Worldsheet from worldline

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We take a step toward a "microscopic" derivation of gauge-string duality. In particular, using mathematical techniques of Strebel differentials and discrete exterior calculus, we obtain bosonic string worldsheet action for string embedded in d+1 dimensional asymptotically AdS space from multi-loop Feynman graphs of quantum field theory of scalar matrices in d-dimensions in the limit of diverging loop number. Our work is building on the program started by 't Hooft in 1974 [1], this time including the holographic dimension which we show to emerge from the continuum of Schwinger parameters of Feynman diagrams.

I. INTRODUCTION

Despite being widely accepted and having passed many non-trivial tests, gauge-gravity duality [2–4] lacks a satisfactory derivation as an equivalence between quantum gauge theory and string theory. In particular, apart from certain low dimensional examples and integrable class of quantum field theories, e.g. [5–9], we are unaware of any "microscopic" derivation of the string world sheet directly from the gauge theory path integral. This derivation would entail, in a concrete sense, completing the program started by 't Hooft's seminal paper [1], that is, to derive the string path integral — this time including the holographic dimension — from the ribboned Feynman graphs of gauge theories.

A promising starting point for this endeavor is Gopakumar's observation [10–12]. See [7, 13–20] for an incomplete list of other examples and alternative approaches. In particular, [12] proposed to use Strebel differentials to map the skeleton graphs of n-point functions of gauge invariant operators in *free* CFTs to the worldsheet of a putative string theory, see also [21, 22]. Furthermore, in [23, 24] a curious similarity between (discretized) open and closed strings and a graphical representation of one- and two-trees that appear in Symanzik polynomial representation of high-loop Feynman graphs, see e.g. [24–28], were observed and it was proposed that embedding of the sting in the holographic dimension arises from a continuum limit of the Schwinger parameters of graphs with large number of loops.

We merge these observations in this work and explicitly construct the string worldsheet action that arises from the continuum limit of Feynman graphs of a quantum field theory in d-dimensions. We show that the target space of the string that correspond to the QFTs we consider in this work is asymptotically AdS_{d+1}.

II. GENERATING FUNCTION

Consider expansion of a QFT generating function in Feynman diagrams. We restrict ourselves to QFTs of N×N matrices that are massless scalars with a generic polynomial type interaction. Contribution of a graph Γ is given by

$$\mathcal{Z}_\Gamma \propto \int \prod_i d^d X_i \prod_{e_{ij}\in\Gamma} G(X_i,X_j) \prod_{v_i\in\mathcal{S}_\Gamma} J(X_i) \quad (1)$$

with the proportionality constant including all factors of the couplings and symmetry factors, e_{ij} and v_i denoting the edges and the vertices and J denoting the sources. We assume, following [1], that Feynman graphs are put in ribboned form and grouped as discretized string worldsheets with genus g. We first rewrite (1) in a form that allows us to read off the corresponding (discretized) worldsheet action.

A key role is played by the Schwinger parametrization of the propagators $G(X_i - X_j)$ with Schwinger parameters σ_{ij}. If there are m edges between a pair of vertices then we can merge them in an effective propagator [11] written in terms of a single Schwinger parameter as

$$G(X_i,X_j)^m = \frac{i^m \Gamma\left(\frac{d}{2}-1\right)^m}{2^{\frac{md}{2}}\Gamma\left(m\left(\frac{d}{2}-1\right)\right)} \cdot \\ \cdot \int_0^\infty d\sigma_{ij}\, \sigma_{ij}^{m\left(\frac{d}{2}-1\right)-1} e^{-\frac{\sigma_{ij}(X_i-X_j)^2}{4}}\,. \quad (2)$$

We assume vanishing masses throughout this paper—see [24] for generalization of this expression to massive scalars. This procedure constructs the so-called "skeleton graph" $\bar\Gamma$ of Γ [11] with all duplicated edges fused into a single one and each skeleton edge assigned a length given by the corresponding σ_{ij}. The partition function is then expressed as

$$\mathcal{Z}[J] = \sum_{\bar\Gamma} \int d[X]d[\sigma]\mathcal{P}_{\bar\Gamma} e^{-\frac{1}{4}\sum \sigma_{ij}(X_i-X_j)^2} \quad (3)$$

where $[X]$ and $[\sigma]$ denote the collection of all vertex positions and graph metrics of the skeleton graph and $\mathcal{P}_{\bar\Gamma}$ is a polynomial in σ_{ij} which depends on the graph structure, coupling constants and sources.

