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Newton on constructions in geometry ☆

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A B S T R A C T

Newton was critical of Descartes's constructivist vision of the foundations of geometry organised around certain curve-tracing principles. In unpublished work, Newton outlined a constructivist program of his own, based on his "organic" method of curve tracing, which subsumes Descartes's emblematic turning-ruler-and-moving-curve construction method as a special case, but does not suffer from the latter's flaw of being unable to trace all conics. This Newtonian program has been little studied and is more thoughtful and technically substantive than is commonly recognised. It also clashes with, and arguably supersedes and improves upon, Newton's perhaps better known earlier statements on the subject.

S A M E N V A T T I N G

Newton was kritisch over Descartes constructivistische visie op de grondslagen van de meetkunde, georganiseerd rond bepaalde principes voor het tekenen van krommen. In ongepubliceerd werk schetste Newton een eigen constructivistisch programma, gebaseerd op zijn 'organische' methode voor het tekenen van krommen, die de emblematische constructiemethode van Descartes met draaiende liniaal en bewegende kromme als een speciaal geval onderbrengt, maar zonder de zwakte van de Cartesiaanse aanpak dat niet alle kegelsneden tekenbaar zijn. Dit Newtoniaanse programma is weinig bestudeerd en is doordachter en technisch inhoudelijker dan algemeen wordt erkend. Het botst ook met, en vervangt en verbetert, de wellicht beter bekende eerdere uitspraken van Newton over dit onderwerp.

1. Introduction

Many 17th-century mathematicians—including Descartes, Huygens, Leibniz, and Jacob and Johann Bernoulli—viewed constructions as foundationally central in geometry. With good reason, they perceived ancient Greek geometry to have been based on constructivist principles, and they were committed to remaining true to these ideals in their own mathematical practice. This essentially meant specifying curve-tracing devices or postulates that could generate the entities they wanted their geometry to include, just as Euclid's ruler and compass postulates had been sufficient to produce the entities that he considered in the *Elements*. (See Bos (1988), Bos (1993), Bos (2001), Blåsjö (2017), Blåsjö (2022).) This tradition may conveniently be called "constructivist," although perhaps with the caveat that it is manifested more in the *de facto* mathematical practice of centering constructions than by any fully articulated philosophical doctrine.

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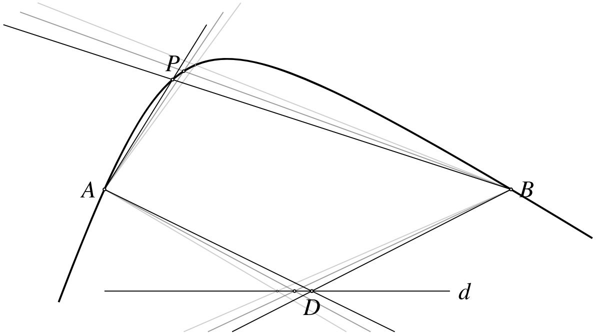

Figure 1. Newton's organic (NOrg) construction of conics. Two V-shaped rulers (each with a fixed angle between its two arms) pivot about two pole points, A and B. One of their intersections, D, is made to move along the "directrix" line d. The other intersection, P, traces a conic section. "If [the directrix] be a straight line, [the described curve] is a conic passing through A and B." (Newton, MP, II.111).

Newton's relation to this constructivist tradition is complex and interesting. In the earlier stages of his career, Newton seems to have had little interest in and perhaps little awareness of constructivist issues. Geometrical constructions did figure in his mathematical practice already in the 1660s in the form of his "organic" construction of conics, cubics, and higher curves by rotating rulers. But the role of these constructions is very different from the constructivist tradition. Constructions devised for foundational purposes were often almost exclusively geared toward theoretical and epistemological goals, and were often of little or no practical utility to the working mathematician. Newton's organic constructions, by contrast, were eminently useful for explorative research. By all indications, Newton did indeed make significant discoveries through actual use of his construction instruments.

Eventually, Newton had occasion to engage more and more with the constructivist tradition. In his 1680s lectures on algebra, which were later published as *Arithmetica Universalis*, Newton railed against some aspects of Descartes's program of constructivist foundations for geometry. I shall argue that Newton's outlook at this early stage was quite simplistic. His remarks seem to be driven by a zeal to reject specific points made by Descartes, while leaving much to be desired in terms of awareness of the subtleties of constructivist geometry more broadly.

In time, Newton was led to try to articulate a more complete view of the role of constructions on geometry. In a group of manuscripts from around the early 1690s for a projected but never completed work on "*Geometria*," Newton engages much more thoughtfully with the legacy of constructions in ancient Greek geometry, and gives a much more mature critique of Descartes. He returns to the organic constructions of his youth and tries to encapsulate and generalise their particular virtues and use this as the basis for his own programmatic vision of the role of constructions in geometry.

2. Newton's organic constructions

Fundamental to Newton's work on constructions was his so-called "organic" (i.e., instrument) construction of conics and its generalisations. Newton developed this idea already in the 1660s (Newton, MP, II.106–159) and it remained a centerpiece of his thinking about constructions throughout his career. I shall use the abbreviation NOrg to refer to Newton's organic construction technique.

Figures 1 and 2 show the NOrg construction of conics. This way of constructing conics has various advantages, including the facility with which the conic through five given points (or four given points and a given tangent, etc.) can be constructed. Newton published such constructions in the *Principia* (Newton, 1687/1999, Book I, Props. 22–27).

Another strength of NOrg is its generalisability. By replacing the directrix line with a more general curve, NOrg can be used to generate higher-order algebraic curves. With this tool, Newton obtained a remarkable range of results that strikingly foreshadow

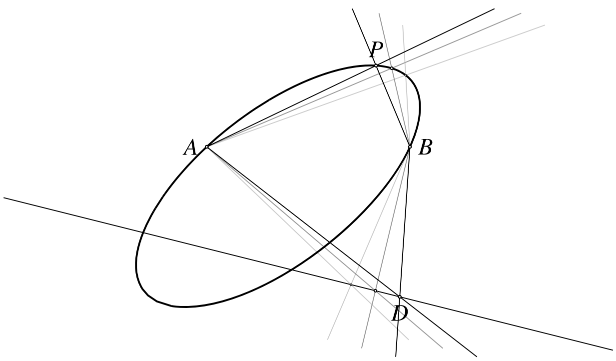

Figure 2. Another example of an NOrg construction of a conic section. Same construction mechanism as Figure 1 but with different parameters.

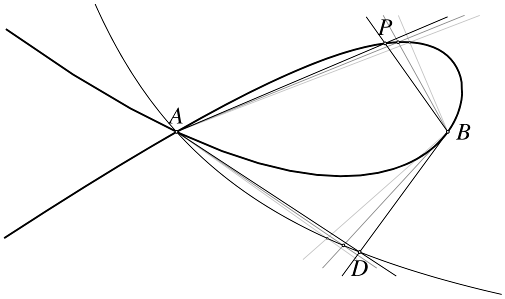

Figure 3. A variant of NOrg. "If [the directrix of D] is a conic passing through A but not B, [the curve described by P] will be a curve of third degree having a double point at A and passing through B." (Newton, MP, II.111).

19th-century algebraic geometry (Shkolenok, 1972). Figures 3 and 4 show some basic examples of the kinds of results involved in these investigations. Indications are that Newton indeed made substantial use of actual constructions as a research tool: "Newton's technique must have been graphical"; "Newton actually made use of a real instrument" (Guicciardini, 2009, 96–97).

3. Constructions in Descartes's geometry

Much of what Newton says about constructions in geometry is directly anti-Cartesian. So to understand Newton it is necessary to keep the following aspects of Descartes's geometry in mind. One of Descartes's key construction principles is that of the "turning ruler and moving curve," which I shall refer to as tRmC for short. As Bos explains:

> Special cases of ... what henceforth I call the "turning ruler and moving curve" procedure ... occur at several places in the *Geometry*. All these cases involve a curve C moving in one fixed direction and a ruler turning around a fixed point O. The two motions are interrelated via a point Q whose position with respect to C is fixed, which means that Q partakes in the rectilinear motion of C; the ruler connects O with Q. During the combined motion the point or points P of intersection of the ruler and the curve trace a new curve C'. (Bos, 2001, 278–279)

Descartes argues that curves traceable by tRmC are acceptable in geometry and susceptible to exact and rigorous mathematical study. Descartes describes the tRmC constructions of a hyperbola (Figure 5; (Descartes, 1637, 320)) and a conchoid (Figure 7; (Descartes, 1637, 322)) in this context. Figure 6 shows another example of a similar type: the cissoid. (The tRmC construction of the cissoid is not explicitly discussed by Descartes in the *Géométrie*, although he notes that the cissoid is an acceptable curve (Descartes, 1637, 317).)

