On rationally connected varieties
over C_1 fields of characteristic 0

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We use birational geometry to show that the existence of rational points on proper rationally connected varieties over fields of characteristic 0 is a consequence of the existence of rational points on terminal Fano varieties. We discuss several consequences of this result, especially in relation to the C_1-conjecture. We also provide evidence that supports the conjecture in dimension 3 for C_1 fields of characteristic 0.

1. Introduction

A field k is called C_1 (or quasi algebraically closed) if every hypersurface of degree at most n in $\mathbb{P}^n_k$ has a k-rational point. Quasi algebraically closed fields were introduced by Artin and first extensively studied by Tsen (see [Ding et al. 1999]) and Lang [1952]. Smooth hypersurfaces of degree at most n in $\mathbb{P}^n$ are Fano and rationally chain connected [Kollár et al. 1992; Campana 1992]. In characteristic 0 they are also rationally connected. Hence, it is natural to study rational points on rationally (chain) connected varieties over C_1 fields. In an unpublished paper of 2000, Lang formulated the following conjecture, also known as the C_1-conjecture.

Conjecture 1.1 (Lang 2000). Every smooth, proper, separably rationally connected variety over a C_1 field has a rational point.

At the time it was known that smooth proper rational curves and surfaces have points over C_1 fields [Colliot-Thélène 1987]. Soon after it was formulated, the conjecture was proven to hold for the following C_1 fields: finite fields [Esnault 2003], function fields of curves over algebraically closed fields [Campana et al. 2002; Graber et al. 2003; de Jong and Starr 2003], fields of formal power series over algebraically closed fields [Colliot-Thélène 2011]. It is still open for the maximal unramified extensions $\mathbb{Q}_p^{nr}$ of p-adic fields. Positive evidence in support of the conjecture for the latter class of fields is given in [Duesler and Knecht 2017], where it is proven that for sufficiently large primes p, smooth projective rationally connected varieties over $\mathbb{Q}_p^{nr}$ with fixed Hilbert polynomial have a rational point. As a consequence of our first main result we manage to improve [Duesler and Knecht 2017]: we replace the dependence on the Hilbert polynomial by dependence only on the dimension, and we remove the smoothness and properness assumptions.

The goal of this paper is to use birational geometry in characteristic 0 to reduce the C_1-conjecture to the case of Fano varieties and to provide evidence for the existence of rational points on rationally connected varieties of dimension 3. Along the way, we also show that for proper rationally connected varieties of fixed dimension over an arbitrary field of characteristic 0, there is a uniform upper bound for the degree of the minimal field extensions where the set of rational points becomes nonempty.

1A. *Birational geometry.* The first main result of this paper reduces the C_1-conjecture for C_1 fields of characteristic 0 to the following conjecture.

Conjecture 1.2. Every terminal $\mathbb{Q}$-factorial Fano variety of Picard rank 1 over a C_1 field of characteristic 0 has a rational point.

More precisely, birational geometry and induction on dimension are used to prove the following statement.

Theorem 1.3. *Let k be a field of characteristic* 0. *For every positive integer n, the following statements are equivalent*:

(i) *Every smooth proper rationally connected k-variety of dimension $\leq n$ has a k-point.*

(ii) *Every terminal $\mathbb{Q}$-factorial Fano variety of dimension $\leq n$ and Picard rank* 1 *over k has a k-point.*

The proof of Theorem 1.3 rests upon the birational invariance of rational connectedness and the fact that the existence of rational points is a birational invariant among proper smooth varieties [Nishimura 1955]. We recall that the use of the minimal model program in dimension ≥ 3 produces singular birational models in general. Results on degenerations of rationally connected varieties [Hogadi and Xu 2009] and induction on dimension are used to transfer the existence of rational points among birationally equivalent varieties with mild singularities. The restriction on the characteristic is due to the use of the minimal model program, of resolution of singularities and of [Hogadi and Xu 2009]. None of these are known to hold in positive characteristic, except for some results in low dimension.

Boundedness of terminal Fano varieties [Birkar 2021] assures that there are only finitely many deformation families of terminal Fano varieties in each fixed dimension over algebraically closed fields of

characteristic 0. As an application of this fact we prove the following corollary of Theorem 1.3, which gives a generalization of [Duesler and Knecht 2017].

Corollary 1.4. *For every positive integer n, there exists a finite set $S(n)$ of prime numbers such that for all prime numbers $p \notin S(n)$, every rationally connected variety of dimension n over $\mathbb{Q}_p^{\mathrm{nr}}$ has a $\mathbb{Q}_p^{\mathrm{nr}}$-point.*

During a workshop in Edinburgh in November 2018, it was brought to the author's attention that the strategy of the proof of Theorem 1.3 is similar to the one used by Prokhorov and Shramov [2016], who prove, among other things, a uniform bound depending only on dimension for the indices of subgroups of finite groups of automorphisms such that the subgroup acts with fixed points. The corresponding statement for rational points is the following theorem, which is proven by combining the proof of Theorem 1.3 with boundedness of terminal Fano varieties.

Theorem 1.5. *For every positive integer n, there exists a positive integer $d(n)$ such that for every field k of characteristic 0 and every proper rationally connected k-variety X of dimension n, there exists a field extension k' of k of degree $\leq d(n)$ such that $X(k') \neq \varnothing$.*

For every positive integer n, let $\tilde{d}(n)$ be the smallest positive integer such that for every field k of characteristic 0 and every proper rationally connected k-variety X of dimension n, there exists a field extension k' of k of degree $\leq \tilde{d}(n)$ such that $X(k') \neq \varnothing$. We prove the following effective bounds.

Theorem 1.6. $$\tilde{d}(1) = 2, \quad \tilde{d}(2) = 6, \quad \tilde{d}(3) \leq 144.$$

The proof of Theorem 1.5 gives the bound

$$\tilde{d}(n) \leq \tilde{d}(n-1) \max\{d'(n), \max_{1 \leq m \leq n-1} \tilde{d}(m)\tilde{d}(n-m)\},$$

where $d'(n)$ is an upper bound for the degree of the minimal field extensions where terminal Fano varieties of dimension n acquire rational points. The existence of $d'(n)$ is a consequence of boundedness of terminal Fano varieties. The effective computability of $d'(n)$ depends on the classification of the Fano varieties that appear in Theorem 1.3(ii). In dimension 4 or higher, the classification is not complete. In dimension 2 the inequality only gives an upper bound $\tilde{d}(2) \leq 12$. The sharp bound $\tilde{d}(2) = 6$ is obtained in Section 4 using the Enriques–Iskovskih–Manin classification of surfaces over nonclosed fields [Iskovskikh 1979]. In Section 5 we recall the classification of the terminal Fano threefolds that appear in Theorem 1.3(ii), and we prove that $d'(3) \leq 24$. We do not expect the bound for $\tilde{d}(3)$ to be sharp.

1B. *Classification of Fano varieties.* Further evidence for Conjecture 1.1 that can be found in the literature is the fact that the following rationally connected varieties have rational points over C_1 fields that admit normic forms of arbitrary degree: complete intersections in weighted projective spaces (see [Kollár 1996] for example) and hypersurfaces of split toric varieties [Guilbot 2014]. We refer to [Lang 1952] for the condition about normic forms and we recall that it is satisfied by fields that admit finite extensions of every degree. In Section 7 we prove similar results for varieties that after extension of the base field to the algebraic closure belong to the following classes: toric varieties, complete intersections in products of weighted projective spaces, some Fano cyclic coverings, some special Fano varieties of dimension 3.

As a consequence of Theorem 1.3 combined with boundedness of terminal Fano varieties [Birkar 2021], the verification of Conjecture 1.1 is reduced to finding rational points on finitely many geometric families of Fano varieties in each given dimension. In general, the $\mathbb{Q}$-factoriality and the Picard rank 1 conditions in Conjecture 1.2 and in Theorem 1.3 are not preserved under base field extensions. So in order to verify Conjecture 1.2 one needs to find rational points also on some Fano varieties that are not geometrically $\mathbb{Q}$-factorial and on some Fano varieties of geometric Picard rank > 1.

We recall that the Gorenstein index of a log terminal Fano variety X is the smallest positive integer $I(X)$ such that $I(X)K_X$ is Cartier. The $\mathbb{Q}$-Fano index of X is then the largest positive integer $q(X)$ such that $-I(X)K_X$ is linearly equivalent to $q(X)H$ for some Cartier divisor H on X. We write $r(X) = q(X)/I(X)$. In general these indices do not need to be invariant under field extension. In Section 5 we show that $r(X) = r(X_{\bar{k}})$ for X defined over a C_1 field of characteristic 0 with algebraic closure $\bar{k}$. In Section 8, we use the classification results of Fujita [1982] and Sano [1996] to find rational points on Fano varieties of large index.