To determine it, we introduce auxiliary variables ξ_i at the vertices v_i and represent each edge connected to a vertex by $i\xi_i$ so that for each propagator between v_i and v_j we must include a factor of $-\xi_i\xi_j$. It is straightforward to show that

$$\mathcal{P}_{\bar{\Gamma}} = \frac{1}{n!}\left(i\prod_i U\left(-i\partial_{\xi_i}\right)\right)\prod_{\bar{e}_{ij}\in\bar{\Gamma}}\sum_m \frac{(-1)^m\Gamma\left(\frac{d}{2}-1\right)^m}{2^{\frac{md}{2}}\Gamma\left(m\left(\frac{d}{2}-1\right)\right)}\cdot$$

$$\cdot\sigma_{ij}^{m\left(\frac{d}{2}-1\right)-1}\frac{(\mathrm{Tr}\left(\xi_i^\dagger\xi_j+\xi_j^\dagger\xi_i\right))^m}{2(m!)}\Bigg|_{\xi=0} \tag{4}$$

gives a non-vanishing contribution only when the number of edges connected to the vertex v_i is precisely equal to the number of derivatives and only when the coordination number of the vertices equal to the one on some diagram Γ whose skeleton diagram is $\bar{\Gamma}$. The symmetry factors of the graph are correctly accounted for by the factors of $1/n!$ and $1/m!$ in this expression. The latter arises from swapping of propagators between the same pair of vertices and the former from swapping of vertices. This gives the correct factor as long as we sum over all nonequivalent permutations of vertices. This sum will be implemented below as part of the integral over the moduli space, which should now be the moduli space of Riemann surfaces with n marked points counting separately all nonequivalent permutations of these points.

We recognize the exponential in (3) as a kinetic term

$$S_K = \frac{1}{4}\sum_{ij}\sigma_{ij}(X_i - X_j)^2\,, \tag{5}$$

and rewrite $\mathcal{P}_{\bar{\Gamma}}$ above as an interaction potential

$$S_I[\xi,\theta] = -\sum \log(iU(\theta_i)) - i\xi_i\theta_i + h(\sigma_{ij},\xi_i\xi_j)\,, \tag{6}$$

where the function h is

$$h(\sigma_{ij},\xi_i\xi_j) = \log\left(\sum_m \frac{\Gamma\left(\frac{d}{2}-1\right)^m(-1)^m}{2^{\frac{md}{2}}\Gamma\left(m\left(\frac{d}{2}-1\right)\right)}\sigma_{ij}^{m\left(\frac{d}{2}-1\right)-1}(\xi_i\xi_j)^m\right)$$

$$= -\log\sigma_{ij} + f\left(\sigma_{ij}^{\frac{d}{2}-1}\xi_i\xi_j\right)\,. \tag{7}$$

To obtain the expression (6) we followed the trick

$$(\cdots)\Big|_{\xi=0} = \int_{-\infty}^{\infty}d\xi\int_{-\infty}^{\infty}d\theta\, e^{-i\xi\theta}\,(\cdots)\,,$$

and performed integration by parts inside the path integral.

III. TRIANGULATION AND CONTINUUM

We view the sum of the kinetic and potential terms in (5) and (6) as discretized worldsheet action of the dual string theory and take the continuum limit to obtain the continuous worldsheet theory. This should be done with care employing the technology of Strebel differentials and Discrete Exterior calculus which we shortly review below, and in more detail in the Supplementary material.

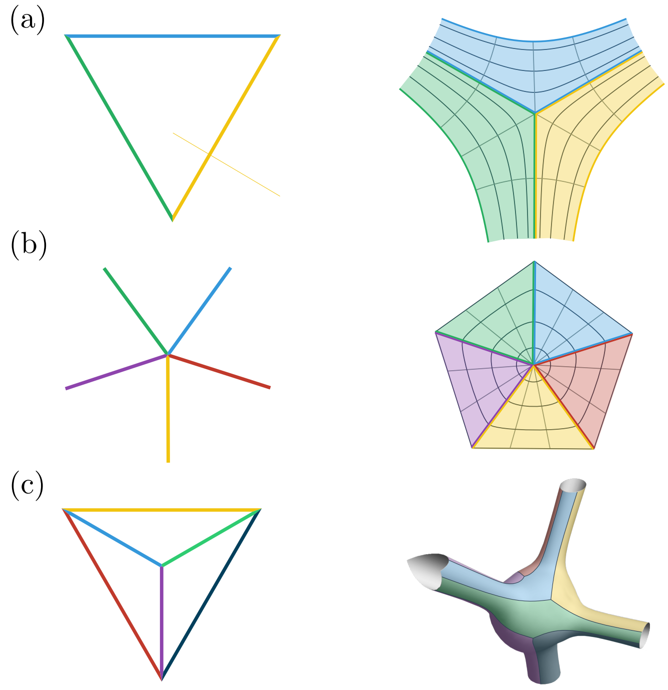

FIG. 1. Gluing of the strips at (a) a face and (b) a vertex. In (c) we show an example of the Riemann surface obtained from a simple graph of genus 0 with 4 vertices corresponding to a sphere with 4 infinite cylinders attached.

