Sixth-Order Adaptive Non-uniform Grids for Singularly Perturbed Boundary Value Problems

Sehar Iqbal and Paul Andries Zegeling

Abstract In this paper, a sixth order adaptive non-uniform grid has been developed for solving a singularly perturbed boundary-value problem (SPBVP) with boundary layers. For this SPBVP with a small parameter in the leading derivative, an adaptive finite difference method based on the equidistribution principle, is adopted to establish 6th order of convergence. To achieve this supra-convergence, we study the truncation error of the discretized system and obtain an optimal adaptive non-uniform grid. Considering a second order three-point central finite-difference scheme, we develop sixth order approximations by a suitable choice of the underlying optimal adaptive grid. Further, we apply this optimal adaptive grid to nonlinear SPBVPs, by using an extra approximations of the nonlinear term and we obtain almost 6th order of convergence. Unlike other adaptive non-uniform grids, our strategy uses no pre-knowledge of the location and width of the layers. We also show that other choices of the grid distributions lead to a substantial degradation of the accuracy. Numerical results illustrate the effectiveness of the proposed higher order adaptive numerical strategy for both linear and nonlinear SPBVPs.

1 Introduction

Boundary-layer phenomena, have many applications in different areas, such as fluid dynamics [1, 10, 17], aerodynamics [26] and mass heat transfer [25]. Nowadays boundary layer problems are addressed mainly with numerical techniques. A large number of numerical techniques have been proposed by various authors for singularly perturbed boundary value problem (SPBVPs). Finite difference approximations on non-uniform grids have been investigated in [13, 27, 29]. Numerical approximations and solutions for SPBVPs have been discussed in [2] and [23]. Theory and applications of singular perturbations in boundary-layer problems and

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multiple timescale dynamics have been reported in [30]. To deal with the steep solutions regions, two different forms of computational non-uniform grids by dividing the region into two or more with different uniform spacing were discussed in [24] for boundary-layer problems. At the beginning of the 90s, special piecewise uniform grids have been introduced by Shishkin [22], in which simple structured grids can be used for the numerical approximation of SPBVPs. A truncation error analysis introduced by the use of non-uniform grids and stretched coordinates for the numerical study of the boundary-layer problems has been reported in [8, 19, 20] and [31] in which numerical results are compared with those obtained on uniform grids. In [21] a stretched grid method was introduced in which the boundary-layer problems are transformed into new coordinates by a smooth mapping which concentrates the grid points in the steep regions without an increase of the total number of grid nodes. This concentration improved the spatial resolution in the region of large variation and enhanced the accuracy in numerical solutions. Adaptive moving grid methods for solving the partial differential equations are discussed in [5, 28]. Adaptive grid methods, graded grid difference schemes and uniformly accurate finite difference approximations for SPBVPs have been presented in [9, 11] and [3], respectively. The rate of convergence of finite difference schemes on non-uniform grids and super convergent grids for two point BVPs have been described in [6, 14, 32]. Further, finite difference approximations of multidimensional steady and unsteady convection- diffusion-reaction problems have been discussed in [7, 15, 16]. All these numerical approaches on non-uniform grids are more accurate than for the uniform case but with the same order of convergence. The supra-convergence phenomenon of the central finite difference scheme is well-known and it was studied rigorously. In literature, the increase from second to higher order accuracy for linear boundary value problems has been reported in [4] and recently in [18]. Second order central finite differences on non-uniform grids are discussed in [18], in which an adaptive numerical method is applied to obtain 4th-order convergence.

