, graphic, or symbolic models" (p. 174).

(3) Progressive development of concrete experiences

The origins of Freudenthal's ideas lie in the observation that mathematics education tends to inverse the development of mathematical ideas by presenting students the final products. He referred to his approach as phenomenological, and—although he insisted on the divergence with Husserl, Hegel, and Heidegger (Freudenthal, 2002b, p. 28)—he was apparently essentially influenced by phenomenological thinking. This influence can be traced in his ideas of developing *common sense*: in "the course of life, common sense generates common habits, in particular, where arithmetic is concerned, algorithms and patterns of actions and thoughts, initially supported by paradigms, which in the long run are superseded by abstractions" (Freudenthal, 2002a, p. 7). So, mathematical abstraction, such as arithmetic, derives from the common sense experience of acting and thinking. Further, those mathematical abstractions support later common sense experiences: "These products of common sense acquire in turn the behavioural status of common sense, while their common sense ancestry may have even been forgotten" (p. 7). Per Freudenthal, good mathematical education develops students' ability to *see* reality mathematically; mathematical symbolism is a lens for this *newly developed common sense*.

We find Freudenthal's idea of developing common sense to be close to the dialectical materialist method of *ascending from abstract to concrete* that Davydov exploited. This method does not mean presenting abstraction from the beginning. As Ilyenkov explains, "the ascent from the abstract to the concrete without its opposite, without the ascent from the concrete to the abstract would become a purely scholastic linking up of ready-made meager abstractions borrowed uncritically" (Ilyenkov, 2008, p. 137-138). So, abstraction starts from concrete experience, as well as progresses towards concrete experience: "the ascent from the concrete to the abstract and the ascent from the abstract to the concrete, are two mutually assuming forms of theoretical assimilation of the world, of abstract thinking" (p. 137). However, those two directions are not forward and backward. Abstraction reveals the latent aspects of initially experienced concrete reality, and those aspects further become salient in the theoretically grounded concrete perception of the objects. So, *ascending from the abstract to concrete* does not mean a detachment from the initial concrete experiences but a transformation of perception towards *seeing concrete objects in a new way—through* the lens of abstraction, facilitated by the artifacts as if superimposed on the perceptual field.

Concluding the theoretical analysis

Answering our first research question, we interpret Freudenthal's and Davydov's positions towards concrete experiences and abstract notions preserved by mathematical artifacts as converging in the following vision of the learning process. Students derive an abstract understanding from the concrete experiences within a specially organized practical problem-solving activity. This practical transformative activity elicits latent aspects of the world, which students fixate in cultural artifacts, such as mathematical models and symbols (movement from concrete to abstract). Having constituted an abstraction supported by the artifacts, students can put these artifacts into action and distinguish new initially latent aspects (movement from abstract to new concrete). In this iteration, students develop their *common sense* (in Freudenthal's words) or *ascend from abstract to concrete* (in Davydov's words) in establishing a theoretical vision of an object. Thus, we see students' development as *a spiral*: from concrete approaching the world in practical activities to abstracting latent aspects and fixating them in mathematical artifacts, and further towards establishing *new-concrete* perception mediated by those artifacts. From this approach, cultural artifacts emerge as reifications of the actions, which have elicited abstract features. Students need to actively constitute those artifacts to preserve the history of initial concrete enactment.

Concrete–abstract–new-concrete in implementing embodied design

Embodied action-based design is one of the quickly developing paradigms related to radical-embodied-enactivist-phenomenological reconsiderations within cognitive science (Abrahamson & Sánchez-García, 2016). The learning sequence in this design genre consists of a few major steps (Alberto et al., 2021; Abrahamson et al., 2020). At first, students are invited to solve a motor problem, i.e. discover a new coordination between their hands based on continuous feedback, and uncover the rule of positive feedback to their actions. Later, artifacts are introduced into the problem space, and students are guided towards the quantification of their experiences. Within this paradigm, the researchers have intensively questioned the position of embodied activities and cultural artifacts within abstract–concrete dialectics. In particular, they consider a sensorimotor scheme as "the epistemological core of mathematical learning and knowing" (Rosen et al., 2016, p. 1509), which can be further developed in both directions: towards abstract notions through semiotic signification by cultural artifacts, and towards concrete situations through providing context. Our empirical analysis of embodied activities through the lens of a joint view of Freudenthal and Davydov hints towards another role of the artifacts in abstract–concrete dialectics and further advances the design framework.