A. Strebel differentials

Theory of Strebel differentials [29] provides an isomorphism between the space of metric ribboned graphs and the moduli space on punctured Riemann surfaces with a positive real number assigned to every puncture. To every propagator with length σ corresponds a vertical strip in the complex plane of width σ [12, 21]. Then, these strips are glued into a manifold as depicted in figure 1 and detailed in the Supplementary material. The vertices of Γ become punctures in the manifold and a positive real number a_i associated to the i-th puncture is given by the sum of all the lengths of edges incident to the corresponding vertex $a_i = \sum_j \sigma_{ij}$. This construction naturally provides a metric for the Riemann surface given by the flat metric on each strip, corresponding to a particular gauge fixing of Weyl and diffeomorphism symmetries. This metric exhibits a double pole around the punctures (vertices of Γ)

$$g_{Str} = \frac{a_i^2}{4\pi^2}\frac{dud\bar{u}}{|u|^2}\,. \tag{8}$$

B. Discrete Exterior Calculus

Theory of Strebel differentials provide the worldsheet as a Riemann surface but it does not determine the worldsheet action from (5) and (6). For this, one first needs

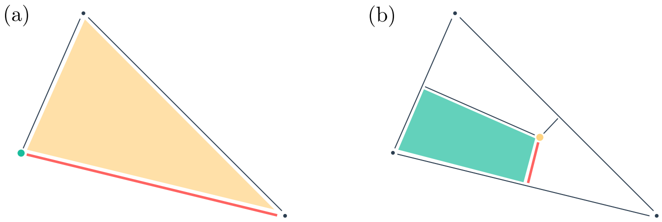

FIG. 2. Depiction of part of the triangulation (a) with its dual shown in (b). Cells painted in the same color are dual to each other.

to extend the map between double poles of the Riemann surface (8) and location of vertices in space time, X_i^μ, to entire functions $X^\mu(u,\bar{u})$ on the Riemann surface. We can address this issue through the machinery of Discrete Exterior Calculus used to solve geometric problems on manifolds through numerical methods where the specified data is necessarily finite [30].

Consider a Riemann surface Σ with a triangulation, see below, in terms of points v_i, intervals e_{ij} and triangles f_{ijk} such that the edges of f_{ijk} are e_{ij}, e_{jk} and e_{ki} and the endpoints of e_{ij} are v_i and v_j. One discretizes data on the manifold by integrating over the relevant simplexes of our triangulation. That is, one integrates a function over the points v_i, a 1-form over the intervals e_{ij} and a 2-form over the triangles f_{ijk}. This allows us to use Stokes' theorem without issue. For instance, take the 1-form df specified over the edges e_{ij}, then one has

$$(df)_{e_{ij}} = \int_{e_{ij}} df = \int_{\partial e_{ij}} f = (f)_j - (f)_i \,. \tag{9}$$

This defines for us a discrete exterior derivative which is unambiguous in the discretization lengths by virtue of Stokes' theorem.

Another crucial operation is the Hodge dual. We first define the dual $\star\sigma^k$ of a k-simplex σ^k by the convex span of the circumcenters of all the simplexes that contain σ^k and σ^k itself, see Figure 2. The hodge dual of a k-form α is then defined by

$$(\star\alpha)_{\star\sigma^k} = \frac{|\star\sigma^k|}{|\sigma^k|}(\alpha)_{\sigma^k} \tag{10}$$

where $|\cdot|$ stands for the volume in the corresponding dimension [1]. The last operation we need is the wedge product [30]

$$(\alpha\wedge\beta)_{\sigma^k\cup\star\sigma^k} = \frac{|\sigma^k\cup\star\sigma^k|}{|\sigma^k|\,|\star\sigma^k|}(\alpha)_{\sigma^k}(\beta)_{\star\sigma^k} \tag{11}$$

where $\sigma^k\cup\star\sigma^k$ is the smallest convex set containing both σ^k and $\star\sigma^k$. If we decrease the size of our triangulation we can expect all these operations to converge [31, 32] since surfaces are locally flat and thus well approximated by subsets of $\mathbb{R}^2$.

[1] Note that $\star\alpha$ is not discretized in the same triangulation as α, but instead in a dual version of it.