Theorem 1.7. *If X is a terminal Fano variety of dimension $n \geq 2$ and index $r(X) > n - 2$ over a C_1 field k of characteristic 0 that admits normic forms of arbitrary degree, then $X(k) \neq \varnothing$.*

The classification of terminal Fano varieties is far from known. However, the current literature on terminal Gorenstein Fano threefolds [Mukai 1995b; Prokhorov 2013b] makes the following result possible.

Theorem 1.8. *Let X be a Gorenstein terminal $\mathbb{Q}$-factorial Fano threefold over a C_1 field k of characteristic 0 that admits normic forms of arbitrary degree. Assume that $\operatorname{rk} \operatorname{Pic}(X) = 1$. Let $g(X) = \frac{1}{2}(-K_X)^3 + 1$. Let $\bar{k}$ be an algebraic closure of k. Then $X(k) \neq \varnothing$ except, possibly, in the following cases:*

(i) *$r(X) = 1$, $\operatorname{rk} \operatorname{Pic}(X_{\bar{k}}) = 1$ and $g(X) \in \{6, 7, 8, 9, 10\}$;*

(ii) *$r(X) = 1$, $\operatorname{rk} \operatorname{Pic}(X_{\bar{k}}) = 1$ and $|-K_{X_{\bar{k}}}|$ has a movable decomposition.*

The varieties in part (i) of the theorem are complete intersections of sections of certain vector bundles on some Grassmannians. To the best of the author's knowledge, Fano varieties as in (ii) are not classified. At the beginning of Section 5A5 we recall the definition of movable decomposition from [Mukai 1995b].

To date there is no complete classification of non-Gorenstein terminal Fano threefolds of index $r \leq 1$. There is, however, a classification of the possible configurations of non-Gorenstein singularities for geometrically $\mathbb{Q}$-factorial non-Gorenstein terminal Fano threefolds of geometric Picard rank 1. See the [Graded Ring Database]. Such Fano varieties X are studied using a Fano index defined as the largest positive integer $f(X)$ such that $-K_X$ is linearly equivalent to $f(X)A$ for some Weil divisor A on X [Suzuki 2004]. In Section 9B we prove that if $f(X_{\bar{k}}) \geq 5$, the variety has rational points over C_1 fields of characteristic 0. We also discuss to what extent the same proof applies to varieties of lower Fano index.

To the author's knowledge there is no classification of non-Gorenstein terminal Fano threefolds of geometric Picard rank > 1 except for [Sano 1995]. The varieties in [Sano 1995] have rational points over C_1 fields of characteristic 0, by Theorem 9.1.

We recall that the condition about the existence of normic forms of arbitrary degree is satisfied by the fields $\mathbb{Q}_p^{\mathrm{nr}}$, as they admit finite extensions of every degree. Their algebraic extensions, however, do not need to satisfy the condition on normic forms, but if Conjecture 1.1 holds for $\mathbb{Q}_p^{\mathrm{nr}}$, then it holds also for all its algebraic extensions, as Weil restriction under finite separable field extensions preserves rational connectedness, smoothness and properness, and the Weil restriction of a variety X from a finite extension k to $\mathbb{Q}_p^{\mathrm{nr}}$ has a $\mathbb{Q}_p^{\mathrm{nr}}$-point if and only if X has a k-point.

Theorem 1.3 and Corollary 1.4 are proven in Section 3, Theorems 1.5 and 1.6 are proven in Section 6, Theorem 1.7 is proven in Section 8, and Theorem 1.8 is proven in Section 9.

2. Notation and basic properties

Let k be a field of characteristic 0, and $\bar{k}$ an algebraic closure of k. We denote by $\mathrm{Br}(k)$ the Brauer group of k.

For us, a k-variety is a separated scheme of finite type over k. We use the words curve, surface, threefold to denote a variety of dimension 1, 2, 3, respectively.

We say that a k-variety X is rationally connected if $X_{\bar{k}}$ is integral and rationally connected in the sense of [Kollár 1996, §IV.3]. We recall that if X is rationally connected, $X \dashrightarrow Y$ is a dominant rational map and Y is proper, then Y is rationally connected. In particular, being rationally connected is a birational invariant of proper varieties.

Weil divisors on an integral normal variety X correspond to Cartier divisors on its smooth locus U. The reflexive sheaf $\mathcal{O}_X(D)$ associated to a Weil divisor D on X is the pushforward of $\mathcal{O}_U(D|_U)$ under the inclusion $U \subseteq X$. A canonical divisor K_X on X is the Weil divisor corresponding to a canonical divisor on U. We say that X is Gorenstein if its canonical divisor K_X is Cartier, and $\mathbb{Q}$-Gorenstein if a positive multiple of K_X is Cartier. We say that X is $\mathbb{Q}$-factorial if every Weil divisor on X has a positive multiple that is Cartier. We recall that $\mathbb{Q}$-factoriality is not invariant under field extension (see [Gongyo et al. 2019, Remark 2.7]).

We refer to [Kollár 2013] for the singularities of the minimal model program. We recall that (log) terminal varieties are normal and $\mathbb{Q}$-Gorenstein by definition. We also recall that the notion of (log) terminal singularities is invariant under separable field extension.

A (log) terminal variety X is Fano if $-K_X$ has a positive multiple which is Cartier and ample. We refer to [Iskovskikh and Prokhorov 1999] for the theory of Fano varieties.

We recall that, given a field k and an algebraic closure $\bar{k}$ of k, the k-forms of a k-variety Y are k-varieties Y' such that $Y'_{\bar{k}}$ is isomorphic to $Y_{\bar{k}}$, and the k-models of a $\bar{k}$-variety Z are k-varieties Z' such that $Z'_{\bar{k}}$ is isomorphic to Z.

3. Reformulation of the C_1-conjecture

In this section we prove that over a C_1 field of characteristic 0 the C_1-conjecture is equivalent to Conjecture 1.2. We start by observing that in characteristic 0 the C_1-conjecture is equivalent to both the

stronger version obtained by removing the smoothness assumption and to the weaker version obtained by replacing the properness assumption by projectivity, as the following lemma shows.

Lemma 3.1. *Let k be a field of characteristic* 0 *and n a positive integer. If all smooth projective rationally connected k-varieties of dimension n have k-points, then every proper rationally connected k-variety of dimension n has a k-point.*

Proof. Let X be a proper rationally connected k-variety of dimension n. Since X is of finite type over k, there exists an open subset U of X which is isomorphic to an affine k-variety. Since resolutions of singularities exist for all varieties in characteristic 0 by [Hironaka 1964a; 1964b], we can find a smooth projective compactification $\overline{U}$ of U over k. Then $\overline{U}$ is rationally connected because it is birationally equivalent to X. So $\overline{U}(k) \neq \varnothing$ by assumption, and we conclude that $X(k) \neq \varnothing$ by [Nishimura 1955]. $\square$

If the base field is large in the sense of [Pop 1996, Proposition 1.1] we can remove also the properness assumption.

Lemma 3.2. *Let k be a large field of characteristic* 0 *and n a positive integer. If all smooth projective rationally connected k-varieties of dimension n have k-points, then every rationally connected k-variety of dimension n has a k-point.*

Proof. For every rationally connected k-variety X we can find a smooth projective compactification X' of a smooth affine open subset of X as in the proof of Lemma 3.1. Since $X'(k) \neq \varnothing$ by assumption, the set $X'(k)$ is dense in X' because k is large, and therefore $X(k) \neq \varnothing$. $\square$

In the proof of Theorem 1.3 we use the following form of the minimal model program, which is a consequence of [Birkar et al. 2010, Corollary 1.3.3]. See [Prokhorov 2013c, Proposition 2.3] for a proof.

Proposition 3.3. *Let k be a field of characteristic* 0. *Let X be a smooth projective rationally connected k-variety. Then there exist a birational map $X \dashrightarrow X'$ and a projective dominant morphism $f : X' \to Y$ such that X' is a projective $\mathbb{Q}$-factorial terminal k-variety, Y is a projective $\mathbb{Q}$-factorial log terminal k-variety, $\dim Y < \dim X'$, $\operatorname{rk} \operatorname{Pic}(Y) = \operatorname{rk} \operatorname{Pic}(X') - 1$, and the generic fiber of f is a terminal Fano variety.*

We can now prove the main theorem of this section.

Proof of Theorem 1.3. For every positive integer n, the implication (i) $\Rightarrow$ (ii) holds by Lemma 3.1 because every terminal Fano variety is rationally connected by [Zhang 2006].

To prove the reverse implication, we fix a positive integer n and assume that (ii) holds. Let X be a smooth proper rationally connected k-variety of dimension $\leq n$. We prove that X has a k-rational point by induction on the dimension of X. If $\dim X = 0$, then $X = \operatorname{Spec} k$. Assume that $\dim X \geq 1$ and that every smooth proper rationally connected k-variety of dimension $< \dim X$ has a k-point. By Lemma 3.1, we can assume, without loss of generality, that X is projective. Let $f : X' \to Y$ be the fibration with X' birationally equivalent to X provided by Proposition 3.3.