In the present study, we consider a singularly perturbed linear elliptic ordinary differential equation with a boundary layer. The goal is to propose an efficient adaptive numerical method which can be solve approximately such SPBVPs with an accuracy independent of the value of the perturbation parameter. Three-point finite-difference methods are, in general, of second order accuracy on a uniform grid. In all the schemes that have been discussed before, we may conclude that such schemes are accurate to 2nd order (uniform and non-uniform grids) and 4th order (non-uniform grids). Here, we extend these results to 6th order approximations not only for linear BVPs, but we are also able to obtain an (almost) 6th order accuracy for nonlinear models. For this, we propose a non-uniform equidistributed grid and show that the second order central finite difference scheme is substantially upgraded to *sixth order* on this refined grid. This supra-convergence is obtained by using an appropriate monitor function, which depends on the lowest derivative. Numerical experiments are discussed for different choices of the monitor functions to confirm the higher order of convergence.

The present manuscript is organized as follows: In Sect. 2, the SPBVP under consideration is presented. In Sect. 3, we present a higher order adaptive numerical method after which a general central finite difference scheme on non-uniform grids is derived in Sect. 3.1. To construct the adaptive grid, the equidistribution principle is explained in Sect. 3.2. Further, in Sect. 4, numerical results for different choices of non-uniform grids, are discussed. Numerical implementation for a nonlinear SPBVP on the proposed adaptive grid, is presented in Sect. 5. Finally, we summarize our results in Sect. 6.

2 A Boundary Value Problem

In this section, we first consider the following singularly-perturbed linear boundary-value problem with inhomogeneous Dirichlet boundary conditions:

$$\epsilon\, u'' - u = 0, \qquad u(0) = \mathrm{e}^{-\frac{1}{\sqrt{\epsilon}}}, \qquad u(1) = 1, \tag{1}$$

which has the exact solution

$$u(x) = \mathrm{e}^{\frac{x-1}{\sqrt{\epsilon}}}. \tag{2}$$

For small values of the perturbation parameter $0 < \epsilon \ll 1$, the steep solution (2) shows a boundary-layer behavior at $x = 1$. This is illustrated in Fig. 1. However, we will proceed further as if the exact solution is unknown. We will use (2) only to access the quality of a solution.

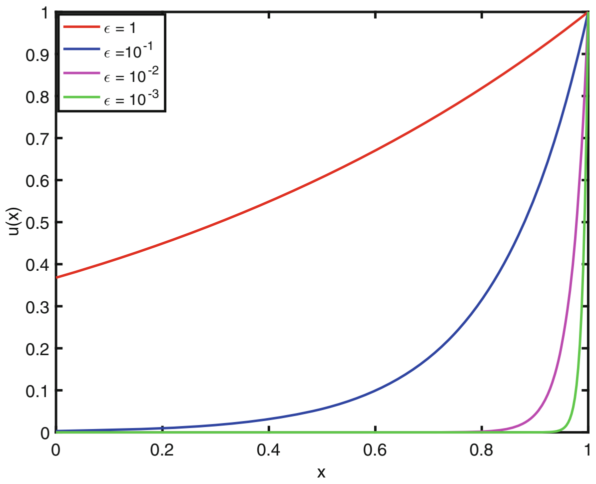

Fig. 1 Exact solutions of model (1) for decreasing values of ϵ

3 An Adaptive Numerical Method

For a higher order of convergence than two (on a three-point stencil), we are going to generate an optimal adaptive non-uniform grid for the convection dominant singularly perturbed BVP (1) as follows:

1. *Discretize problem* (1) *on non-uniform grid and transform the discretized system from the physical coordinate x to a computational coordinate ξ.*
2. *Generate an optimal adaptive non-uniform grid based on the equidistribution principle.*
3. *Compute the solution of the discretized system of given problem* (1) *on the optimal adaptive non-uniform grid, obtained from previous the step to establish a higher order of convergence.*