C. Triangulation

Finally, we need to provide a triangulation of the Riemann surface—which we obtained using Strebel differentials above—on which discrete exterior calculus can operate. This is tricky, as the punctures are located at the endpoints of semi-infinite cylinders, see Fig. 1. We can, however, remove these cylinders by a Weyl transformation with a factor of $\epsilon^2\,|u|^2$ in the region $0<|u|<1$ on the Strebel metric (8). This leaves a flat punctured disk which can be completed by restoring the marked point. Disks corresponding to different poles are identified at the horizontal curves at $|u|=1$ defining a continuous Weyl transformation on the entire manifold. The resulting manifold is a collection of sections of flat disks with opening angles $\frac{2\pi\sigma_{ij}}{a_i}$ and radii $\frac{a_i\epsilon}{2\pi}$ identified along their curved edges, see Fig. 3. The factor of ϵ enables us to adjust the volume V of the manifold, i.e. to keep it fixed even when we introduce more poles or change the a_i. One finds

$$V = \sum_i \frac{\epsilon^2 a_i^2}{4\pi} \qquad \Rightarrow \qquad \epsilon = \sqrt{\frac{4\pi V}{\sum a_i^2}} \sim \frac{1}{\sqrt{n}}\,, \tag{12}$$

What we achieved so far is not quite a triangulation because sections of discs are not simplexes, but we can split each section further into small triangles that approximate the discs arbitrarily well. Note that at strong coupling, where diagrams with $n\gg 1$ dominate the partition function, the mesh describing our manifold becomes finer and finer.

We will need the following geometric data, which we can easily deduce from Fig. 3. Denoting the poles by indices i and j and the auxiliary vertices introduced to make the discs into triangles by an index a, we obtain $|e_{ia}| = a_i\epsilon/2\pi$ for length of the auxiliary edges and $|\star e_{ia}| = \epsilon\delta\sigma/2$ for their duals. Their convex span has area $|e_{ia}\cup\star e_{ia}| = a_i\delta\sigma\epsilon^2/8\pi$. The area of the dual of a pole is $|\star v_i| = a_i^2\epsilon^2/16\pi$ and for the auxiliary vertices lying between disk i and disk j we find $|\star v_a| = 3(a_i+a_j)\delta\sigma\epsilon^2/16\pi$.

D. Continuum

We are ready to take the continuum limit of the discrete action (5) and (6). Discrete exterior derivative of X^μ is given by

$$(\mathrm{d}X^\mu)_{e_{ia}} = \frac{a_i}{a_i+a_j}(X_j - X_i) \tag{13}$$

where the vertex v_a lies between the disks i and j and we chose to linearly interpolate the known values of X^μ to

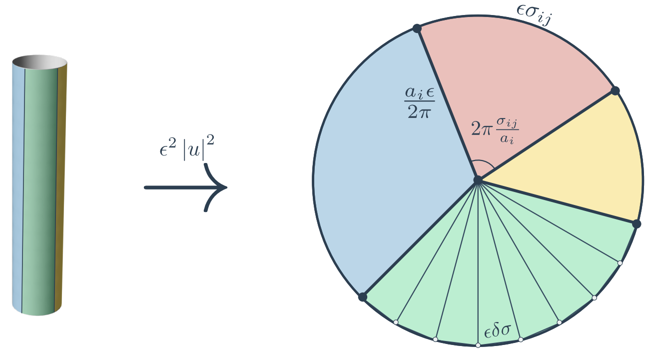

FIG. 3. After we perform a Weyl transformation the infinite cylinders become flat disks split into sections that are identified along its curved edge. We triangulate the manifold by splitting this sections into small triangles.

the auxiliary vertices on the edges of the disk. Following the definitions above one finds

$$(adX^\mu \wedge \star dX_\mu)_{e_{ia}\cup\star e_{ia}+e_{ja}\cup\star e_{ja}} = \frac{\pi\delta\sigma}{4}(X_j - X_i)^2 \quad (14)$$

Summing over the entire manifold yields

$$S_K = \frac{1}{\pi}\int d^2z\sqrt{\det g}\, a\, \nabla_\alpha X^\mu \nabla_\beta X^\nu g^{\alpha\beta}\eta_{\mu\nu}\,, \qquad (15)$$

up to corrections of order $\epsilon \sim 1/\sqrt{n}$.

The continuum limit of the potential term (6) is trickier, as we need to find a function σ such that

$$(\sigma)_{v_a} = \sigma_{ij} \qquad (16)$$

so that we can express it in terms of the data on the manifold. Because the Strebel differential is unique for given puncture locations and residues a_i, this function should be expressible in them. To find it, we compute the second functional derivative of S_K directly from the continuum expression and using the discretized form 5). Equating the two yields,

$$a(z) - n\sigma(z)\int_{S_z}\star f_p = 0 \qquad (17)$$

where $\sigma(z)$ assigns to every point the σ_{ij} of the strip it belongs to and S_z refers to the strip that contains z. The function f_p is defined through its Hodge dual by $\star f_p = \frac{1}{n}\sum_i \delta(z - z_i)$ giving the density of poles normalized to 1. To evaluate this integral we assume that f_p has some finite width and is distributed preserving the symmetry of the disk around v_i, then we obtain

$$\int_{S_z}\star f_p = \frac{\sigma_{ij}}{na_i} + \frac{\sigma_{ij}}{na_j} = \left(\frac{2\sigma}{na}\right)_{v_a} \Rightarrow \quad \sigma(z) = \frac{a(z)}{\sqrt{2}}\,. \quad (18)$$

