Algebra Education and Digital Resources: A Long-Distance Relationship?

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B. Pepin et al. (eds.), *Handbook of Digital Resources in Mathematics Education*,
Springer International Handbooks of Education,
https://doi.org/10.1007/978-3-030-95060-6_16-1

Abstract

This chapter considers the general question of how technology impacts mathematical contents for the case of algebra and addresses it from the conceptual notion of *instrumental distance*. It first provides an overview of perspectives on the teaching and learning of algebra in primary and secondary education. Next, it presents the theoretical lens of instrumental distance, used to study how these algebraic contents are transformed by digital tools. Using three examples of digital resources that cover different school levels and different types of design, the chapter shows that this lens indeed reveals how digital technology may create distance with respect to the regular algebraic contents and procedures carried out in the paper-and-pencil environment. The results thus raise awareness not only of new potentialities, but also of the complexity generated by technology. They suggest that being aware of the instrumental distance would be beneficial while developing digital resources and professional development programs that aim to integrate digital technologies into the teaching and learning of algebra.

Keywords

Instrumental distance · Algebra · Digital technology · Scratch · Excel · GeoGebra

Introduction

How do digital resources transform the mathematical contents that are to be taught? In this chapter, we examine this general question for the specific case of *algebra*. Since the late 1980s, many digital resources around the world have been specifically created to enhance the learning of algebra. Some other tools, imported from outside education, such as CAS (computer algebra systems) or spreadsheets have been analyzed as also promoting algebra learning. Last, some educational tools designed for other teaching areas, such as programming games like LightBot, have also been considered as interacting with algebra learning (Bråting and Kilhamn 2021). In all cases, research in mathematics education has shown potentialities and new possibilities offered to algebra education. Therefore, one could wonder about the teachers' slow uptake of these technologies in algebra education.

In 2006, Ferrera, Pratt, and Robutti carried out a review over the three previous decades of PME[1] research focusing on the impact of technologies with respect to three algebraic notions: expressions, variables, and functions. Their review raised some research questions, among which was the way the use of technology tends to re-define these school algebraic contents. Extending this question of the technology impact on algebraic objects, examined in isolated ways, our study aims at taking into account the relationships that these objects maintain in the teaching of algebra and the fact that technology also affects these coherencies. Doing so, this chapter shows the effects of technology not only through its didactic potentialities, but also through that of the complexity and constraints it generates, a complexity that will have consequences on teaching practices and may explain the difficulties of integrating these technologies into the teaching of algebra.

To address this question, we use the notion of *instrumental distance* (Haspekian 2005) as a theoretical lens. It will be used to analyze three cases of digital resources that may affect school algebra – bloc programming, spreadsheets, and dynamic geometry – and that cover diverse school levels and types of design. The construct of instrumental distance is based on Rabardel's (2002) concept of instrumentation, a process that highlights the non-neutrality of tools on subjects' conceptualizations: Using a tool to perform a mathematical activity affects the conceptualizations of the mathematics at play. As a consequence, the didactic possibilities offered by a new technology affect the usual ways of approaching the school mathematical contents. The instrumental distance designates this qualitative gap between the ways of approaching coherent school mathematical contents of traditional teaching in the paper-and-pencil environment, compared to those in the digital environment. In the teaching of algebra, as we shall see, different approaches currently coexist in the paper-and-pencil environment. The distance generated by an instrument is studied regarding these referential approaches. Therefore, before further developing this theoretical notion (section "Instrumental Distance: Theorizing the Impact of Digital Tools on Mathematics"), the following section gives an overview of the objects and techniques' characteristic of these algebraic approaches, discussing them from each of these teaching perspectives. The ways Scratch, Excel, or GeoGebra affect these various interconnected objects and techniques of school algebra are then analyzed with respect to these different perspectives (Sections "Early Algebra with Scratch," "Generalized-Arithmetic Approach with Excel," and "A Functional Perspective on Algebra: Equation Solving with GeoGebra").

Algebra and Algebraic Thinking

While many characterizations of algebra (Carraher and Schliemann 2007; Kaput 2008; Kieran 2007; Mason et al. 1985), and algebraic thinking (Kieran 2004, 2018; Stephens et al. 2017), have been advanced by researchers for the domain of school algebra, the definition offered by Radford (2018, p. 8) is central to this chapter:

[1] Psychology of Mathematics Education

Algebraic thinking

- *Resorts to:*
  - *Indeterminate quantities*
  - *Idiosyncratic or specific culturally and historically evolved modes of representing/symbolizing these indeterminate quantities and their operations*
- *Deals with:*
  - *Indeterminate quantities in an analytical manner*

The above definition reflects the historical development of algebra, with its equations containing letters that represent unknown numbers (e.g., $x^2 + 10x = 39$), solvable by syntactic methods inspired by Al-Khwarizmi and formalized by Viète and Descartes. Radford's use of the term *indeterminate quantities* comprises, more broadly speaking, unknowns, variables, generalized numbers, and parameters (all of these terms defined in section "Variables"). While Radford clearly includes alphanumeric symbolism for the representation of indeterminate quantities, he stresses that such symbolism is not necessary – indeterminate quantities can be signified with natural language, concrete materials, unconventional symbols, and even with numbers used in a generic manner. Lastly, the assertion that algebraic thinking resorts to dealing with indeterminate quantities in an analytical manner means that the indeterminate quantities and their operations are handled as if they were known – they are added, subtracted, multiplied, and divided, just as is done with determinates.

When algebra was incorporated into school curricula, it was referred to as generalized arithmetic – the generalization of ways of operating with numbers, in accordance with the basic properties of arithmetic and equality (Kilpatrick and Izsák 2008). The mathematical notion of function – associated with graphical curves – which developed historically much later than that of algebra, was initially viewed as a minor part of school algebra. With their gain in prominence in the 1960s – in the aftermath of the Bourbaki and "new math" movements – and heightened further in the 1980s with advances in graphing-capable digital tools (e.g., Fey 1984), functions came to be a full-fledged component of school algebra. The result was that there were now two rather different perspectives on the objects and techniques of school algebra, albeit with each having its own internal coherence: a generalized arithmetic perspective and a functional perspective.

The ensuing presence of these two components within school algebra created challenges for students when moving from one perspective to the other. Sierpinska (1992) has argued that, even if some objects are common to these two perspectives, "the attention focuses on different aspects of them and assigns them different roles" (p. 37).

The Objects and Techniques of Algebra

In elaborating on the various objects and techniques of school algebra, we discuss each from the dual perspective of generalized arithmetic and function. The objects to be presented are variables, expressions, and equations; the techniques are those related to equation solving – all within paper-and-pencil environments.

Variables

While the modern notion of variable, according to Wikipedia, refers simply to a symbol representing a mathematical object that either is unknown, or may be replaced by any element of a given set (e.g., the set of real numbers), algebra educators and researchers have enlarged on this characterization so as to be able to address more subtle distinctions and include the following under the umbrella term of "variable" (e.g., Arcavi et al. 2017): unknown, placeholder, varying quantity, parameter, and generalized number. (Note that, having opted for the umbrella term of "variable" rather than that of "indeterminate," we then decided to use the phrase, *varying quantity,* in order to avoid the terminology of *variables that vary.*) As will be seen, algebraic variables involve a shift in interpretation as one moves not only between the two main perspectives on algebra, but also within a given perspective, depending on the task at hand (Philipp 1992).

An *unknown*, usually represented by a letter that signifies a fixed value, is found in an equation (e.g., $2x + 5 = x - 6$), which upon being solved yields the sought-for, specific, numerical value (or values, depending on the equation). However, it is also the case that equations, such as the quadratic $3x^2 + 8x + 5 = 0$, can be solved more generally with the use of the *placeholders* a, b, and c to stand for the coefficients and constants. These placeholders represent nonspecific values, as in $ax^2 + bx + c = 0$, where the literal solution for the unknown x shows at once the solutions for an entire set of equations (i.e., the quadratic formula), and where the process used to arrive at that symbolic solution lays bare an underlying general method for solving such equations (Ely and Adams 2012).

In the functional perspective, variables are typically interpreted as *varying quantities* – more specifically, the idea of an independent variable varying freely while the dependent variable changes in accordance with the value of the independent variable, within a systematic functional relationship, such as $y = 2x + 7$ (Freudenthal 1982). Variables that are interpreted as *parameters* usually occur within the same functional perspective on algebra – parameters being defined as nonspecific values that identify a collection of distinct cases and that can take on particular numerical values, such as the parameter a in the function $y = 2x + a$. In much of the graphing activity involving linear functions, parameters are treated as constants; however, activity exploring the role of parameters could require shifting the interpretation to that of an unknown (Arcavi et al. 2017).

Variables that are interpreted as *generalized numbers* can occur in different contexts. One context is the expression of fundamental properties of arithmetic and equality (e.g., $a + b = b + a$), and more generally any identity; another is that of polynomial expressions (e.g., $3x^2 + 2x + 5$) whose letter terms have no fixed value. Note, however, that while the variables of a polynomial expression are usually considered generalized numbers, if such an expression were part of an equation to be solved, the variables would be interpreted as unknowns. Furthermore, within the functional perspective, the same expression might be viewed as an implicit function, whose variable terms are interpreted as varying quantities.