Descartes's tRmC construction of the Cartesian cubic parabola (Figure 8) is fundamental. It is the crucial "next" construction beyond conics both in his treatment of the Pappus problem (Descartes, 1637, 337) and in his theory of equations (Descartes, 1637, 404–405), and furthermore Descartes maintains that higher-degree problems should be dealt with by generalising this paradigm example (Descartes, 1637, 413).

Descartes's tRmC is a special case of his more general notion of which kinds of curve-tracing are legitimate in geometry. He regards lines and circles as traced by ruler and compass, and holds that the proper generalisation thereof are curves "described by a continuous motion or by several successive motions, each motion being completely determined by those which precede" (Descartes, 1637, 316). The latter is a single-degree-of-freedom requirement: a curve-tracing instrument should be determined by one single "input" motion. This rules out for example the quadratrix and the Archimedean spiral (Descartes, 1637, 317), which are defined in

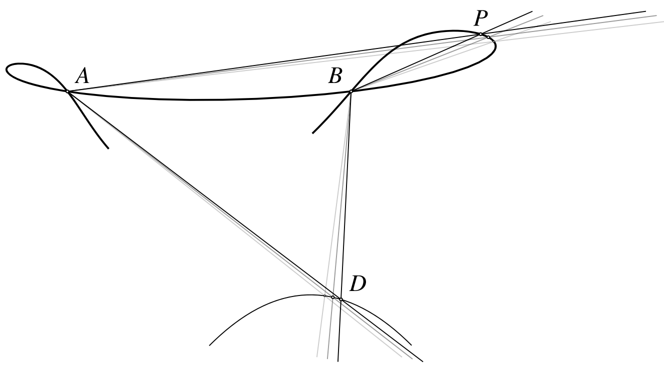

Figure 4. A variant of NOrg. "If [the directrix of D] is a conic passing through neither of A or B, [the curve described by P] will be a quartic having two multiple points, one each at A and B." (Newton, MP, II.111).

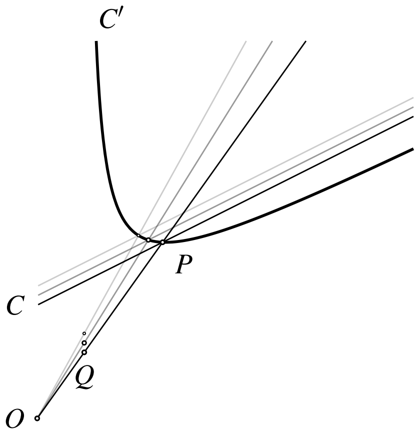

Figure 5. Hyperbola traced by tRmC. The curve C (here a line) and the point Q participate in the same linear motion (here vertical). The motion of Q causes the ruler OQ to turn about its fixed point O. The intersection $P = C \cap OQ$ traces the curve C' (here a hyperbola).

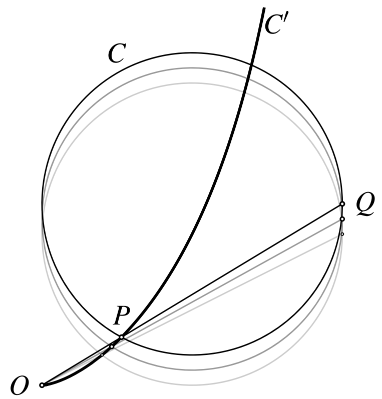

Figure 6. Cissoid traced by tRmC. The circle C: $(x-1)^2 + (y-t)^2 = 1$ and the point Q: $(2, t)$ participate in the same linear motion. The motion of Q causes the ruler OQ to turn about its fixed point O: $(0, 0)$. The intersection $P = C \cap OQ$ traces the cissoid C': $x(x^2 + y^2) = 2y^2$.

terms of two separate primary motions. Besides tRmC, examples of constructions that do live up to Descartes's general criteria of legitimate tracing are Descartes's mesolabe (Descartes, 1637, 317–318) and his string-and-ruler construction of ovals (Figure 9).[1]

This general notion of legitimate curve-tracing is arguably the official foundation of Descartes's geometry, and indeed the basis for his delineation of geometricity altogether. Sometimes, however, Descartes offers pointwise constructions, which on the face of it goes against these official foundations. But this is better understood as a pragmatic device that is expedient for some purposes but does not ultimately challenge or diminish the importance of his curve-tracing ideal as the sole authoritative foundations of geometry.

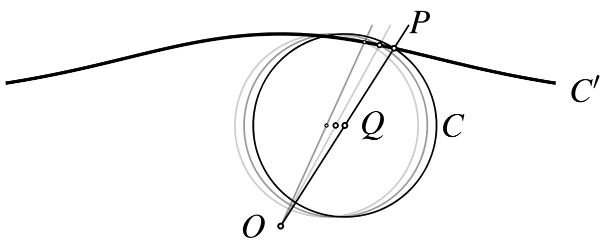

Figure 7. Conchoid traced by tRmC. The circle C: $(x-t)^2 + (y-b)^2 = a^2$ and the point Q: (t, b) participate in the same linear motion. The motion of Q causes the ruler OQ to turn about its fixed point O: $(0, 0)$. The intersection $P = C \cap OQ$ traces the conchoid C': $(y-b)^2(y^2 + x^2) = a^2 y^2$.

[1] The oval constructed in Figure 9 is a "Cartesian oval" with $CF + nCG =$ constant. However, note that the purpose of double-wrapping the string around CK is not to create the $n = 2$ case of a Cartesian oval $CG + nCK =$ constant (cf. (Guicciardini, 2009, 49)). Such a way of realising a Cartesian oval condition $CG + nCK =$ constant by strings would only work for integer values of n. Descartes gave his more complicated construction since it does not need n to be an integer. With F, A, K, L, G along the x-axis at x-values 0, $\frac{n+1}{2}$, $(\frac{n+1}{2} + n)/2$, n, $n+1$ respectively, the curve traced is the Cartesian oval $CF + nCG = \frac{(n+1)^2}{2}$. See also (Warusfel, 2010, 361–362), (Farouki, 2022).

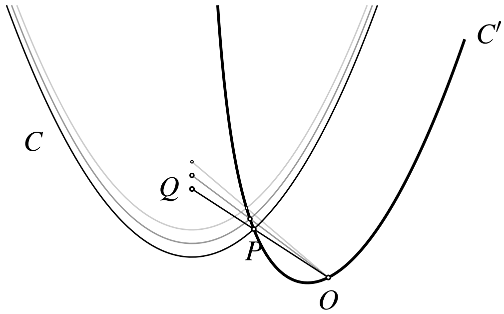

Figure 8. Cartesian cubic parabola traced by tRmC. The parabola C: $y = x^2/p + t$ and the point Q: $(0, a + t)$ participate in the same linear motion. The motion of Q causes the ruler OQ to turn about its fixed point O: $(b, 0)$. The intersection $P = C \cap OQ$ traces the Cartesian cubic parabola C': $pxy = x^3 - bx^2 - apx + abp$.

Indeed, Descartes makes the foundational status of his pointwise constructions rest on his curve-tracing principles, arguing that his pointwise constructions of algebraic curves are legitimate precisely because "this method of tracing a curve by determining a number of its points taken at random applies only to curves that can be generated by a regular and continuous motion" (Descartes, 1637, 340). Thus pointwise constructions do not themselves have primary foundational status, but are merely a resort of convenience in cases where foundationally proper curve-tracing "seems to you awkward" (Descartes, 1637, 407).

When, in the context of the Pappus problem, Descartes needs to construct conic sections, he explicitly uses the construction propositions of Apollonius's *Conics* (Descartes, 1637, 329, 331, 332). Why did Descartes not give his own tracing methods for the conics, in the manner of the tRmC hyperbola example that he does discuss (Figure 5)? "It is remarkable that … he did not do so," as (Bos, 2001, 325) says.

In fact, in a sense Descartes could not have done so, because tRmC cannot generate the other conics in the manner that it generates the hyperbola. We can prove this as follows. Consider any instance of tRmC where the curve C is a straight line. Without loss of generality, we can choose $O = (0, 0)$, and let the direction of motion of the curve C be along the y-axis. Then, if the curve C is the line $y = mx + b$, its subsequent positions can be parametrised by $y = mx + b + t$, and the position of the moving Q can be parametrised as $(w, h + t)$. The turning ruler OQ is then $y = \frac{h+t}{w}x$. The locus of the intersection point $P = C \cap OQ$ is obtained by eliminating t from the equations for C and OQ. This gives the equation $xy - mx^2 + (h - b)x - wy = 0$ for the curve C'. This equation has positive discriminant regardless of the choices of the constants m, h, b, w. Hence the curve is always a hyperbola (except in degenerate cases, including $w = 0$, or C being parallel to the direction of motion, or $Q \in C$, when C' is a line). We can also see this by observing that the curve C' has a vertical asymptote at $x = w$ and another asymptote parallel to C. Thus, when the moving curve C is a line, the tRmC procedure can *only* generate hyperbolas, not parabolas or ellipses.