The other quantity in (6) for which we need a continuum approximation for is $\xi_i\xi_j$. It is straightforward to show that

$$\xi_i\xi_i = \left(\xi^2 + \frac{8\xi}{a\pi}\frac{\nabla\xi\cdot\nabla a}{nf_p} - \frac{4}{\pi}\frac{(\nabla\xi)^2}{nf_p}\right)_{v_a} + \mathcal{O}\left(\frac{1}{n^{\frac{3}{2}}}\right)\,. \qquad (19)$$

Using the definition of f_p to turn (6) into an integral we can express its continuum limit as

$$\begin{aligned} S_I = &-n\int\left(\log(iU(\theta)) - i\xi\theta + h\left(\frac{a}{\sqrt{2}},\xi^2\right)\right)\star f_p \\ &-\int \partial_{\xi^2}h\left(\frac{a}{\sqrt{2}},\xi^2\right)\left(\frac{8\xi}{a\pi}\nabla\xi\cdot\nabla a - \frac{4}{\pi}(\nabla\xi)^2\right)\star 1 \end{aligned} \qquad (20)$$

up to $\epsilon \sim 1/\sqrt{n}$ corrections. For more details of this computation we refer to [33].

Putting everything together we find the following expression for the generating function

$$\mathcal{Z}[J] = \sum_n \frac{1}{n!}\int \mathcal{D}X^\mu\mathcal{D}\xi\mathcal{D}\theta\mathcal{D}a\mathcal{D}[g]\left|\frac{\delta\sigma}{\delta(a,g)}\right| e^{-S_K - S_I} \qquad (21)$$

where S_K and S_I are given by (15) and (20). The path integrals are understood to be discretized on the triangulation described above and $\mathcal{D}[g]$ refers to an integral over the moduli space of the Riemann surfaces with marked points.

Leaving the Jacobian $|\delta\sigma/\delta(a,g)|$ aside for a moment, we now consider the large n limit of (21). Clearly the path integrals over ξ and θ will be dominated by the classical saddle of (20). These can be further be solved in powers of a around $a \to \infty$ which is also a natural limit to take, as it corresponds to the UV limit of the QFT, which we expect to be a CFT—see Discussion below. Restricting to the case of real fields with a single trace potential given by $U = \mathrm{Tr}\left(\sum \frac{\lambda_k}{k!}\Phi^k + J^T\Phi\right)$, we find the following saddles for ξ and θ

$$\xi = \frac{\alpha}{\sqrt{N}}\left(\frac{a}{\sqrt{2}}\right)^{-\frac{d-2}{4}} + \mathcal{O}\left(a^{-\frac{d}{4}}\right) \qquad (22)$$

$$\theta = -\frac{ik\sqrt{N}}{\alpha}\left(\frac{a}{\sqrt{2}}\right)^{\frac{d-2}{4}} + \mathcal{O}\left(a^{\frac{d}{4}-1}\right) \qquad (23)$$

where k is the order of the highest interaction in U and α is a numerical constant that satisfies

$$2\alpha^2 f'(\alpha^2) = k \qquad (24)$$

which has solutions for real α [2]. More general situations can be considered just as easily with similar results. Using this solution in (20) we obtain

$$S_I = C_0 + C_1\int d^2z\sqrt{\det g}\frac{\nabla_\alpha a\nabla^\alpha a}{a^2} \qquad (25)$$

where the constant C_0 is undetermined and $C_1 = \left(1/4\pi + k(2d^2 + 7d - 22)/16\pi\right)$ is not universal but depends on our choice to discretize in (13). We further assume that $a(z) = 1$ at all the zeroes. Eqs. (25) and (15)

[2] This holds only for $d \geq 3$ otherwise the terms that dominate the contribution from the potential are no longer the highest order interactions.

constitute our final result for the worldsheet action in the large n and $a \to \infty$ limits. We find see that together with an AdS$_{d+1}$ sigma model $ds^2 = du^2 + e^{-2u}\eta_{\mu\nu}dX^\mu dX^\nu$ with $a(z) = \exp(-2u(z))$ near the boundary. Deviations from this limit relate geometry toward the interior to the RG flow of the QFT. We leave study of such deviations to future work.