Expressions and Equations

Algebraic expressions consist of variables, coefficients, and constants, along with the operation signs of addition, subtraction, multiplication, division, and exponentiation (e.g., $2x - 5$, $3x^2 - 14x + 11$, $4x + 3y$, and $(x + 5)(x - 3)$), but no equal sign. Rational expressions, which are a type of algebraic expression, are those expressions that can be written as rational fractions (e.g., $(x^2 - 1)/(x^3 - 1)$). Note that arithmetic expressions (e.g., $150 + 23 - 17$) do not contain variables.

Algebraic equations consist of two algebraic expressions (or an algebraic expression and an arithmetic expression), with an equal sign between them (e.g., $7x + 5 = 2x - 3$). As already mentioned, within the generalized arithmetic perspective, equations are usually considered as objects to be solved. However, not all equations are of the sort, "equations to be solved." For example, well-known geometric formulas are expressed in the form of equations, such as the formula for area of a rectangle, $A = L\text{x}W$, or for area of a square, $A = S^2$. In such formulas, the variables can be considered as generalized numbers, or as varying quantities, or as unknowns, depending on the nature of the task or problem situation. Then, there are also the equations that are associated with the functional perspective, generally written in the form of y="the expression of the functional relation" – the functional equation typically being an object that is to be graphed, but also sometimes to be solved as when two functional relations are equated. Research has documented (e.g., Chazan and Yerushalmy 2003) the many challenges faced by algebra students in attempting to sort out these various types of equations and the different meanings given to the variables – dilemmas, in fact, exacerbated by the two very distinctive generalized arithmetic and functional components of school algebra.

Equation Solving

Solving an equation in school algebra means finding all the values (numerical or symbolic) for the unknown that satisfy the given equation. Finding the solutions to an algebraic equation can involve several different methods, depending on the nature of the equation. While we distinguish between arithmetic and algebraic methods of equation solving – arithmetic methods including the use of known number facts, "covering-up," and trial-and-error step-by-step substitution (see Kieran 1992) – our focus here is algebraic methods.

Maintaining equivalence is at the heart of equation solving. While equivalence has both a computational and structural dimension (Kieran and Martínez-Hernández 2022), within the algebraic world the structural is at the forefront. The structural transformations that maintain top-down equivalence from one step of the solving process to the next are of two types: (i) Subexpressions on either side can be operated upon by applying properties and (ii) the equation itself can be operated upon by performing the same operation on both sides. Both types of structural transformations are often needed to arrive at the solution – a solution that when substituted back into the initial equation yields a numerical identity. As Freudenthal (1983) reminds us, even if the value of the unknown remains the same throughout the equation-solving process – as does the truth value of the equation – the application of the latter type of transformation changes the numerical value of the left and right sides from one equation to the next.

When, however, the functional perspective on algebra is dominant, the idea of equation solving does not immediately present itself. In fact, graphing is the main algebraic technique within the functional perspective – even if for certain types of functions, some symbolic manipulation may be needed for determining salient points to be graphed. The functional equation $y = 7x + 5$, for example, is not an equation to be solved. It is an object to be graphed. If this function is to be compared with another, such as $y = 2x - 3$, it too is graphed. If one were then to ask, "What is the point of intersection of these two graphs?," one would usually try to locate within the intersecting graphs of the two functions the value of x for which the two functions have the same y-value – but we note that for non-integer values of x, it is only an approximate value. For an exact solution to the point-of-intersection problem, one would need to turn toward the solving techniques used within the generalized arithmetic perspective (sometimes referred to as the symbolic approach, so as to differentiate it from the graphing approach). This necessary interweaving of perspectives would involve equating the expressions of the two functional relations (e.g., $7x + 5 = 2x - 3$) and solving for a variable that has now become an unknown. Within the functional perspective, the solutions to such equations are sometimes referred to as zeroes of the function (as would be illustrated if the eq. $7x + 5 = 2x - 3$ were to be transformed to $5x + 8 = 0$, such a transformation suggesting that this equivalent equation can also be interpreted as the equating of the functions $y = 5x + 8$ and $y = 0$). In this somewhat unique example of bringing together the two main perspectives on school algebra, although the solving technique is patently that of generalized arithmetic, the interpretation of the expressions that form each of the equivalently transformed equations is clearly that of the functional perspective.

Early Algebra

While the above content of school algebra is typically encountered in secondary school (from about the age of 13 years), the recently emerging field of research and practice, referred to as early algebra (e.g., Kaput et al. 2008; Kieran et al. 2016; Kieran 2022), has aimed at finding ways to build meaning for the objects and ways of thinking to be encountered within later algebra from the earliest years of primary school (grades 1–6). More specifically, the instructional focus is oriented toward moving students away from a purely arithmetical way of thinking by developing awareness of the structural, relational, and general aspects of certain algebraic objects and techniques – aspects that are often taken for granted within secondary school algebra. Notwithstanding the argument by Carraher et al. (2008) that early algebra is not algebra early, the essence of the objects and techniques being dealt with is basically the same. While the representations and discourse may be more accessible, young students do experience variables that are unknowns and variables that are varying quantities, as well as equations that are to be solved and equations that represent functional relations. A few examples will suffice to illustrate the nature of the connections of early algebra activity with the generalized arithmetic and functional perspectives on algebra sketched above.

Both numerical and figural patterning have been used as a means of sensitizing young students to the generalizing inherent to algebraic activity, with many studies centering on growing figural patterns (Rivera 2013). With respect to numerical patterning, some studies (e.g., Mason et al. 2009) have focused on pattern recognition and detecting underlying structures, such as 121, 1221, and 12,221. However, most have been aimed at encouraging students to produce some form of expression for the generalized functional relation underpinning the pattern. For example, Moss and London McNab (2011) describe the use of a function machine (handmade out of cardboard) where 8-year-old students took turns to try and figure out the rule created by a classmate (e.g., "double the number and add 3 more"). The pattern of input and output values was recorded in a T-table. The authors stress that when the values are recorded in a sequence where the inputs increase by 1, students tend to guess an arithmetically oriented recursive rule (e.g., "keep adding 2 as you go down"); however, when the pairs of values are recorded in a non-sequential way, students are more likely to deduce a correspondence rule for their generalization – that is, a functional correspondence relating output to input.

Introducing the notion of variable has been another current of early algebra research – variables that are either varying quantities/numbers or unknowns, and often involving unconventional means of symbolizing. With respect to representing varying quantities, Carraher and Schliemann (2018), for example, introduced fourth graders to situations such as "Mike has $8 in his hand and the rest of his money is in his wallet; Robin has exactly 3 times as much money as Mike has in his wallet. What can you say about the amounts of money Mike and Robin have?," which led the study participants to generate the expressions, "Mike's money, *M*, is $N+8$, and Robin's money, *R*, is $3\text{x}N$." For studies acquainting students with unknowns, and the associated ideas of equations and equation solving, some researchers have used simple, multi-operation equations, such as $7+6 = \square+5$, and have engaged students in expanding their views of the equal sign from an operational to a relational symbol (e.g., Carpenter et al. 2003). Others have employed word problems as a means of bringing forward not only the idea of representing unknowns within an equation but also the notion of an equation describing the word problem situation with the operations that have been stated in the problem. For example, Usiskin (1988) has used the problem, "*When 3 is added to 5 times a certain number, the sum is 48; find the number,*" to illustrate the shift required by young students in moving from immediately finding the answer, often mentally, by subtracting 3 from 48 and then dividing by 5 (i.e., the arithmetical approach of inversing), toward first representing the problem with an equation containing an unknown, along with the problem's given operations of addition and multiplication (e.g., $5x + 3 = 48$), before going about solving the equation. Still, other researchers (e.g., Radford 2022) have used story problems represented by concrete and iconic semiotic systems to introduce students as young as 8 and 9 years of age to the solving of equations having occurrences of the unknown on both sides of the equation and to the technique of performing the same operation on both sides.

As a closing remark to this section, we reiterate that the above discussion of the different perspectives for thinking about, and acting with, the objects and techniques of algebra were situated within paper-and-pencil environments. The sections that follow present the construct of instrumental distance and then study the distances that Scratch, Excel, and GeoGebra present, respectively, to one or the other of these perspectives, analyzing, within each, the impact of digital tools on ways of conceptualizing these same objects and techniques.

Instrumental Distance: Theorizing the Impact of Digital Tools on Mathematics

Origin and Theoretical Roots

The handwriting and paper support environment invites the techniques described above of literal transformations of equations, graphing of functions, and symbolic manipulation of variables. Digital environments may change the conceptualizations of these objects and techniques and disrupt the usual connections described in section "Algebra and Algebraic Thinking," which may then be reconfigured: New ways to solve similar tasks are introduced, in addition to or replacing the usual ones, and new tasks and objects may appear.