We first prove that $X'(k) \neq \varnothing$. If $\dim Y = 0$, then X' is a $\mathbb{Q}$-factorial terminal Fano variety of Picard rank 1. So $X'(k) \neq \varnothing$ by (ii). If $\dim Y > 0$, then Y is rationally connected and has dimension $< \dim X$.

Hence, $Y(k) \neq \varnothing$ by the induction hypothesis combined with Lemma 3.1. Let $y \in Y(k)$. By [Hogadi and Xu 2009, Theorem 1.2], the fiber of f over y contains a projective rationally connected k-subvariety Z. Since $\dim Y > 0$, $\dim Z < \dim X' = \dim X$. So $Z(k) \neq \varnothing$ by the induction hypothesis combined with Lemma 3.1. Then $X'(k) \neq \varnothing$.

Let $x \in X'(k)$. Let $h : X'' \to X'$ be a resolution of singularities. By [Hogadi and Xu 2009, Theorem 1.3], the fiber of h over x contains a rationally connected k-subvariety W. Since h is a proper birational morphism, W is a proper rationally connected k-variety with $\dim W < \dim X'' = \dim X$. Therefore $W(k) \neq \varnothing$ by the induction hypothesis combined with Lemma 3.1. Hence, $X''(k) \neq \varnothing$, and $X(k) \neq \varnothing$ by [Nishimura 1955]. $\qquad\square$

3A. *A consequence of boundedness of Fano varieties.* We show that Birkar's boundedness of Fano varieties [2021] implies boundedness of Hilbert polynomials for terminal Fano varieties over nonclosed fields of characteristic 0.

Proposition 3.4. *Let k be a field of characteristic 0 and n a positive integer. Then there exist a positive integer N and finitely many polynomials $f_1, \ldots, f_s \in \mathbb{Q}[t]$ such that for every terminal Fano variety X of dimension n, there exists an embedding $X \subseteq \mathbb{P}_k^N$ such that X has Hilbert polynomial f_i for some $i \in \{1, \ldots, s\}$.*

Proof. Let $\bar{k}$ be an algebraic closure of k. By [Birkar 2021, Theorem 1.1], there are finitely many projective morphisms of $\bar{k}$-varieties, say $\{V^i \to T^i\}_{i \in I}$ for a finite set I, such that for every terminal Fano k-variety X of dimension n, there exists $i \in I$, a point $t \in T^i(\bar{k})$ and an isomorphism $V_t^i \cong X_{\bar{k}}$ of $\bar{k}$-varieties. Up to replacing each T^i by a suitable finite stratification, we can assume that T^i is smooth, $V^i \to T^i$ is flat, $K_{V^i}|_{V_t^i} = K_{V_t^i}$ for all $t \in T^i(\bar{k})$, and there exists $m \in \mathbb{Z}_{>0}$ such that $-mK_{V^i}$ is Cartier and relatively very ample over T^i for all $i \in I$. We can use the complete linear system $|-mK_{V^i}|$ to embed V^i into a projective space $\mathbb{P}_{T^i}^{N_i}$ over T^i for all $i \in I$, where N_i is the projective dimension of the linear system $|-mK_{V^i}|$ for all $i \in I$. Then for every $i \in I$ the Hilbert polynomial f_i of V_t^i with respect to the projective embedding given by $|-mK_{V_t^i}|$ is independent of the choice of $t \in T^i$ by [Hartshorne 1977, Corollary III.9.13]. Let s be the cardinality of I and write $I = \{1, \ldots, s\}$. Let $N = \max_{1 \leq i \leq s} N_i$. Then for every terminal Fano variety X of dimension n over k, there exists $i \in \{1, \ldots, s\}$ such that X has Hilbert polynomial f_i under the embedding $X \subseteq \mathbb{P}_k^{N_i}$ defined by the complete linear system $|-mK_X|$. Choose a linear embedding $\mathbb{P}_k^{N_i} \subseteq \mathbb{P}_k^N$. Then X has Hilbert polynomial f_i under the induced embedding $X \subseteq \mathbb{P}_k^N$. $\qquad\square$

3B. *Application to rationally connected varieties over $\mathbb{Q}_p^{\mathrm{nr}}$.* We apply the previous results to prove Corollary 1.4.

Proposition 3.5. *Let $f \in \mathbb{Q}[t]$ and $N \in \mathbb{N}$. Then there exists a finite set $S(f, N)$ of prime numbers such that for every prime number $p \notin S(f, N)$, every projective rationally connected variety $X \subseteq \mathbb{P}_{\mathbb{Q}_p^{\mathrm{nr}}}^N$ with Hilbert polynomial f has a $\mathbb{Q}_p^{\mathrm{nr}}$-point.*

Proof. Replace [Kollár 1996, Theorem IV.3.11] by [de Fernex and Fusi 2013, Proposition 2.6] in the proof of [Duesler and Knecht 2017, Theorem 1.3]. $\qquad\square$

Proof of Corollary 1.4. Let n be a positive integer. Combining Propositions 3.4 and 3.5 we obtain a finite set $S(n)$ of prime numbers such that for every prime number $p \notin S(n)$ every terminal Fano variety of dimension n over $\mathbb{Q}_p^{\mathrm{nr}}$ has a $\mathbb{Q}_p^{\mathrm{nr}}$-point. Then by Theorem 1.3 every smooth proper rationally connected variety of dimension n over $\mathbb{Q}_p^{\mathrm{nr}}$ with $p \notin S(n)$ has a $\mathbb{Q}_p^{\mathrm{nr}}$-point. We conclude by Lemma 3.2, as $\mathbb{Q}_p$ and $\mathbb{Q}_p^{\mathrm{nr}}$ are large fields by the Implicit Function Theorem over local fields by [Serre 1992, p. 73] and [Pop 1996, Proposition 1.2] respectively. $\square$

4. Del Pezzo surfaces

In this section we collect some properties related to the existence of rational points on rationally connected curves and surfaces. We recall that terminal varieties of dimension ≤ 2 are smooth [Kollár and Mori 1998, Corollary 5.18], and that the birational classification of smooth rationally connected curves and surfaces is well understood.

Remark 4.1. We recall that isomorphism classes of Severi–Brauer varieties of dimension n over k are in bijection with $(n+1)$-torsion elements of $\mathrm{Br}(k)$ [Gille and Szamuely 2006, Theorem 5.2.1], and the projective space corresponds to the neutral element. Moreover, a Severi–Brauer variety over k is a projective space if and only if it has a k-rational point [Gille and Szamuely 2006, Theorem 5.1.3]. In particular, a Severi–Brauer variety of dimension n acquires a rational point over a base field extension of degree at most $n + 1$.

Smooth curves are rationally connected if and only if they are Fano. Hence, terminal rationally connected curves are precisely the smooth conics.

Lemma 4.2 [Gille and Szamuely 2006, §1.3, Example 5.2.4]. *For a field k, the following are equivalent*:

(1) $\mathrm{Br}(k)$ *has nontrivial 2-torsion.*

(2) *There is a conic C over k with $C(k) = \varnothing$.*

Smooth surfaces are rationally connected if and only if they are rational. Hence, terminal rationally connected surfaces are rational conic bundles or del Pezzo surfaces [Iskovskikh 1979].

Lemma 4.3. *Let X be a rational conic bundle surface over a field k. Then*:

(i) *There exists a finite extension k' of k of degree at most 4 such that X has a k'-point.*

(ii) *There exists a rational conic bundle over $\mathbb{Q}$ that has no rational points on any finite extension of $\mathbb{Q}$ of degree at most 3.*

Proof. The conic bundle structure is given by a morphism $f : X \to C$ where C and the fibers of f are conics. Then C acquires a rational point over a suitable quadratic extension of k and the fiber of f over that point acquires a rational point over a suitable further quadratic extension. Moreover, given two conics C_1 and C_2 without rational points over $\mathbb{Q}$ and with distinct splitting fields, for example, $C_1 = \{x^2 + y^2 + z^2 = 0\}$ and $C_2 = \{x^2 + 2y^2 + 5z^2 = 0\}$, by [Lam 2005, Example III.2.13], then $C_1 \times C_2$ has no rational points on any finite extension of $\mathbb{Q}$ of degree ≤ 3. $\square$

Del Pezzo surfaces are the Fano varieties of dimension 2, and they are classified according their degree, i.e., the self-intersection number of the canonical class, which is an integer between 1 and 9.