Consider finally the Jacobian in (21). Luckily, this was already obtained by Kontsevich [34], see also [35, 36]

$$\prod d\sigma_{ij} = \left(\prod \frac{da_j}{4}\right) \times \frac{\Omega^{3g-3+n}}{(3g-3+n)!} 2^{-5g+5} \tag{26}$$

with Ω given by

$$\Omega = \sum a_i^2 \omega_i \tag{27}$$

and ω_i being the first Chern class of the manifold evaluated at the i-th marked point. While this contribution does not seem to affect our conclusion above, it is desirable to provide its physical interpretation, see the Discussion below.

IV. DISCUSSION

We note that the same ω_i in (27) appear in topological 2d gravity [12]. It turns out that the physical operators of this theory are operators $\mathcal{O}_n$ indexed by the cohomology classes of the moduli space $\mathcal{M}_{g,n}$. This suggest that one may replace insertions of ω_i by insertions of $\mathcal{O}_n$. If that is the case, the non-trivial measure can be written as a collection of topological terms while simultaneously introducing a ghost sector to the theory as would be expected from the fact that we are only integrating over the moduli space of metrics in our derivation.

At first sight it is surprising that we obtain AdS target space from a generic class of large N QFT in the large loop, large a limit. There are two hidden assumptions behind our result however. First, we ignored issues of renormalization and assumed that the field mass continues to vanish in the full quantum theory. One can show [33] that issues of renormalization do not alter our results, see also [24], as long as the QFT is sufficiently close to a fixed point. Second, such a fixed point is indeed found, precisely when the large loop limit is well defined, see for example [37]. Indeed, Zamolodchikov considered the so-called "fishnet" theories and showed that large vertex limit exists at a specific coupling where the theory becomes a CFT for $(d, k) = (3, 6)$, $(4, 4)$ and $(6, 3)$, precisely the combinations argued to possess a continuum limit in [24]. In [38] the $(4, 4)$ Fishnet theory has been shown to lead to string propagation in AdS$_5$ using integrability techniques. Our derivation provides an alternative without resorting to integrability.

What happens beyond the CFT limit is of utmost importance. This can be obtained by going beyond the large a limit in the generic expressions (15), (20). On the other hand, it is straightforward, yet cumbersome, to include fermions and gauge fields in our derivation that is essentially based on the Schwinger parametrization of the generating function, see for example [25]. Further generalizing our method to include renormalization properly, one could, in principle be able to derive the worldsheet action dual to QCD!

Finally, we note that our computation is valid for arbitrary genus. In particular, (21) holds even though the Strebel metric varies yet remaining unique. It is tempting to ask whether we can extend our analysis to non-perturbative quantum gravity. In principle, our starting point, eqs. (5) and (6) remains valid for arbitrarily small N. See [24] for a discussion on this issue.

There are various other exciting questions to explore in the future, for instance string propagation on blackholes through extending our analysis to thermal field theory and dynamical gravity through, perhaps, generalizing to ensemble of QFTs.

V. ACKNOWLEDGMENTS

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Appendix A: Supplementary material: Strebel Differentials

Let Σ be a Riemann surface of genus g with n marked points $\{p_1, \dots, p_n\}$ equipped with a complex structure. The basic objects we are interested in are quadratic meromorphic forms, that is, objects $\chi \in \Omega_1^{(1,0)}(\Sigma) \otimes \Omega_1^{(1,0)}(\Sigma)$ which in holomorphic coordinates locally take the form

$$\chi = \phi(z)\mathrm{d}z^2 \tag{A1}$$

with $\phi(z)$ being a holomorphic function that transforms as $\phi(w) = \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 \phi(z)$ under holomorphic changes of coordinates. Using these differentials one can define a special set of curves γ such that $\chi(\dot\gamma, \dot\gamma) > 0$. We call such curves horizontal since for $\Sigma = \mathcal{C}$ with the usual metric and choosing $\chi = \mathrm{d}z^2$ they correspond to lines of constant imaginary part. We similarly denote by vertical curves those for which $\chi(\dot\gamma, \dot\gamma) < 0$. At regular points of ϕ the vertical and horizontal curves form a grid which specify a set of coordinates for which $\chi = \mathrm{d}z^2$. Near zeroes or poles of ϕ this is no longer true and we depict the local structure of vertical and horizontal lines in Figure 4. Note that from a quadratic differential one can define a metric for the manifold given by

$$g_{Str} = |\phi(z)|\mathrm{d}z\mathrm{d}\bar{z} \tag{A2}$$

In the special coordinates defined by vertical and horizontal curves this metric reduces to patches of the Euclidean metric in $\mathcal{C}$ along with curvature localized at the zeroes and poles of ϕ.