Nor can tRmC generate all conics by using a circle instead of a line as the moving curve C. If C is a circle, then in the course of its motion it traces out an infinite rectangular strip (with its diameter as width). All intersection points P, and hence the entire traced curve C', must clearly lie inside this strip. But it is evident that no such strip can contain for example a parabola. Hence a parabola cannot be traced by tRmC using either a straight line or a circle as the moving curve C.

The fact that tRmC cannot generate all conics is briefly noted by Whiteside (Newton, MP, II.9) but seems to have been somewhat neglected in more recent literature. For instance, Bos argues that, in 1631-32, "Descartes probably believed that … all loci which satisfied the requirement of a Pappus problem … could be traced by the iteration of the 'turning ruler and moving curve' procedure" (Bos, 2001, 403–404). When assessing the plausibility of this hypothesis, it is surely relevant to know that even a simple parabola is a counterexample to this belief, but this is not highlighted by Bos anywhere in his book. In any case, at least by 1637, Descartes surely knew that tRmC could not generate all such curves, which is why he resorted to the Apollonian construction propositions instead of tRmC in his systematic constructions of conic solutions to the Pappus problem.

Similarly, the following statement perhaps also seems to be suggesting that the absence of tRmC constructions of parabolas in Descartes is merely an omission rather than a mathematical necessity:

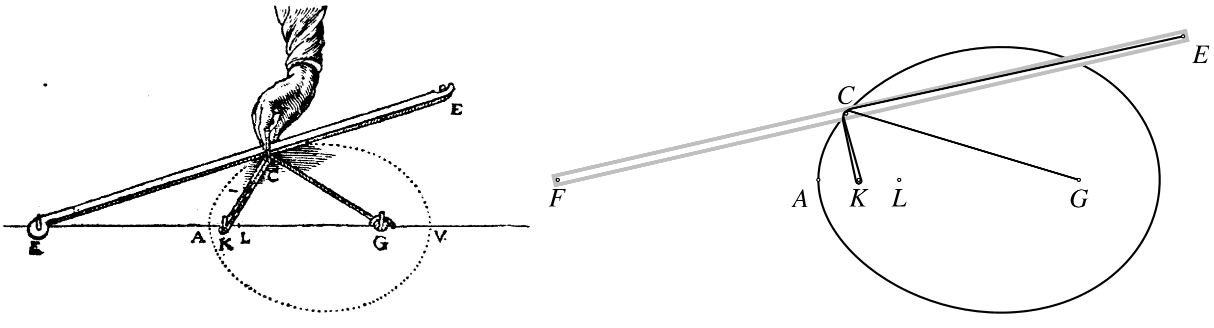

Figure 9. Descartes's construction of ovals. The point C is determined by the condition that the string length $EC + CK + KC + CG$ is constant. (Descartes, 1637, 356).

> Descartes does not provide in *La Géométrie* the construction of the parabola by the "turning ruler and moving curve procedure" like he does for the hyperbola or for the so-called Cartesian parabola. However, such a construction raises no problem according to Descartes' criteria of "coordinated continuous motion". It is provided for instance in the work of Descartes' leading disciple Frans van Schooten, *De Organica Conicarum Sectionum in Plano Descriptione* (Leiden, 1646), 73–77. (Maronne, 2010, 541)

It is true that Van Schooten gave constructions for conics that conform to Descartes's general precept of coordinated continuous motion. However, these are not tRmC constructions, but rather *ad hoc* constructions unique to each case.

The tRmC is the closest thing Descartes has to a systematic and general curve tracing method. It is a blemish on his system that it fails to be completely general even in the simple case of conics. The fact that, with much effort and ingenuity, *ad hoc* workaround constructions can be found in particular cases does not alleviate this systemic issue. In this respect, Newton's NOrg succeeds precisely where Descartes and his followers had failed (as Whiteside (Newton, MP, II.8) remarks), thereby pointing the way to Newton's own NOrg-based constructivist program as we shall see below.

4. Newton's Arithmetica Universalis critique of Descartes

In his Lucasian lectures on algebra of around the early 1680s (later published as *Arithmetica Universalis*), Newton included a section where he burst into a harsh critique of Descartes's views on constructions. This section is largely disconnected from the rest of the lectures. The ostensible reason for including it in these algebra lectures is the issue of "the construction of equations" (Bos, 1984), which was concerned with translating results found by algebra into constructions comparable to those used by Greek geometers for problems such as cube duplication and angle trisection. But Newton's take on this issue can hardly be said to be in continuity with that tradition, which he does not appear to be very sensitive to or interested in. The emphasis on constructions in the ancient and Cartesian traditions was surely strongly connected to issues of epistemological rigour (Blåsjö, 2022). Newton, by contrast, rather bafflingly frames the purpose of constructions in solely practical terms:

> It now remains merely to explain how to extract the roots of equations numerically .... Here, the especial difficulty lies in obtaining their first two or three figures. That is most conveniently accomplished by some construction or other of the equation, be it a geometrical or a mechanical one. (Newton, MP, V.423) ≈ (Newton, 1720, 227)

> My immediate concern is not for a construction which is geometrical, but for one of any sort whereby I may attain a numerical approximation to the roots of equations. (Newton, MP, V.429) ≈ (Newton, 1720, 230)

It can hardly be seriously maintained that this ultra-pragmatist conception had been the goal of constructions in ancient and Cartesian geometry. Yet this idiosyncratic perspective is the ground on which Newton goes on to criticise Descartes and even construe ancient tradition.

Newton claims ancient precedence for his view as follows.

> The Ancients ... feeling that constructions [using conic sections] are, because of the difficulty of describing conics, of no practical use, they looked for easier constructions by means of the conchoid and cissoid, the extending of threads and any kind of mechanical application of figures: as we learn from Pappus, mechanical usefulness was preferred to useless geometrical speculation. Thus the mighty Archimedes ignored the trisection of an angle by means of conics expounded by his predecessors in geometry, and in his Lemmas taught how to cut an angle into three by [neusis instead]. (Newton, MP, V.469–471) ≈ (Newton, 1720, 248)

These claims are not individually indefensible; they are some among many possible interpretations that can be imposed on the ancient record, since very little explicit testimony has survived regarding the grounds on which some particular construction method was preferred to another. However, insofar as Newton is insinuating that this means that his pragmatist conception of constructions is in line with ancient tradition, this implication is surely highly misleading. For example, ancient mathematicians gave many constructions similar to the neusis trisection of an angle for the problem of doubling the cube (or finding two mean proportionals) as well. But, numerically, this geometrical problem is equivalent to cube root extraction, and the Greeks knew perfectly well how to find cube roots numerically with much greater accuracy and efficiency than that of the elaborate geometrical constructions they provided for this problem (Blåsjö, 2022, §4.6.8). So it is evident that they had other purposes in mind with their constructions than finding approximate numerical solutions. Indeed, Newton's specious framing of the purpose of constructions as an aid to numerical equation solving was not to recur in his later, more thoughtful take on the subject in the 1690s.

But in the Lucasian lectures it serves Newton's anti-Cartesian ends well to ignore this and pretend that his pragmatist perspective is a fair standard by which to judge Descartes's constructivist program. Indeed, Newton's claim that the ancients preferred neusis, conchoids, and cissoids to conics for their greater practical value is a direct counterpoint to Descartes. For, according to Descartes, constructions should be classified by their algebraic degree, and hence conics, which are of degree 2, are preferable to neusis, conchoids, and cissoids, which are all of higher degree.

But Newton was able to score this point against Descartes only by committing himself to an opportunistic and one-sided reading of ancient geometry and constructivist tradition. If it was not for his desire to reject Descartes, it is hardly very plausible that Newton would have arrived at this conception of constructions through open-minded and dispassionate reflection on past geometry. Yet it is interesting that the view he is articulating fits not only a preconceived anti-Cartesian bias but also his own very pragmatically oriented use of constructions in his own previous mathematical practice (which he seems to have pursued largely in blissful obliviousness as far as the philosophical subtleties of the constructivist geometrical tradition are concerned).

To be sure, Newton is right that Descartes's principle of using algebraic degree as the sole measure of complexity of a geometrical construction is not perfect. Others, such as Jakob Bernoulli, also pointed out that of course excessive focus on reducing problems to algebraic equations of the lowest degree can lead to clumsy solutions where geometrically simpler ones are available (Bos, 1984, 358–359). Likewise it is obviously true, as Newton passionately argues, that the circle must be counted as simpler than the conics even though they have the same degree, and simpler than the parabola even though the parabola has the simpler equation (Newton, MP, V.425, V.471), (Newton, 1720, 248, 251). But these rather trifling points were already acknowledged by Descartes himself (Descartes, 1637, 323).

