Introducing Density Histograms to Grades 10 and 12 Students: Design and Tryout of an Intervention Inspired by Embodied Instrumentation

Lonneke Boels and Anna Shvarts

Abstract Density histograms can bridge the gap between histograms and continuous probability distributions, but research on how to learn and teach them is scarce. In this paper, we explore the learning of density histograms with the research question: How can a sequence of tasks designed from an embodied instrumentation perspective support students' understanding of density histograms? Through a sequence of tasks based on students' notions of area, students reinvented unequal bin widths and density in histograms. The results indicated that students had no difficulty choosing bin widths or using area in a histogram. Nevertheless, reinvention of the vertical density scale required intense teacher intervention suggesting that in future designs, this scale should be modified to align with students' informal notions of area. This study contributes to a new genre of tasks in statistics education based on the design heuristics of embodied instrumentation.

Keywords Density histograms · Design-based research · Embodied design · Statistics education

1 Introduction

Histograms are ubiquitous in newspapers, textbooks and research, as well as in online blogs, vlogs and television broadcasts. In education, histograms are considered as supporting the transition to continuous probability distributions (Wild, 2006), and density histograms are of key importance for this transition (Behar,

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2021; Derouet & Parzysz, 2016). Research on how to teach density histograms is scarce (Boels et al., 2019a; Reading & Canada, 2011). Besides its role in statistics and probability, density is also important for other topics taught in Grades 10–12:

> The notion of density is commonly met in other matters, such as geography (population density), physics (voluminal mass), biology (cell density), and so on. In statistics, it is implicitly at play in histograms with unequal classes, but this feature remains often implicit or is not taken into account, the more so that spreadsheet [*sic*] cannot produce such histograms and hence can be no help (Parzysz, 2018, p. 70)

In density histograms, density is along the vertical axis and bin widths may vary (Fig. 1). In statistics, density is often given in percentages instead of absolute numbers (e.g., Freedman et al., 1978), but relative frequencies—a number between 0 and 1—can also be used. The latter is always used in probability density functions, such as a normal or Poisson distribution. Density can be calculated as:

$$age\ density = \frac{(relative)\ frequency\ or\ percentage\ of\ people}{age\ group\ \ period}$$

In density histograms, the area of a bar is proportional to the number of cases in that bar. Consider the example of the number of people per ten-year age groups (Fig. 1). The density along the vertical axis is the number of people in thousands per ten-year age group and is approximately 154,000 for the age group 15–19 or [15–20) in mathematical notation. Since this bin width is half of 10 years, 154,000 must be divided by two to get an estimated number of 77,000 people in this age group.

The aim of the present study is to explore how students can be supported in understanding density histograms. To this end, we designed a sequence of tasks inspired by ideas on embodied instrumentation—an approach that theorizes the role of artifacts such as technological tools, symbols, and graphs in teaching and

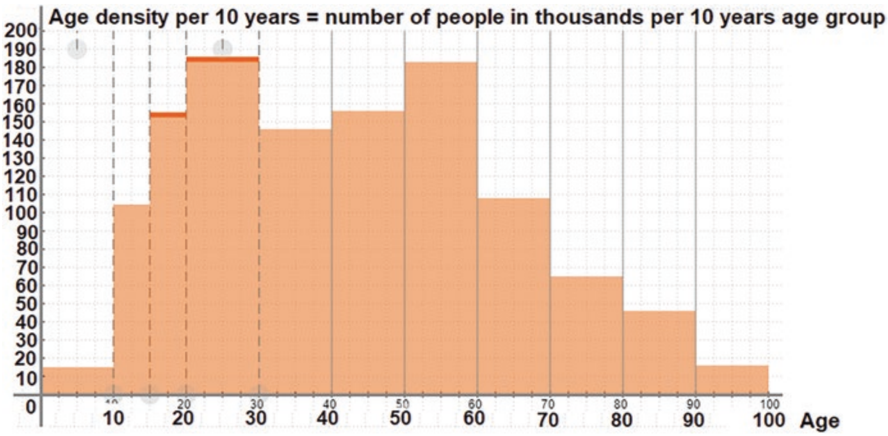


Fig. 1 Example of a density histogram with unequal bin widths. Number of people tested positive for COVID-19 in a country

learning mathematics from an embodied perspective (Drijvers, 2019; Shvarts et al., 2021). It is aligned with our intention to understand how new artifacts such as density histograms can be meaningfully introduced by taking advantage of the opportunities provided by technological environments. Therefore, we investigate the question: How can a sequence of tasks designed from an embodied instrumentation perspective support students' understanding of density histograms?

The learning trajectory is designed for Grades 10–12 pre-university track students (15–18 years old) in the Netherlands. A design research approach is used in a multiple case study with five students. Design research is an approach that aims to both test theories and contribute to further theorization of the field (Bakker, 2018), in our case, theories on how the learning of density histograms can be promoted and students' conceptions of density in histograms can be developed. Suggestions for redesign are an essential part of design research.

We discuss the results of an empirical study of a design on density histograms. Based on these results, we suggest ideas for redesign and elaborate on theoretical outcomes. With this study, we aim to inform educational practice and to add to the literature on how to teach density histograms. In addition, we hope to contribute to further usage and design of tasks from the theoretical perspective of embodied cognition and instrumentation.

When designing tasks, both form and content are relevant. In the succeeding section, we first review the literature on density histograms. Next, we address key elements of theories on embodied cognition and instrumentation.

2 Theoretical Background

2.1 Review of the Literature on Density Histograms

Histograms can support students in understanding key statistical concepts such as distribution. Density histograms are a special type of histogram. Students and teachers alike often misinterpret regular histograms (e.g., Boels et al., 2019a, b, c; Roditi, 2009). Density histograms are considered of key importance for the transition from regular histograms to continuous probability distributions (Behar, 2021; Bakker, 2004). However, the literature on how to teach density histograms is scarce, and density histograms are barely taught (e.g., Derouet & Parzysz, 2016; McGatha et al., 2002). For example, in France, teachers spend on average only 1 hour each year on histograms (Roditi, 2009) and many of them think that teaching histograms is not necessary in mathematics education.