As Lagrange et al. (2003) noted, the changes introduced by technology were precisely one of the initial arguments in the research literature for using technologies in mathematics courses. Yet, from the 1990s, it became clear that the integration of digital environments in mathematics education was far from simple, despite instructions and resources, and required new frameworks (Artigue 2020). The concepts of instrumentation and instrumental genesis, developed in cognitive ergonomics (Verillon and Rabardel 1995), then found favorable ground in didactic theories (Trouche 2016). This led to the Instrumental Approach to Didactics (see also chapter ► "Introduction to Section: Roles of Theory, Methodology, and Design of Digital Resources in Improving Mathematics Education" in this book) – a frame that diverse authors have contributed to, such as Artigue, Drijvers, Lagrange, and Trouche. This approach allowed the authors to shed light on some complex phenomena in the instrumental geneses of computer algebra systems (CAS). The instrumentation of CAS changes the usual algebraic concepts and moves the techniques far away from the usual institutional techniques. Instruments are non-neutral on learners' conceptualizations. For example, Drijvers (2003) showed that the students' instrumental geneses of CAS impact on their understanding of the notion of parameter. Similarly, research revealed that using a graphing calculator leads to schemes that impact on the students' conceptualizations of the notions of limit (Guin and Trouche 1998) and of tangent line (Guin et al. 2004).

Thus, in the domain of algebra learning, evidence of a certain distance was beginning to emerge. For example, Tabach and Friedlander (2008) explored how spreadsheets have an impact on algebraic transformational tasks and indicated that spreadsheets may alter the early learning of symbolic transformations. For Hoyles et al. (2020, p. 80): "the distance between the syntax of Spreadsheet formulas and algebraic syntax may be a hallmark of weak mathematical fidelity." A little later in the text, they return to the idea of distance to emphasize that not only are the objects changed by spreadsheets, but also the techniques: “the spreadsheet method for solving word problems (. . .) is **far** *[our emphasis]* from the Cartesian method of solving problems” (p. 81).

Haspekian (2005), also addressing the gap between spreadsheet and paper-and-pencil algebra, has used the same term “distance” and proposed, within the frame of the instrumental approach to didactics, the construct of *instrumental distance* – a tool that examines the impact of technology not only on mathematical contents but also on the connections that these contents usually have in mathematics teaching.

The Notion of Instrumental Distance

The mathematical activity of a subject depends on the environment used to conduct that activity. Solving a task with digital technology can differ and be quite distant from solving the same task in the paper-and-pencil environment. This idea of “distance” finds a theoretical basis in the instrumental approach framework, which highlights, through the concept of *instrumental genesis*, the non-neutrality of instruments on the subjects’ conceptualizations (Rabardel 2002). By applying this to mathematical activity, the notion of instrumental distance (ID in the following) points to the qualitative gap between the mathematics “living” within a digital environment, compared to those in the usual paper-and-pencil one.

As seen in the case of algebra, school mathematics contents do not live in an isolated way. They maintain connections between them, giving rise to different coherent conceptual perspectives. To take into account the fact that technology may affect these usual connections, the notion of ID also relies on Vergnaud’s (2009) concept of *mathematical conceptual field*: a coherent network of mathematically interconnected tasks, techniques, and objects, with their vocabulary, definitions, and representations – linguistic and symbolic. The perspectives described in section “Algebra and Algebraic Thinking” are examples of portions of conceptual fields for algebra learning. Each of them, as a whole, encompasses its coherent set of tasks, objects, techniques, representations, etc., according to the institutional context (grade level, culture, epoch, etc.). Using a digital environment may impact on these conceptual fields, bringing out a different algebraic framework, with its own contents and coherences. The ID aims at identifying and documenting all the differences that may occur between the mathematical framework that the tool invites and the one of the usual institutional conceptual field. These differences may appear at the level of the field’s constituents (objects, representations, vocabulary, type of typical tasks, and resolution techniques), as well as their connections. The ID of a new environment is thus examined relative to a given institutional context, with its usual conceptual fields in the paper-and-pencil environment. In this context, the ID between a digital environment and paper-and-pencil environment, or, more generally between two environments, is defined as the overall technical and conceptual differences that emerge in the instrumental geneses of these two environments.

For example, if we consider the conceptual perspective of the generalized arithmetic approach to algebra (section “The Objects and Techniques of Algebra”), its usual algebraic contents (the main concepts, symbolizations, vocabulary, and solving techniques) are all changed in spreadsheets (see section "Generalized-Arithmetic Approach with Excel"). As a result, the whole algebraic framework offered in Excel is quite distant from that in the paper-and-pencil environment.

The paper-and-pencil environment (p&p in the following) does not play a role symmetrical to the others. It has a referential status in the traditional context of mathematics teaching and learning. School knowledge is institutionally defined in relation to p&p. The transformation of these referential contents into contents that are more or less distant raises the question of their articulation, especially in the case of too significant an ID. The contents can be transformed in a too complex way. They can also, conversely, be too simplified; the environment can, for example, automate and consequently eliminate the mathematical solving of some tasks, which were formative in themselves. We may recall the example of the log tables that have disappeared with calculators or the debates on the use of calculators, which supplant the technique of division, whereas this technique is formative in itself (it allows, for example, to understand why certain divisions never end, why certain rational numbers are not decimal, and why their decimal part is periodic).

Thus, from a teaching and learning perspective, too great an ID helps to explain teachers' resistance to integrating new technologies. On the other hand, the affordances and didactical potential of a tool depend on how the algebraic contents are transformed by the tool. Therefore, a certain ID is needed for a tool to present opportunities for learning.

Factors Generating Instrumental Distance

The ID carried by a tool has different sources and different ways of impacting the mathematical activity. One source of ID is "computational transposition" (Balacheff 1994): a phenomenon that transforms mathematical concepts due to computers' internal and external (interface) representations. More generally, the instrumentalization of the new environment could require technical knowledge that can also interfere with algebra (due to the design of the tool, its functionalities, the new vocabulary, new objects, etc.). Other sources are the algebraic techniques and associated schemes that may differ within the instrumentation process.

Another source of ID arises from whether or not the environment has been designed for the purpose of teaching mathematics. Thus, the use of a mathematical tool such as a spreadsheet will generate a greater distance than the use of a dynamic geometry software. A tool designed with a non-educational aim may have a distance, for example, at the level of the vocabulary involved, which can be distant from the usual mathematical vocabulary. This puts demands on the teacher in linking the terms to the usual mathematical vocabulary. While the geometry software refers to points, lines, circles, etc., the spreadsheet refers to cells, rows, and columns. How do we relate them to the concept of variable in mathematics?

The mathematical domain that is considered is another factor. The use of a dynamic geometry software to teach algebra will generate a greater distance than its use for geometry. The use of a spreadsheet to teach algebra will generate a more significant distance than its use in statistics.

The ID carried by a tool, regarding a given conceptual field, manifests itself in two ways. The tool can transform each of the components of this field with respect to the usual algebraic contents: objects, vocabulary, symbolizations, etc. It can also put forward contents, which, even if not modified, prove to be unusual with respect to this specific field.

Resulting from the points above, we synthesize here the different directions in which one can look for possible sources of ID, some very focused, others broader:

- The vocabulary
- The representations, symbolizations
- The tasks, the solving techniques
- The contents involved are as follows: Are the usual objects modified? Does the environment introduce new objects? Are they interfering with the mathematics?
- The algebraic framework that the digital environment induces in general or when solving some tasks

One could examine each of these points in the new environment and compare them to the p&p. For example, one could compare the p&p technique with the corresponding one in the new environment for the same mathematical task.

The comparisons between the main characteristics of the algebraic frameworks invited by an environment with the p&p ones for the same tasks may be presented in a table as in Table 1.

In the following, we will examine various tasks emblematic of these algebraic perspectives: solving an arithmetic word problem, generalization/patterning task, solving an equation (with parameters or not), and see how the contents, as much as the connections usually put forward in p&p for these tasks, are affected.

Early Algebra with Scratch

Nowadays, the use of programming environments is requested in several countries' curriculum. In early grade levels, block programming languages such as Scratch[2] may be used for developing computational thinking but also mathematical thinking

Table 1 ID between algebraic frameworks in p&p and in another environment

Main characteristics of the algebraic framework	p&p	Other environment
Fostered objects		
Fostered processes of solving tasks		
Other relevant points (depending on the mathematics at stake) as: Fostered representations Pragmatic potential/major field of problems Nature of solutions, etc.		

[2] https://scratch.mit.edu

as these languages bring certain important mathematical content into play (Benton et al. 2017). Some of these contents strongly relate to algebra. Focusing on algebraic concepts such as equality, variable, or function within three different types of programming environments, Bråting and Kilhamn (2021) draw attention to a certain critical distance between the domain of algebra and that of programming: "We reveal potential conflicting interpretations when these concepts appear in the different systems of representation of these domains" (p. 171). Here, we focus on Scratch and illustrate the ID it generates in the context of algebra learning in the early grades.

At this school level, algebra teaching is often supported by the study of structures and relationships arising in arithmetic, viewed in connection with the generalized arithmetic or functional perspectives on algebra (see section "Early Algebra"). To show the ID of Scratch regarding these views in p&p, we first focus on a simple arithmetic word problem and thereafter a problem based on the Fibonacci sequence.