Stage 1. Action-based abstraction: Seeing new structures in concrete embodied experience

When solving a motor problem, students discover new abstract qualities,—at first at the embodied level and later in conversations with tutors—such as a proportional relation between the length of two bars, or a coordination of a unit circle circumference and an x-coordinate of a sine graph. Solving a motor problem is coherent with Freudenthal's and Davydov's ideas about abstraction as emerging from a goal-oriented practical actions. Technological environments restrict students' degrees of freedom, thus facilitating quasi-investigation, as Davydov would insist. Although restricted, the students appear to come up with a multiplicity of personal strategies and perceptual orientations, thus meeting Freudenthal's idea of reinventing rather than exposing culture. So, initial embodied enactment with concrete, tangible objects enables the discovery of abstract mathematical relations.

Stage 2. New-concrete: Looking through the lens of emerging cultural artifacts

While the phase of solving a motor problem has been extensively studied, researchers paid relatively less attention to the stage when artifacts are introduced. Therefore, we bring forth a small classroom episode from an experimental tryout of an embodied action-based design for proportions in an ordinary third grade (8-9 years old) classroom in the Netherlands (see a description of the data collection in Alberto, van Helden, Bakker, submitted). Four student pairs were video recorded and the following episode is selected to provide the best insights on the use of mathematical artifacts in establishing new understanding. Two girls (Iris and Frida) collaborated in the tablet-based activity: they manipulated two bars on a screen, which turned green when their lengths were in a particular fixed ratio. The girls were required to keep the bars green while moving and later "to guess a code" that determines the bars' green color, thus describing the proportional relation between the green bars' lengths. In the analysis, we contrast the use of two cultural artifacts: **a dice**, which was spontaneously appropriated by the students, and **a grid**, which was imposed by the educators.

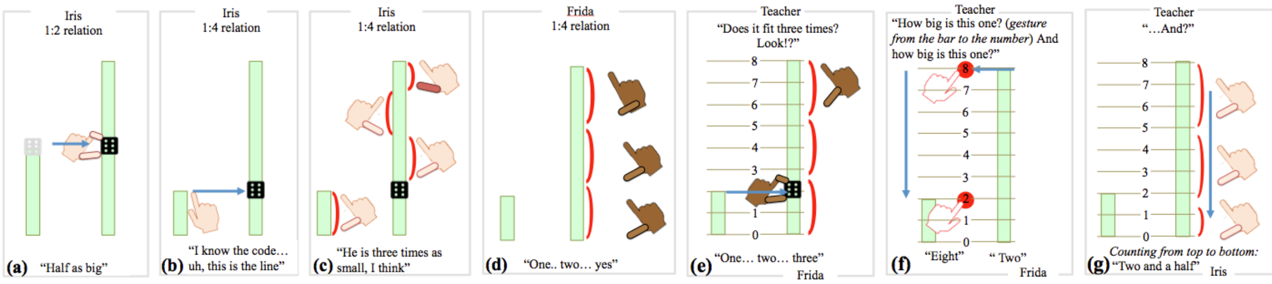

Fig. 1. (a, b, c, d): Seeing proportional relation through a dice. (e, f, g,): Missing proportional relation when applying a grid

By the moment of the episode, the girls have already solved the task with the bars being in ratio 1:2. In order to see that the length of one bar is doubled in the length of another bar, the girls spontaneously used a dice that was occasionally lying on a table: They positioned the dice in the middle of the big bar, marking that small bar fits in it two times (Fig. 1a). In the next task, the bars turned green at a ratio 1:4, but the girls did not know this yet. Adjusting one hand upwards somewhat slower than the other one, Iris found several green positions. She exhibited a general abstract strategy of maintaining two lengths in the same proportional relation, which needs to be concretized in quantifying the exact relation. Iris again took a dice and marked the length of the shorted bar on the longer bar (Fig. 1b), thus marking a unit of measurement that would help assess how many times the small bars would fit into the big one—a concretization of the abstract relation of "fitting into the other one." Then both girls tried to measure with their fingers how many times the length of the short bar would fit into the long bar (Fig. 1 c, d). The relation between two bars is seen through the lens of a cultural artifact, a dice, which here means *marking down a measurement unit*. The dice served as a reification of previous sensory-motor coordination; it allowed for distinguishing a *new-concrete*, new previously invisible aspects of the bars, namely a measurement unit that helps assess how many times one bar fits in another bar. A common sense action, in Freudenthal's terms, developed for Iris towards seeing the big bar as containing some number of small bars. Iris ascended from abstract coordination of two lengths to concrete quantification of their relation mediated by a dice (in a very physical sense). Unfortunately, a dice was barely an appropriate artifact for marking the small bar length: its own size distorted the measurement. As a result, the girls efficiently exploit an abstract idea of proportional relation as "fitting some number times in" but miscalculated the relation as 1:3 (combining two possible mistakes, see Fig 1 c, d).