In a French study Derouet and Parzysz (2016) analyzed Grade 10 textbooks on their usage of histograms—including density histograms—as well as how density histograms in Grade 12 textbooks were used as a preparation for the introduction of probability density curves. According to the authors, histograms are not taught in Grade 11. They found several misinterpretations regarding density histograms in the textbooks, including: (1) usage of a histogram with unequal bin widths, no scaling, and the word absolute frequency along the vertical axis instead of frequency per something; (2) using area or nothing along the vertical axis; (3) showing the frequency at the top of a bar in a histogram with unequal bin widths. In addition, French textbooks rarely give students the opportunity to summarize data in a table and construct a density histogram from this table. Furthermore, when a formula is presented to students to calculate density—height of a bar = absolute frequency/bin width—the word density never appears, and this formula is not justified. Derouet and Parzysz (2016) propose a learning trajectory from histograms to probability curves and functions to integrals. They give several suggestions for task design of which the following are taken into account in the trajectory presented here: proportional calculation such as for the heights of bars for unequal bin widths, comparing areas, and choosing and changing bin widths.

In another study aiming to inform future design of a learning trajectory on collecting and analyzing data, seventh graders were assessed on their initial understandings (McGatha et al., 2002). The task was to analyze survey data from 30 students about the number of hours per week they watched television. Students were asked to summarize the data, and most groups used a graph. One group used unequal bin widths (e.g., 11–20 and 21–25) with the number of students on the vertical axis. This seems to have gone unnoticed during class discussion. “Students are primarily concerned with school-taught conventions for drawing graphs”, instead of what the graphs signify (p. 348).

University students’ textbooks can also cause confusion (Huck, 2016). For example, besides the incorrect definition of histograms as bar charts for which the rectangles touch, a Gilmartin and Rex (2000) textbook concentrates on details of bin widths instead of density and area:

> Most histograms have intervals that are the same size but occasionally you may be asked to draw one with intervals of differing sizes. It is very important that you always check the size of each interval, as the width of each rectangle should correspond to its interval size. (Gilmartin & Rex, 2000, p. 21)

University students in an introductory statistics course were not able to correctly interpret areas in frequency histograms, nor were they able to compute these when a change in intervals at the tails of the graphs made it necessary (Batanero et al., 2004). In another study, graduate students taking a non-compulsory introductory statistics class constructed histograms based on a frequency table but “failed to hold […] the intervals constant across the x-axis” (Kelly et al., 1997, p. 87).

How density histograms can best be introduced is not clear from the literature we found. Biehler (2007) compares density in a histogram with density in a boxplot. Gratzer and Carpenter (2008) explain to mathematics teachers what a density histogram is and when it should be used. Finzer (2016) had students investigate the effect of changing bin widths on the shape of a histogram with equal bins. Lee and Lee (2014) claimed that density histograms are often overlooked in curricula materials.

Based on experience in classrooms, Lee and Lee expected that many students would not be able to use area to estimate a proportion from a histogram if frequencies were not given, as this does not align with the procedures learned from textbooks.

Taken together, the literature on density histograms suggests that:

- students' understanding of density histograms needs to be developed;
- there is limited research on which to build the current study;
- tasks need to include at least choosing and changing intervals;
- interpretation of area plays a role but might be difficult in histograms;
- tasks need to focus on what a graph signifies rather than on the procedure of drawing a graph or calculating heights of the bars.

Several findings from a review study on students' difficulties with histograms (Boels et al., 2019a) were used in designing tasks for the present study. Two examples are: (1) graphs without context should be avoided (e.g., Cooper & Shore, 2010), so all our tasks have context; (2) as density histograms appear impossible to create in most professional statistical software, we used tailor-made tasks in Numworx ©.[1]

2.2 *Embodied Cognition and Instrumentation*

Our task design is inspired by an embodied view on mathematics teaching and learning. There are different variations of these theories (e.g., Abrahamson et al., 2020), but they share the idea that thinking is based on sensorimotor experiences such as movements, touch, vision, hearing. For example, most people who can ski cannot explain exactly how they do it; their bodies adapt.

As mathematical cognition is inextricably linked to artifacts usage, we also rely on an instrumental approach, which argues that artifacts are a constitutive part of instrumental actions (Artigue, 2007; Trouche, 2014; Verillon & Rabardel, 1995; Vygotsky, 1997). Learning cannot be separated from acquiring the use of a new artifact—skiing without skis makes no sense. Similarly, key concepts in statistics—such as density distribution—cannot be learned without statistical graphs such as histograms (Bakker & Hoffmann, 2005). This requires instrumental genesis, the process in which the user learns to interact with the artifact (e.g., tool, symbol, graph) by which it becomes an instrument for further actions. An instrument here is, therefore, a mixed entity that includes both the artifact and knowing when and how to use it—in order to ski, one puts down the skis, clicks the boots into the bindings, and heads down the hill. In addition, instrumental genesis also involves selecting or even designing suitable artifacts. Think of choosing snowshoes that have a large footprint for easier walking in snow or using flippers to swim faster. Similarly, histograms need to be selected or designed and further used in a mathematically appropriate manner.

Core to the *embodied* instrumentation approach is that learning is seen as the development of a body-artifacts functional system (Shvarts et al., 2021). In such a system, there is no homunculus inside of the head who regulates movements step-by-step. Instead, new forms of instrumental actions emerge when using skis to learn

[1] https://www.numworx.nl/en/

to ski down a hill: the learner barely can say how exactly the skies are used, yet a body can manage it well; the body and the skis form a body-artifact functional system. In an embodied instrumentation approach, statistical understanding emerges from *intentional* bodily actions—solving problems through multimodal sensorimotor interactions with the environment that includes artifacts—and reflection on those interactions. We briefly outline the main theoretical statements of embodied instrumentation and explicate the six design heuristics driven by those statements (e.g., Boels et al., 2022b). First, every human activity unfolds in response to some problem in the environment (Wilson & Golonka, 2013). For example, even descending a small hill on skis initially is a motor problem (Bernstein, 1940/1967) for which a new bodily functionality develops: a new functional dynamic system of skiing. Similarly, learning mathematics can be reconsidered as solving motor and perceptual problems as a learner learns to move in a mathematical way and see the world mathematically (Abrahamson, 2019; Abrahamson & Sánchez-García, 2016; Radford, 2010). The design heuristic based on this principle is to provide students with a set of problems with which they *productively struggle* while actively seeking a solution.

Second, a particularity of a functional system is that it never has a strict answer to the problem but rather exhibits an emergent, self-organized behavior that allows for flexibility across environments while still being able to achieve the target state (Bernstein, 1967). This development is fostered by presenting tasks in different environments and contexts—digital and on paper (*transfer tasks*).