An Arithmetic Word Problem in Scratch

This part concerns the use of the equal sign and its mathematical formalization, the concept of variable and its mathematical meaning, and the solving technique.

An Arithmetic Word Problem Involving a Two-Step Calculation

Anna buys three cookies for SEK 12 each. Then, she buys a glass of lemonade for SEK 15. How much does she have to pay?

In p&p, young students sometimes solve this kind of task with calculations written one after the other, employing what has been referred to (in Kieran 1981) as a "running total" use of the equal sign:

$$3 \cdot 12 = 36 + 15 = 51 \quad (1)$$

Answer: Anna pays SEK 51.

The answer is correct but mathematically the equal sign to the left in the solution process is not used in the conventional way.

This points to a well-known difficulty for students in the passage from arithmetic to algebra. In an arithmetical context, young students tend to consider the equal sign from a computational or operational perspective ($5 + 3$ *gives* 8) rather than relationally or structurally (see section "Early Algebra"). In a teaching context, this error could be an opportunity for a teacher to explain the meaning of the equal sign from a structural dimension.

Yet, the solution process (1) above reflects how we in real life solve this kind of task mentally: We would probably first have multiplied 3 by 12 which equals 36 and then added 15 to 36. However, in mathematics there is no given syntax that *directly* corresponds to these steps of solving. You do not erase "$3 \cdot 12=$", which you actually do when solving in your head. Instead, in mathematics the most common way to write down the equality is in the synthesized form:

$$3 \cdot 12 + 15 = 51 \tag{2}$$

Note that, unlike (1), this process does not include 36, the intermediate result, coming from a first calculation used when calculating mentally. The point here is that in real life, we solve this kind of task *stepwise*, which corresponds to the process (1) but not to the mathematical equality (2).

A mathematical technique more similar to the stepwise approach (1) can be written by dividing the synthesized calculation "3 · 12 + 15" into two independent calculations: In the first, the intermediate value 36 is calculated and then carried over to the second that calculates the final result:

$$3 \cdot 12 = 36 \tag{3}$$

$$36 + 15 = 51$$

Solving the Task in Scratch

In programming, everything is performed stepwise. Therefore, a programming environment such as Scratch seems interesting for this kind of task. Let us examine what a solution process can look like there.

Scratch is a block-based visual programming language adapted primarily for children. Using a block-like interface, users can program a sprite (an object, or a character) to perform functions controlled by scripts. Figure 1 shows a typical way of solving our task. The program is simple in the sense that it uses only a succession of one-step instructions. It uses a variable *moneyToPay*, which is initially given the value 0 by the assignment block "*set [variable] to.*" By using the block "*change*

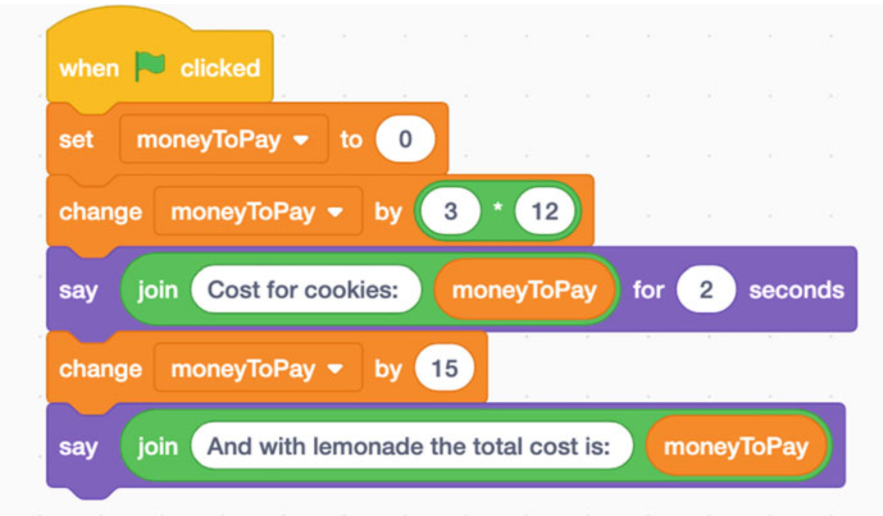

Fig. 1 A Scratch code for solving the task

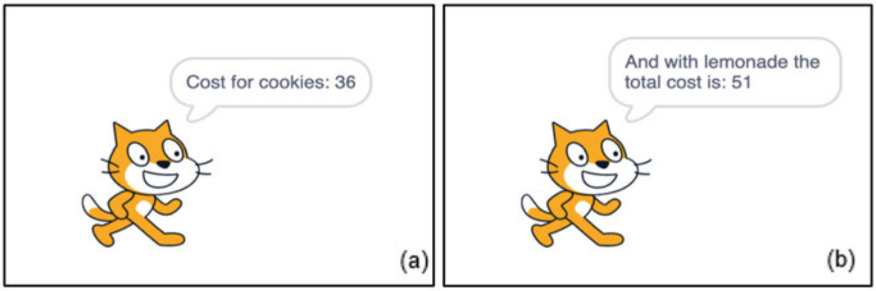

Fig. 2 Sprite saying the cost for cookies (**a**) and sprite saying the total cost (**b**)

[variable] by," the multiplication "3 · 12" is then added to 0 (and the result is assigned to the same variable). Thereafter, the sprite displays the intermediate result 36 (Fig. 2a) before the program adds 15 (again assigning the result to the variable *moneyToPay*). Finally, the sprite displays the last value stored in *moneyToPay* (Fig. 2b).

First, we can notice that Scratch shows significant differences in the solving technique and its representation, particularly concerning the equal sign. Unlike in p&p, the solving approach does not provide the opportunity to discuss the meaning of the equal sign since it is *completely absent* in the program. Instead, we find some new objects, which do not exist in p&p solving approaches, as the variable blocks "*set [variable] to [value]*" and *"change [variable] by [value].*" These two instrumented notions do not transfer to p&p. The connection between the techniques carried by the two environments is therefore weak.

Second, Scratch adds an algebraic content that is not present in p&p approaches for this type of word problem: the variable concept. The code (Fig. 1) includes the variable *moneyToPay*, whose value is updated throughout the program. In p&p mathematics, we cannot update variables in that way. On the other hand, it is possible to algebraically manipulate variables without any specific value being assigned. This is not possible in programming languages such as Scratch, Python, and JavaScript since every operation involving a variable operates on its assigned value (Bråting and Kilhamn 2021).

In this kind of task, in p&p mathematics, we store intermediate values such as 36 as a part of the calculation. The variable *moneyToPay* can perhaps be seen as hidden within these calculations. The point is that there is no mechanism in mathematics that temporarily can hold a value[3]; instead, a temporary value needs to be explicitly included within the calculation. Thus, the Scratch solving approach provides an opportunity to introduce a variable concept already in early grades. However, it is not an easy task for the teacher as the vision of this concept offered by

[3] If the solving process is more complex, we may need to introduce one or several variables corresponding to important subgoals of the broader calculation.

Scratch differs from that in p&p. As seen above, there is an ID between the variable concept in Scratch and in p&p mathematics.

In conclusion, using Scratch to solve this task does not afford the occasion to discuss students' difficulties in understanding the equal sign. In addition, Scratch brings new objects into the scene, in particular the concept of variable, which is moreover considerably transformed compared to the mathematical concept of variable. The variable concept is further discussed in our next example.

Pattern Generalizing and Variables

An important part of early algebra is to develop the idea of generalizing, where one common way is to use patterns and number sequences (see section "Early Algebra"). In our next example, we will consider a task based on a well-known number sequence:

Consider the Fibonacci numbers: 0, 1, 1, 2, 3, 5, 8, . . .

I. *How does the sequence of numbers continue?*
II. *Let the numbers in the sequence be denoted* $F_1, F_2, F_3, \ldots$.
How can you get the next number in the sequence? Try to describe it mathematically
III. *Can you write a program in Scratch that calculates any number in the Fibonacci sequence?*

In the first task, the students are supposed to find out that each number in the Fibonacci sequence is the sum of the two preceding ones. The second task is more difficult since it can only be solved if the students are familiar with recursive formulas, which in this case can be expressed as $F_{n+2} = F_{n+1} + F_n$ for $n > 1$, and $F_0 = 0$ and $F_1 = 1$ (where F is short for Fibonacci). Since the task is intended to be solved by young students, we can assume that this is problematic due to the complicated notation. Let us move on to the third task. In Fig. 3, we show an example of what a program in Scratch that calculates a specified number of terms in the Fibonacci sequence can look like.

The program includes five variables; *prev, curr, next, countEnd,* and the loop variable *count*. Note that in programming the name of a variable often reflects its content and gives the programmer a better understanding of the purpose of the variable. In this case, we avoid the mathematical notation $F_{n+2} = F_{n+1} + F_n$ and instead use the more intuitive notation *next* = *curr* + *prev*. The program starts by assigning the variables *prev* and *curr* and *count* their initial values 0, 1, and 0. Then, the program asks how many numbers in the Fibonacci sequence are to be calculated by using the sensing block "*ask and wait.*" Finally, the Fibonacci numbers are calculated one after the other in the loop block "*repeat until.*" For each iteration, in the loop the sprite writes the current Fibonacci number on the screen. Now, let us take a closer look at the different characteristics of the variables in the program and compare this with the mathematical view.