3.1 Discretizations on Non-uniform Grids

In order to deal with the appearance of steep boundary layers in the given model (1), non-uniformly distributed grids could be used to obtain more efficient and more accurate numerical solutions. Non-uniform discretization of the second derivative in (1) is given by

$$u_j'' \approx \frac{\frac{u_{j+1}-u_j}{p} - \frac{u_j-u_{j-1}}{q}}{h_j},$$

where the p, q and h_j are computed as:

$$p := x_{j+1} - x_j, \qquad q := x_j - x_{j-1}, \qquad h_j := \frac{p+q}{2}.$$

Approximating the derivatives in model (1) by using these expressions, yields the following numerical approximation:

$$\epsilon \frac{\frac{u_{j+1}-u_j}{p} - \frac{u_j-u_{j-1}}{q}}{h_j} - u_j = 0 \tag{3}$$

with boundary conditions $u_0 = \mathrm{e}^{-1/\sqrt{\epsilon}}$, $u_J = 1$. The idea is to choose a non-uniform central finite difference method such that the steep parts in the solution can be resolved. We will do this by performing the following steps. First, we transform the original model from the physical domain $I = [0, 1]$ to a computational domain I^*.

Let x and ξ denote the physical and computational coordinates, respectively. Without loss of generality, we define a composed function $v(\xi) := u \circ x = u(x(\xi))$

and coordinate transformation between x and ξ as follows:

$$x = x(\xi), \qquad \xi \in I^* = [0, 1],$$

where $x(0) = 0$ and $x(1) = 1$. The computational domain I^* can be discretized into J equal segments $\{\xi_j = j\ \Delta\xi\}_{j=0}^{J}$ with $\Delta\xi = 1/J$. The first derivative is then transformed as

$$u' = \frac{du}{dx} = \frac{dv}{d\xi}\frac{d\xi}{dx} = \frac{dv}{d\xi}\frac{1}{x_\xi}.$$

The SPBVP (1) on the computational domain I^* can be written as

$$\frac{\epsilon}{x_\xi}\frac{d}{d\xi}\left(\frac{1}{x_\xi}\frac{dv}{d\xi}\right) - v(\xi) = 0, \qquad v(0) = \mathrm{e}^{-\frac{1}{\sqrt{\epsilon}}}, \qquad v(1) = 1. \tag{4}$$

We also assume that the Jacobian $\mathbb{J}(\xi)$ of mapping $x(\xi)$ is bounded from below and above by some positive constant: $0 < \mathbb{J} := dx/d\xi < \infty$. Equation (4) equivalent with:

$$\frac{\epsilon}{\mathbb{J}}\frac{d}{d\xi}\left(\frac{1}{\mathbb{J}}\frac{dv}{d\xi}\right) - v(\xi) = 0 \tag{5}$$

with the same boundary conditions $v(0) = \mathrm{e}^{-1/\sqrt{\epsilon}}$, $v(1) = 1$. Equation (5) can be discretized on the uniform grid h as follows:

$$\frac{\epsilon}{\mathbb{J}_j}\left(\frac{v_{j+1} - v_j}{\mathbb{J}_{j+\frac{1}{2}}} - \frac{v_j - v_{j-1}}{\mathbb{J}_{j-\frac{1}{2}}}\right) - v_j = 0, \qquad v_0 = \mathrm{e}^{-\frac{1}{\sqrt{\epsilon}}}, \qquad v_J = 1, \tag{6}$$

where the Jacobian $\mathbb{J}$ is computed as:

$$\mathbb{J}_{j+\frac{1}{2}} := \frac{p}{\Delta\xi}, \qquad \mathbb{J}_{j-\frac{1}{2}} := \frac{q}{\Delta\xi}, \quad \mathbb{J}_j := \frac{\mathbb{J}_{j+\frac{1}{2}} + \mathbb{J}_{j-\frac{1}{2}}}{2\Delta\xi}.$$

Scheme (6) is equivalent to (3).

