A Mysterious Tensor Product in Topology

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The students of my generation had to survive without the internet and mobile phones, and depended on books and real paper to write on. As undergraduates in mathematics, we were always carrying yellow books around, and Springer-Verlag had a big part in our mathematical development. A little later, when I was a PhD student, two Springer Lecture Notes had a lasting influence on my own mathematical work: *Homotopy Invariant Algebraic Structures on Topological Spaces* by Boardman and Vogt [1] and *The Geometry of Iterated Loop Spaces* by Peter May [6]. These two books together shaped the foundation of the theory of operads about which I will write below.

My contacts with Catriona go back to the preparation and publishing of "Sheaves in Geometry and Logic" with Saunders MacLane in the early 1990s. This period was also the beginning of the use of e-mail, and it is interesting and entertaining to see how e-mail customs and etiquette have changed over the years. I had my first e-mails to Catriona typed by a secretary, and Catriona had several people working for her who wrote on her behalf, all using an e-mail account and address under the name "Byrne". A few years after that, together with Albrecht Dold, Catriona helped me to get SLN 1616 into publishable shape. In more recent years, she remained most helpful in several matters, and I wish to thank her for that.

Now I would like to come back to Boardman and Vogt, and May, and talk about some mathematics. To begin with, let me remind you that a (coloured) *operad* P consists of a set $C = \text{colours}(P)$ of *colours*, and for each sequence $(c_1, \dots, c_n; c)$ of elements of C (where $n \geqslant 0$) a set of *operations* $P(c_1, \dots, c_n; c)$, to be thought of as taking inputs of "types" $c_1, \dots, c_n$, respectively, to an output of type c. Moreover,

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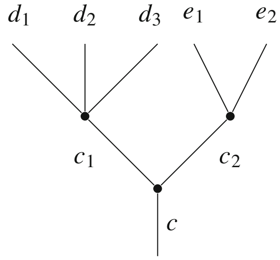
P is equipped with several structure maps for symmetry and composition, such as

$$P(c_1, c_2; c) \xrightarrow{\sim} P(c_2, c_1; c)$$

$$P(c_1, c_2; c) \times P(d_1, d_2, d_3; c_1) \times P(e_1, e_2; c_2) \to P(d_1, d_2, d_3, e_1, e_2; c)$$



and a unit element $1_c \in P(c; c)$ for each colour c. These are to satisfy several natural conditions, such as an associativity law for composition. If $P(c_1, \dots, c_n; c)$ is empty unless $n = 1$, this simply defines the notion of a (small) category. And if C consists of just one element $*$, one calls P uncoloured and writes $P(n)$ for $P(c_1, \dots, c_n; c)$ where each c_i and c are necessarily $*$.

For a coloured operad P, a *P-algebra A* is a family of sets $\{A_c : c \in C\}$ equipped with maps

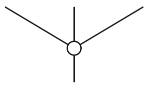
$$P(c_1, \dots, c_n; c) \times A_{c_1} \times \cdots \times A_{c_n} \to A_c,$$

(for all sequences of colours $c_1, \dots, c_n, c$), compatible with the structure of P mentioned above (symmetry, associativity, units). For example, if P is the uncoloured operad for which each $P(n)$ consists of a single point, a P-algebra is simply a commutative monoid, and one usually writes **Comm** for this operad. The collection Σ_n, $n \geqslant 0$ of symmetric groups also has the structure of an operad; the composition $\Sigma_n \times \Sigma_{k_1} \times \cdots \times \Sigma_{k_n} \to \Sigma_k$ for $k = k_1 + \ldots + k_n$ is defined by replacing the non-zero entries in an $n \times n$ permutation matrix representing an element of Σ_n by the permutation matrices representing given elements of $\Sigma_{k_1}, \dots, \Sigma_{k_n}$ respectively, thus yielding a $k \times k$-permutation matrix. An algebra for this operad is an associative monoid, and one usually writes **Ass** for this operad (although Σ would obviously have been a good name as well).