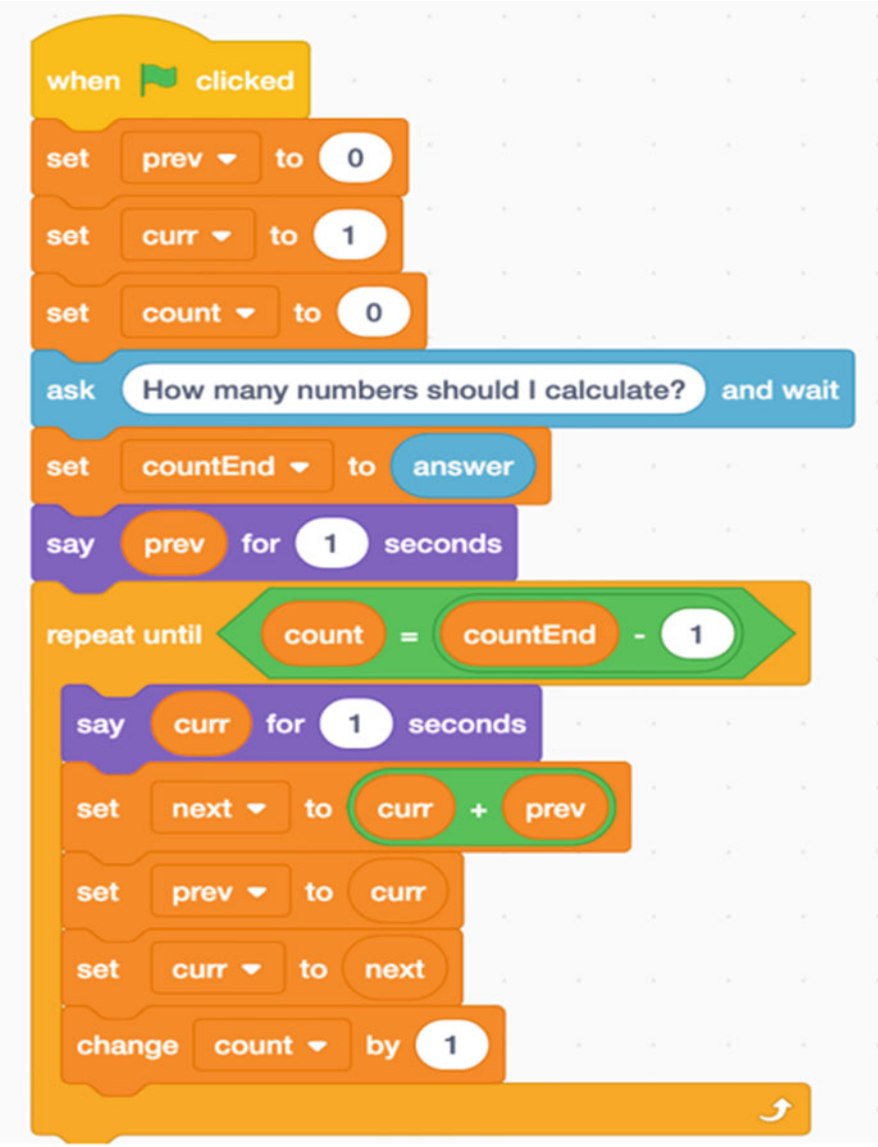

Fig. 3 A program that calculates a specified number of terms in the Fibonacci sequence

The variable *countEnd* is assigned a value through the assignment block "*set* countEnd *to answer*" where "answer" is the value entered by the user of the program. The answer block is a temporary variable, although not identified as a Scratch variable in an orange block since it is tightly bounded to the blue sensing block "*ask and wait.*" Note that the value of *countEnd* is never reassigned, it keeps its value throughout the whole program; that is, it can be seen as a constant. This is different from the other variables that are changing their values throughout the program. In the loop, the next value in the Fibonacci sequence is calculated through the assignment block "set *next* to *curr* + *prev.*" In order to prepare for the next iteration of the loop, the values of *prev, curr,* and *next* are thereafter shifted. Finally, the variable *count* increases its value by one in order to keep track of the number of iterations in the loop. As in the previous task, these variables change value within the same solution process which again differs from a mathematical variable.

Another difference is that in school mathematics the Fibonacci numbers cannot be described without introducing an infinite sequence of indexed variables, where each indexed variable can be calculated recursively from the initial conditions. This clearly differs from our Scratch program where we only have three main variables that are always well defined. That is, they are assigned initial values and then reassigned several times as the procedure takes us stepwise through the Fibonacci sequence.

Algebraic Framework Invited by Scratch for the Early Algebra Contents

As seen in section "Early Algebra," early algebra focuses on introducing algebraic thinking, for example, by taking advantage of word problem situations and patterning tasks. Word problems are used by teachers to help build meaning for the objects encountered within later algebra, especially those shared with arithmetic. This concerns both the equal sign, for which the students' conceptualizations must evolve from operational toward structural aspects, and the idea of an equation to represent not only unknowns but also a way of describing a word problem situation. As for numerical and figural patterning tasks, they are exploited to develop the idea of generalizing.

The Scratch analyses above show that these algebraic contents are strongly impacted. The example of the "Anna and the cookies" word problem has no link with the equal sign. Instead, new algebraic content is introduced, such as the use of variables in the solving of an arithmetic word problem. The patterning of the Fibonacci example introduces new objects linked to the computer science domain, as in the concept of loop, and the multiple different meanings of the notion of variable. Hence, building meaning for the objects encountered within later algebra is not at all obvious, and developing the idea of generalizing must be done by finding new ways, which exploit some computer science knowledge. Table 2 synthesizes the ID to early algebra carried by Scratch.

Regarding the design, Scratch is an educational tool, but it is not designed for algebra teaching. As shown in Table 2, the algebraic framework invited by Scratch is distant from that of p&p in Early Algebra. If the task is oriented loosely toward the generalized arithmetic perspective (section "An Arithmetic Word Problem in Scratch") or more directly toward the functional perspective with a patterning task (section "Pattern Generalizing and Variables"), it requires important adaptations in order to be used with the same goals as is usually the case for p&p.

Table 2 Different algebraic frameworks invited by p&p versus Scratch in early algebra

Main characteristics	p&p	Scratch
Fostered objects	Equal sign and operational signs (particularly addition and subtraction signs used in arithmetic) and meaning of equality as a relation rather than a sign indicating a process Unknowns Possible use of letters for unknowns	Equal sign used in loops for coding conditions Unknowns not apparent, instead: introduction of different types of variables Possible use of letters for variables Introduction of new instrumented objects such as loops and blocks and new notions such as "assignment"
Pragmatic potential/major field of problems	Introducing algebraic thinking as a tool for representing and solving Word problem situations, in relation to arithmetical thinking (relation with arithmetic, focus on structural aspects of expressions, etc.)	Relation with arithmetic differs from that in p&p and is not as obvious as it was in p&p with such arithmetic word problems
	Introducing algebraic thinking as a tool for representing and solving Numerical and figural patterning tasks in relation with the idea of generalizing	Scratch coding fosters computational aspects rather than structural Pattern generalization can be related to loops, yet this may amplify the differences for the notion of variable Developing the idea of generalizing must be done by finding new ways
Fostered processes of solving tasks	Analysis/synthesis process Applying syntactic rules respectful of algebraic properties of the signs	Stepwise problem solving Using an intermediary: the programming of a sprite to make the machine carry out the solution

Generalized Arithmetic Approach with Excel

Using the list of ID elements given in section "The Notion of Instrumental Distance," we examine here some elements of ID of the Excel spreadsheet, when considering its use in the context of a generalized arithmetic approach to algebra. The basic connection of cells by "formulas" links spreadsheets to algebra and is the reason why many studies from different countries (see summary in Haspekian 2005) gave spreadsheets a positive role for entering algebra, identifying them as arithmetic-algebraic tools. However, Haspekian (2014) showed that spreadsheets modify algebraic contents and solving strategies by creating new action modalities and objects. Spreadsheets impact algebraic concepts, solving techniques, symbolizations, nature of solutions, and vocabulary, and these impacts interfere with usual teaching. The algebraic world experienced in spreadsheets compared to p&p (variable and formulas arise, unknowns and equations disappear, new instrumental knowledge as the copy functionality comes into use, etc.) is quite different. Next, we summarize these analyses.

Variables and Copying Down Formulas

Haspekian (2014) highlighted different objects in Excel, such as the "variable-cell," that interferes with the p&p concept of a "varying quantity." In Excel, cells but also columns or rows may play such a role, while bringing new representations and meanings. Moreover, a cell with a formula may have a two-faced status: both a formula and a possible variable for the formula of another cell. The recopy functionality complicates even more the situation. For instance, a formula, when recopied downwards, generates a new object "variable column," but also a "formula column," for which the usual operational invariance is not translated by a syntactic invariance in the Excel. Then, how does this invariance make sense for the student? Is it through the gesture of copying down?