3.2 Adaptive Non-uniform Grid Generation

The aim of the *equidistribution principle* is to concentrate the non-uniformly distributed grid points in the steep regions of the solution (see [5, 12, 28, 33] and references therein). In this principle, the desired mapping $x(\xi)$ is obtained as a solution of the nonlinear problem:

$$\frac{d}{d\xi}\left[\omega(x)\frac{dx}{d\xi}\right] = 0, \qquad x(0) = 0, \qquad x(1) = 1, \tag{7}$$

where $\omega(x)$ is the so-called *monitor function*. The name equidistribution has to do with the fact that we would like to 'equally distribute' the positive valued and sufficiently smooth function $\omega(x)$ on each non-uniform interval. For this, we first define the grid points

$$0 = x_0 < x_1 < x_2 < \cdots < x_{J-1} < x_J = 1.$$

Next, we determine the grid point distribution such that the contribution of ω on each subinterval $[x_{j-1}, x_j]$ is equal. A discrete version of (7), after integrating once, reads:

$$\int_{x_j}^{x_{j+1}} \omega(x)\,\mathrm{d}x = \frac{1}{J}\int_0^1 \omega(x)\,\mathrm{d}x, \qquad j = 0, 1, \ldots, J-1.$$

In practice, one has to choose the monitor function $\omega(x)$ and solve the nonlinear (7) to obtain the required mapping $x(\xi)$. Equation (7) can be discretized using central finite differences as:

$$\frac{1}{\Delta\xi}\left[\omega_{j+\frac{1}{2}}\,\frac{p}{\Delta\xi} - \omega_{j-\frac{1}{2}}\frac{q}{\Delta\xi}\right] = 0, \qquad j = 1, 2, \ldots, J-1, \tag{8}$$

with boundary conditions $x_0 = 0$, $x_J = 1$. Equation (8) is called a *discrete equidistribution principle*. The discrete system can be solved efficiently using a tridiagonal matrix algorithm. The iterations are continued until convergence for a prescribed tolerance has been achieved.

4 Numerical Results

The aim of this section is to point out that it is possible to develop a central three-point finite-difference scheme on non-uniform grids which exhibits a higher order accuracy than expected and known until now. We solve the discretized system of SPBVP (1) on adaptive non-uniform grids based on the equidistribution principle. We establish these higher-order optimal grids (4th order and 6th order) with the help of a local truncation error analysis of the discretized system of (1). Numerical experiments show that the other choices of grid distribution lead to a substantial degradation of the accuracy. Numerical results illustrate the effectiveness of the proposed numerical strategy for linear and nonlinear SPBVPs. We measure the accuracy of the numerical solution by computing its distance to the reference solution

$$\|\varepsilon_h\|_\infty \equiv \|u_h - u\|_\infty = \max_{1\le j\le J} |u_j - u(x_j)|, \tag{9}$$

and the order of convergence can be calculated numerically as:

$$order = \log_2 \frac{\|\varepsilon_h\|_\infty}{\|\varepsilon_{h/2}\|_\infty}. \tag{10}$$

4.1 Case 1: Fourth Order of Convergence

The discretization of model (1) by approximating the second derivative on non-uniform grids, is given by

$$\epsilon\, \frac{\frac{u_{j+1}-u_j}{p} - \frac{u_j-u_{j-1}}{q}}{\frac{1}{2}(p+q)} - u_j = 0. \tag{11}$$

Scheme (11) can be equivalently rewritten as:

$$\epsilon\, \left(2u_j - \frac{2p}{p+q}u_{j-1} - \frac{2q}{p+q}u_{j+1}\right) + pqu_j = 0. \tag{12}$$

We rewrite expression (12):

$$\epsilon\, \left(2u_j - R_1u_{j-1} - R_2u_{j+1}\right) + S_0u_{j-1} + S_1u_j + S_2u_{j+1}) = 0, \tag{13}$$

where

$$R_1 = \frac{2p}{p+q}, \quad R_2 = \frac{2q}{p+q}, \quad S_0 = 0, \quad S_1 = pq, \quad S_2 = 0.$$

For 4th order of convergence, the discretized system is defined as:

$$\epsilon\, \left(2u_j - R_1u_{j-1} - R_2u_{j+1}\right) + S_1u_j = 0. \tag{14}$$

This is equivalent to scheme (11).