Lemma 4.4. *Let X be a del Pezzo surface of degree d over a field k. Then*:

(i) *X acquires a rational point over a suitable finite extension of k of degree at most d.*

(ii) *If $d = 9$, then X acquires rational points over a suitable finite extension of k of degree at most 3.*

(iii) *If $X_{\bar{k}} \cong \mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}}$, then X acquires rational points over a suitable finite extension of k of degree at most 4.*

(iv) *If $d = 5$, or $d = 7$, or $d = 8$ and $X_{\bar{k}} \not\cong \mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}}$, then X is k-rational and $X(k) \neq \varnothing$.*

(v) *There exists a del Pezzo surface of degree 6 over $\mathbb{Q}$ that has no rational points on any finite extension of $\mathbb{Q}$ of degree smaller than 6.*

Proof. Part (i) is a consequence of the fact that the base locus of the anticanonical linear system has dimension at most 0. In (ii), X is a Severi–Brauer surface, and we conclude by Remark 4.1. Part (iii) follows from Lemma 4.3 if $\mathrm{rk}\,\mathrm{Pic}(X) = 2$, and from Lemma 5.10 if $\mathrm{rk}\,\mathrm{Pic}(X) = 1$. Part (iv) follows from [Swinnerton-Dyer 1972; Skorobogatov 1993] for $d = 5$, and from [Manin 1966, Theorem (3.7)] together with [Várilly-Alvarado 2013, §1.4] for the other cases.

For (v), let $K = \mathbb{Q}(\sqrt{29})$ and $L = \mathbb{Q}(\zeta_7 + \zeta_7^{-1})$, where ζ_7 is a primitive 7-th root of unity. Let E be the smallest number field containing both K and L. Then K, L and E are cyclic Galois extensions of $\mathbb{Q}$ of degree 2, 3 and 6, respectively. Let $b = \frac{1}{2}(\sqrt{29} - 5)$ and $q = (\zeta_7 + \zeta_7^{-1})^2 - 2$. For $F \in \{K, L, \mathbb{Q}\}$ we denote by $N_{E/F}$ the norm of E over F and by $\mathrm{Br}(E/F)$ the Brauer group of F relative to E. Since the extensions are cyclic, by [Gille and Szamuely 2006, Corollary 4.4.10] we can write $\mathrm{Br}(E/F) = F^\times / N_{E/F}(E^\times)$. Under this identification, the corestriction maps $\mathrm{Br}(E/K) \to \mathrm{Br}(E/\mathbb{Q})$ and $\mathrm{Br}(E/L) \to \mathrm{Br}(E/\mathbb{Q})$ are induced by the norms $N_{K/\mathbb{Q}}$ and $N_{L/\mathbb{Q}}$ respectively; see [Elman et al. 2008, Example 99.6]. Let $B \in \mathrm{Br}(E/K)$ be the class of b and $Q \in \mathrm{Br}(E/L)$ be the class of q. Computations (e.g., via the software [SageMath]) show that

$$b \notin N_{E/K}(E^\times), \quad q \notin N_{E/L}(E^\times), \quad N_{K/\mathbb{Q}}(b), N_{L/\mathbb{Q}}(q) \in N_{E/\mathbb{Q}}(E^\times).$$

Let X be the del Pezzo surface of degree 6 associated to (B, Q, E) by [Blunk 2010, Theorem 3.4]. By [Blunk 2010, Corollary 3.5], X has a rational point on an extension k' of $\mathbb{Q}$ if and only if both B and Q are split over k'. This happens if and only if $E \subseteq k'$, which implies $[k' : \mathbb{Q}] \geq 6$. $\qquad\square$

5. Fano threefolds

We recall the classification of the terminal Fano threefolds that are mentioned in Conjecture 1.2 and in Theorem 1.3(ii), and we investigate some of their properties related to the existence of rational points for the proof of Theorem 1.6.

Unless stated otherwise, in this section X denotes a terminal Fano threefold over a field k of characteristic 0. We distinguish between Gorenstein (for example, smooth) and non-Gorenstein varieties. We recall

from Section 1B that Fano varieties are classified according to the index $r(X_{\bar{k}})$, which is a positive rational number up to $\dim X + 1$; see [Iskovskikh and Prokhorov 1999, Corollary 2.1.13]. In the Gorenstein case, $r(X_{\bar{k}})$ is a positive integer. We denote by H a fundamental divisor of $X_{\bar{k}}$; i.e., $H \in \mathrm{Pic}(X_{\bar{k}})$ such that $-K_{X_{\bar{k}}} = r(X_{\bar{k}})H$.

The classification of Fano threefolds uses the following invariants:

- the geometric Picard rank $\mathrm{rk}(\mathrm{Pic}(X_{\bar{k}}))$;

- the index $r(X_{\bar{k}})$;

- the degree H^3, i.e., the top self-intersection of the fundamental divisor H.

We recall that $(-K_{X_{\bar{k}}})^3 = r(X_{\bar{k}})^3 H^3$.

We consider first Gorenstein Fano threefolds, which are well studied and mostly classified, while the classification of the non-Gorenstein ones is very far from being complete.

Lemma 5.1. *Let X be a Gorenstein terminal Fano threefold over a field k of characteristic 0.*

(i) *If the base locus of $|-K_X|$ is nonempty, then $r(X_{\bar{k}}) = 1$ and X acquires a rational point over a suitable base field extension of degree at most 2.*

(ii) *If $|-K_X|$ is basepoint-free, then X acquires a rational point over a suitable base field extension of degree at most $(-K_X)^3$.*

Proof. Part (i) follows from the fact that if nonempty, the base locus of $|-K_X|$ either consists of one k-point or it is a smooth conic by [Shin 1989, Theorem 0.5]. Part (ii) is a consequence of Bertini's theorem. $\square$

The rest of the section is devoted to improving the bound in Lemma 5.1(ii). We start by collecting a few facts about indices of Fano varieties.

Lemma 5.2. *Let X be a log terminal Fano variety over a field k of characteristic 0. Let $\bar{k}$ be a separable closure of k. Let $H \in \mathrm{Pic}(X_{\bar{k}})$ such that $-K_{X_{\bar{k}}} = r(X_{\bar{k}})H$. Then $H \in \mathrm{Pic}(X_{\bar{k}})^{\mathrm{Gal}(\bar{k}/k)}$. Moreover,*

(i) *$I(X) = I(X_{\bar{k}})$ and $q(X) \mid q(X_{\bar{k}})$;*

(ii) *if $\mathrm{Br}(k) = 0$ or if $X(k) \neq \varnothing$, then $r(X) = r(X_{\bar{k}})$.*

Proof. Let $q := q(X_{\bar{k}})$ and $I := I(X_{\bar{k}})$, so that $r(X_{\bar{k}}) = q/I$. Since the canonical divisor class is invariant under field extension, IK_X is a Cartier divisor class on X. Thus $I(X) = I$ and $qH = -IK_X \in \mathrm{Pic}(X)$. Let $g \in \mathrm{Gal}(\bar{k}/k)$. Since $qH = gqH = qgH$, the element $H - gH$ of $\mathrm{Pic}(X_{\bar{k}})$ is torsion. But $\mathrm{Pic}(X_{\bar{k}})$ is free by [Iskovskikh and Prokhorov 1999, Proposition 2.1.2]. Hence, $gH = H$. Thus H is invariant under the action of $\mathrm{Gal}(\bar{k}/k)$ on $\mathrm{Pic}(X_{\bar{k}})$. The divisibility condition $q(X) \mid q$ comes from the definition of $q(X)$ and $q(X_{\bar{k}})$ and the fact that $\mathrm{Pic}(X_{\bar{k}})$ is free. If $\mathrm{Br}(k) = 0$ or $X(k) \neq \varnothing$, the exact sequence [Colliot-Thélène and Sansuc 1987, (1.5.0)] gives $\mathrm{Pic}(X) = \mathrm{Pic}(X_{\bar{k}})^{\mathrm{Gal}(\bar{k}/k)}$. $\square$

Lemma 5.3. *Let X be a projective geometrically integral variety over a field k. Let $\bar{k}$ be a separable closure of k. Let $H \in \mathrm{Pic}(X_{\bar{k}})^{\mathrm{Gal}(\bar{k}/k)}$. Then there exists a field extension k'/k of degree at most $\dim_{\bar{k}} H^0(X_{\bar{k}}, \mathcal{O}_{X_{\bar{k}}}(H))$ such that $H \in \mathrm{Pic}(X_{k'})$.*

Proof. By [Kollár 2016, (7.4), Aside 32], the exact sequence [Kollár 2016, Proposition 69] associates to H a Severi–Brauer k-variety P of dimension $\dim_{\bar{k}} H^0(X_{\bar{k}}, \mathcal{O}_{X_{\bar{k}}}(H)) - 1$ such that P splits over a finite extension k' of k contained in $\bar{k}$ if and only if $H \in \text{Pic}(X_{k'})$. Then we conclude by Remark 4.1. □

5A. *Gorenstein of geometric Picard rank* 1. In this subsection let X be a Gorenstein terminal Fano threefold of geometric Picard rank 1 over a field k of characteristic 0.