The purpose of Descartes's rule of degrees was not to perfectly reflect mathematicians' intuitive judgements of simplicity in every conceivable case. Rather, it was to meet the necessary challenge for a foundational geometrical program of imposing a hierarchy on geometrical methods that broadly respects tradition and common sense, while also mirroring a natural hierarchy in terms of the construction postulates permitted in the theory. Already the Greeks (as made explicit by Pappus, (Sefrin-Weis, 2010, 145, 271–275)) had conceived a hierarchy of geometrical methods, going from the Euclidean ruler and compass, to conic sections, to more complex methods, along with a requirement to keep to the lowest possible rung in this order for any given problem. This is a way to ensure that the introduction of new (and more questionable) principles and postulates necessary to treat more advanced problems does not undermine or nullify previous efforts to rigorously establish more basic parts of mathematics (such as Euclid's *Elements*) using a more minimalistic and secure set of assumptions. Rather than the quite banal objections offered by Newton, a serious challenge to Descartes's program should offer a systematic alternative solution to the overarching need for such a global organisation of geometrical method. As we shall see, Newton was later to develop a very serious alternative to Descartes that accepted that it was precisely in this more comprehensive setting that any honest attack on Descartes had to be mounted. But, in his lectures on algebra, Newton's critique of Descartes remains in an unsophisticated state.

Newton's apparent ignorance of the broader stakes of the philosophy of constructions in geometry is also reflected in another similarly naive argument that he raises against Descartes in the lectures, namely the following.

> Were the cycloid to be accepted into geometry, it would be allowable by its aid to cut up an angle in a given ratio. Could you then, if someone were to use this line to divide an angle in an integral ratio, see anything reprehensible in this and contend that this line is not defined by an equation [i.e., the cycloid is a transcendental curve], but that lines defined by equations [i.e., algebraic curves] need to be employed? We would in consequence, were the angle to be divided into (for instance) 10001 parts, be compelled to bring into play a curve defined by an equation of more than a hundred dimensions: this, however, no mortal would be capable of describing, let alone comprehending and valuing above the cycloid—a curve which is exceedingly well known and very easily described through the motion of a wheel or a circle. How absurd this is, any one may see. (Newton, MP, V.427) ≈ (Newton, 1720, 229)

It is true that 10001-section of an angle by Cartesian methods will involve an algebraic equation of such high degree as to be completely impracticable. But this argument misses the point. By this logic, one could just as well argue that Euclid's construction of a square in *Elements* I.46 is idiotic on the grounds that, if all the steps of the preliminary propositions on which it depends are spelled out in full, it involves an absurd number of steps and is completely out of touch with how anyone would construct a square for any practical purpose. (Sidoli (2018, 437) discusses and visualises how convoluted Euclid's construction of a square is from this point of view.) But of course the glory of Euclid's construction is not its usability but its rigorous reduction, ***in principle***, of the construction of a square to a minimum of impeccable construction principles. In the same way, the purpose of Descartes's program is to show how angle multisection and many other problems can in principle be reduced to elemental construction principles, not to provide the most practically workable solution to such problems. By assuming otherwise, Newton's critique seems to reveal more about his own ignorance of the constructivist tradition than it does about the alleged "absurdity" of Descartes's geometry.

Newton's suggestion of accepting the cycloid to solve construction problems is highly questionable and arguably incongruous with ancient tradition. The cycloid is transparently susceptible the same criticism that the ancients directed at the quadratrix. The quadratrix is defined in terms of a rectilinear motion and a circular motion, which are assumed to be coordinated so that the rectilinear motion covers the radius of the circle in the same time it takes the circular motion to cover a quarter of the circumference of the circle. By exploiting this assumed link between circular and rectilinear motion, the quadratrix can be used to turn the problem of n-secting an angle into the (trivial) problem of n-secting a line segment. As Pappus objects (Sefrin-Weis, 2010, 132), this is arguably not so much a solution of the angle multisection problem as a rather blatant assumption in thinly veiled form of effectively exactly what the problems asked for in the first place, namely, the intertranslatability of circular arcs and straight line segments. It is very plausible that many ancient geometers were sympathetic to this critique. The fact that numerous angle trisection methods are recorded in ancient geometry that postdate the quadratrix, even though the quadratrix trivially "solves" the problem of angle trisection, can be seen as evidence of widespread dissatisfaction with quadratrix angle multisection.

Assuming the cycloid comes to much the same thing. Like the quadratrix, it "solves" angle multisection problems only by effectively postulating the intertranslatability of circular arcs and straight line segments. Indeed, the distance between the endpoints of one arch of a cycloid equals the circumference of its generating circle. In other words, the cycloid effectively "unrolls" the circumference of a circle into a straight line. But if one has the ability to unroll circular arcs into straight lines then one does not need the cycloid, since allowing strings to be wrapped around circles and then stretched into lines already itself immediately trivialises the angle multisection problem without the unnecessary detour of the cycloid, whence the latter arguably contributes nothing more to the solution than to obscure behind a smoke screen the essential dependence on the string rectification assumption. Indeed, (Huygens, 1673, 10–11) and (Leibniz, 1706, Figure 2) pointed out that the most realistic method for drawing a cycloid accurately is precisely to wrap a string around the generating circle and use the unrolling of this string to control the generating motion of the cycloid (to ensure the mathematically necessary condition that the motion of the generating circle is a pure rolling, without slipping).

From this point of view, it makes no sense to allow the cycloid as a means of constructing classical problems such as angle multisection. Rather, accepting the cycloid is tantamount to collapsing the entire ancient tradition of construction problems into trivial non-problems. If strings can be wrapped around circles and then stretched into lines, then it hardly makes any sense to speak of angle multisection or circle quadrature as problems at all, since these become immediately trivial. Clearly, then, this is not what the ancients had in mind when they pursued the classical construction problems.

Later Newton himself seems to have realised this perfectly well, and in fact even used this as an argument against Descartes. As we shall see, in the *Geometria*, Newton attacks Descartes's string construction of ovals on the grounds that allowing the operation of wrapping strings along curves trivialises the entire construction tradition. Thus, in this more mature and sophisticated work, Newton condemns string constructions, while in the algebra lectures he explicitly mentions approvingly the ancients' use of string constructions, as quoted above.

These arguments against the permissibility of strings and cycloids are what (Blåsjö, 2012, 415) called retro-consistency arguments: declarations on the foundations of geometry should not contradict the mathematical practice of the great masters of the past. In his algebra lectures, Newton was evidently not yet very alert to the force of such arguments. But later in his *Geometria* he was to use such arguments extensively and effectively. There he repeatedly argues against particular constructions on the grounds that "this postulate eliminates the vast majority of problems" (Newton, MP, VII.387) or "admit such constructions and all ancient geometry will be put out of joint" (Newton, MP, VII.297)—that is to say, he invokes retro-consistency as a key guide to evaluating foundational proposals. Indeed, Newton's historical awareness and sophistication as an interpreter of Greek geometry had increased drastically by then. For instance, at one point he rattles off a carefully selected list of a dozen specific propositions from various works of Archimedes to claim ancient precedent for his position (Newton, MP, VII.303).

Newton's liberal use of the cycloid in the algebra lectures seems to be consistent with how he treated—or perhaps paid lip service to—the need for construction postulates in some unpublished works from around this time. In a manuscript dated by Whiteside to c. 1670, Newton gives three very generally worded postulates based on the exceedingly vague notion that "any extended object be allowed to move in any given respect" (Newton, MP, II.453). This evidently includes a variety of "complex motions" (Newton, MP, II.499) though the precise scope of the postulates is not spelled out by Newton. In another manuscript treatise from c. 1680, Newton again offers similar postulates that grant in very general and vague terms that "any line [i.e., curve] may move in any geometrical fashion whatever" and that any designated point on such a curve, or any intersection of such moving curves, can be taken to generate a new curve (Newton, MP, IV.426–429). Although the postulates are vaguely worded, it seems plausible that the cycloid is an example of a curve that can be generated according to these postulates. Indeed, the latter treatise mentions spirals and quadratrices (Newton, MP, IV.485), so, whatever the precise meaning of Newton's postulates is supposed to be, it seems that they evidently do not entail a restriction to curves counted as "geometrical" by Descartes. We shall see below that the mature Newton argued that construction postulates must not be too liberal or general in admitting curves traced by "any" motion. This can plausibly be read as a reprimand to his younger self.