We study the approximation properties of scheme (11) on general non-uniform grids. For this, we need to work out Taylor expansions and compose the finite differences which appear in (11):

$$\begin{aligned} \frac{u_{j+1}-u_j}{p} &= u' + \frac{p}{2!}u'' + \frac{p^2}{3!}u''' + \frac{p^3}{4!}u^{(4)} + \frac{p^4}{5!}u^{(5)} + \mathscr{O}(p^5), \\ \frac{u_j-u_{j-1}}{q} &= u' + \frac{q}{2!}u'' + \frac{q^2}{3!}u''' + \frac{q^3}{4!}u^{(4)} + \frac{q^4}{5!}u^{(5)} + \mathscr{O}(q^5). \end{aligned} \tag{15}$$

Assume further that the mapping $x(\xi)$, is sufficiently smooth. Note that the grid difference functions p and q can be written as:

$$\begin{aligned} p &= x_{j+1} - x_j = \Delta\xi\ x_\xi + \frac{\Delta\xi^2}{2!} x_{\xi\xi} + \frac{\Delta\xi^3}{3!} x_{\xi\xi\xi} + \mathscr{O}(\Delta\xi^4), \\ q &= x_j - x_{j-1} = \Delta\xi\ x_\xi - \frac{\Delta\xi^2}{2!} x_{\xi\xi} + \frac{\Delta\xi^3}{3!} x_{\xi\xi\xi} + \mathscr{O}(\Delta\xi^4). \end{aligned} \tag{16}$$

We obtain the asymptotic expression:

$$\epsilon\ u'' - u - \epsilon \left(\frac{p-q}{3} u''' - \frac{p^2 - pq + q^2}{12} u^{(4)} - \frac{(p-q)(p^2+q^2)}{60} u^{(5)} + \mathscr{O}(\Delta\xi^4) \right) = 0. \tag{17}$$

We transform the system $x \mapsto \xi$ so, from (17), one gets

$$-\epsilon \frac{\Delta\xi^2}{3} \left[x_{\xi\xi} u''' + \frac{1}{4} x_\xi^2 u'''' \right] + \mathscr{O}(\Delta\xi^4) = 0. \tag{18}$$

In general, the scheme (11) is second order accurate. However, we notice that the scheme will be fourth order accurate, if the mapping $x(\xi)$ satisfies the following equation:

$$x_{\xi\xi} u''' + \frac{1}{4} x_\xi^2 u'''' = 0, \tag{19}$$

where $u(x)$ is a solution of (1). Equation (19) can be rewritten as:

$$(u''')^{\frac{3}{4}} \left[(u''')^{\frac{1}{4}} x_\xi \right]_\xi = 0.$$

Since $u''' \propto u'$, we obtain $\left[(u')^{1/4} x_\xi\right]_\xi = 0$. For our numerical simulations, we make the following choice of the monitor function to illustrate the use of the equidistribution principle (see Sect. 3.2):

$$\omega = (u')^{\frac{1}{4}}. \tag{20}$$

In the equidistribution method, we obtain the mapping $x(\xi)$ from (7). For SP-BVP (1), we can write the monitor function in a generalized form:

$$\omega_\eta(x) = (u'(x))^\eta. \tag{21}$$

We take the optimal choice of the monitor function (20) to establish the 4th order of convergence for model (1). The numerical experiments are performed for different choices of η in (21) and $\epsilon = 0.01$.

We slightly change the monitor function by changing the values of η and observe that the convergence order also changes. Table 1 shows clearly that the optimal result is obtained for $\eta = 1/4$. This is the *optimal* choice to get the higher order of convergence 4 for the scheme (11). On the other hand, for other choices of the monitor functions $\omega(x)$ with different η in (21), we obtain 2nd order of convergence. However, for the choice $\omega = (u')^2$, the convergence of order falls down to $\approx 1/2$, which is, of course, to be avoided for practical numerical simulation. As mentioned above, by taking different choices for the monitor functions ω, we observe a difference in accuracy and convergence order of the numerical solutions. An even higher order of convergence can be found by an appropriate choice of the monitor function in the next section.