5A1. *Index* 4. If $r(X_{\bar{k}}) = 4$, then $H^3 = 1$, $(-K_{X_{\bar{k}}})^3 = 64$, and X is a Severi–Brauer variety of dimension 3. Hence, X is smooth and acquires a rational point over a suitable field extension of degree at most 4 by Remark 4.1.

5A2. *Index* 3. If $r(X_{\bar{k}}) = 3$, then $H^3 = 2$ and $(-K_{X_{\bar{k}}})^3 = 54$.

Lemma 5.4. *Let X be a Gorenstein terminal Fano threefold over a field k of characteristic 0. Assume that $\text{rk}(\text{Pic}(X_{\bar{k}})) = 1$ and $r(X_{\bar{k}}) = 3$. Then X is a quadric hypersurface in $\mathbb{P}^4_k$. In particular, X acquires a rational point over a suitable base field extension of degree at most* 2.

Proof. By [Fujita 1982, Theorem 0] (see also [Iskovskikh and Prokhorov 1999, Theorem 3.1.14]), we know that $X_{\bar{k}} \xrightarrow{|H|} \mathbb{P}^4_{\bar{k}}$ is a quadric hypersurface. By Lemma 5.2, H is invariant under the $\text{Gal}(\bar{k}/k)$ action on $\text{Pic}(X_{\bar{k}})$. Hence, by [Liedtke 2017, Theorem 3.4], there is a Severi–Brauer k-variety P of dimension 4 and a morphism $X \to P$ that is a k-model of the inclusion $X_{\bar{k}} \subseteq \mathbb{P}^4_{\bar{k}}$. Let α_H be the image of H in $\text{Br}(k)$ under the morphism α from [Kollár 2016, (69.1)]. Since $X_{\bar{k}}$ is a quadric in $\mathbb{P}^4_{\bar{k}}$, we see that $2\alpha_H = 0$, but also $5\alpha_H = 0$ (e.g., by [Kollár 2016, Proposition 44]), from which we conclude that $\alpha_H = 0$ and $P \cong \mathbb{P}^4_k$. Then X is a quadric hypersurface in $\mathbb{P}^4_k$. □

5A3. *Index* 2. If $r(X_{\bar{k}}) = 2$, then H^3 is an integer between 1 and 5, and $(-K_{X_{\bar{k}}})^3 = 8H^3$. We recall the classification from [Shin 1989, Corollary 0.8] (see also [Iskovskikh and Prokhorov 1999, Theorems 3.2.5 and 3.3.1] and [Prokhorov 2013a, Theorem 1.7]).

- If $H^3 = 1$, $X_{\bar{k}}$ is a sextic hypersurface in the weighted projective space $\mathbb{P}_{\bar{k}}(3, 2, 1, 1, 1)$.
- If $H^3 = 2$, $X_{\bar{k}} \xrightarrow{|H|} \mathbb{P}^3_{\bar{k}}$ is a double cover ramified along a smooth quartic surface.
- If $H^3 = 3$, $X_{\bar{k}} \xhookrightarrow{|H|} \mathbb{P}^4_{\bar{k}}$ is a smooth cubic hypersurface.
- If $H^3 = 4$, $X_{\bar{k}} \xhookrightarrow{|H|} \mathbb{P}^5_{\bar{k}}$ is a smooth complete intersection of two quadrics.
- If $H^3 = 5$, $X_{\bar{k}} \xhookrightarrow{|H|} \mathbb{P}^6_{\bar{k}}$ is an intersection of five quadrics. If X is smooth, $X_{\bar{k}}$ is a section of the Grassmannian $\text{Gr}(2, 5) \subseteq \mathbb{P}^9_{\bar{k}}$ by a linear space of codimension 3 under the Plücker embedding.

Lemma 5.5. *Let X be a Gorenstein terminal Fano threefold over a field k of characteristic 0. Assume that $\text{rk}(\text{Pic}(X_{\bar{k}})) = 1$ and $r(X_{\bar{k}}) = 2$. Then X acquires a rational point over a suitable field extension of degree at most $d_1(H^3)$, for*

$$(H^3, d_1(H^3)) \in \{(1, 1), (2, 8), (3, 3), (4, 16), (5, 1)\}.$$

Proof. By [Iskovskikh and Prokhorov 1999, Remarks 3.2.2(ii)], we know that $\dim_{\bar{k}} H^0(X_{\bar{k}}, H) = H^3 + 2$. Hence, by Lemmas 5.2 and 5.3 there exists a finite extension k' of k of degree at most $H^3 + 2$ such that $H \in \mathrm{Pic}(X_{k'})$. By [Iskovskikh and Prokhorov 1999, Proposition 3.2.3] the general member of $|H|$ is a del Pezzo surface of degree H^3 over k', and hence acquires a rational point over a further extension of degree H^3 by Lemma 4.4(i). If $H^3 = 5$, then $X_{k'}$ has a k'-point by Lemma 4.4(iv).

Let α_H be the image of H in $\mathrm{Br}(k)$ under the morphism α from [Kollár 2016, (69.1)]. Since $2H = -K_X \in \mathrm{Pic}(X)$, we have $2\alpha_H = 0$ in $\mathrm{Br}(k)$. But we have also $(H^3 + 2)\alpha_H = 0$ (e.g., by [Kollár 2016, Proposition 44]). Hence, if $H^3 \in \{1, 3, 5\}$, we conclude that $\alpha_H = 0$, and we can choose $k' = k$. If $H^3 = 4$, we can choose k' of degree at most 4 over k by [Kollár 2016, Theorem 53, Corollary 54]. $\square$

5A4. *Smooth of index* 1. If X is smooth and $r(X_{\bar{k}}) = 1$, then $H = -K_X \in \mathrm{Pic}(X)$, and H^3 is an even integer between 2 and 22, with $H^3 \neq 20$. Let $g := \frac{1}{2}(-K_X)^3 + 1$. Then g is an integer between 2 and 12, with $g \neq 11$. According to [Mukai 1995b], $|H|$ is basepoint-free and it induces a morphism $X \to \mathbb{P}_k^{g+1}$. We recall the classification from [Mukai 1995b, Theorem 1.10; Iskovskikh and Prokhorov 1999, Theorem 5.2.3 or §12.2]. See [Pieropan 2021] for the Galois descent of complete intersections.

- If $H^3 = 2$, then $g = 2$ and $X \xrightarrow{|H|} \mathbb{P}_k^3$ is a double cover ramified along a sextic surface.
- If $H^3 = 4$, then $g = 3$ and $X \xrightarrow{|H|} \mathbb{P}_k^4$ is either an embedding as quartic hypersurface, or a double cover of a quadric hypersurface in $\mathbb{P}_k^4$.

If $g \geq 4$, H is very ample.

- If $H^3 = 6$, then $g = 4$ and $X \xhookrightarrow{|H|} \mathbb{P}_k^5$ is a complete intersection of a quadric and a cubic.
- If $H^3 = 8$, then $g = 5$ and $X \xhookrightarrow{|H|} \mathbb{P}_k^6$ is a complete intersection of three quadrics.
- If $H^3 = 10$, then $g = 6$ and $X \xhookrightarrow{|H|} \mathbb{P}_k^7$ is a k-form of a complete intersection of a cone over the Grassmannian $\mathrm{Gr}(2, 5)$ under the Plücker embedding with a quadric and a linear space of codimension 3 (see [Iskovskikh and Prokhorov 1999, Examples 5.2.2(i)]).

If $g \geq 7$, a general hyperplane section of the embedding $X \xhookrightarrow{|H|} \mathbb{P}_k^{g+1}$ cuts a smooth K3 surface S in X by [Kollár 1997, §7.7] and [Mukai 1995b, Proposition 7.8].

- If $H^3 = 12$, then $g = 7$. Denote by $\mathcal{N}_{S/\mathbb{P}_k^7}^*$ the dual of the normal bundle of $S \subseteq \mathbb{P}_k^7$. In the proof of [Mukai 1995b, Theorem 4.7], the vector bundle $\mathcal{N}_{S/\mathbb{P}_k^7}^* \otimes \mathcal{O}_S(2)$ determines a $\mathrm{Gal}(\bar{k}/k)$-equivariant embedding $X_{\bar{k}} \subseteq \mathrm{Gr}(5, 10)$ whose image is a linear section of an orthogonal Grassmannian $\mathrm{OGr}(5, 10, Q) \subseteq \mathbb{P}_{\bar{k}}^{15}$, where Q is a $\mathrm{Gal}(\bar{k}/k)$-invariant quadric hypersurface Q in $\mathbb{P}_{\bar{k}}^9$ by a version of [Mukai 1995a, Corollary 2.5] for S.

If $g \geq 8$, a general hyperplane section of $S \xhookrightarrow{|H|_S|} \mathbb{P}_k^g$ is a smooth curve of genus g which determines a Mukai–Lazarsfeld bundle E on S (see [Aprodu 2013, §1.3]).