The activity progressed towards the next stage of introducing cultural artifacts (Abrahamson et al., 2020), in which the girls were asked to confirm their code using an imposed grid (Fig. 1 e, f, g). With the help of a teacher, two bars were positioned at lines 8 and 2 and the bars were green. The teacher asked: "What does it mean?" expecting that 2:8 relation was obvious enough to dissolve students' 1:3 hypothesis. "Three times smaller" was the answer. The teacher invited the students to check: "Does it fit in three times? Look?" Frida did not use the grid, but took the dice (!), and marked the length of a small bar on the large bar by a horizontal gesture (Fig. 1e). Despite ignoring the imposed grid, by this horizontal gesture, Frida *re-invented* the functionality of grid's horizontal lines, as both artifacts serve the same function of marking equal units of measurement. The dice was big, and an approximate measurement led to the answer 1:3 again. Supporting the use of the imposed mathematical artifact, the teacher guided the students' perception towards the grid (Abrahamson & Sánchez-García, 2016): She gestured the horizontal alignment of the large bar and number 8 and then pointed at number 2, Frida read the numbers (Fig. 1f). However, their relation did not guide further enactment. Following the sequence of the teacher's gestures from top to bottom, Iris made a new measuring attempt counting from the top without clear measurement unit (Fig. 1g). She came up with an answer 2,5. The initial abstraction of "fitting in" was lost, and the students could not concretize (quantify) abstract proportional relation using the grid.

The dice was a natural continuation of the students' thinking and bodily enactment (see Shvarts et al., 2021 for conceptualization of this situation as a body-artifact functional system), and it allowed the girls' common sense development. By exploiting the dice, the girls could mark a measurement unit and concretize an abstract embodied idea of proportional relations in the given situation. Contrary, an imposed grid was not reinvented and stayed alien to the emerging abstraction. The teacher could see the bars' proportional relation naturally through the grid, while the girls could use the grid in this way. Their phenomenological realm did not the grid, contrary, a dice that became a mediator for a *new concrete*, i.e. for distinguishing new mathematical aspects of reality.

Towards a new design heuristic

As the theoretical and empirical analyses reveal, a cultural artifact might become a reification of practical actions, which helped to distinguish an abstract relation—a proportional relation between the bars, tangible as "small bar fitting into the big one." Importantly, an imposed mathematical artifact (a grid, see Abrahamson et al., 2020) did not fulfill this function, even with the teacher's guidance. Another artifact (a dice) spontaneously was appropriated by the students to reify an action of distinguishing a unit of measurement and served as an instrument in concretizing the ratio. Yet, this other artifact was barely appropriate for fulfilling this function. A design solution to this dilemma might be in creating an environment where students could spontaneously find suitable materials for creating the target artifacts. Such material might include thin sticks to mark the horizontal position of a small bar, paper stripes for creating a measuring unit and overlaying it on the big bar, or even a ruler, which was spontaneously and efficiently appropriated by some other pairs in the study. This way, a classroom can be enriched by appropriate means for progressive mathematizing/modeling of the situation, which could support establishing the perception and use of new mathematical aspects of concrete situations, thus distinguishing *new concrete*.

Concluding remarks

Freudenthal and Davydov are unique figures by the scale of their influence on the mathematics education and educational psychology communities. However, the program of each of them does not flourish nowadays in the countries where they were working despite intensive and successful experiment-based elaborations. Our analysis brings forth the complexity of their understanding of mathematical abstraction and concrete experience. Those approaches aim to develop in students a new ability to see an object concretely within its mathematical interrelations, i.e., developing *a new common sense*. From a concrete action-based experience, students come to distinguish abstract relations that are later reified in cultural artifacts. Further, the artifacts come to illuminate their *new-concrete* experiences. We hope that contemporary technologies can support students and teachers in fulfilling the aim of learning to see the world mathematically. This type of mathematics learning is in particular valuable for the 21st century with routine calculations being outsourced to the machines and increasing importance of skills such as mathematical modeling and recognizing mathematical patterns in everyday situations (Gravemeijer et al., 2017).

Looking back at the interaction of theoretical ideas and design heuristics, we notice that we used two prominent theoretical approaches as a way of backing a design idea that has been already emerging in our design work and empirical data. We uncovered the essential coherence of two approaches in seeing cultural artifacts as instruments that transform students' concrete experiences. The fact that those approaches are widely recognized as highly valuable strengthens the design heuristic of re-inventing mathematical artifacts and its potential for curriculum design.