4.2 Case II: Sixth Order of Convergence

Instead of only using the grid values (x_j, u_j) for the approximation of the linear reaction term in (1), we consider now the case $S_0 \neq 0$ and $S_2 \neq 0$ in (13), which means that the reaction term will be approximated on a three-point stencil (x_{j-1}, u_{j-1}), (x_j, u_j) and (x_{j+1}, u_{j+1}). We expand the various terms of expression (13) in a Taylor expansion as mentioned in (15) and (16). We obtain the following:

$$
\begin{aligned}
(2 - R_1 - R_2 + S_1)u_j + \Delta\xi(R_1 q - R_2 p)u'_j \\
+ \Delta\xi^2\left(-\frac{R_1}{2}q^2 - \frac{R_2}{2}p^2 + S_0 + S_2\right)u''_j \\
+ \Delta\xi^3\left(\frac{R_1}{3!}q^3 - \frac{R_2}{3!}p^3 - S_0 q + S_2 p\right)u'''_j \\
+ \Delta\xi^4\left(-\frac{R_1}{4!}q^4 - \frac{R_2}{4!}p^4 + S_0\frac{q^2}{2!} + S_2\frac{p^2}{2!}\right)u''''_j \\
+ \cdots = 0.
\end{aligned}
\tag{22}
$$

We now determine the values for S_i $(i = 1, 2, 3)$ for *Case II*, where the coefficients R_1 and R_2 are similar to the ones in *Case I*. We can rewrite expression (22) in the following way:

$$
T_0 u_j + \Delta\xi T_1 u'_j + \Delta\xi^2 T_2 u''_j + \Delta\xi^3 T_3 u'''_j + \Delta\xi^4 T_4 u''''_j + \cdots = 0,
$$

Table 1 Maximum error and convergence order for *Case I* for different choices of ω and $\epsilon = 0.01$

J	$\omega = 1$		$\omega = (u')^{\frac{1}{4}}$		$\omega = (u')^{\frac{1}{5}}$		$\omega = (u')^2$	
	Error	Order	Error	Order	Error	Order	Error	Order
10	0.0170	–	9.2012e-05	–	0.0099	–	0.4733	–
20	0.0041	2.0518	5.3023e-06	4.1171	0.0027	1.8745	0.3353	0.4973
40	0.0010	2.0356	3.1052e-07	4.0939	6.7831e-04	1.9929	0.2332	0.5239
80	2.4523e-04	2.0278	1.8629e-08	4.0591	1.6817e-04	2.0120	0.1575	0.5662
160	6.0609e-05	2.0165	1.1383e-09	4.0326	4.1676e-05	2.0126	0.0989	0.6713
320	1.5062e-05	2.0086	9.6514e-11	4.0328	1.0365e-05	2.0075	0.0664	0.5748

where the coefficients of $\Delta\xi$ are set as:

$$\begin{aligned} T_0 &= 2 - R_1 - R_2 + S_1 = 0, \\ T_1 &= R_1 q - R_2 p = 0, \\ T_2 &= -\frac{R_1}{2} q^2 - \frac{R_2}{2} p^2 + S_0 + S_2 = 0, \\ T_3 &= \frac{R_1}{3!} q^3 - \frac{R_2}{3!} p^3 - S_0 q + S_2 p = 0, \\ T_4 &= -\frac{R_1}{4!} q^4 - \frac{R_2}{4!} p^4 + S_0 \frac{q^2}{2!} + S_2 \frac{p^2}{2!} = 0. \end{aligned} \tag{23}$$

Making use of the coefficients $T_1, \ldots, T_4$ and R_1, R_2 from *Case I*, we obtain the following three expressions for the unknowns coefficients for *Case II*:

$$\begin{aligned} S_0 &= \frac{p}{6(p+q)}(q^2 + pq - p^2), \\ S_1 &= pq - S_0 - S_2, \\ S_2 &= \frac{q}{6(p+q)}(p^2 + pq - q^2). \end{aligned} \tag{24}$$