In sum, I maintain that Newton's arguments about constructions in the Lucasian lectures are underwhelming. A short-sighted desire to criticise Descartes, it seems, led Newton to make assertions that are not seriously tenable as an overall philosophy of constructions in geometry. I shall argue below that Newton's tirade in these lectures is worlds apart from his later take on the subject in the *Geometria*, which is much more historically aware and philosophically refined. My interpretation is that Newton himself later realised that his arguments in these early lectures were naive and untenable. Although Newton himself did not spell this out, I shall argue below that Newton's later writings can be used to argue that he had changed his mind especially regarding the cycloid argument. The interpretation that Newton's critique of Descartes in these relatively early lectures is rash and poorly justified also squares with the well-known fact that Newton was strongly ill-disposed to Descartes for a number of independent and extrinsic reasons (including for instance theological considerations).

Newton's lectures were later published as *Arithmetica Universalis* (1707). This book thus appeared at a time when I claim that Newton had already abandoned the views expressed therein. Nevertheless, this does not necessarily disprove my interpretation, due to the special circumstances of publication. The book was published at the initiative William Whiston, Newton's successor as Lucasian Professor. Newton seems to have disapproved of the publication. According to Gregory, "[Newton] was forced seemingly to allow of it. ... He has not seen a sheet of it, ... nor does he well remember the contents of it." (Newton, MP, V.9–10) The book appeared anonymously, without Newton being identified as the author.

In his own copy of the book, Newton made notes for corrections. Notably, all the philosophical material discussed above was marked for deletion by Newton (Newton, MP, V.14, V.422). Nevertheless this material remained after all in the 1722 edition that Newton himself was behind (thought it was still published only anonymously). But the main purpose of the revised edition seems to have been "merely to print a corrected text to oust" the first edition (Newton, MP, V.17), which contained some glaring technical errors (Newton, MP, V.13).

It is possible, then, that Newton let the remarks on constructions stand, even in the revised edition, despite having outgrown those views. Removing this entire section, as he had indicated in his private copy that he wished to do, would have been contrary to the purpose of printing an edition that merely fixed some outright mathematical mistakes. And revising the section in accordance with his mature views on constructions would be a major undertaking and would force him to have to finalise some version of the several manuscript drafts of the *Geometria* that he had evidently struggled considerably to complete to his satisfaction. For these reasons I do not think that the (anonymous) publication of the *Arithmetica Universalis* should be taken to indicate that Newton at that point still held the views expressed therein regarding constructions.

5. Newton's account of the role of constructions in the Geometria

Newton set out his mature view on constructions in geometry in the early 1690s, in a series of manuscripts under the title "*Geometria*." In these unfinished manuscripts, Newton appears to have tenuously attempted a systematisation of geometry in line with his increasing veneration for the ancients while also incorporating much of his own earlier mathematical work. Geometrical constructions are one of the recurrent themes of these manuscripts.[2] Newton's remarks on constructions in the *Geometria* are scattered and disrupted by the stops and starts of his various draft versions. I shall attempt to give a unified interpretation of them, with the caveat that this necessarily involves some reconstructive interpretation since the source text is not systematic and unequivocal.

Newton seems to be juggling two opposite poles, neither of which he wanted to give up. On the one hand, Newton is keen to stress that geometry is free to study any curves and is not bound by constructibility constraints such as those Descartes used to exclude spirals, quadratrices, and cycloids from geometry. Thus for example:

> It ... does not matter how [figures] shall be described. ... The purpose of ... geometry is neither to form nor move magnitudes, but merely to measure them. (Newton, MP, VII.289–291)

On the other hand, Newton is well aware that such an absolutists anti-constructivist position is clearly incompatible with geometrical tradition. The great ancient geometers evidently *did* think it mattered "how figures shall be described"—for example, this was a prominent focus of the classical traditions of angle trisection and cube duplication.

To accommodate these opposites, Newton opts for a two-tier account of geometry: a constructivist approach is appropriate for "easier problems" that are within the reach of viable and useful constructions, whereas more advanced geometrical subject matter can be "merely propounded" or "speculated upon" without a rigid constructivist foundation:

> [Geometry] demands that operations for the construction of easier problems be in the power of its disciples and consequently commands at will that they be performed; it thinks it absurd, however, to postulate that novices should control the more difficult operations, and accordingly does not dictate these for all [its disciples] but merely takes them within its scope and obliquely propounds on them. ... It solves all problems, some in a practical manner, the others speculatively, some by laying down precepts and the rest by instruction without precept. ... Geometry commands ... geometrical construction ..., while it has no power over ... mechanical construction ... but merely speculates upon it. (Newton, MP, VII.303)

Thus "no figure can be defined which [geometry] shall not regard, no problem proposed which [geometry] does not accomplish" (Newton, MP, VII.303) since any exact figure can be regarded by geometry, in any case at least "speculatively." Figures that can be treated by "geometrical construction" should be, but figures that cannot are no less the legitimate purview of geometry.

One might object that constructions have long been seen as essential to geometry, as embodied by Euclid's ruler and compass postulates. On the face of it, the role of these postulates in the *Elements* seems to suggest that a primary and indispensable task of the geometer is to construct the entities that one is to speak about. It is to this line of argument, I take it, that Newton objects as follows.

> The description of straight lines and circles, which is the foundation of geometry, appertains to mechanics. Geometry does not teach how to describe these straight lines and circles, but postulates such a description. ... To describe straight lines and to describe circles are problems, but not problems in geometry. (Newton, 1687/1999, 381–382)

> Geometry postulates because it knows not how to teach the mode of effection. (Newton, MP, VII.291)

This revisionist reading of Euclid thus factors constructions out of geometry proper. This enables Newton to maintain, on the one hand, that a geometry without constructions—that merely postulates whatever it needs at will—is as legitimate as anything and perfectly in keeping with ancient tradition, but also, on the other hand, that the study of geometrical constructions nevertheless has a respectable place as well.

Thus Newton divides geometry into one part that is founded on constructions and one part that is not. In the non-constructive part of geometry, anything goes as far as descriptions or definitions of curves are concerned:

[2] These manuscripts expand in a much fuller form Newton's brief remarks on constructions in the preface of the *Principia* (1687). The *Principia* preface has sometimes been read as endorsing a constructivist tradition (Garrison, 1987), (Dear, 1995, Ch. 8). But this view has been countered, rightly in my view (cf. (Blåsjö, 2017, §3.3)), by (Domski, 2003) (whose view is also endorsed by (Kaplan, 2018, 456)). Instead, (Domski, 2003) and (Guicciardini, 2009, §13.1) emphasise that Newton's remarks about constructions in the preface and the *Geometria* are directly anti-Cartesian, which fits well with the reading that I shall offer here.

> We are free to describe [figures] by moving rulers around, using optical rays, taut threads, . . . , points separately ascertained, the unfettered motion of a careful hand, or . . . any mechanical means whatever. Geometry makes the unique demand that they be described exactly. (Newton, MP, VII.289)

Hence, in this part of geometry, there is no control and no restrictions on constructions having to meet any particular standard in terms of such considerations as usefulness, practicability, epistemological primitiveness, intuitability, etc.

Yet in the constructive part of geometry Newton expressly forbids several of the types of descriptions that he just expressly permitted in the non-constructive part. An example of this is constructions by strings. We just saw Newton argue that "we are free to describe [figures] by . . . taut threads" in non-constructive geometry, but in constructive geometry the rules of the game are different and there this type of construction should be rejected.

> Postulates which allude to stretched threads are scarcely to be admitted. . . . [Descartes admitted this type of construction in the case of threads stretched along straight "rods."] But the principle of this description will be the same when the rod . . . is straight and when it is curved . . . ; when, however, it is curved, the description of the line [by the unwinding string] has nothing in common with geometry. (Newton, MP, VII.385)

In other words, if one accepts for example string constructions of conics (which (Descartes, 1637, 340) mentioned approvingly, and which were expounded in detail by his disciple (Van Schooten, 1646)) then there is no reason to not also accept the unwinding of a string from a circle, for example. So a line segment equal to the circumference of a circle, or π, is just as constructible as conics: an undesirable conclusion that is obviously inconsistent with ancient tradition. Descartes had effectively anticipated this objection. He wanted to use some string constructions but he knew that constructions where strings are wrapped around or unwound from curves such as circles amount to allowing the rectification of such curves to be constructively given, which he certainly did not want to accept. He therefore declared that string constructions where strings are placed along straight lines are fine but others not (Descartes, 1637, 340). But this is more an *ad hoc* assertion than a convincing argument from first principles, so Newton's critique is well taken.

In my view, Newton has at this point surely abandoned his old suggestion that cycloids should be accepted as a construction device for such problems as angle multisection. Although Newton does not explicitly say this, in light of his more considered view in the *Geometria*, the cycloid is surely blatantly objectionable for much the same reason that string constructions are. Such things can of course be studied with full geometrical legitimacy, according to Newton, which is why he expressly permits these kinds of things (and more) in his non-constructive geometry. Yet it would be nonsensical to conflate (as Newton rather naively did in his earlier algebra lectures) this kind of permissibility with viability as foundational constructions in constructive geometry. The latter is subject to much more stringent standards, including epistemological and retro-consistency considerations. Newton rightly invokes such stringent standards to reject various constructions, including Descartes's string constructions, and by the same logic he should (and most likely did) reject his own earlier cycloid argument.