The main result of the theory of Strebel differentials is given by the following theorem

Theorem 1 *Given a Riemann surface Σ of genus g with n marked points $\{p_1, \dots, p_n\}$ and a set of n positive real numbers $\{a_1, \dots, a_n\}$ one can find a unique quadratic meromorphic form χ such that*

- *χ locally takes the form $\phi(z)\mathrm{d}z^2$ with $\phi(z)$ holomorphic on $M/\{p_1, \dots, p_n\}$*
- *$\phi(z)$ has a double pole at every p_i*
- *The union of all non-compact horizontal curves is a set of measure 0*

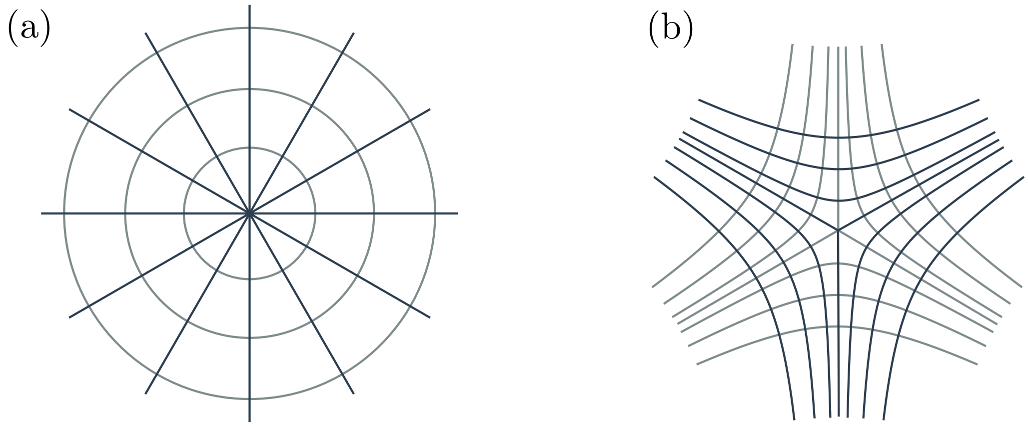

FIG. 4. Structure of a Strebel differential near a double pole (a) and a single zero (b). The darker lines correspond to vertical curves and the lighter ones to horizontal curves.

- *Every compact horizontal curve γ is a simple loop around some p_i and it satisfies*

$$a_i = \int_\gamma \sqrt{\phi(z)} \mathrm{d}z \tag{A3}$$

Such a quadratic form is called a Strebel differential and the set of non-compact horizontal curves is called its critical graph. These corresponds to horizontal lines connecting two zeroes such that the critical graph can be regarded as the embedding of some abstract graph in our manifold.

In light of theorem 1 one can deduce that Strebel differentials parametrize the space $\mathcal{M}_{g,n} \times \mathbb{R}_+^n$ where $\mathcal{M}_{g,n}$ is the moduli space of Riemann surfaces of genus g with n punctures and one can take g_{Str} as a representative of the classes of metrics up to Weyl and diffeomorphism transformations for some fixed collection of residues at every marked point.

From a metric ribboned graph Γ one can construct a Riemann surface with its corresponding Strebel differential such that its critical graph is given by Γ^*, that is, the dual graph of Γ obtained by replacing vertices by faces and vice-versa. The construction goes as follows.

Let Γ be an arbitrary metric ribboned graph with all faces bounded by at least three distinct edges. For every edge e_i of Γ with length σ_i let us take an infinite strip in $\mathbb{C}$ oriented along the imaginary axis and of width σ_i. On these strips $\mathrm{d}z_i^2$ defines a holomorphic quadratic form. Take any vertex v_i. Since Γ is ribboned we have an ordering of the edges incident to this vertex, then we can glue the corresponding strips into a cylinder according to this ordering. This can be done by mapping all these strips into a single disk according to the map

$$u_i(z_k) = \exp\left(2\pi i \frac{z_k + \sum_{j<k} \sigma_j}{\sum_j \sigma_j}\right) \tag{A4}$$

Pulling back the quadratic differentials on the strips to the u-disk we obtain

$$\mathrm{d}z_k^2 = -\frac{(\sum \sigma_j)^2}{4\pi^2} \frac{\mathrm{d}u_i^2}{u_i^2} \tag{A5}$$

so that the form develops a second order pole at the points $u_i = 0$. Here k runs over all the vertices incident to v_i according to the ordering induced by the ribbon structure of G. We also do something similar for the faces. Take any face f_i with $m \geq 3$ edges bounding it. We now glue the strips of the edges forming the loop into a disk by mapping them with