The terms R_1 and R_2 from *Case I* and (24) define *Case II*. For a higher order accuracy, we expand several more terms of the Taylor expansion of (13) and then estimate the terms asymptotically as mentioned in (15) and (16), respectively. Finally, (13) yields:

$$-\epsilon \frac{\Delta\xi^4}{20}\left(x_{\xi\xi} u^{(5)} + \frac{1}{12} x_\xi^2 u^{(6)}\right) + \mathcal{O}(\Delta\xi^6) = 0.$$

For *Case II*, we obtain a higher order of accuracy (supra-convergence), if the transformation $x(\xi)$ satisfies the following relation:

$$x_{\xi\xi} u^{(5)} + \frac{1}{12} x_\xi^2 u^{(6)} = 0.$$

From this follows the equidistribution principle:

$$\left[(u^{(5)})^{\frac{1}{12}} x_\xi\right]_\xi = 0. \tag{25}$$

It is easily checked from (2), that $u^{(5)} \propto u'$, and we find for *Case II* the equivalent equation:

$$\left[(u')^{\frac{1}{12}} x_\xi\right]_\xi = 0. \tag{26}$$

Next, we numerically solve the system (13) with the monitor function $\omega = (u')^{1/12}$ from (26). As indeed follows from the theory, we get more accurate results and a 6th order accuracy (see Table 2). By considering other choices for the monitor function, we observe that by taking $\eta = 1/2$ in (21), scheme (13) suddenly drops to the 2nd order of accuracy, the same as or the uniform grid case (see Fig. 2). Also, for this choice of the monitor function in *Case I* (see Table 1) it gives 4th order accurate solutions. We also demonstrate this by taking slightly changed values of $\eta = 1/11$ or $\eta = 1/13$: we obtain second order of accuracy for non-optimal grids. The power $\eta = 1/12$ in (26) gives the optimal sixth order accuracy, which is the *maximum* order that can be obtained on a three-point non-uniform stencil. No further improvement of the order can be reached. This follows directly from the analysis of systems (23) and (24).

5 Numerical Implementation for a Nonlinear Problem

To show the effects on a *nonlinear* model, we finally consider the SPBVP:

$$10^{-2}u'' - \sin(u) = 0, \qquad u(0) = \mathrm{e}^{-10}, \qquad u(1) = 1. \tag{27}$$

Equidistribution equations (8) with monitor functions $\omega = 1$, $\omega = (u')^{1/4}$, and $\omega = (u')^{1/12}$, respectively, are being solved iteratively in combination with (13). We cannot find an exact optimal grid transformation as for the linear case. Therefore, we approximate model (27) by approximating $\sin(u) \approx u$. Optimal non-uniform grids, for obtaining fourth and sixth order accuracy for the linear case are given by grid transformation (21) with $\eta = 1/4$ and $\eta = 1/12$, respectively. A reference solution of model (27) has been obtained by applying a uniform grid with $J = 1281$ and the routine $fsolve$ from Matlab. The exact solution (2) of the linearized model (1) has been chosen as the initial guess for the iterative procedure. The numerical results can be found in Table 3.

We clearly observe an almost fourth order (≈ 3.5) and almost sixth order (≈ 5.4) accuracy of the proposed non-uniform grid methods. The full orders of four and six cannot be reached, since we approximate the nonlinear function in SPBVP (27) by a linear one. Despite of the linearization, a significant gain in accuracy can be realized for the nonlinear case as well.

6 Conclusion

In the present article, we proposed higher order adaptive non-uniform finite difference grid, to solve convection-dominated singularly perturbed linear and non-linear boundary value problems with boundary-layers. Traditionally, three point central finite differences on a uniform grid produce a second order of accuracy.