It is also possible to name a group of cells, for instance, "n" for the group "A2: A5," and then use "n" in formulas, as "=n^2" in B2. This generates another notion of variable, this time closer to the traditional one. This variable refers to a finite number of cells, having each the characteristics of a variable-cell. The fact that they are all linked by this same name "n" gives a numeric multiplicity dimension to this notion of variable.

Technique of Equation Solving

This usual p&p algebraic technique of solving equations is transformed in Excel into a "trial-and-refinement" technique.[4] Moreover, this technique presents some differences with the p&p trial and refinement: In Excel, it is organized in a table, planned, and automated (for its calculations). Several organizations and functionalities (inducing different formulas and instrumented knowledge) are possible. Figure 4 shows three possibilities for solving the eq. $2x + 4 = 3x - 10$.

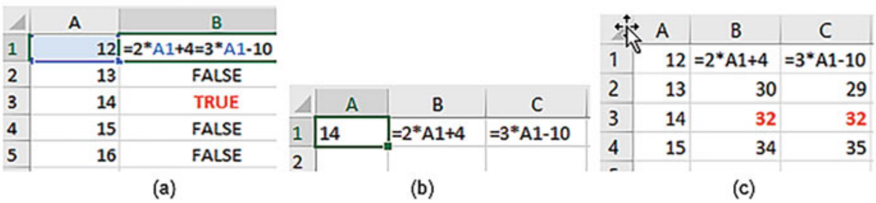

Fig. 4 Three examples of the instrumented trial-and-refinement technique in Excel

[4] We do not mention here the "solver" function of spreadsheets.

In (a), the formula, copied downward in column B, uses a double "=" sign – an instrumental functionality rather unusual in resources, which raises difficulties about the different meanings of the equal sign in the formula: "$=2*A5+4=3*A5-10$." This syntax could also be misleading for students if we refer to the classic error mentioned in the Scratch section of writing a wrong succession of equal signs. Yet, unlike (b) and (c), the equation is here present and in a form that is close to its p&p form (even if not usual). In (b) and (c), the equation is no longer present (if only in a form splitting its two members). The (b) solution uses a cell with a value (playing the role of x) and 2 cells (each having a formula) referring to this value. The trial and refinement consist of testing different values in A1 until reaching equal results in B1 and C1. This technique is very close to the p&p one, yet it uses the algebraic prerequisite of the notion of formula, which is not the case in the p&p arithmetic technique. Compared to the (b) solution, (c) automatizes the technique using three columns instead of three cells.

Algebraic Framework Invited in Excel for the Generalized Arithmetic Approach

From an institutional point of view, the changes described above have different impacts depending on the algebraic approach considered (section "Algebra and Algebraic Thinking"). For example, in the French lower secondary curriculum, spreadsheets are presented as good instruments to introduce algebra. Yet, this introduction in France (as in many other countries) is mainly based on solving equations in a generalized arithmetic approach. The algebraic contents usually emphasized in this approach include equations, unknowns, and algebraic techniques for solving equations in terms of exact solutions. In Excel, different elements come into play such as variables (varying quantities), formulas, and trial-and-error techniques that lead to approximate solutions (numerical or graphical). We move, here, from algebra as a tool for solving word problems with well-defined rules toward a more functional aspect of algebra, seen as an experimental tool of numerical or graphical conjectures and approximate solutions. Table 3 summarizes the significant ID between the generalized arithmetic approach to algebra and the algebraic framework characteristic of Excel, mainly coming from its design (commercial tool, with no link with the teaching of algebra) and from the computational transposition of the mathematics in play.

A Functional Perspective on Algebra: Equation Solving with GeoGebra

In section "Algebra and Algebraic Thinking," we distinguished the generalized arithmetic and the functional perspectives on algebra. In this section, the focus is on the latter. In a function, the independent variable acts as a changing quantity that runs through a domain set and causes covariation of the dependent variable in the

Table 3 Instrumental distance between the algebraic frameworks offered by p&p versus spreadsheets for a generalized arithmetic approach

Main characteristics	p&p	Excel
Fostered objects	Symbolizations Unknowns Equations Equal sign indicating a relation of equivalence	Symbolizations not necessarily visible Variables (cell/column/line), Formulas (with loss of the syntactic invariance) New meaning of the equal sign (indicating a formula) coexisting with the standard one
Pragmatic potential/major field of problems	Tool for solving world problems (sometimes involving proof)	Tool for generalization or optimization (problems of generalization/patterns, problems of optimization or model)
Fostered processes of solving tasks	Structural transformations that maintain top-down equivalence	No literal solving process, arithmetical process of trial and refinement
Nature of solutions	Exact	Approximate

range set. In school algebra, functions can be represented by algebraic expressions; vice versa, algebraic expressions generate functions as soon as we assign one variable as the independent one. In this way, algebraic expressions and functions are closely connected, even if functions also come with other representations such as graphs. A functional perspective on algebra, therefore, highlights the connections between algebraic expressions, functions, and graphs.

In combining different representations of function, mathematics educational software offers powerful tools. Digital technology such as GeoGebra[5] integrates an algebra module, a spreadsheet module, a graphing module, a CAS module, and a geometry module. Together, these tools offer simultaneous, connected graphical and algebraic representations. Clearly, this invites a functional perspective on algebra. In this section, we explore the impact of these tools for the case of equation solving with GeoGebra. First, we consider two tasks and compare the p&p approach of algebraic manipulation to the functional perspective GeoGebra invites (leaving apart the options for a functional perspective using p&p). Next, we describe the differences between these two approaches through the lens of instrumental distance.

Solving an Equation with GeoGebra

Suppose a fictional 15-year-old student is working on the task to solve the equation $x^2 = 2x + 3$. She knows how to factor quadratic expressions and how to use the quadratic formula. Also, she has quite some experience in graphing functions with GeoGebra (or a graphing calculator). In this task, she recognizes the variable x as the unknown to be found and starts the usual p&p procedure: She first subtracts $2x + 3$

[5] https://www.geogebra.org

from both sides of the equation, resulting in $x^2 - 2x - 3 = 0$. Next, she either factors the left-hand side or applies the quadratic formula. Both techniques lead to the solutions $x = -1$ and $x = 3$, which can be easily checked through substituting the values back into the unknown. The functional approach, to graph the functions defined by the left-hand side and the right-hand side of the equation and search for intersection points of the two graphs, is somewhat laborious in paper-and-pencil and is therefore not invited.

In case, this student would decide to use her GeoGebra experience, and the equation-solving procedure might look different. The functionalities in GeoGebra directly orient the attention toward a graphical approach, rather than algebraic transformations. Therefore, she considers the left-hand side and the right-hand side of the equation as functions of x and enters them in GeoGebra's algebra window. If the graph window is properly set, the two graphs appear immediately. Solving an equation is considered finding the intersection points of the two graphs, or, to phrase it more algebraically, to find input values for which the two function values are equal. In this case, this comes down to intersecting a parabola and a line. The intersection points can be approximated with the dynamic trace option and are automatically generated through a geometric intersection procedure, which shows the solutions in the algebra window. As was the case for the p&p approach, the solutions may be checked through substitution, either by-hand or in the algebra window. Figure 5 summarizes this approach.

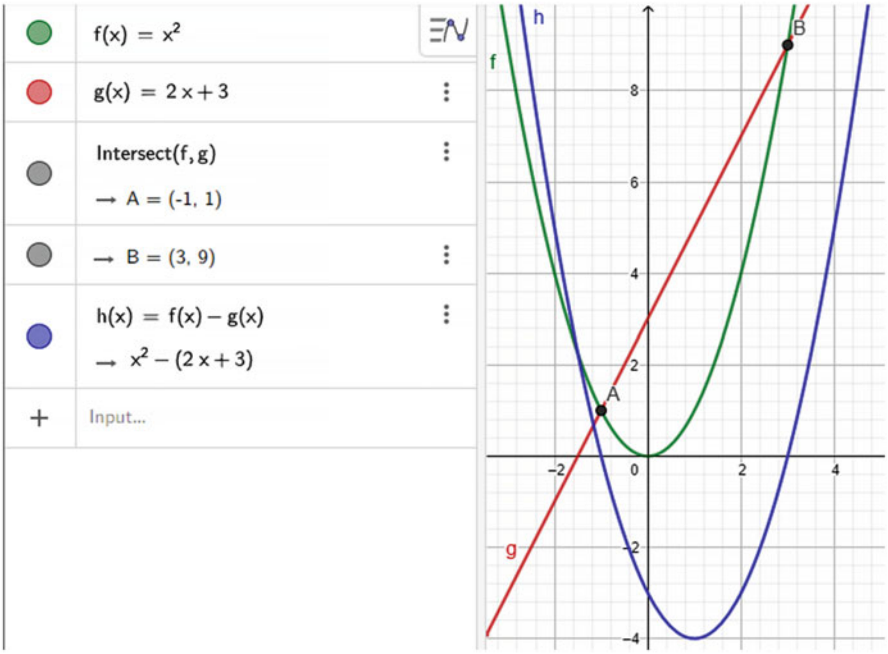

Fig. 5 A graphical approach to equation solving using GeoGebra

As an aside, the expression x^2 can be entered using the equation editor, or as x^2, as is the case of Excel (see section "A Functional Perspective on Algebra: Equation Solving with GeoGebra"). Both ways are more cumbersome than writing the expression on paper. Also, if the equation is more exotic, setting the dimensions of the graphing window might not be straightforward, as would be the case for p&p graphing.