Third, cognition develops in an interaction between humans and the environment in response to an underlying problem (Varela et al., 1991). This interaction can be traced as a system of sensorimotor processes and perceptual action loops, such as noticing a bump on a ski slope and going around it. Another example is projecting a height of a bar in a graph onto the vertical axis to assess its height (Boels et al., 2022a). For this reason, we provided students with an environment that allowed for sensorimotor processes and direct mathematical actions with mathematical objects, rather than having students manipulate sliders or enter numbers and let the digital environment perform the mathematical actions. Therefore, *students perform these mathematical actions.*

Fourth, while in solving a motor problem (skiing down a hill), a new movement is enough, in learning mathematics, students also need to reflect on their performance and articulate how it was done (Abrahamson et al., 2020; Alberto et al., 2022). To make sure that students go beyond manipulation and *reflect* on their enactment, we supplemented all tasks with reflective questions requiring students to explain why such a result is achieved.

Finally, mathematical artifacts are an essential part of mathematical thinking and understanding. They are part of students' perception-action loops for mathematical problem solving (Shvarts et al., 2021). We consider artifacts as crystallized histories of mathematics, hence, as efficient reification of cultural practices (Leontyev, 2009; Radford, 2003). When students encounter those artifacts in their learning, the artifacts need to be embedded into their body-functional systems. But when the artifacts are ready-made, they can be embedded without understanding, as we may never think about why skies have a particular shape or how a smartphone functions. Yet, understanding mathematical artifacts requires students to *reinvent* those artifacts by their own actions: artifacts need to become crystallizations of meaningful procedures, thus come to reify actions (Shvarts et al., 2022). For example, as students try to visualize the frequency for different outcomes, they might reinvent histograms. This last theoretical statement unfolded for us in two design heuristics. First, we reflected on what actions formed the *target artifact*—density histograms. Second, we *melted—decomposed—this artifact back into actions* (Shvarts & Alberto, 2021), the hypothetical source of its historical development.

When applying this last design heuristic to density histograms, we came to an overview of the main actions with previously acquired artifacts that led to the constitution of density histograms according to our logical-historical reconstruction (Fig. 2). We made sure that the mathematical *problems we exposed students to promote those actions as solutions*, thereby allowing *reinvention* by the students of the target artifact. In summary, our design heuristics are: (1) distinguish the target knowledge and melt this back to its constituting actions (2) create motor-control or perception problems in which students reinvent artifacts (3) create productive struggle, (4) have students perform the (digital) actions, (5) include reflection, (6) create

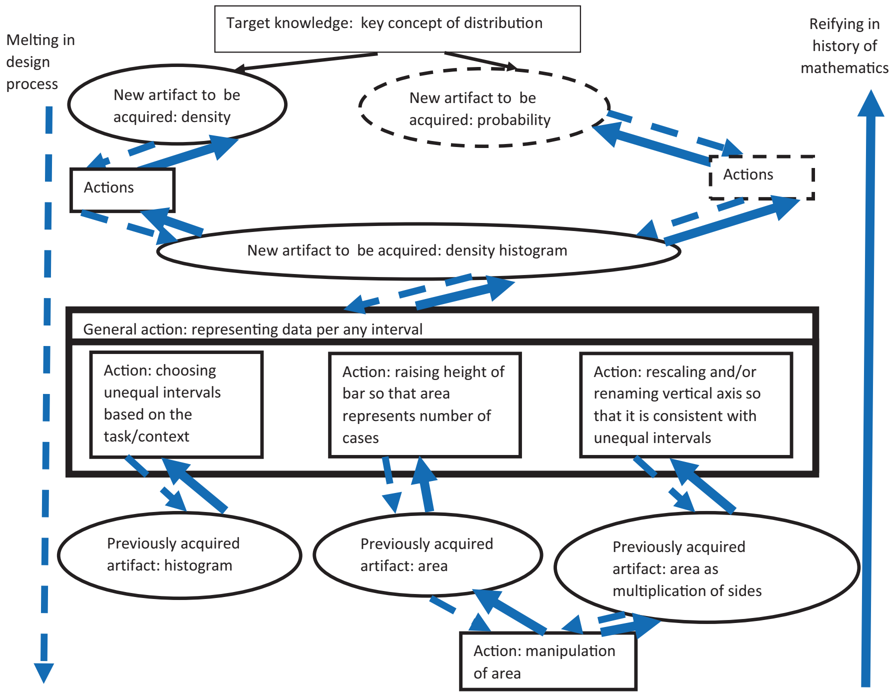


Fig. 2 Melting (decomposing) artifacts to actions that constituted them in the history of mathematics. The tasks in our design concentrate on the general action of representing data per any interval

possibilities for transfer. Those heuristics allowed us to develop a technological educational environment and a sequence of statistical problems. Aiming to solve these problems by sensory-motor interaction within the environment, the students would reinvent the target action and further target artifacts in their own sensory-motor activity of problem solving.

In this paper, we focus on the target action of *presenting data per any interval* to reinvent a density histogram artifact. Previously acquired artifacts are not part of our design since we are building on previous education. For example, in primary education, students worked with area and we assumed they acquired it as an artifact, and in the learning trajectory preceding the present one, students acquired knowledge of a histogram.

3 Materials and Method

The present study represents one part of a larger design research study, which showed that students could successfully reinvent the role of the horizontal and vertical scale in histograms and reinvent regular histograms (Boels et al., 2022b).[2] Some reflection tasks belonging to the digital tasks were presented in the lesson materials on paper. For all lesson materials (in Dutch), readers can contact the first author. Furthermore, the italic words in the task description refer to the design heuristics discussed in the theoretical background with numbering corresponding to the summary at the end of that section.

3.1 Reinventing Bin Widths

In this task, students completed a histogram (Fig. 3) by adding bars between 10 and 30 years using the given frequency table. The age group 15–19 years contains all people from the age of 15 up to the day before they turn 20. The aim of this task was that students should perceive that bin width is a choice, see Fig. 2, in line with our design heuristic to *distinguish the target knowledge and melt this back to its constituting actions (1)*. By the design of this task, students were given the choice of bin widths of 5 or 10 years for the age group 5–29 years old or [5, 30) although even bin widths of 15, 20 or 25 were possible—in line with our design heuristic to *create a perception problem in which students reinvent artifacts (2)*. When choosing bin widths of 5 years, a *productive struggle (3)* could occur because this age group is unlikely to contain all students from a 10-year age group. *Reflection (5)* was stimulated by questions posed in the lesson material on paper, such as: “How can you

[2] All digital tasks can be found here: https://embodieddesign.sites.uu.nl/activity/histograms/