Table 2 Maximum error and convergence order for *Case II* for different choices of ω and $\epsilon = 0.01$

J	$\omega = (u')^{\frac{1}{2}}$		$\omega = (u')^{\frac{1}{11}}$		$\omega = (u')^{\frac{1}{12}}$		$\omega = (u')^{\frac{1}{13}}$	
	Error	Order	Error	Order	Error	Order	Error	Order
10	0.0092	–	0.0025	–	2.1171e-05	–	0.0083	–
20	0.0025	1.8797	5.8710e-04	2.0902	3.8724e-07	5.7727	0.0020	2.0531
40	6.2523e-04	1.9995	1.4199e-04	2.0478	6.7322e-09	5.8460	4.7668e-04	2.0689
80	1.5463e-04	2.0156	3.4757e-05	2.0304	1.1312e-10	5.8952	1.1680e-04	2.0290
160	3.8305e-05	2.0132	8.5887e-06	2.0168	1.8454e-12	5.9378	2.8851e-05	2.0173
320	9.5462e-06	2.0045	2.1343e-06	2.0087	2.8758e-14	6.0038	7.1685e-06	2.0189

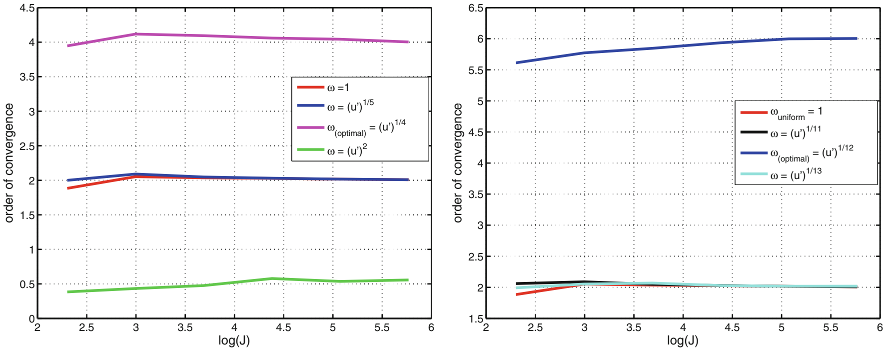

Fig. 2 Convergence order for several choices of the monitor function ω for *Case I* (left) and *Case-II* (right). We observe numerical evidence of the theoretically predicted convergence order, depending on the power in the monitor function. The convergence order can be *two* for standard choices (left and right) but also *four* (left) and even *six* (right) for special monitor functions, yielding supra-convergence

Table 3 Maximum error and convergence orders for different choices of the grids for model (27): a uniform grid (second order), non-uniform grids with $\eta = 1/4$ ($\approx$ fourth order) and $\eta = 1/12$ ($\approx$ sixth order)

	Uniform: $\omega = 1$		$\omega = (u')^{1/4}$		$\omega = (u')^{1/12}$	
J	Error	Order	Error	Order	Error	Order
20	0.0153	–	3.4663e-04	–	2.6304e-05	–
40	0.0038	2.0194	4.1124e-05	3.1663	8.4274e-07	4.9641
80	9.4621e-04	2.0057	4.3215e-06	3.2740	2.2635e-08	5.2185
160	2.3584e-04	2.0013	3.8656e-07	3.4591	5.6042e-10	5.3359
320	5.8774e-05	2.0049	3.4502e-08	3.4967	1.3462e-11	5.3998

However, we have presented higher order of accurate adaptive non-uniform grids approximations based on the equidistribution principle. We provided several numerical experiments for different choices of the monitor functions, which demonstrate the effectiveness of the proposed adaptive numerical method. We have also described an *optimal* choice of the adaptive non-uniform grid. We presented a detailed discussion, to get higher order of accuracy by considering a special way to discretize the given system. The proposed method on *optimal* adaptive non-uniform grids, performed exceptionally. We established numerically the 6th order of accuracy by considering a three point central finite differences. Numerical results confirmed this behavior. Comparisons between numerical results illustrate that to achieve the same accuracy, the proposed method needs approximately a factor of 5–10 fewer grid points than the uniform case. This depends, of course, on the value of the small parameter ϵ in the model.

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