What new light can the graphical approach through GeoGebra shed on the p&p method presented above? After the first by-hand step, the software's speed and ease to graph a function might invite the student to also graph the left-hand side of the new equation $x^2 - 2x - 3 = 0$. Apparently, the algebraic manipulation graphically comes down to intersecting the graph of the difference in the two initial functions with the x-axis. The equivalent equation results from "pushing down intersection points to become zeroes." The next step, the by-hand factoring to $(x + 1)(x - 3) = 0$, suggests that the parabola representing the difference function is the product of two linear functions. Therefore, the solutions are the zeroes of the two lines, which once more can be checked through substitution.

To summarize, using a digital tool such as GeoGebra invites a functional perspective on equation solving. Solving an equation, in this view, means intersecting the two graphs. The variable is still an unknown to be found through an intersect procedure, but can also be seen as a varying quantity moving toward the intersection point. The two sides of the equation are considered as algebraic function representations. Even the p&p transformation of the equation through subtracting $2x + 3$ can be graphically understood as moving from the intersection points of two graphs to the zeroes of the graph of the difference function.

Extending to a Parametric Equation

Now the next task is to change 3 into 8 and to solve the equation $x^2 = 2x + 8$. Our student could repeat the above procedures and find $x = -2$ and $x = 4$, but she might also generalize the task and look at the parametric case: $x^2 = 2x + a$. The p&p algebraic method would be to first subtract $2x + a$ and then apply the quadratic formula or complete the square. This is not an easy job, as she needs to carefully distinguish the different roles of the two variables involved. The graphical method is hardly doable through p&p, as it would require a three-dimensional graph.

In GeoGebra, a functional perspective invites an extension of the procedure for the specific case through the introduction of a parameter a. This generates a slider bar (see Fig. 6), suggesting that the parameter is a varying quantity acting at a higher level than x. Changing the value of a through this slider bar moves the line up or down. Similar to the previous task, GeoGebra easily finds the intersection points of the two graphs for $a = 8$.

Problem solved, but the environment invites new explorations. For example, one might notice that the graph of the difference function always seems a parabola with a vertex for $x = 1$ and that the two solutions seem to coincide for $a = -1$. This suggests symmetrical zeroes in $x = 1$.

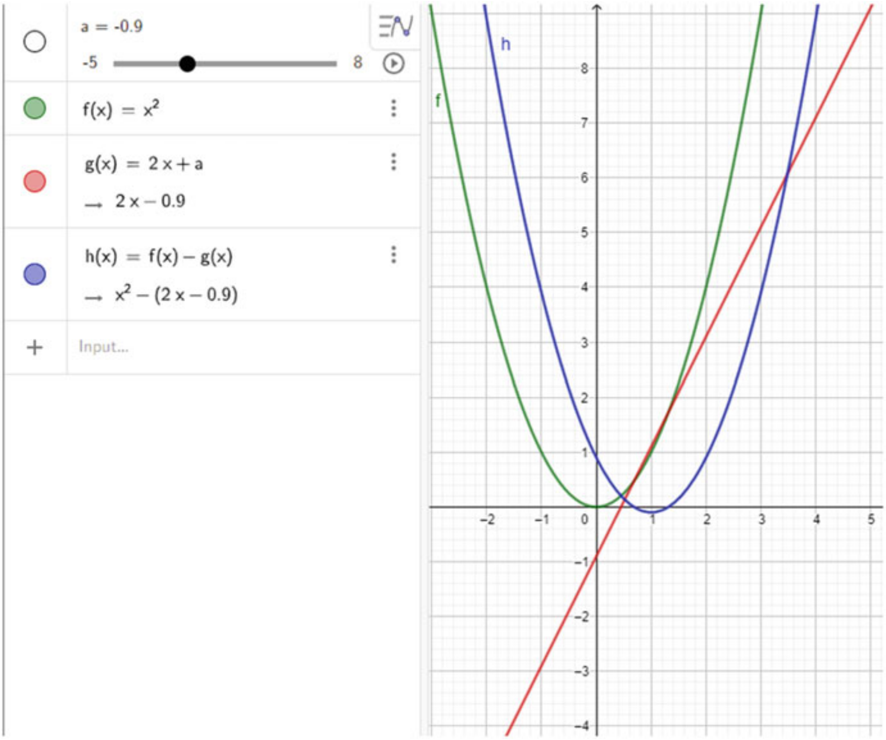

Fig. 6 Slider use suggesting symmetrical zeroes for x = 1

To check this symmetry conjecture, our student realizes that GeoGebra's CAS module seems a better tool for working with more variables. Therefore, she opens a CAS window. There she enters the solve command, adds the parametric equation, and specifies the unknown with respect to which to solve – even if the latter is not needed in this case, as GeoGebra considers x as the default unknown. This initially may give numerical solutions, based on the slider value of parameter a. However, once the slider is removed, or a different parameter is used, GeoGebra gives parametric solutions, which are no longer numbers but algebraic expressions. She adds up the two solutions and divides the sum by 2. The result is 1 indeed (see Fig. 7), which confirms that the two zeroes are always symmetric in $x = 1$.

Finally, the symmetry can be confirmed graphically: Let us trace the midpoint of the two intersection points while moving the line vertically. To do so, the geometry embedded in GeoGebra is used once more through constructing the midpoint of the two intersection points. The result shown in Fig. 8 is a vertical trace indeed.

To summarize, solving this parametric equation with GeoGebra invites the integration of different tools, including graphing options, slider use, CAS commands, and geometrical construction tools. The parameter is represented as a

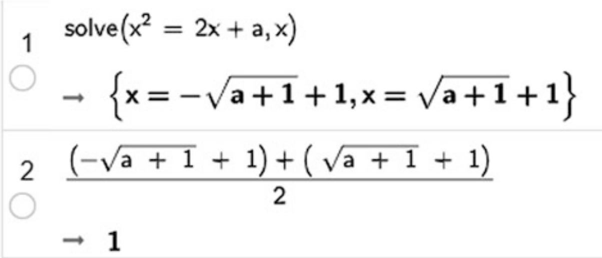

Fig. 7 Using the CAS to prove the symmetry of the zeroes algebraically

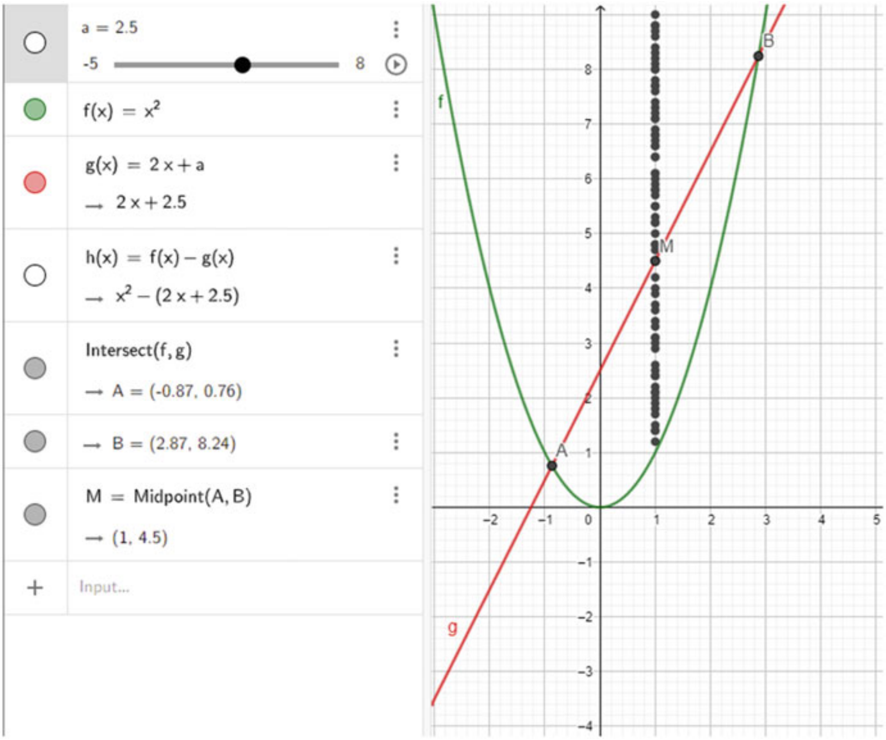

Fig. 8 Vertical trace of the midpoint of the two intersection points while the line moves up and down

quantity changing through a slider bar, and the symbolic work that might be done by-hand can be outsourced to the CAS module. Also, the environment may stimulate further exploring the mathematical situation beyond just solving the equation, in this case on the position of the solutions for different parameter values.