Continuing in the same vein, just a few pages after emphatically stating that anything goes in non-constructive geometry, Newton is keen to legislate further and very restrictively as to what is permissible in constructive geometry.

> The postulate of moving any lines whatever according to appointed laws and by their intersections describing new lines is to be rejected for many reasons. It is exceedingly complex and akin to an infinity of postulates. (Newton, MP, VII.295)

Indeed, when doing geometry in a constructive mode it is naturally desirable to keep the number of postulates low (as Newton argues in (Newton, MP, VII.291), (Newton, 1687/1999, 382)), while the curves it is permissible to study in non-constructive geometry are obviously not subject to any cardinality restriction. The "many reasons" for rejecting this too liberal construction postulate include retro-consistency, as Newton made clear in another version of the argument:

> Should anyone . . . postulate that 'given lines may move in any assigned manner whatever so that by their intersections fresh lines are described', he will postulate that in innumerable cases unknown points are given the discovery of which all geometers count as problems, and hence he will do away with problems. (Newton, MP, VII.387–389)

Indeed, to permit construction postulates that are too broad and liberal is effectively the same thing as to not do constructive geometry at all; that is to say, to trivialise and nullify the ancient constructivist tradition. One cannot go against "all geometers" like that. This again shows why Newton's cycloid proposal, which would do exactly this, cannot be maintained. Indeed, in these passages Newton is in effect criticising his earlier self, since, as we saw above, he had earlier (Newton, MP, II.453), (Newton, MP, IV.426–429) given postulates of the type he now rejects. Newton has in the meantime become aware of "many reasons" for why his youthful position is untenable, leading to a much more sophisticated and mature outlook.

Pointwise construction is yet another construction principle that Newton expressly permitted, as we saw, in non-constructive geometry, yet expressly rejects in constructive geometry:

> Even if you have found a thousand points through which a curve shall pass, you will yet never be able so to find them all individually but must at length trace the separate intervals from one point to the next by a chance drawing of the hand. (Newton, MP, VII.385)

Indeed, Newton is right that pointwise constructions can hardly be seriously accepted in constructive geometry for reasons further discussed in (Blåsjö, 2022, §§4.2.6, 4.4.3, 4.6.7), (Blåsjö, 2017, 16–17), (Bos, 2001, 177, 189). (Perhaps Newton's most immediate reason for bringing up and condemning pointwise constructions was to cast yet more shade on Descartes. But as we noted above, Descartes himself arguably essentially acknowledged the dubiousness of pointwise constructions and took care not to let anything foundational rest upon them, so Newton's attacks on this point do little damage to the Cartesian program.)

Altogether, Newton's two-tier account of geometry in the *Geometria* allows for one mode of mathematics where any geometrical relations are studied without regard for constructivist foundations, and another mode of mathematics that is constrained to proceed in accord with the ancient constructivist tradition. Newton's vision is locally well-argued and consistent. However, a compelling principled reason to accept such a disunified vision of geometrical method overall is lacking.

6. Newton's constructivist program in the Geometria

In a little-studied part of the *Geometria*, Newton finally offers his own full-blown proposal for a constructivist foundation of geometry. It is one thing to criticise Descartes on isolated points, as Newton had done so many times. But without a viable rival system of constructive geometry, the Cartesian vision would likely remain the status quo, its imperfections not withstanding. Chipping away at details would not change the fact that the Cartesian vision based on algebraic degree provided a systematic framework that included the bulk of both ancient and modern geometry, was reducible to a small number of construction principles roughly comparable to Euclid's ruler and compass, and imposed a hierarchy of complexity on geometrical curves that roughly reflected classical geometrical practice and most of the qualitative judgements of the great geometers of the past—for example, classifying curves by algebraic degree "explains" why conics are the "next" curves after ruler and compass, why neusis (conchoid) angle trisection is preferable to quadratrix angle trisection, why the conchoid and the cissoid are among the most basic higher curves and certainly much simpler than for example the cycloid, and so on. Many of Newton's criticisms of Descartes can be seen as arguments that the Cartesian program fails in various specific respects to fully meet these goals. But what is the alternative? The Cartesian program may not be perfect but no other system of geometry that could meet these goals was readily available. Such a rival system—a systematic foundation of constructive geometry that is comprehensive and substantive enough to rival Descartes's—is what Newton now proposes for the first time.

To this end, Newton offers three construction postulates (Newton, MP, VII.389). In effect, they amount to taking NOrg (including its various generalisations) to be the legitimate constructions of geometry. Figures 1, 2, 3, 4, 11, 12 are examples of constructions subsumed by Newton's postulates.

Somewhat perplexingly, Newton declares that "in these three postulates all legitimate geometry is, I think, embraced" (Newton, MP, VII.389). Surely we must understand here that Newton's two-tier conception of geometry is still in effect and that "all" geometry here means merely the constructivist part of geometry. Otherwise this statement would be a dramatic negation of Newton's earlier point that non-constructivist geometry is as legitimate as anything, which he argued so passionately and persistently. Indeed, later in the same document Newton goes on to imply that "allowable" geometry extends beyond the scope of his construction postulates:

> It remains for us to discuss what is describable by means of these postulates. But … I shall in the first instance consider indiscriminately all regular lines which it is allowable to conceive. I call a line 'regular' when its every point is determined by the same principle. (Newton, MP, VII.395)

The manuscript cuts off before the promised conclusion, but we may understand that curves "describable by means of these postulates" are algebraic curves. Meanwhile, "regular lines" include transcendental curves "such as the Archimedean spiral, quadratrix and cycloid" (Newton, MP, VII.397), which are not traceable by Newton's construction methods. Yet it is of course "allowable to conceive" these curves (non-constructively)—much in accordance with Newton's two-tier division of geometry into a constructive part (restricted to "what is describable") and a non-constructive part (studying all curves "indiscriminately").

Newton's postulates are the following.

1. To draw a straight line from a given point to a given point.

2. To draw through two given points a line congruent to a given line at a distance not less than that of the points, an ordained point of which shall coincide with one or other given point.

My interpretation of the somewhat cryptic second postulate is as follows (see Figure 10). We are given two points, T and R, and a "line" (that it to say, a previously constructed curve or figure) γ. We "ordain" (at will) a point T' on γ as the point that is to coincide with T. We then move γ until T' coincides with T, and then rotate it until R is on γ (which is possible since we required the distance between T and R to be not too great in relation to γ—this, I take it, is the meaning of "… a distance not less …"). In other words, the postulate grants that a "given line" (curve, figure) γ can be moved into a new position determined by a *translation target point* T (that says where a particular point T' of γ is to be moved) and an *rotation target point* R (that determines how much γ is to be rotated or pivoted about the translation point T after the translation has taken place).

Equivalently, instead of speaking of moving γ, we can say that we have constructed a congruent version of it in this new position, which is Newton's phrasing in the postulate. But the connection with motion was very much on Newton's mind. Indeed, Newton even speaks in terms of such a moving curve lying in a "mobile plane" sliding on top of a "stationary plane" (Newton, MP, VII.467). Thus

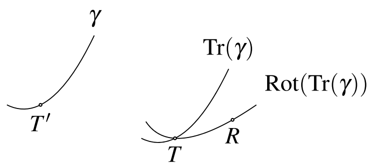

Figure 10. Postulate 2 of (Newton, MP, VII.389). Given is the curve γ. We designate a translation point T' on the curve, a translation target point T, and a rotation target point R. The postulate grants us that we can draw the curve Rot(Tr(γ)) that is congruent to γ but has been translated so that T' coincides with T and then rotated about T until the point R is on the curve.

points T and R are in the stationary base plane, while γ and T' are in the moving plane. The fact that γ remains congruent means that the motion of the moving plane is a rigid motion or isometry (indeed, a translation followed by a rotation, as we just said).

Finally, Newton's third postulate is:

3. To draw any line on which there shall always fall a point which is given according to a precise rule by drawing from points to points lines congruent to given ones.

By "lines congruent to given ones" we should understand moving or rotating curves or figures in the manner of Figures 1, 2, 3, 4, 12. That these are "drawn from points to points" seems to mean that their positions (at any stage of their motion) are determined in the manner of Postulate 2. A new curve is generated by tracing the path of a particular point defined ("according to a precise rule") in terms of these figures (for example, the tracing point may be defined as an intersection point, as in Figures 1, 2, 3, 4, or as a particular constructed point on one of the curves, as in Figures 11, 12).