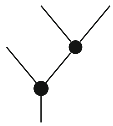
These P-algebras form a category Alg(P), or Alg(P, **Sets**) to emphasize that we consider algebras in the category of sets. One can similarly define a category Alg($P, \mathcal{E}$) of algebras in any category with products, as long as expressions like "$P(c_1, \dots, c_n; c) \times A_{c_1} \times \cdots \times A_{c_n}$" occurring in the definition of P-algebra make sense in $\mathcal{E}$. (This is the case, for example, when it is possible to view the set $P(c_1, \dots, c_n; c)$ as an object of $\mathcal{E}$ through a suitable embedding of sets as "discrete objects” in $\mathcal{E}$.) So, if Q is another operad with set of colours $D = \text{colours}(Q)$, one can construct a category $\text{Alg}(P, \text{Alg}(Q, \mathbf{Sets}))$. This category is itself a category of algebras over a new operad $P \otimes Q$ with set of colours $C \times D$; in other words,

$$\text{Alg}(P \otimes Q, \mathbf{Sets}) = \text{Alg}(P, \text{Alg}(Q, \mathbf{Sets})).$$

This operad $P \otimes Q$ is known as the “Boardman–Vogt tensor product” of P and Q and was first introduced in SLN [1]. It is possible to describe $P \otimes Q$ explicitly by generators and relations. In particular, if P and Q are *free* operads defined by trees S and T, i.e. $P = \text{Free}(S)$ and $Q = \text{Free}(T)$, then $P \otimes Q$ is defined by glueing free operads $\text{Free}(R)$ together, where R ranges over all the *shuffles* of the two trees S and T. A minimal example can be pictured as follows:

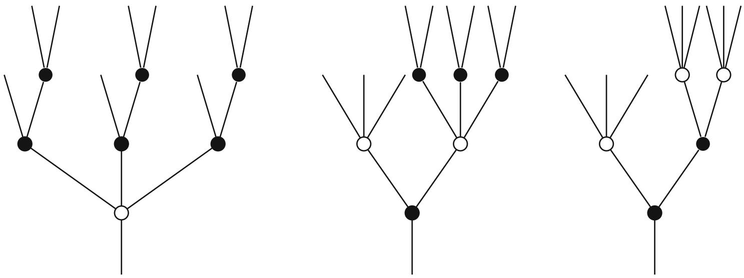


$P = \text{Free}(S)$



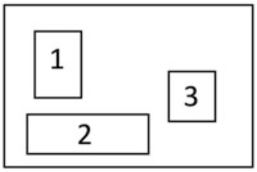
$Q = \text{Free}(T)$

$$P \otimes Q = \text{Free}(R_1) \cup \text{Free}(R_2) \cup \text{Free}(R_3)$$



See [5] for details. The case of *categories* (viewed as operads with unary operations only, as above) corresponds to shuffling linear trees. For two trees with n and m vertices, respectively, there are $\binom{n+m}{n}$ such shuffles, as everybody who encountered products of simplicial complexes will be aware of. However, it seems impossible to find a nice closed formula for the number of shuffles of trees that aren’t linear (see *loc. cit.*)

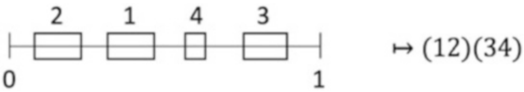
Turning to operads in topological spaces, the most famous ones are probably the (uncoloured) operads $\mathcal{C}_d$ of “little d-cubes” (for $d \geqslant 1$ the dimension of the cubes involved). Specifically, $\mathcal{C}_d(n)$ is the space of sequences of n rectilinear embeddings of a d-dimensional cube into the unit cube $[0, 1]^d$, with disjoint interiors. Here is a picture of a point in the space $\mathcal{C}_2(3)$:



By composing such embeddings, one obtains maps

$$\mathcal{C}_d(n) \times \mathcal{C}_d(k_1) \times \cdots \times \mathcal{C}_d(k_n) \to \mathcal{C}_d(k)$$

for $k = k_1 + \ldots + k_n$, representing the composition operation of the operad $\mathcal{C}_d$. These operads derive their fame from the fact that $\mathcal{C}_d$-algebras (in spaces) describe d-fold loop spaces, as discussed in detail in [6]. Since the d-fold loop space of a space which is itself an e-fold loop space is evidently a $(d + e)$-fold loop space, it is natural to expect that $\mathcal{C}_d \otimes \mathcal{C}_e$ is closely related to $\mathcal{C}_{d+e}$. This is indeed the case, as these two operads have been proved to be equivalent up to homotopy, a result known as Dunn's additivity theorem [3]. Although a positive result, Dunn's additivity theorem show at the same time that the tensor product of topological spaces behaves rather badly under weak homotopy equivalence. For example, there is a map $\mathcal{C}_1 \to$ **Ass** or operads, assigning to a point in $\mathcal{C}_1(n)$, i.e. a sequence of n numbered disjoint intervals in the unit interval, the permutation representing the order in which there intervals occur:



This map is weak homotopy equivalence of operads, i.e. each $\mathcal{C}_1(n) \to \mathbf{Ass}(n) = \Sigma_n$ is one of spaces. On the other hand, $\mathbf{Ass} \otimes \mathbf{Ass} = \mathbf{Comm}$ by the Eckmann–Hilton trick, while $\mathcal{C}_1 \otimes \mathcal{C}_1 \simeq \mathcal{C}_2$ describes double loop spaces and is very different from **Comm**.

There are variations of the operad $\mathcal{C}_d$ which also describe d-fold loop spaces up to homotopy, leading to a notion of "E_d-operad": An operad P is said to be an E_d-operad if it can be related to $\mathcal{C}_d$ by a zigzag of weak homotopy equivalences between operads,

$$\mathcal{C}_d \leftarrow \cdot \rightarrow \cdot \leftarrow \cdot \rightarrow \ldots \leftarrow P.$$

These E_d-operads often arise as combinatorial versions of $\mathcal{C}_d$, for example the one used by McClure and Smith in their proof of the Deligne conjecture [7, 8]. From a mathematical point of view, however, the notion of an E_d-operad is a rather unusual one, as it does not give any structural properties for an operad to be an E_d-operad. This becomes particularly awkward when one considers the tensor product of E_d-operads, since the tensor product is not invariant under weak equivalence, as we have just seen. So one may ask for which particular "models" of E_d-operads the additivity of Dunn holds. This is problem to which Rainer Vogt devoted much of his work (see e.g. [4]), but which remains largely unsolved.

An alternative approach is to replace the Boardman–Vogt tensor product of topological operads by a "derived" one which is invariant under weak equivalence. Denoting such a derived tensor product by $\widehat{\otimes}$, one way to construct it explicitly is as the pushout

$$\begin{array}{ccc} w_!(w^*(P)) \vee w_!(w^*(Q)) & \longrightarrow & w_!(w^*(P) \otimes w^*(Q)) \\ \downarrow & & \downarrow \\ P \vee Q & \longrightarrow & P \widehat{\otimes} Q. \end{array}$$

Here P and Q are topological operad with sets of colours C and D, say, and $P \vee Q$ denotes the coproduct in the category of operads with $C \times D$ as set of colours (where we first pull P and Q back along the two projections). Furthermore, the adjoint functors $w_!$ and w^* are the ones establishing a Quillen equivalence between topological operads and dendroidal sets [2]. The symbol $\otimes$ in the diagram refers to the tensor product of dendroidal sets. Since this tensor product is much better behaved, especially for "closed" operads like $\mathcal{C}_d$ where there is a unique nullary operation (of each colour), one can prove that $P \widehat{\otimes} Q$ is invariant under weak equivalence in each variable separately, at least for operads with free Σ_n-action on $P(n)$ (respectively $Q(n)$), for each n; see [2]. So $P \widehat{\otimes} Q$ describes quite a good tensor product, from the point of view of topology. It comes equipped with a map $P \widehat{\otimes} Q \to P \otimes Q$ expressing that $P \widehat{\otimes} Q$ is a "thick" version of the original Boardman–Vogt tensor product. It would be interesting to know for which operads this derived tensor product is equivalent to the original one of Boardman and Vogt. (This is the case for "cofibrant" operads P and Q, but cofibrant operads are hard to come by and rarely occur naturally.) It would also be interesting to know whether this derived tensor product $\widehat{\otimes}$ satisfies Dunn's additivity property for certain types of models of E_d-operads.

Thus, natural as the tensor product may seem from the point of view of algebra (remember the equation $\mathrm{Alg}(P, \mathrm{Alg}(Q, \mathbf{Sets})) = \mathrm{Alg}(P \otimes Q, \mathbf{Sets})$), it is surrounded by many unanswered questions: combinatorial ones about the number of shuffles, questions about invariance under weak equivalence, and questions about additivity.

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