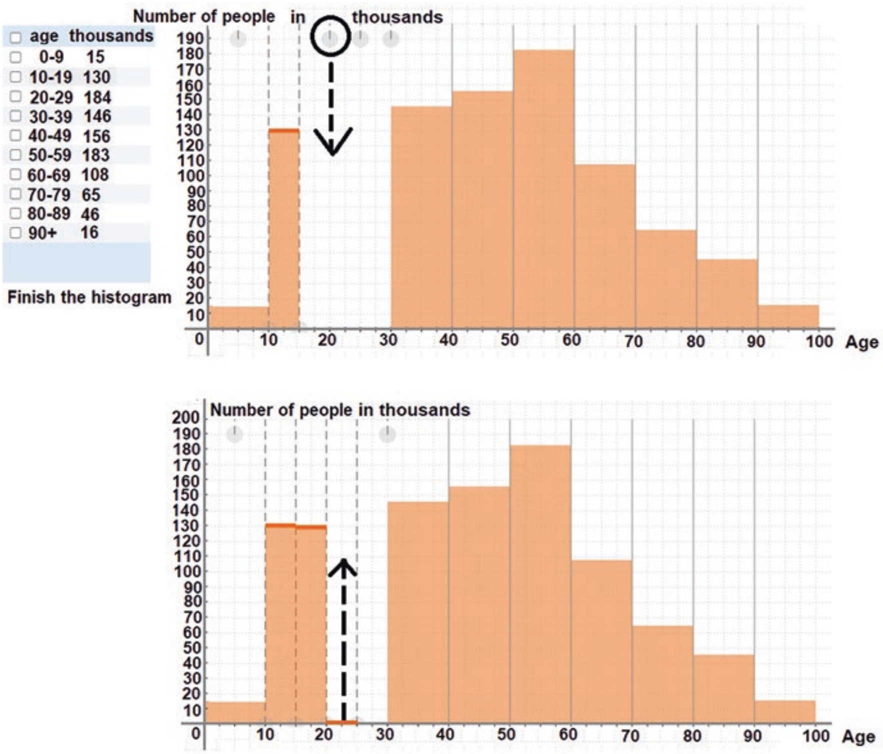


Fig. 3 Screenshots for completing a histogram from a frequency table. Students draw down sliders—indicated by the black circle (top)—after which a strip appears on the bottom of the bin. Next, they pull up this strip to the desired height (bottom). Black circle, arrows, translation, and thicker numbers added for readers convenience

check by yourself if you have performed this task correctly? Why? Perform this check."

What is new in the design of this and the next task is that students moved their hands in a specific way to *create (4)* things happening, such as raising a strip up to build a histogram instead of selecting the correct option in the software and seeing things happening or—even more common—the result of the hidden action. In a technological environment, we can constrain the possibilities and, therefore, provide better support for students so that their *struggle* remains *productive (3).*

The role of area comes into play when students compared the bars to the previous task. In this previous task, not reported here, students created a regular histogram with 10-year wide bars from the same frequency table as in the present task. In this table, the number of people per five years is not given while in the present task—by design—students were invited to make bars per five years. The total area of two bars together—for example, for the 10–19 years age group—is as large as the area of the 10-year wide bar in the previous task.

3.2 Reinventing Density in a Histogram

The aim of this task was that students *reinvent (2)* density in a histogram. Students needed to *perform two actions (4)*: raise the height of the bar so the area represents the number of cases and change what is along the vertical axis so it is consistent with the unequal bin widths—in line with the design heuristic *distinguishing the target knowledge and melt this back to actions (1)*. The finished previous task was intended to reappear for this task, but this did not technically work. Therefore, students first needed to complete the histogram once again (see Fig. 3).

Next, students were given the context of a school principal who wants to know if different COVID-protection measures should be taken for the age groups 10–14 and 15–19 years. For this, new numbers were reported for these specific age groups: 51,903 and 77,820. Students were asked to split the bar for the 10–19 year age group into two, proportional to the number of students in the 5-year age groups, in such a way that the total area of the histogram does not change (Fig. 4). The intention of this split was, in combination with the question that follows, to *create a productive struggle (3)* and *induce reflection (5)*. The first question was what now is represented along the vertical axis. Students could choose the following options that were given on paper: "Number of people in thousands, or this per five-year age group, per one-year age group, or per ten-year age group". The next question was: "Why?"

The last questions concerned what to advise the school principal about different COVID-measures per age group and what the student discovered about a histogram with unequal bin widths. Note that in this task students do not yet work with density

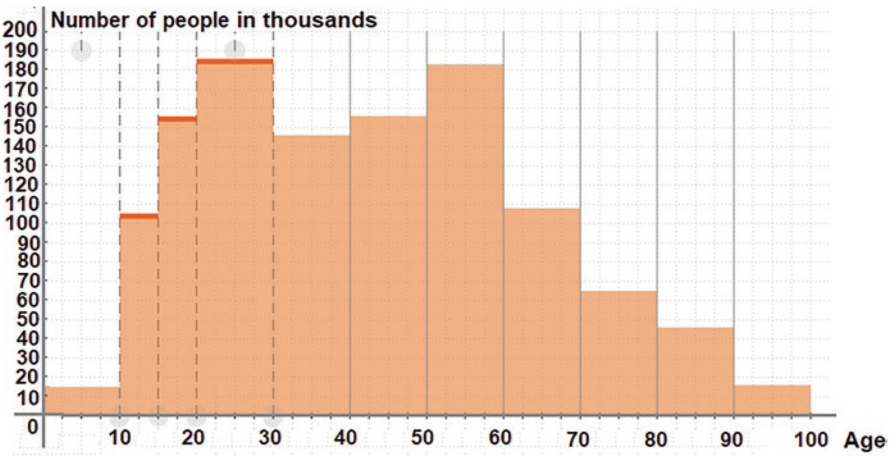


Fig. 4 Completion of the histogram after proportional splitting. Thicker numbers added for convenience of the reader

in percentages or relative frequencies but instead use people per age group to avoid possible difficulties with proportions or percentages.