- If $H^3 = 14$, then $g = 8$ and E induces a $\mathrm{Gal}(\bar{k}/k)$-equivariant embedding of $X_{\bar{k}}$ in $\mathrm{Gr}(2, 6)$ such that if $\mathrm{Gr}(2, 6) \subset \mathbb{P}_{\bar{k}}^{14}$ is the Plücker embedding, $X_{\bar{k}}$ is a linear section of $\mathrm{Gr}(2, 6)$ by a $\mathrm{Gal}(\bar{k}/k)$-invariant linear subspace of codimension 5.

- If $H^3 = 16$, then $g = 9$ and E induces a $\mathrm{Gal}(\bar{k}/k)$-equivariant embedding of $X_{\bar{k}}$ in $\mathrm{Gr}(3, 6)$ such that if $\mathrm{Gr}(3, 6) \subseteq \mathbb{P}^{19}_{\bar{k}}$ is the Plücker embedding, $X_{\bar{k}}$ is a complete intersection of $\mathrm{Gr}(3, 6)$ with the zero locus of a $\mathrm{Gal}(\bar{k}/k)$-invariant global section of the second exterior power of the dual of the tautological bundle of $\mathrm{Gr}(3, 6)$ and a $\mathrm{Gal}(\bar{k}/k)$-invariant linear subspace of codimension 3 (see [Mukai 1995b, Example 5.1]).

- If $H^3 = 18$, then $g = 10$ and E induces a $\mathrm{Gal}(\bar{k}/k)$-equivariant embedding of $X_{\bar{k}}$ in $\mathrm{Gr}(5, 7)$ such that if $\mathrm{Gr}(5, 7) \subseteq \mathbb{P}^{20}_{\bar{k}}$ is the Plücker embedding, $X_{\bar{k}}$ is a complete intersection of $\mathrm{Gr}(5, 7)$ with the zero locus of a $\mathrm{Gal}(\bar{k}/k)$-invariant global section of the fourth exterior power of the dual of the tautological bundle of $\mathrm{Gr}(5, 7)$ and a $\mathrm{Gal}(\bar{k}/k)$-invariant linear subspace of codimension 2 (see [Mukai 1995b, Example 5.2]).

- If $H^3 = 22$, then $g = 12$ and E induces a $\mathrm{Gal}(\bar{k}/k)$-equivariant embedding of $X_{\bar{k}}$ in $\mathrm{Gr}(3, 7)$ such that $X_{\bar{k}}$ is the zero locus in $\mathrm{Gr}(3, 7)$ of three linearly independent $\mathrm{Gal}(\bar{k}/k)$-invariant global sections of the second exterior power of the dual of the tautological bundle of $\mathrm{Gr}(3, 7)$ (see [Mukai 1995b, §5, p. 15]).

Now we investigate some properties of linear sections of orthogonal Grassmannians that we can use to study rational points in the case $g = 7$.

Given a nonsingular quadric hypersurface $Q \subseteq \mathbb{P}^9$, we denote by $\mathrm{OGr}(5, 10, Q)$ the orthogonal Grassmannian of isotropic 4-dimensional linear subspaces of $\mathbb{P}^9$. It is also known as the spinor variety in $\mathbb{P}^{15}$. We refer to [Ranestad and Schreyer 2000] and [Kuznetsov 2018] for a detailed description.

Lemma 5.6. *Let k be a field of characteristic 0 and let $\bar{k}$ be a separable closure of k. Let $Q \subseteq \mathbb{P}^9_k$ be a nonsingular quadric hypersurface. Let $Y \subseteq \mathbb{P}^{15}_k$ be a k-model of the spinor embedding $\mathrm{OGr}(5, 10, Q) \subseteq \mathbb{P}^{15}_{\bar{k}}$. Let $L \subseteq \mathbb{P}^{15}_k$ be a linear subspace of codimension 7 such that $X := L \cap Y$ is geometrically irreducible of dimension 3. Then X acquires a rational point over a suitable quadratic extension of k.*

Proof. We denote by $\mathbb{P}^{15*}_k$ the dual projective space as in [Kuznetsov 2018, Notation 3.1] and by $Y^* \subseteq \mathbb{P}^{15*}_k$ the dual of $Y \subseteq \mathbb{P}^{15}_k$. The dual L^* of L is a 6-dimensional linear subspace of $\mathbb{P}^{15*}_k$, and thus $L^* \not\subseteq Y^*$ by [Kuznetsov 2018, Theorem 3.2]. Let v be a k-point on L^* that is not contained in Y^*, and let $L_v \subseteq \mathbb{P}^{15}_k$ be the corresponding hyperplane. By [Kuznetsov 2018, Lemma 5.10], the intersection $L_v \cap Y$ contains a quadric Q_v of dimension 6 defined over k. Then $L \cap Q_v$ is a nonempty k-subvariety of degree 2 in L, it is contained in X, and it acquires a rational point over a suitable extension of k of degree at most 2. $\square$

Lemma 5.7. *Let X be a smooth Fano threefold over a field k of characteristic 0. Assume $\mathrm{rk}(\mathrm{Pic}(X_{\bar{k}})) = 1$, $r(X_{\bar{k}}) = 1$ and $H^3 = 22$. Then X acquires a rational point over a suitable field extension of degree at most 6.*

Proof. The Fano variety of conics of X is a Severi–Brauer surface by [Kuznetsov et al. 2018, Proposition B.4.1]. Hence, X contains smooth conics defined over a cubic extension of k by Remark 4.1, and has rational points over a further quadratic extension. $\square$

5A5. *Singular Gorenstein of index* 1.

Definition 5.8. The anticanonical linear system $|-K_X|$ of a Gorenstein Fano variety X of index 1 is said to have a movable decomposition if there are Weil divisors A and B such that the linear systems $|A|$ and $|B|$ have positive dimension and $A+B$ is linearly equivalent to $-K_X$. If $|-K_{X_{\bar{k}}}|$ has no movable decomposition, for an algebraic closure $\bar{k}$ of the base field k, we say that X is indecomposable. Otherwise, we say that X is decomposable.

Indecomposable Gorenstein terminal Fano threefolds of index 1 and geometric Picard rank 1 are classified in [Mukai 1995b, Theorems 1.10, 6.5] (we remark that [Mukai 1995b, Proposition 7.8] and [Mella 1999, Theorem 1, Theorem 2.4] show that the classification is exhaustive), and the bounds from Section 5A4 apply.

Decomposable terminal Gorenstein Fano threefolds of index 1 and geometric Picard rank 1 are not completely classified. However, by [Iskovskikh and Prokhorov 1999, Proposition 4.1.12; Przyjalkowski et al. 2005, Theorem 1.5] (see also [Prokhorov and Shramov 2017, Proposition 6.1.1]), we know that the cases with $g \in \{2, 3, 4, 5\}$ are completely classified in [Mukai 1995b, Theorem 6.5].

Lemma 5.9. *Let X be a Gorenstein terminal $\mathbb{Q}$-factorial Fano threefold over a field k of characteristic 0. Assume that $\mathrm{rk}(\mathrm{Pic}(X_{\bar{k}})) = 1$ and $r(X_{\bar{k}}) = 1$. Then X acquires a rational point over a finite extension of k of degree at most* 18.

Proof. By [Namikawa 1997; Jahnke and Radloff 2011], $X_{\bar{k}}$ admits a smoothing that preserves the Picard group and the degree $(-K_{X_{\bar{k}}})^3$. Hence, $(-K_X)^3 \in \{2, 4, 6, 8, 10, 12, 14, 16, 18, 22\}$. If $(-K_X)^3 = 22$, the only decomposable cases we are interested in are classified in [Prokhorov 2016, Theorem 1.3], and their singular locus consists of a rational point. We conclude by Lemmas 5.1 and 5.7. $\square$

5B. *Gorenstein of geometric Picard rank* > 1. In this subsection let X be a Gorenstein terminal Fano threefold of geometric Picard rank > 1 and Picard rank 1 over a field k of characteristic 0.

5B1. *Products of projective spaces.*

Lemma 5.10. *Let k be a field of characteristic 0, and let X be a twisted form of $\prod_{i=1}^{n} \mathbb{P}_k^m$ over k such that $\mathrm{rk}(\mathrm{Pic}(X)) = 1$. Then there is an extension $\tilde{k}$ of k of degree at most $n!(m+1)$ such that $X_{\tilde{k}} \cong \prod_{i=1}^{n} \mathbb{P}_{\tilde{k}}^m$.*

Proof. Let $e_1, \ldots, e_n$ be the standard generators of $\mathrm{Pic}(X_{\bar{k}})$. From the exact sequence

$$1 \to \prod_{i=1}^{n} \mathbf{Aut}(\mathbb{P}_k^m) \to \mathbf{Aut}\left(\prod_{i=1}^{n} \mathbb{P}_k^m \right) \to S_n \to 1$$

and the induced exact sequence in étale cohomology, there exists a unique finite Galois extension k' of k of degree between n and $n!$ such that e_i is $\mathrm{Gal}(\bar{k}/k')$-invariant for all $i \in \{1, \ldots, n\}$. For every $i \in \{1, \ldots, n\}$, let α_i be the image of e_i in $\mathrm{Br}(k')$ under the morphism α from [Kollár 2016, (69.1)]. Then $\alpha_1, \ldots, \alpha_n$ form an orbit under the conjugation action of $\mathrm{Gal}(k'/k)$ (see [Gille and Szamuely 2006, Construction 3.3.12]). Hence, they split over the same field extension $\tilde{k}$ of k', which has degree at most $m+1$ by Lemma 5.3.