In prototype NOrg cases such as Figures 1, 2, 3, 4, the rotation target point of the second postulate is the parametrisation or directrix point D. In all of these cases, the position of the moving curves (the two V-shaped rulers) at any stage is determined by their fixed pole points (the translation target points of Postulate 2) and the position of the moving directrix point D (the rotation target point of Postulate 2). Thus it is Newton's second postulate that grants that the state of the machine at every such stage can be constructed.

Alternatively, the translation target of Postulate 2 can be the point moving along a directrix while the rotation target is stationary, as in the cissoid example of Figure 12. In this example again the second postulate enables the state of the machine for any given D to be constructed by a rigid motion of the L-shaped ruler (namely, translate the ruler until the endpoint coincides with D, then rotate until O falls on the other leg).

Another classical construction obtainable as a special case of Newton's postulates is the conchoid (Figure 11). Newton discusses this construction in (Newton, MP, V.429). Unlike Descartes's construction of the conchoid (Figure 7), Newton's construction is in effect exactly the instrument construction of the conchoid proposed by Nicomedes in antiquity (as reported e.g. by Pappus, *Collection*, Book IV, Prop. 23).

As Newton points out (Newton, MP, 389), Euclid's second and third postulates are also contained in Newton's postulates. Newton doesn't spell out how exactly, but this can be explicated as follows. To extend a line segment AB, use Postulate 2 to place a copy $A'B'$ of AB with some point on AB as the translation target to which A' is moved, and one of the endpoints of AB in the original position as rotation target. Now let A' move along AB (and eventually its constructed extensions) as a directrix line. Then a point on the moving segment traces the infinite extension of AB. To draw a circle, take a straight line OR, and use Postulate 2 to place a copy $O'R'$ of it with O as translation target and some point D as rotation target. Now let D move along a straight line (not through O) as a directrix line. Then the marked point R' on the moving segment $O'R'$ ($= OR'$) traces a (semi)circle.

The dependence on the second postulate for controlling the motions makes Newton's third postulate more restricted than it might seem at first sight. For example, Newton observes (Newton, MP, VII.395), the classical trammel construction of ellipses is not covered by the postulates. In this construction, two intersecting axes AOB and COD are given and fixed, and a line segment EF of fixed length is placed and moved in such a way that each of its endpoints fall on one of the axes. A marked point P on EF traces an ellipse

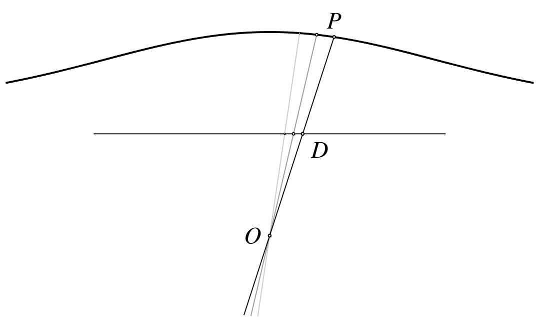

Figure 11. NOrg conchoid. The straight line ODP is moved with D as a translation target moving along a directrix line and the fixed O as rotation target. The endpoint P traces the conchoid.

in the course of this motion. The position of EF for a given E is defined by rotating EF around E until it hits a given *line* (one of the axes), whereas the second postulate only enables us to rotate curves until they hit a given *point*. Newton does not provide a very clear justification for why only the latter and not the former should be admitted in geometry. Indeed, on the contrary, Newton immediately goes on to explicitly "admit nonetheless" that the former type of postulate (the one needed for the trammel) could indeed in principle be accepted, but he argues that since his postulates are sufficient for geometry (for example for generating the same ellipse that the trammel produces) it is better to stick to this minimal set of assumptions.

It is especially remarkable that Newton rejects the trammel construction since this time he is not censoring Descartes but himself. Newton had himself earlier advocated for the use of the trammel as a construction tool (Newton, MP, V.479), presenting it as a superior alternative to Descartes's constructions. This is yet another indication that, at the late stage of his *Geometria*, Newton has reached a much more refined position that supersedes his rash and hasty anti-Cartesian proclamations in his algebra lectures. Indeed, Newton's rejection of the trammel can be related to a systematic vision as we shall now see.

The significance of Postulate 2 in determining the motion of the moving plane can be formulated as follows: the position of the moving configuration at any stage is determined solely by the geometry of the moving configuration and the two base plane points T and R; it does not depend on any other aspects of the geometry of the base plane, and in particular it does not depend on the geometry of the directrix curve. This would no longer be true for the trammel or other constructions where the placement of the moving figure at any given stage depends on entire curves or figures in the stationary base plane (such as the two trammel axes).

This principle is arguably useful and powerful, and is concretely reflected in the algebra of the corresponding parametrisation. To see this, let us consider a parametric description of the NOrg construction of Figures 1, 2, 3, 4. Without loss of generality, we can take the two angular rulers of the NOrg to be rotating about pole points $(0,0)$ and $(c,0)$, and let their angles be given as $\arctan(A)$ and $\arctan(B)$. Let the directrix point D be (s,t), and let $P=(x,y)$ be the "output" tracing point. The angle of intersection of two lines with slopes m_1, m_2 is $\arctan((m_1-m_2)/(1+m_1m_2))$, so

$$A=\frac{\frac{y}{x}-\frac{t}{s}}{1+\frac{ty}{sx}} \quad \text{and} \quad B=\frac{\frac{0-y}{c-x}-\frac{0-t}{c-s}}{1+\frac{(0-t)(0-y)}{(c-s)(c-x)}}$$

In these equations we can eliminate y and solve for x or vice versa. This gives the parametrisation

$$x=\frac{ABc^2t-ABcst-Act^2-Bc^2s+Bcs^2+cst}{ABct+Acs-As^2-At^2-Bcs+Bs^2+Bt^2+ct}$$
$$y=\frac{-ABc^2s+ABcs^2+Acst-Bc^2t+Bcst+ct^2}{ABct+Acs-As^2-At^2-Bcs+Bs^2+Bt^2+ct}$$

(as also noted in (Shkolenok, 1972, 32)). Note that this holds regardless of whether the directrix (given by the relation between s and t) is a straight line or some other curve. This corresponds to the fact that the position of each moving figure is completely determined by the pole point and the directrix point, which correspond to the translation and rotation targets of Newton's Postulate 2.

This makes parametrisations obtained in accordance with Postulate 2 flexible and powerful. They can immediately be used for generalised cases such as Figures 3 and 4 (indeed, I made use of this very convenient fact when plotting my NOrg figures: I used the same parametrisation for all of them, and varied from straight to curved directrices by simply plugging in different expressions for t in terms of s). The same thing happens in the case of the cissoid construction of Figure 12. With pole point $O=(0,0)$, directrix point $D=(s,t)$, and tracing point P halfway between D and $C=(x,y)$, and taking $CD=1$ as unit length, we have the Pythagorean relationships $x^2+y^2+1=t^2+s^2$ and $(s-x)^2+(t-y)^2=1$, which give the parametrisation

$$C=(x,y)=\left(\frac{s^3+st^2-s-\sqrt{s^2t^2+t^4-t^2}}{s^2+t^2},\frac{s^2t+t^3-t+\sqrt{s^4+s^2t^2-s^2}}{s^2+t^2}\right)$$

from which the parametrisation for $P=((x,y)+(s,t))/2$ follows. Again, to derive these equations we have not made any use of the directrix curve. To get the standard cissoid, we take the directrix curve to be the horizontal line $(s,t)=(s,1)$. Then the above parametrisations simplify to $C=\left(\frac{s^3-s}{s^2+1},\frac{2s^2}{s^2+1}\right)$ and hence $P=(\frac{s^3}{s^2+1},\frac{s^2}{s^2+1}+\frac{1}{2})$, which is indeed a standard parametrisation for the cissoid.

In the trammel, by contrast, expressing P in terms of E or F selected as a parametrisation point necessarily involves explicit dependence on the trammel axis configuration, meaning that the construction arguably corresponds to a less generalisable and perhaps less convenient parametrisation.

Newton perhaps appears to have been sensitive to precisely the issue of his postulates corresponding to simple or useful parametrisations, judging by his critique of another construction that does not fall within the scope of his postulates: Descartes's construction of ovals (Figure 9), which Newton criticises as follows.