3.3 Confirming learning: Transfer

Between the reinventing density task and the present task was a transfer task not reported here that involved the construction of a regular histogram on paper based on a frequency table in a realistic context. It *created possibilities for transfer* (6) as both the environment—paper—and the context were new. This context was reused in the present paper task, but now a histogram was provided with more fine-grained income groups. In the present task, we decided to explore students' notions of area in a histogram by providing students with a histogram without frequencies listed (Fig. 5) as suggested by Lee (1999). As there is no label on the vertical axis, students could solve this task by reasoning with only the area of the given histogram. This was new to them and, therefore, also a transfer task. The context and first question in this task was:

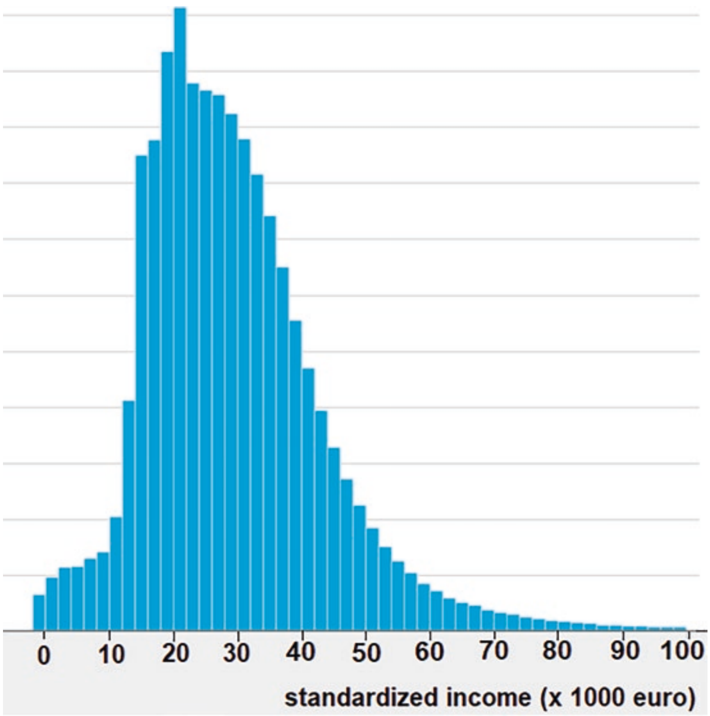


Fig. 5 Distribution of net annual income [in thousands of euros] in the Netherlands. Thick numbers and text added for convenience of the readers. (Figure source: Statistics Netherlands (CBS))

> In reality, the distribution [of annual income, Figure 5] is much more skewed than you might think given the histogram from the preceding task [task 18]. About what percentage of households has less than 27000 euros of disposable income per year? Explain how you arrived at your answer.

3.4 Method

3.4.1 Participants

The materials were tested with six pre-university Grades 10 and 12 students over two mornings. Each morning, two students worked in a pair and one student worked alone. One alone working student was excluded from the data analysis, as special educational needs made it impossible for this student to work with the materials. The five remaining students were 15–18 years old, two males and three females. They all took mathematics A, which stands for applied mathematics and includes an introduction to statistics. Students were recruited from different schools in Utrecht. The trajectory presented here took roughly 40–50 minutes. A 30-euro fee was given to the students for their participation, and approval from the Science-Geosciences Ethics Review Board and consent from the participants and legal representatives—if necessary—was obtained. Age, grade, math level and mathematics mark were collected. Furthermore, students' pre knowledge was tested with a questionnaire. Discussions between students and with the teacher-researcher (first author) and observer (second author) were audio- and videotaped. Students' grid papers and writings on lesson materials were also collected.

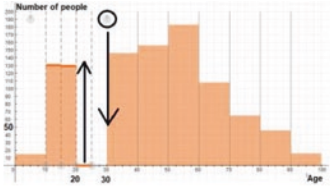
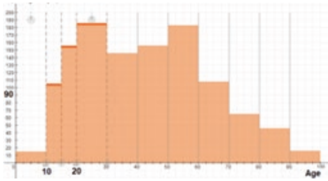
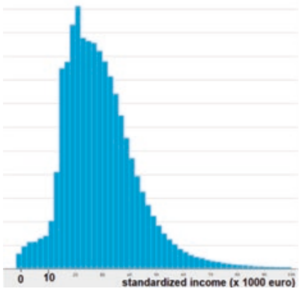
3.4.2 Intervention in the Teaching and Learning Lab

Students worked in the Teaching and Learning Lab (TLL) of the Freudenthal Institute at the Utrecht University. Lesson materials were partly on paper but the embodied design tasks were on a digital whiteboard, which was connected to a laptop. All students finished almost all 22 tasks of the whole learning trajectory as well as the questionnaire on their pre knowledge.

3.4.3 Data Analysis—Conjectured Learning Trajectory

For the data analysis, we used a conjectured learning trajectory (e.g., Simon & Tzur, 2004) which starts after students used messy dotplots and reinvented histograms (Boels et al., 2022b). Sequence is an important part of our design (Bakker, 2018). In the present learning trajectory (Table 1), students started with reinventing bin widths while completing a histogram from a given frequency table (task 16) and reinvented a density histogram from the same data in task 17. Furthermore, we added a task (19) on paper.

Table 1 Summary of the conjectured learning trajectory

Step	Task	Student activities	Conjecture
1 Reinventing bins	Task 16 	Students choose bin widths by deciding which slider to pull down.	H16: By dragging down (dotted) lines which create bins, students perceive that bin width is a choice.
2 Reinventing density histogram	Task 17 	By splitting one bin into two bins, students need to rethink what the vertical axis is depicting	H17a: By keeping the total area of two smaller (5 years) bins equal to the area of the former larger (10 years) bin, students discover that the vertical axis in a density histogram does not depict frequencies. H17b: By keeping the total area of two smaller bins equal to the area of the former larger bin, students question what the vertical axis depicts and, in this way reinvent a notion of density (e.g., number of people per 5 years) in a histogram with unequal bin widths.
3 Confirming learning: transfer task	Task 19 (Data: CBS) 	Students estimate percentages from a total area	H19a: By estimating the percentage of a population in a more fine-grained version of a previously drawn graph, students notice that the total area of all bars represents 100% of the data points and that this can be used to estimate the percentage for a proportion of the total area (i.e., the population). H19b: In addition, students perceive that it is impossible to give an exact value for the percentage or proportion of people if a measured value is cutting a bar into two portions.

4 Results

4.1 Reinventing Unequal Bin Widths Task 16

Most students had no difficulties with the task itself. However, they did not seem to reinvent bin widths. Students S1, S2, S4 and S5 worked together in two pairs and created bin widths of 5 years age groups (Fig. 6). After creating these bins by pulling all sliders down, they discussed how to pull the strips up:

S5: This is from 10 to … [age; horizontal axis]
S4: 130 [the number in thousands; vertical axis]
S5: but what is the age? 10 to 19
S4: yes
S5: but that's both of these [both bars, namely from 10–14 and from 15–19]
S4: yes, you have to slide both of those then. To 130.
S5: ok. Oh no, not this one, this one. Come on guys. 130
S4: yes
S4: [long pause]. Well! [Sounds little bit frustrated] I want to be exact [bars' heights do not align perfectly due to some technicalities in the software]
S5: No, [inaudible] don't have to look perfect ha ha [laughs a bit]

Student S3 created bins of 10 years. Although this student did not encounter any difficulties in completing the histogram, he also easily stepped over the possibility of creating smaller bins. The students that used smaller bins did not struggle with it. They all almost immediately pulled up both smaller bars for 5 years age groups to the number that belonged to the corresponding 10 years age group. Taken together, the written materials and videos do not provide evidence that supports the conjecture (H16) that students perceive bin widths as a choice. Instead, their notions of keeping the area for two bars together the same as for one wider bar seemed to be so strong that they do not question the height of the bar.