In particular, $e_1, \ldots, e_n \in \mathrm{Pic}(X_{\tilde{k}})$, and the morphism $X_{\tilde{k}} \to \prod_{i=1}^{n} \mathbb{P}_{\tilde{k}}^{m}$ induced by the product of the projections corresponding to $e_1, \ldots, e_n$ is an isomorphism. $\qquad\square$

5B2. *Smooth.* Let X be a smooth Fano threefolds of geometric Picard rank > 1 and Picard rank 1. We recall the classification from [Prokhorov 2013b, Theorem 1.2, §2].

If $\mathrm{rk}\,\mathrm{Pic}(X_{\bar{k}}) = 2$:

- $r(X_{\bar{k}}) = 1$, $H^3 = 12$, and $X_{\bar{k}}$ is a divisor of bidegree $(2, 2)$ in $\mathbb{P}_{\bar{k}}^2 \times \mathbb{P}_{\bar{k}}^2$; see [Pro 2013b, Case 1.2.1 a), p. 421].

- $r(X_{\bar{k}}) = 1$, $H^3 = 12$, and X is a double cover $\pi : X \to V$ of a variety V that belongs to the case [Pro 2013b, (1.2.4)] with branch locus a member of $|-K_V|$; see [Pro 2013b, Case 1.2.1 b), p. 421].

- $r(X_{\bar{k}}) = 1$, $H^3 = 20$, and $X_{\bar{k}}$ is a complete intersection of three divisors of bidegree $(1, 1)$ in $\mathbb{P}_{\bar{k}}^3 \times \mathbb{P}_{\bar{k}}^3$; see [Pro 2013b, (1.2.2)].

- $r(X_{\bar{k}}) = 1$, $H^3 = 28$, and $X_{\bar{k}}$ is a blowup of a quadric in $\mathbb{P}_{\bar{k}}^4$ along a twisted quartic curve; see [Pro 2013b, (1.2.3)].

If $\mathrm{rk}\,\mathrm{Pic}(X_{\bar{k}}) = 3$:

- $r(X_{\bar{k}}) = 2$, $H^3 = 6$, and X is a k-model of a divisor of bidegree $(1, 1)$ in $\mathbb{P}_{\bar{k}}^2 \times \mathbb{P}_{\bar{k}}^2$; see [Pro 2013b, (1.2.4)].

- $r(X_{\bar{k}}) = 1$, $H^3 = 12$, and $X_{\bar{k}}$ is a double cover of $\mathbb{P}_{\bar{k}}^1 \times \mathbb{P}_{\bar{k}}^1 \times \mathbb{P}_{\bar{k}}^1$ ramified along an element of $|-K_{\mathbb{P}_{\bar{k}}^1 \times \mathbb{P}_{\bar{k}}^1 \times \mathbb{P}_{\bar{k}}^1}|$; see [Pro 2013b, (1.2.5)].

- $r(X_{\bar{k}}) = 1$, $H^3 = 30$, and $X_{\bar{k}}$ is a complete intersection of three divisors in $\mathbb{P}_{\bar{k}}^2 \times \mathbb{P}_{\bar{k}}^2 \times \mathbb{P}_{\bar{k}}^2$ of tridegrees $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$, respectively; see [Pro 2013b, (1.2.6)].

- $r(X_{\bar{k}}) = 2$, $H^3 = 6$, and $X_{\bar{k}} \cong \mathbb{P}_{\bar{k}}^1 \times \mathbb{P}_{\bar{k}}^1 \times \mathbb{P}_{\bar{k}}^1$; see [Pro 2013b, (1.2.7)].

If $\mathrm{rk}\,\mathrm{Pic}(X_{\bar{k}}) = 4$:

- $r(X_{\bar{k}}) = 1$, $H^3 = 24$, and X is k-model of a divisor of multidegree $(1, 1, 1, 1)$ in $\mathbb{P}_{\bar{k}}^1 \times \mathbb{P}_{\bar{k}}^1 \times \mathbb{P}_{\bar{k}}^1 \times \mathbb{P}_{\bar{k}}^1$; [Pro 2013b, (1.2.8)].

Lemma 5.11. *Let $n \in \{1, \ldots, 8\}$, and let X be a smooth Fano threefold over a field k of characteristic 0 as in [Pro 2013b, (1.2.n)]. Then X acquires a rational point over a suitable base field extension of degree at most $d_2(n)$, for*

$$(n, d_2(n)) \in \{(2, 8), (3, 4), (4, 6), (6, 18), (7, 12)\}.$$

Proof. If $n = 2$, we have $\mathrm{Pic}(X_{\bar{k}}) \cong \mathrm{Pic}(\mathbb{P}_{\bar{k}}^3 \times \mathbb{P}_{\bar{k}}^3)$ by [Hartshorne 1970, Corollary IV.3.3] and $-K_X$ is an element of bidegree $(1, 1)$. By Lemma 5.10, there is an extension k' of k of degree at most 8 such that $X_{k'} \subseteq \mathbb{P}_{k'}^3 \times \mathbb{P}_{k'}^3$ is defined by three bihomogeneous polynomials f_1, f_2, f_3 of bidegree $(1, 1)$ with coefficients in k' in a set of variables $(x_0, x_1, x_2, x_3; y_0, y_1, y_2, y_3)$. Evaluating f_1, f_2, f_3 at $y_0 = y_1 = y_2 = y_3 = 1$ gives three linear forms in four variables, which have a nontrivial common zero $(\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \neq (0, 0, 0, 0)$ over k'. Then $((\tilde{x}_0 : \tilde{x}_1 : \tilde{x}_2 : \tilde{x}_3), (1 : 1 : 1 : 1))$ is a k'-point on X.

For $n = 3$, let $\varphi : X_{\bar{k}} \to \mathbb{P}^4_{\bar{k}}$ be the morphism induced by the blowup and let H be the hyperplane class in $\mathrm{Pic}(\mathbb{P}^4_{\bar{k}})$. By [Pro 2013b, pp. 426–427, 432], there is a quadratic extension k' of k such that $\varphi^*(H)$ is invariant under the $\mathrm{Gal}(\bar{k}/k')$ action on $\mathrm{Pic}(X_{\bar{k}})$. By [Liedtke 2017, Theorem 3.4], there is a Severi–Brauer k'-variety P of dimension 4 and a morphism $\varphi' : X_{k'} \to P$ that is a k'-model of φ. Let α_H be the image of H in $\mathrm{Br}(k')$ under the morphism α from [Kollár 2016, (69.1)]. Since the image of φ is a quadric in $\mathbb{P}^4_{\bar{k}}$, we see that $2\alpha_H = 0$, but also $5\alpha_H = 0$, from which we conclude that $\alpha_H = 0$ and $P \cong \mathbb{P}^4_{k'}$. Then $X_{k'}$ is the blowup of a quadric Q in $\mathbb{P}^4_{k'}$ along a suitable quartic curve. In particular, there is a suitable quadratic extension k'' of k' such that Q has a k''-point x and φ' is an isomorphism around x.

If $n = 4$, we have $\mathrm{Pic}(X_{\bar{k}}) \cong \mathrm{Pic}(\mathbb{P}^2_{\bar{k}} \times \mathbb{P}^2_{\bar{k}})$ by [Hartshorne 1970, Corollary IV.3.3], and H is an element of bidegree $(1, 1)$. By Lemma 5.10, there is an extension k' of k of degree at most 6 such that $X_{k'} \subseteq \mathbb{P}^2_{k'} \times \mathbb{P}^2_{k'}$ is defined by a bihomogeneous polynomial f of bidegree $(1, 1)$ with coefficients in k' in a set of variables $(x_0, x_1, x_2; y_0, y_1, y_2)$. Evaluating f on $y_0 = y_1 = y_2 = 1$ yields a linear form in three variables, which has a nontrivial zero $(\tilde{x}_0, \tilde{x}_1, \tilde{x}_2) \neq (0, 0, 0)$ over k'. Then $((\tilde{x}_0 : \tilde{x}_1 : \tilde{x}_2), (1 : 1 : 1))$ is a k'-point on X.