References

Alberto, R., Shvarts, A., Drijvers, P., & Bakker, A. (2021). Action-based embodied design for mathematics learning: A decade of variations on a theme. *International Journal of Child-Computer Interaction*, 100419. https://doi.org/10.1016/j.ijcci.2021.100419

Alberto, R., van Helden G., Bakker, A. (submitted). Action-based embodied design for proportions: from the laboratory to the classroom setting. *Implement. and Replic. Studies in Math. Education*.

Abrahamson, D., Nathan, M. J., Williams-Pierce, C., Walkington, C., Ottmar, E. R., Soto, H., & Alibali, M. W. (2020). The future of embodied design for mathematics teaching and learning. *Frontiers in Education*, *5*, 147. https://doi.org/10.3389/feduc.2020.00147

Abrahamson, D., & Sánchez-García, R. (2016). Learning is moving in new ways: The ecological dynamics of mathematics education. *Jour. of the Lear. Sci.*, *25*(2), 203–239. https://doi.org/g8bp

Bakker, A. (2018). Design research in education. A practical guide for early career researchers. Routledge. https://doi.org/10.4324/9780203701010

Bikner-Ahsbahs, A., & Prediger, S. (2014). Networking as research practices: Methodological lessons learnt from the case studies. In *Networking of Theories as a Research Practice in Math. Education* (pp. 235–247). Springer, Cham. https://doi.org/10.1007/978-3-319-05389-9_14

Coles, A., & Sinclair, N. (2019). Re-thinking 'concrete to abstract' in mathematics education: Towards the use of symbolically structured environments. *Canadian Journal of Science, Mathematics and Technology Education 2019 19:4*, *19*(4), 465–480. https://doi.org/g8bk

Davydov, V. V. (1990). *Types of generalization in instruction: Logical and psychological problems in the structuring of school curricula* (J. Teller (ed.)). National Coun. of Teachers of Math.

Freudenthal, H. (1974). Soviet research on teaching algebra at the lower grades of the elementary school. *Educational Studies in Mathematics*, *5*(1), 391–412. https://doi.org/10.1007/BF00684710

Freudenthal, H. (1977). Bastiaan's experiments on archimedes' principle. *Educational Studies in Mathematics*, *8*(1), 3–16. https://www.jstor.org/stable/3482020

Freudenthal, H. (1979). Lessen van Sovjet rekenonderwijskunde [Lessons of Soviet mathematics educational science]. *Pedagogische Studiën*, *56*, 17–25.

Freudenthal, H. (2002a). *Revisiting Mathematics Education*. Springer. https://doi.org/fvhh2c

Freudenthal, H. (2002b). Natural numbers. In *Didactical Phenomenology of Mathematical Structures* (pp. 73–132). Kluwer Academic Publishers. https://doi.org/10.1007/0-306-47235-X_4

Gravemeijer, K., Lin, F. L., Stephan, M., Julie, C., & Ohtani, M. (2017). Reconsidering mathematics education for the future. In *Proceedings of the 13th International Congress on Mathematical Education* (pp. 659-660). Springer. https://doi.org/10.1007/978-3-319-62597-3_93

Ilyenkov, E. V. (2008). *The dialectics of the abstract and the concrete in Marx's Capital* (S. Syrovatkin (ed.)). Aakar Books.

Nelissen, J. M. C. (1987). *Kinderen leren wiskunde: een studie over constructie en reflectie in het basisonderwijs* . De Ruiter. https://www.librarything.com/work/3858598

Oers, B. van. (2019). The double move in meaningful teaching revisited. In A. . Edwards, M. . Fleer, & L. Bøttcher (Eds.), *Cultural-Historical Approaches to Studying Learning and Development.* (Vol. 6, pp. 119–133). Springer. https://doi.org/10.1007/978-981-13-6826-4_8

Rosen, D., Palatnik, A., & Abrahamson, D. (2016) Tradeoffs of situatedness: iconicity constrains the development of content-oriented sensorimotor schemes. In: M.B. Wood, E.E. Turner, M. Civil, J.A. Eli (Eds.), *38th Annual Meeting North Americal PME*, 2016: pp. 1509–1516

Shvarts, A., & Bakker, A. (2021). Vertical analysis as a strategy of theoretical work: From philosophical roots to instrumental and embodied branches (pp. 1–4). *A paper presented at 14th International Concress of Mathematics Education,* Shanghai, China.

Shvarts, A., Alberto, R., Bakker, A., Doorman, M., & Drijvers, P. (2021). Embodied instrumentation in learning mathematics as the genesis of a body-artifact functional system. *Educ. Studies in Mathematics*, *107*(3), 447–469. https://doi.org/10.1007/s10649-021-10053-0