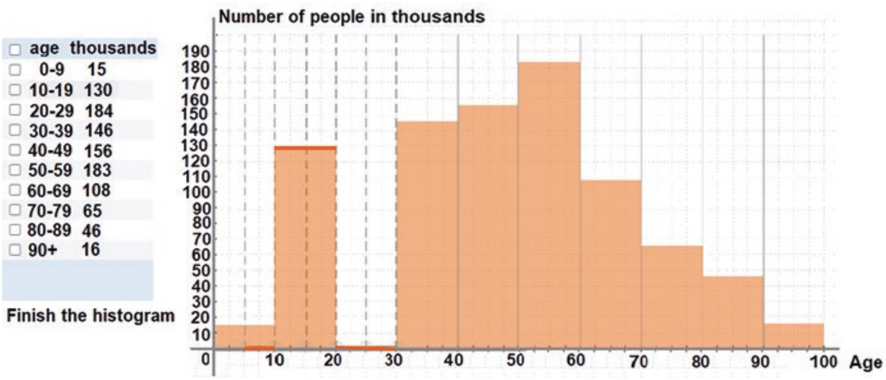


Fig. 6 Students S4 and S5 created bins of 5 year at the start of task 16

4.2 Reinventing Density Histogram Task 17

Again, students S1, S2, S4 and S5 dragged down all sliders, thus creating five bins each of 5-year widths. S3 used two bins of 10-year widths. Task 17 caused a lot of struggles. An example of S1 and S2 who worked together is below. A long intervention by the teacher was needed to have them reinvent the density scale along the vertical axis, although the first intuition of student S2 when questioned by the teacher was correct (Fig. 7, left):

S1: Yes. I would just set it to 50. […] [the second bar is dragged down to approx. 78][…] [reads task aloud. Then the teacher intervenes.]

T: Yes, but you haven't quite done the task yet, because it also says split the bar for this age group into two bars, and in such a way, way that the **total area** [stresses these words] of the histogram doesn't change.

S1: hm?

S2: Um. So basically, we need to have these two… So […] you have to double or something? [moves left and right bar up to double height …]

S2: no, but wait. Look, look, look. A moment ago we had from here to there [moves mouse from left to right in the middle of the two bars around the height of 130] 130 and now this average is also 130, right?

S1: yes, but then the numbers don't match the number of infections.

S2: uhm.

S1: because with the rest you don't have to divide by two and with that one bar you do. That would be weird. That doesn't match anyway.

What equal area means seemed clear to students S1 and S2. When the teacher continued asking questions about what the vertical scale now represented, these students became very confused and dragged the sliders back to the position they had first chosen (see Fig. 7, right).

T: If you look at the next question that's there: what variable is along the vertical axis? On this paper there are four possibilities. What does it say now do you think? [Pause]

S1: the number of people. [Pause] Right?

T: so in the age group of 10–14 there are, 100 what is it, 103 thousand students

S2: um, no. [pause].

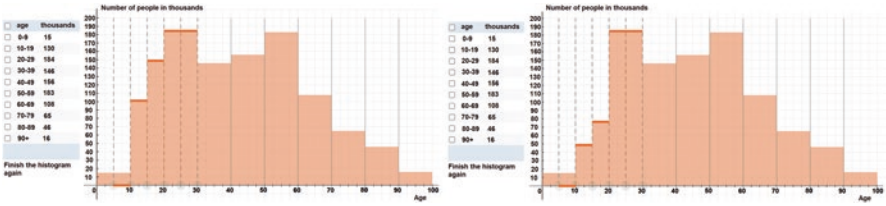


Fig. 7 Correct heights of bars for five-year age groups by students S1 and S2 (left) and their modification (halving) when asked about the scaling (right)

T: ok, what about then?
S2: Yes, even [equal] in terms of percentage but then the um… Hmm. [takes the mouse and lowers both bars back to their halves]. Then it changes, the whole thing.
T: the area.

Meanwhile, these students seemed to understand that area represented all people.

T: but we just agreed that this area wasn't right [points to the board with the two bars that were too short because halved].
S2: no.
T: and in a histogram, no matter how you make the layout, the amount of people should
S1: remain the same
T: yes [gestures with two hands and points with them to the whole area of the histogram]. The total area is what percent of the people?
S1: 100%
T: so that total area, yes we didn't change anything about the total number of people. [pause]
S1: but then shouldn't you just divide everything by two?
S2: um, no.
T: and if you were to do that, what do you get?
S1: well, then you get the same number of people, don't you? […]
S2: but then the total area of the histogram does change?

Building on the halving bars, they had another good idea that was not possible in this environment: to split all bars and then rescale the vertical axis.

S1: Oh, no, wouldn't you? [Points to the board. Sounds engaged] Eh. Wait. If you now because we've now done 1 bar, divided by 2 and split that. If you split each bar into two… if you split each bar,
T: yes
S1: then it's right.
S2: but we don't know how to split it, do we?
S1: yes, because you know the number of people, then you just can
S2: oh, you just divide it into two.

Students then discussed whether this would change the area:

S2: but then your total area changes, right?
S1: your area does but you still have the same number of people. The number of people is still 100%…. Right? […]
T: So the number of people is still 100% then. So I, I um, I would think that's a very good idea. We haven't made that possible. And, what have. If you think on that for a moment, huh, what you just said,
S1: yes
T: so you're actually going to do every class [bin] by half. […] then in one class you have the number of people per … what?
S2: five years.

After about 8 minutes of discussion, students seemed almost there. However, the system did not allow halving all bars both vertically and horizontally. Therefore, the heights of the 5 year bars needed to be adjusted.