GeoGebra's Instrumental Distance to Paper-and-Pencil Equation Solving

The two equation tasks show the different techniques that emerge in GeoGebra and that are used in conventional p&p strategies. GeoGebra may offer new views on regular by-hand procedures and also invites a functional perspective.

To capture GeoGebra's instrumental distance to p&p, Table 4 summarizes differences between equation solving with p&p and with GeoGebra, and distinguishes between different views on algebraic objects and different techniques.

Overall, the p&p column in Table 4 reflects the symbolic approach to equation solving, whereas the GeoGebra column reflects a functional approach. Some rows show aspects of considerable instrumental distance. For example, the constraints of

Table 4 GeoGebra's instrumental distance to p&p equation solving: objects and techniques

Algebraic objects	**Paper-and-pencil**	**GeoGebra**
Equation	The question to find (exact) values of the unknown that make true <expr1> = <expr2>	The question to find (approximate) intersection points of the graphs of $f(x)$ = <expr1> and $g(x)$ = <expr2>
Solution	An exact value for the unknown that makes the equation true	Graphing view: An approximate value of the first coordinate of an intersection point of two graphs CAS view: an exact value or expression
Equivalent equations	Transform an equation into another one according to algebraic transformation rules	Transform two graphs into two easier ones, for example, one being the x-axis, so that intersection points turn into zeroes
Parameter	A second-order variable	A slider bar; its dragging affects the graphs. Option of automated animation
Parametric equation	An equation in more than one variable, in which you treat the parameter(s) as constant	An equation in which solutions are not numbers but algebraic expressions
Solution of a parametric equation	A solution is an algebraic expression	Graphing view: a solution depends on the value of the parameter; CAS view: a solution is an algebraic expression
Algebraic technique	**Paper-and-pencil**	**GeoGebra**
Equation solving	Primarily a matter of algebraic transformation of the equation	Graphical intersection technique or CAS solve command
Entering an expression	Hand writing	Using a formula editor or linear input, with possible mistakes
Algebraic manipulation	Is laborious and sensitive to mistakes	Is quick in the CAS module and free of errors, but involves syntax
Variation and generalization	Usually is hard work	Is easy to do and requires automated recalculation of previous steps
Exploring new situations	Requires starting over the work	Is relatively easy to do

entering expressions are typically a form of instrumental distance. Another example concerns the nature of solutions, which in p&p algebra usually are exact solutions, whereas the graphical approach in GeoGebra may lead to approximate numerical results. Most of the rows in Table 4, however, show that GeoGebra techniques may offer new views on the algebra at stake in a natural way. The shift from symbolic to functional (e.g., see the row Equation) and the changing nature of solutions impact on students' view on algebra.

To conclude this section, our analysis of the use of GeoGebra for solving equations shows mixed results about the consequences of the instrumental distance the software offers to p&p conventional mathematics. On the one hand, this distance is small: GeoGebra does support the conventional concepts and techniques, even if they may be slightly changed. On the other hand, GeoGebra use invites extending the landscape of concepts and the repertoire of techniques and in this way reshapes the "world of algebra." It invites new views through the availability of tools for connecting and integrating symbolic and graphical representations. As such, rather than a purely symbolic perspective, the use of GeoGebra invites a functional perspective on algebra, which benefits from the interplay between the symbolic, graphical, and even geometrical techniques the software offers.

Discussion

In this chapter, we address the question of *how digital tools impact on the teaching and learning of algebra* through the lens of instrumental distance, which highlights the changes generated by the use of technology on the contents of school algebra. In the case of algebra, instrumental distance deals with algebraic objects (expressions, equations, variables, etc.), algebraic techniques (solving an equation symbolically or graphically, etc.), symbolic representations (of equal sign, of variables, etc.), but also with the way these contents are usually connected in p&p activities. We illustrated this with three cases of digital resources – Scratch, Excel, and GeoGebra – that provide an overview of different algebraic contents, from primary to lower and upper secondary school levels, while covering the main perspectives relating to the teaching and learning of algebra.

The results raise awareness of new potentialities, but also the complexity, constraints, and difficulties generated by technology, indicating that using the didactic opportunities coming from technology is neither easy nor straightforward. As illustrated with the examples herein presented, the usual connections and coherences in algebra teaching can be more or less disrupted depending on the tool in question. They can be recast, as in the case of GeoGebra , which, when faced with an equation-solving task in a *symbolic approach*, invites rather a *functional approach*. They can also be recomposed by mixing different approaches, as in the case of Excel, which combine elements of a *generalized arithmetic approach* with elements of other approaches such as the "variable quantity" object, rather linked to the *functional approach*, and the trial-and-error technique, usually associated with the *arithmetical domain*. Finally, they can also be disrupted either with elements from outside of mathematics (like computer objects as the loop in Scratch, or the slider in GeoGebra), or with the transformation of some crucial usual contents (as the new uses of the equal sign in spreadsheets for equation solving), or even with the disappearing of crucial contents (as the equal sign in Scratch for an arithmetic problem). We suggest that such ID differences may be dependent on the design: whether the digital tool is educational (like GeoGebra or Scratch), or not (like Excel), but also, whether its educational aims purposely include algebra (GeoGebra was specifically designed with the purpose of linking geometry and algebra; see Hohenwarter 2006), or not (like Scratch, designed with the purpose of developing algorithmic thinking).

The above analyses highlight different degrees of ID. For the chosen tasks, there is a significant distance for Excel and Scratch, consisting of substantial modifications of the usual objects and techniques. In contrast, GeoGebra presents less ID. Favoring the visual aspects, it invites a *functional approach* to algebra, in which the contents are quite close to their p&p analogs. The mathematical framework invited by Excel also tends to favor a functional interpretation of the objects of algebra. As for Scratch, as we have seen, this environment favors a step-by-step computational approach that is close to a form of arithmetic thinking rather than algebraic. By its design, Scratch is not specifically oriented toward one or the other of the two main perspectives on algebra. Its potential algebraic character would seem to depend strongly on the choice of tasks. In our Scratch examples, the tasks were illustrative of some early algebra tasks with the solving of an arithmetic problem and the generating of a recursive rule for a particular numerical sequence. With other types of tasks, Scratch activity with young students could be tied more explicitly to either the generalized arithmetic or functional perspectives on algebra. Indeed, the setting up of an equation with an unknown could be approached as a process of generalizing an algebraic problem situation (see Hoyles et al. 2020) – despite the solving being carried out computationally by means of successive numerical substitutions to calculate the trials. Kilhamn et al. (2022) provide an example, where Scratch activity is oriented toward the development of functional thinking, with a task involving variables, functional correspondence rule, and table of values. With tasks such as these, it is possible to reduce the Scratch ID with respect to the two main perspectives on algebra, within the context of early algebra.

The current study clearly has its limitations. We have focused on three digital environments and a limited number of tasks. Obviously, this calls for its extension in the direction of other types of tasks, emblematic of the teaching and learning of algebra, or other environments for algebra learning, specifically for other types of new technologies, incorporating additional new functionalities, for instance, linked to body (gestures, embodiment, virtual reality, etc.), but also the consideration of multiple environments and the resulting ID. Likewise, analyzing other domains of mathematics via the ID lens is another extension. A recent example is the study by Bakos (2022), who uses ID to analyze *TouchTimes*, a software for teaching and learning multiplication, which provides visual, tactile, and symbolic retroactions to the fingertip actions of young students on a digital screen.

Lastly, the different impacts on mathematical school contents shown in this chapter naturally turn in the direction of teachers, asking the question of how to manage the ID and the transformed contents. Responding to this question resonates with Hoyles et al. (2020, p. 87) who conclude that a central problem of integrating technology in school mathematics is the issue that there are "many teachers, who (. . .) don't see how technology matches with official curricula." Our three modest studies help to understand teachers' difficulties in integrating digital technologies into the mathematics classroom. These difficulties are often explained by the institution or research, by a lack of equipment, funding, personal investment, or time. The notion of ID shows that the reasons are more complex and go beyond these usual reasons to reach directly the mathematics itself, here algebra, for which the usual school contents and coherences are modified by technology. It draws attention to the fact that integrating a new environment requires reconsidering the way in which these contents are taught and learned: New mathematical and didactic organizations have to be created, more or less distant from the usual organizations of teachers. This new workload is underestimated by both curricula decision-makers and designers. Official prescriptions present technologies as transparent with respect to mathematics, bringing forward content that is assumed to be of high fidelity to the usual targeted content. The ID makes us aware that these contents, on the contrary, are "detached" from the targeted content by a greater or lesser distance. This disruption on the one hand and its non-consideration by the institution on the other hand are essential factors that constitute the difficulties teachers have in integrating technologies into mathematics teaching. Our study calls for a better awareness of the kind of ID that a tool presents regarding the mathematical contents it is supposed to serve. It suggests that it could be beneficial to develop both professional training programs and resources that incorporate analyses of tools in terms of ID with respect to the given tasks and to the aim of the teaching, and use these analyses so as to help teachers integrate digital technologies into the teaching and learning of algebra.

Cross-References

► Networking of Theories: An Approach to the Development and Use of Digital Resources in Mathematics Education

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