> This description is foreign to geometry not merely because a thread is employed [which alone is cause for rejection as discussed above], but also because, when some one of the lines in the figure is assumed either in position or length, the finding of the rest from it is a problem whose solution by a thread-stretching would shrink away from geometry. (Newton, MP, VII.387)

In other words, Newton seems to be saying that the oval construction is bad because it is mathematically impractical, specifically because it does not entail a workable parametrisation. "Some one of the lines in the figure assumed either in position or length"

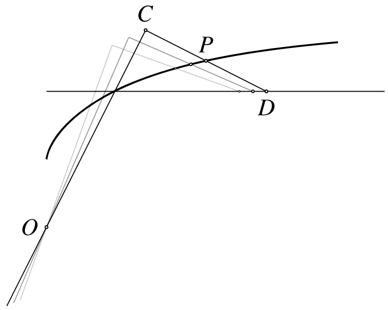

Figure 12. NOrg cissoid. An L-shaped ruler is made to move so that one of its endpoints D moves along a horizontal line while its other leg passes through a fixed point O. The point P (the midpoint of the leg CD) traces a cissoid. (Newton, MP, V.219, V.467, VII.391).

presumably means taking a particular length or angle as a parameter (t or θ) in terms of which all other entities are to be expressed. Many constructions—such as Van Schooten's string constructions of conics or the constructions of Figures 7, 6, 8—are naturally expressible as parametrisations. This means that one can recreate the instantaneous state of the machine corresponding to any specific condition (any length or angle of the various parts of the machine). Namely, by using the given condition to find the corresponding value of the parameter t, and then finding all the rest by filling this specific value of t into the parametrisations for the various points or other entities involved.

In the case of Descartes's oval, the string condition does not translate naturally into a convenient parametrisation, so it cannot be used this way. For example, let $F = (0,0)$, $K = (a,0)$, $G = (b,0)$, $C = (x,y)$, $FE = R$, $EC + CK + KC + CG = L$, and take $FC = t$ as the parameter, and suppose we want to express x in terms of t. The string length condition

$$L = (R - t) + 2\sqrt{(x-a)^2 + y^2} + \sqrt{(x-b)^2 + y^2}$$

and the Pythagorean relation $t^2 = x^2 + y^2$ combined give

$$L = (R - t) + 2\sqrt{a^2 - 2ax + t^2} + \sqrt{b^2 - 2bx + t^2}$$

which when the roots are expanded becomes

$$\left(16(L - R + t)^2\right)\left(a^2 - 2ax + t^2\right) = \left(4a^2 - 8ax - b^2 + 2bx + L^2 - 2LR + 2Lt + R^2 - 2Rt + 4t^2\right)^2.$$

Here a, b, L, R are known constants, and we need to solve for $x(t)$. Since this is arguably impracticable, the geometer has perhaps no other choice than to resort to a "solution by a thread-stretching," which is of course very imperfect. So while some string constructions (such as those of conics) directly encode technically useful parametrisation relationships, the string construction of ovals does not. So this construction is divorced from useful mathematical practice.

Newton's critique of Descartes's curve tracings by strings should of course be contrasted with his own NOrg construction method, which he indeed goes on to discuss almost immediately in the same document. Implicit in Newton's parametrisation critique of Descartes is the insinuation that NOrg performs better in this regard, and Newton is arguably right about this as we saw above.

Newton explicitly points out that various generalised forms of the basic NOrg are included in his postulates (Newton, MP, VII.393). One permissible generalisation is that exemplified in Figures 3 and 4: "if in place of the straight line [the directrix] you employ the conics just now described, the [tracing] point ... will describe innumerable other curves, ... which may in turn be used to replace the conics, and so on indefinitely." The other generalisations mentioned by Newton are allowing "curves already described" as the moving curves, and moving pole points. Both of these principles are used for example in Newton's cissoid construction (Figure 12). As this example indicates, these generalisations are still subject to the Postulate 2 restriction which is in turn reflected in the type of parametrisation that results from the construction, as discussed above.

Altogether, Newton's three postulates correspond to a very clearly delineated class of constructions. Postulate 2 in particular entails a clear and substantive restriction on how the positions of the tracing configurations in these constructions are determined (namely, a given curve can be moved by a translation followed by a rotation, determined by a single translation target point and a single rotation target point respectively). This restriction is reflected in an algebraically clear and unequivocal way in the parametrisations that the constructions give rise to (namely, the position of the tracing point at any stage of the motion can be expressed in terms of moving curves and the translation and rotation target points, without reference to the directrix curve along which the translation or rotation targets are moving). The particular types of constructions and parametrisations obtained in this way are arguably particularly useful and powerful compared to other standard constructions, at least in some notable respects. This perspective can be seen as giving technical teeth to Newton's mature proposal, in contrast with his earlier critiques of Descartes's constructivist program which have been portrayed as being merely "aesthetic" (Guicciardini, 2009, 65), (Kaplan, 2018, 457) and lacking in algebraic definitiveness (Guicciardini, 2009, 74).

It is notable that Newton's generalised NOrg constructions include Descartes's tRmC as a special case (a fact which Newton does not divulge explicitly). If one starts with a V-shaped ruler as in Newton's prototype cases of NOrg and makes the angle between the legs zero, then one gets a Cartesian turning ruler. And Descartes's tRmC and Newton's NOrg are likewise the same in that they both allow the moving curve to be any previously constructed curve moving, as it were, in an independent plane superimposed on a base plane. In Descartes's tRmC, the motion is a rectilinear translation. Newton's NOrg is more general since it allows the translation to be composed with a rotation (an alignment with a fixed point, such as O in Figure 12). But it is easy to obtain any desired rectilinear motion from Newton's principle as well: take the directrix to be a line, take the alignment point on this line, and add a line coinciding with the directrix line to the moving figure. Then Cartesian rectilinear motion is obtained as a special case of Newton's construction principles. Indeed Newton himself arranges rectilinear motion in exactly this way (by adding an extra line to the moving figure) in one of his examples (the "compound line $GDFC$" of (Newton, MP, VII.393): the directrix point guiding the motion is C and the segment DG or dg is needed solely for alignment purposes, to restrict the motion of $GDFC$ to a pure horizontal translation without rotation).

In other words, in the notation of Section 3, Descartes's tRmC can be obtained from Newton's postulates as follows. Let Q be a point moving along a straight-line directrix. The position of the ruler OQ is determined by taking the fixed point O as the translation target and the moving point Q as the rotation target (in the sense of Newton's Postulate 2). Consider the intersection Q' of the curve C with the line QQ' of motion of Q. Extend the curve C by adding the line QQ' to it, making a composite figure that we may call C^+. The position of C^+ is determined by taking the moving point Q as the translation target and the initial position of Q as the rotation target. Then C' is the curve traced by the intersection P of OQ and C in the course of the motion of Q.

This gives us reason to qualify the view that "Newton found all means for curve construction proposed in [Descartes's] *Géométrie* to be unacceptable" (Guicciardini, 2009, 103). Descartes's paradigm construction method, the tRmC, is indeed acceptable by Newton's own standards, in the sense that it is subsumed by his own postulates. (The conclusion that Newton rejected Descartes's tRmC appears to be based largely on his statement (already discussed above) that "the postulate of moving any lines whatever according to appointed laws and by their intersections describing new lines is to be rejected for many reasons" (Newton, MP, VII.295). Both (Guicciardini, 2009, 104) and (Kaplan, 2018, 456) take this to be a rejection of a postulate used by Descartes, which presumably means that they view tRmC as being based on this postulate. However, Descartes's tRmC is vastly more restricted than the extremely general postulate criticised by Newton. Even Descartes's most general statements are much more restricted than the postulate Newton criticises, due to Descartes's single-degree-of-freedom requirement that disqualifies quadratrices and spirals.)

Thus Newton cannot and does not criticise Descartes's tRmC on acceptability grounds, for to do so would be to criticise his own proposal which includes the tRmC as a special case. Newton can (and arguably does), however, criticise Descartes's tRmC on the grounds that it lacks generality. This is plausibly the point of Newton's critique of some unnamed "others" who "want all lines which are exact to be considered as geometrical, but are as yet unwilling to postulate the description of all" (Newton, MP, VII.295). In other words, Descartes wanted all algebraic curves to be considered legitimate, but his only systematic method of securing legitimacy of curves of any degree by construction—the tRmC—is insufficient to generate them all. The tRmC even fails to generate all conics, as noted above: a fact that Newton may likely have been well aware of. If this is Newton's intended point, then this is a counterexample to the statement by (Bos, 2001, 405) that Descartes was not criticised by 17th-century readers for failing to give a general tracing method for any algebraic curve.

In the end, Newton's sophisticated and well-crafted final position on the role of constructions in geometry—not to be conflated with his immature earlier writings—is a work of considerable conscientiousness and virtuosity that deserves more respect than it has hitherto received. Newton's foray into the philosophy of constructions seems orthogonal to the bulk of his mathematical practice and should hardly be seen as a confession of his innermost beliefs. Perhaps he took up the subject mostly for the sake of argument, as it were, having been led to it inadvertently because of his veneration of the ancients and especially his aversion to Descartes rather than any notable intrinsic interest in constructivism of his own. But, if so, Newton did constructivism better out of spite than many of his contemporaries did out of primary conviction.

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