T: Yes. But I'd like to know now what to do if I have a width of 10 for some and a width of 5 for others. [Pause] And we agree that the group of 30–40 that that's correct, right?
S2: hmhm [confirming] […] but I had just now done this number times two.
T: yes
S2: only… But then why wasn't that correct?
T: who said it wasn't right?
S2: [indignant, louder] I had that just now!!! […]
T: But you started adjusting it yourselves.
S2: Oh, hmpf. [moves the bars up again to the correct position, taking the ratios of the bars into account] […] Well, now you have the same…now it's the same area again. Only…the numbers don't match. Right?
T: yes, you have the same area
S2: on average. Average that would be, but average would make 130 […] But say what is wrong with this graph now?
S1: The area
S2: Yes [she means: no] it is [correct], the area that is correct. […]
T: you just agreed on that. So the area is correct.[…] And then with these numbers along the vertical axis. What does it say? Because that was the point where you adjusted it. When I asked that question. Because then you said: number of people. But if it's the number of people, you have a problem. Because from 10 to 15 there's no … 100 … what is it? what did you put it at?…. 104 people. 104 thousand
S2: but this is just going to be uhmm, number of people per 1000 … and then age group of … let's see [pause] […] of … 10 years [louder]. C seems to me. Because in steps of … Oh no, wait. […] then I think five years makes the most sense because with that you can read the other ones too
T: hmhm. So from 30 to 40 you have 146 thousand every five years.
S2: [pause] No… [hesitantly]
S1: no, you have to divide those by two then anyway

Note that student S2 gave the correct answer (C) but then changed her mind. It then took another 8 minutes to get the students back on track. The teacher tried different approaches, building on the students' ideas. S2, for example, moved the mouse horizontally at 104 thousand and 130 thousand (the mean for the 10–14 and 15–19 years of age group together) but when the teacher tried to build on that idea, the students got confused. The teacher then decided to apply the same idea on a bar that had not been split: the 40–50 years of age group.

T: Then we're going to apply it to this what you just said. Here we had? [draws the horizontal line segment at the top of the bar, see Fig. 8]
S2: 156

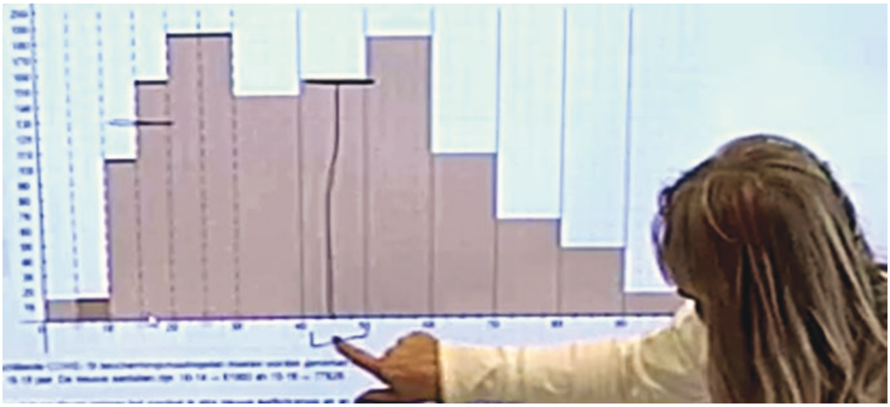


Fig. 8 The teacher halved a bar in line with an idea from a student

T: 156 thousand. And now we're going to split it in mind [draws a vertical line that splits the bar].

S1 and S2: Yes.

T: both the same amount. Then how many are in this one here? [left bar in split bar]

S1: half of 156 thousand […]

T: only what does it say? Then what does the 156 thousand mean? Because there are no 156 thousand in here [points at left bar in the vertically split bar] you just said [waves at S1]. You said: there is only half in it.

S1: yes that's the total actually of those […] of those ten

T: yes, of that age group of [draws a tray under 40 to 50 years, Fig. 8]

S1: [of] ten years, of five years

T: of ten years, you said. Yes. So this number [points at the top of the bar at 156 thousand] belongs to the age group of ten years. […]

T: so what is now along the vertical axis then?

S1: the total of the 10-year age group?

T: yes, and if you have to choose from the answers that are there? [on paper] […]

S1: the number of people by age group of ten years. […]

T: and why is that?

S2: because you read it off by ten years.

T: yes. So even though you then make bins that are smaller, you still have to keep reading it as if it were per ten years.

In the discussion with students S1 and S2, it also became clear that they wanted to change the scale and halve all other bars, too. Therefore, on the second day, task 17 was available on paper for the students throughout the teacher-students-intervention. Moreover, students S3, S4 and S5 joined together in front of one whiteboard during this teacher intervention. Student S3 became engaged and used

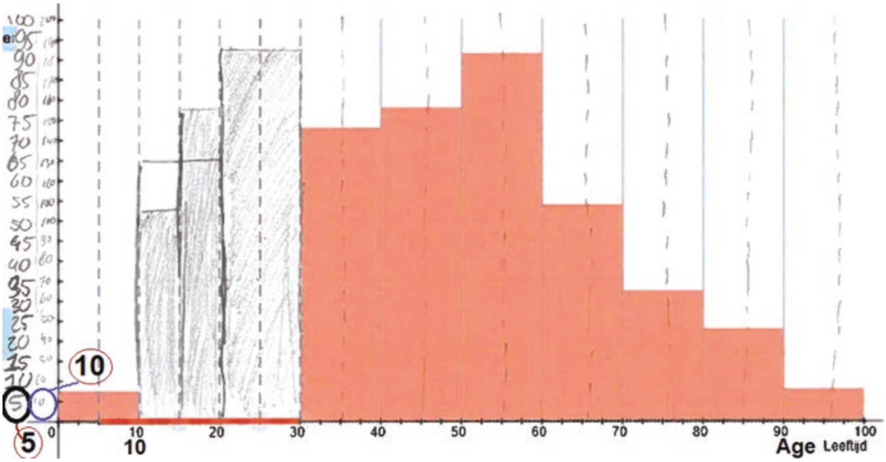


Fig. 9 Student's S3 new scaling for the vertical axis (left: 5 instead of right: 10). Circles and thicker numbers added for convenience of the reader

the paper. He first wrote down the numbers from the digital task and then halved all numbers along the vertical axis, as well as all bars in the density histogram (see vertical dotted lines in Fig. 9). He wrote about this scale:

> As a result, you are looking at groups of 5 years. With a modified scale (halve the bars), then you halve the numbers [along the vertical scale]. By also halving the groups [all other bars], the ratio is correct again. With the groups of 10–20 (etc.), everything is the same, with the groups where it is in 5 years, it also fits proportionally.

For the question regarding which variable the vertical axis is now representing, students could choose from four options given on paper (see Materials section). S3 and S4 chose option A ("per five year age groups") in line with the new scaling above. Student S5 chose an incorrect option D ("Number of people"). Students S1 and S2 chose option C ("per ten year age groups"), which is in line with the scale given in the digital environment. Taken all together, the results suggest that four students fulfilled our expectation formulated in H17a. Given their answers on the written materials, there are also some indications that students started to grasp the idea of density (H17b). Nevertheless, as can also be inferred from the excerpts of the transcripts, these ideas are still fragile and not yet well articulated.