$$w_i(z_k) = e^{\frac{2\pi i k}{m}} z_k^{\frac{2}{m}} \tag{A6}$$

Now the quadratic differentials are mapped to

$$\mathrm{d}z_k^2 = \frac{m^2}{4} w_i^{m-2} \mathrm{d}w_i^2 \tag{A7}$$

so that they develop a zero of order $m-2$ at the points $w_i = 0$. The collection of the strips with coordinates z_i for the edges, the disks with coordinates u_i for the vertices and the disks with coordinates w_i for the faces define an atlas with holomorphic transition functions given by $u_i(z_k)$ and $w_i(z_k)$. Thus they completely characterize a Riemann surface with the same genus as the graph. Moreover the form χ with local expression on the strips given by $\mathrm{d}z_i^2$ is the unique Strebel differential with marked points given by $u_i = 0$ and residues $\sum_{e_j \in v_i} \sigma_j \equiv a_i$. In Figure 1 we give a drawing of how the strips are glued at vertices and faces and an example of the resulting Riemann surface from a simple graph.

There is no obstacle to following the same construction for diagrams that have two-edge loops, however the strips corresponding to the two edges get glued together into a single strip without any zero and the information that they were originally separate is lost.

This construction is actually an isomorphism between the set of skeleton ribboned graphs with lengths assigned to each edge with $\mathcal{M}_{g,n} \times \mathbb{R}_+^n$ with the positive numbers a_i being the coordinates on $\mathbb{R}_+^n$. Note additionally that we can construct a triangulation of our Riemann surface with 0-cells given by the zeroes and the poles, 1-cells given by the horizontal curves connecting zeroes to zeroes and the vertical curves connecting zeroes to poles and 2-cells given by half-strips split at $\mathrm{Im}(z_i) = 0$.

Finally in the main text we have used some properties of Strebel differentials for large number of punctures that we now prove. In particular, directly using the solution for the saddles (22-23) on equation (20) a term of the form $n \int \log a \star f_p$ appears that contributes to the kinetic term for $a(z)$ by using the following identity

$$-4\pi \left(n f_p - \frac{m}{2} f_q \right) = \nabla^2 \log a \tag{A8}$$

where m is the number of zeroes of the Strebel differential (including multiplicities) and f_q is the density of zeroes defined similarly to f_p replacing poles by zeroes. Note that the term with f_q vanishes when integrated against $loga$ due to the assumptions that $a(z) = 1$ at the zeroes. To show that this is true when we have a large number of punctures we note that the holomorphicity of the Strebel differential implies

$$\sqrt{\mathrm{det} g_0}\ \nabla_0^2 \log |\phi(z)| = -4\pi \sum_{p \in \mathrm{poles}} \delta^{(2)}(z - p) + 2\pi \sum_{q \in \mathrm{zeroes}} \delta^{(2)}(z - q) \tag{A9}$$

This is all expressed in the metric given by $g_0 = \mathrm{d}z\mathrm{d}\bar{z}$ where z are the coordinates that make the Strebel metric take the form of equation (A2). We would like to use instead the metric g where our triangulation has been defined which is related to g_0 and g_{Str} by a Weyl transformation. In particular we have

$$\sqrt{\mathrm{det} g} = e^{2\omega} |\phi(z)| \sqrt{\mathrm{det} g_0} = e^{2\omega} \sqrt{\mathrm{det} g_{Str}} \tag{A10}$$

where ω is the parameter of some Weyl transformation that we need to find. Using the transformation properties of the Ricci scalar under Weyl transformations we can find

$$\nabla^2 \omega = -\frac{R}{2} + 2\pi \left(n f_p(z) - \frac{m f_q(z)}{2} \right) \tag{A11}$$

Which can be directly solved using the Green's function to find

$$|\phi(x, y)| = |C| \exp\left(-4\pi \int \left(n f_p(z') - \frac{m}{2} f_q(z') \right) G(z'; z) \mathrm{d}^2 z' \right) \tag{A12}$$

for some complex constant C. This must have the right residue at each of the poles and imposing such a condition is the way to find f_q. Then, if we want to compute the residue at some pole p_i we can restrict the integral that computes $\log |\phi|$ to integrate only over $|z - p_i| > \epsilon$ and take the limit $\epsilon \to 0$ this excludes precisely the contribution that gives the pole of interest and nothing more. Demanding that the absolute value of ϕ reduces to the residue when we remove the contribution from a pole we obtain the relation

$$\lim_{\epsilon \to 0} |C| \exp\left(-4\pi \int_{|z' - p_i| > \epsilon} \left(f_p(z') - \frac{1}{2} f_q(z') \right) G(z'; p_i) \mathrm{d}^2 z' \right) = \frac{a(p_i)}{2\pi} \tag{A13}$$

Assuming that the triangulation is dense enough for f_p and f_q to be sufficiently continuous we directly obtain equation (A8) by taking a logarithm and a Laplacian.