Transfer Task 19. S1, S2 and S4 answered "50%" to the question given on paper for this task: "About what percentage of households has less than 27,000 euros of disposable income per year?" S3 drew a vertical line and braces along the horizontal axis but did not come up with a percentage. S5, who worked together with S4, answered "About 60% large part is below 27000 euros." Four out of five students did not seem to have any trouble with this task. For these four students, the results seem to support conjecture H19a. As 27,000 euros is in the middle of a bar and all students but S3 seemed to have no problems with this, H19b is likely to be supported as well. However, as we did not explicitly address this in either the task or in the discussion, we cannot be sure.

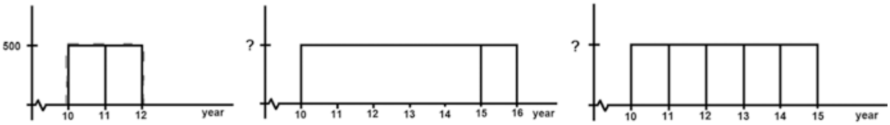


Fig. 10 Ideas for area tasks in preparation for tasks 16 or 17

5 Ideas for Redesign

In retrospect, task 17 needs to be done in previous activities. First, we could add a check button to task 16 to reassure the students that their initial construction is correct and to make sure that they use 5 year bin widths. The students' visual intuition about the histogram might be so strong they did not question the smaller bins at all. Therefore, we would ask them how many students there are in the 10–19 years age group. Next, we would ask how many there are in the 10–14 years group. We expect this would trigger the intuition that it is half in the latter, although we cannot be sure.

Additionally, we would redesign task 17 in such a way that students need to join two bars instead of splitting one. Area as a multiplication might be an easier access to density than the division we used. Moreover, the area task 19 might have been a good preparation for density histograms and histograms with unequal bin widths. In a next cycle, we would, therefore, consider either constructing a digital area task and/or giving this task earlier in the trajectory.

Furthermore, before task 16, we suggest introducing simpler tasks using area and number of people (Fig. 10). Possible information and questions are as follows: First, there are 500 students in the [10, 11) year age range (left). "What is along the vertical axis? How many people are in the [11, 12) year age range?" Both bars are now combined into one bar (dashed line). "How many students are together in that new, dashed bar? What is the 500 along the vertical axis now depicting?" Second, together there are 1000 students in the [10, 15) year age range (middle). "How many students are there in the [15, 16) year age range? What number should be along the vertical axis at the question mark?" Third, together there are 1000 students in the [10, 15) year age range, equally spread across each year (right). "How many students are in each year? What number should be along the vertical axis at the question mark? What is this number depicting?"

6 Conclusions and Discussion

Theoretical analysis of an embodied instrumentation approach led to the six design heuristics described in Sect. 2.2 that would facilitate conceptual understanding. Using these six heuristics we designed three tasks for supporting students' understanding of density histograms. We used these design heuristics to unravel the statistical key concept distribution into artifacts that need to be understood (here: density histogram). Next, we melted this artifact back to actions that historically constituted this artifact, such as choosing unequal bin widths, raising the height of a bar so that the area of that bar represents the number of cases, and (re)scaling or renaming the vertical axis so that it is consistent with the unequal bin widths.

The research question was: How can a sequence of tasks designed from an embodied instrumentation perspective support students' understanding of density histograms? We found it was important to include tasks targeting students' notions of area, and tasks in which students can reinvent unequal bin widths and density in a histogram. Both halving bin widths without changing the ratios of bins and using area of bars to estimate a proportion of the total area in a histogram without a vertical scale seemed to have worked well in our multiple-case study. Reinventing how to proportionally split a bar into two and then determine what is along the vertical scale in a density histogram required intense teacher intervention.

After the teacher's intervention, students started have a tentative sense of what density is. It might be good to introduce a density scale in simpler situations such as with only two bars and round numbers to better align with students' notions of areas. In addition, having students combine bars might align better their notions than the splitting bars approach we took. As students wanted to change the vertical scale when reinventing density, the second day task (17) was available on paper for students and used by them in their discussions.

A limitation of this study is that we worked with only five students in a laboratory setting. Therefore, further research is needed to explore how this design works for other students and in classrooms. To facilitate use in classrooms the tasks are freely available on the web and require a computer with a mouse only. Introduction in a classroom is a big step and will require (alongside resources) attending to how teachers can be trained to work with embodied pedagogies and tutoring multiple students at the same time (Abrahamson et al., 2021).

As our study is in an early stage of design, our empirical tryout revealed directions for further development rather than an end product. A contribution of our work is first that it provides early researchers insight into the step of rethinking the design based on empirical results. Further, in contrast to statements of Lee (1999), four out of five students seemed to have no trouble understanding that the whole area in a histogram is 100% and that area in a histogram can be used for estimating a proportion of a population.

In addition, reporting on not-so-successful parts of the design is important as published research tends to be biased toward successful implementations. As design research is becoming more and more accepted as a method, it is important to describe how to overcome difficulties in the design based on empirical tryouts of the design. This includes rethinking what actions students need to perform and how theory can be put into practice, paying close attention to the step of rethinking the design. In this way we hope to contribute to further application of design research methods in the field of statistics and mathematics education.

In discussing how our work fits into a chain of other works, we built on artifacts acquired in primary education such as area (e.g., constant area task, Shvarts, 2017) and artifacts acquired in the learning trajectory that preceded this learning trajectory such as histograms (Boels et al., 2022b). As a possible follow-up to these tasks, an embodied design for probability (Abrahamson, 2009; Abrahamson et al., 2020) or a design preparing for probability density (Van Dijke-Droogers, 2021) can be considered. Van Dijke-Droogers' intervention was not developed from an embodied design framework but nevertheless shares some ideas and similarities with the design of Abrahamson and colleagues.

Our six design heuristics can be applied to other key concepts and artifacts within and outside statistics education. Particularly new is that we decompose a key concept—distribution—into artifacts (e.g., density histogram) and further melt these down into actions that have constituted them in the history of mathematics (e.g., manipulate area). The six design heuristics open a new way for task design. We, therefore, hope that our study will contribute to a new genre of tasks in statistics education based on the heuristics of embodied instrumentation designs.

Our study is also important for teachers and designers of tasks and textbooks, as the literature on how to teach and learn density histograms is scarce. By describing in detail what actions the teacher took to support students, we hope to provide insight into students' thinking and how teaching of density can be done. Our description of tasks and suggestions for redesign aim to contribute to further development of tasks that advance students' learning of the key concept density as well.

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