If $n = 6$, we have $\mathrm{Pic}(X_{\bar{k}}) \cong \mathrm{Pic}(\mathbb{P}^2_{\bar{k}} \times \mathbb{P}^2_{\bar{k}} \times \mathbb{P}^2_{\bar{k}})$ by [Hartshorne 1970, Corollary IV.3.3], and H is an element of tridegree $(1, 1, 1)$. By Lemma 5.10, there is an extension k' of k of degree at most 18 such that $X_{k'} \subseteq \mathbb{P}^2_{k'} \times \mathbb{P}^2_{k'} \times \mathbb{P}^2_{k'}$ is defined by three trihomogeneous polynomials f_1, f_2, f_3 of tridegrees $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$, respectively, with coefficients in k' in a set of variables $(x_0, x_1, x_2; y_0, y_1, y_2; z_0, z_1, z_2)$. Evaluating f_1 and f_2 at $z_0 = z_1 = z_2 = 1$ gives two linear forms $\tilde{f}_1, \tilde{f}_2$ in two distinct sets of variables. Let $(\tilde{y}_0, \tilde{y}_1, \tilde{y}_2) \neq (0, 0, 0)$ be a nontrivial zero of $\tilde{f}_1$ over k'. Evaluating f_3 at $y_i = \tilde{y}_i$ for $i \in \{0, 1, 2\}$ gives a linear form $\tilde{f}_3$. Let $(\tilde{x}_0, \tilde{x}_1, \tilde{x}_2) \neq (0, 0, 0)$ be a nontrivial common zero of $\tilde{f}_2$ and $\tilde{f}_3$ over k'. Then $((\tilde{x}_0 : \tilde{x}_1 : \tilde{x}_2), (\tilde{y}_0 : \tilde{y}_1 : \tilde{y}_2), (1 : 1 : 1))$ is a k'-point on X.

For $n = 7$, we conclude by Lemma 5.10. $\qquad\square$

5B3. *Singular Gorenstein.* By [Pro 2013b, Theorem 6.6], singular Gorenstein terminal Fano threefolds of geometric Picard rank > 1 and Picard rank 1 have the same description $(1.2.n)$ as in Section 5B2 for $n \in \{1, 2, 3, 5, 6, 8\}$.

Lemma 5.12. *Let $n \in \{1, \ldots, 8\}$, and let X be a singular Gorenstein terminal Fano threefold over a field k of characteristic 0 of type $(1.2.n)$ as in [Pro 2013b, Theorem 6.6]. Then X acquires a rational point over a suitable base field extension of degree at most $d_3(n)$, for*

$$(n, d_3(n)) \in \{(1, 12), (2, 20), (3, 18), (5, 12), (6, 17), (8, 17)\}.$$

Proof. Let $\bar{k}$ be an algebraic closure of k, and let X' be a smoothing of $X_{\bar{k}}$. Since the Picard group and the degree $(-K_{X_{\bar{k}}})^3$ are preserved under smoothing by [Namikawa 1997; Jahnke and Radloff 2011], $X_{\bar{k}}$ contains at most

$$N(X) = 20 - \mathrm{rk}\,\mathrm{Pic}(X_{\bar{k}}) + h^{1,2}(X')$$

singular points by [Prokhorov and Shramov 2017, (6.3.3)]. Combining Lemma 5.1 with the fact that the singular locus of $X_{\bar{k}}$ is defined over k, we conclude that X acquires a rational point over suitable base

field extension of degree at most $\min\{N(X), (-K_X)^3\}$. We compute $N(X)$ by consulting [Iskovskikh and Prokhorov 1999, §12.3–12.5]. □

5C. *Non-Gorenstein terminal.*

Lemma 5.13. *Let X be a non-Gorenstein terminal Fano threefold over a field k of characteristic 0. Then X acquires a rational point over a suitable field extension of degree at most 16.*

Proof. Let $\bar{k}$ be an algebraic closure of k. The orbit of a non-Gorenstein $\bar{k}$-point on $X_{\bar{k}}$ under the action of $\mathrm{Gal}(\bar{k}/k)$ has cardinality at most 16 (the computation can be found in the proof of [Prokhorov and Shramov 2017, Lemma 4.2.1]). □

6. Field of definition of rational points

In this section we prove Theorems 1.5 and 1.6.

Proof of Theorem 1.5. For the first statement we proceed by induction on n. If $n = 0$ there is nothing to prove. Let $n > 0$. Since the degree of a projective variety is encoded in the Hilbert polynomial, by Proposition 3.4 there exists a positive integer d' such that for every terminal Fano variety X of dimension n, there exists an embedding $X \subseteq \mathbb{P}_k^N$ of degree $\leq d'$. By Bertini's theorem, n general hyperplanes of $\mathbb{P}_k^N$ intersect X in a smooth subvariety of dimension 0 that acquires a rational point after a field extension of degree $\leq d'$. We retrace the proof of Theorem 1.3 to show that we can take

$$d(n) = d(n-1) \max\{d'(n), \max_{1 \leq m \leq n-1} d(m)d(n-m)\}, \tag{6-1}$$

where $d'(n)$ is a positive integer (for example d' above) such that for every field k of characteristic 0, every terminal $\mathbb{Q}$-factorial Fano variety of dimension n and Picard rank 1 over k acquires rational points over a suitable field extension of degree at most $d'(n)$ over k. Let X be a proper rationally connected variety of dimension n. By the argument in the proof of Lemma 3.1 we can assume without loss of generality that X is smooth and projective. Let $f : X' \to Y$ be the fibration with X' birationally equivalent to X provided by Proposition 3.3, and let $X'' \to X'$ be a resolution of singularities. Then there exists a field extension $k \subseteq k'$ of degree at most $\max\{d'(n), \max_{1 \leq m \leq n-1} d(m)d(n-m)\}$ such that $X'(k') \neq \varnothing$. Indeed, if $\dim Y = 0$, then X' acquires a rational point after a suitable field extension of degree at most $d'(n)$, as X' is a terminal $\mathbb{Q}$-factorial Fano k-variety of dimension n and Picard rank 1. If $\dim Y = m > 0$, then, by the induction hypothesis, Y acquires a rational point after a suitable field extension of degree at most $d(m)$ of k and the fiber of $X' \to Y$ over such a point acquires a rational point after a further suitable field extension of degree at most $d(n-m)$ by [Hogadi and Xu 2009, Theorem 1.2] and the induction hypothesis. By [Hogadi and Xu 2009, Theorem 1.3] and the induction hypothesis there exists a finite field extension $k' \subseteq k''$ of degree $\leq d(n-1)$ such that $X''(k'') \neq \varnothing$. Then $X(k'') \neq \varnothing$ by [Nishimura 1955]. □

Proof of Theorem 1.6. The bound $\tilde{d}(1) = 2$ is immediate, because all conics acquire rational points on suitable quadratic extensions and $\{x^2 + y^2 + z^2 = 0\}$ defines a conic without rational points over $\mathbb{Q}$. For $n = 2$, by resolution of singularities, by [Nishimura 1955], and by the birational classification of

proper smooth surfaces [Iskovskikh 1979], it suffices to find a bound for del Pezzo surfaces and rational conic bundles. Hence, $\tilde{d}(2) = 6$ follows from Lemmas 4.3 and 4.4.

For $n = 3$, let $d'(3)$ be the bound for terminal Fano threefolds of Picard rank 1. Then $d'(3) \leq 24$ by the results in Section 5, and (6-1) gives $\tilde{d}(3) \leq 6 \max\{24, 12\} = 144$. $\qquad\qquad\square$

7. Varieties with points over C_1 fields

In this section we find rational points on a number of rationally connected varieties over C_1 fields. The results in this section will be used for the proof of Theorem 1.8.

Remark 7.1. We recall that C_1 fields have cohomological dimension ≤ 1 and hence trivial Brauer group [Serre 2002, §II.3.2]. In particular, projective spaces have no nontrivial forms over C_1 fields [Gille and Szamuely 2006, §5]. We also recall that algebraic extensions of C_1 fields are C_1 [Lang 1952, Corollary to Theorem 5].

7A. *Toric varieties.* We prove that k-forms of toric varieties have rational points over C_1 fields k of characteristic 0.

Proposition 7.2. *Let k be a field of characteristic 0 of cohomological dimension ≤ 1. Let $\bar{k}$ be a separable closure of k. Let X be a k-variety such that $X_{\bar{k}}$ is isomorphic to a proper $\mathbb{Q}$-factorial toric $\bar{k}$-variety. Then X is an equivariant compactification of a k-torus and has a smooth k-point.*

Proof. Let Σ be a fan in $\mathbb{Z}^n$ and Y the split toric k-variety associated to $\Sigma \subseteq \mathbb{Z}^n$. Assume that Y is proper and $\mathbb{Q}$-factorial (i.e., Σ is simplicial and $\bigcup_{\sigma \in \Sigma} \sigma = \mathbb{R}^n$). By [Serre 2002, Proposition III.1.3.5], the k-forms of Y are classified by $H^1_{\text{ét}}(k, \mathbf{Aut}_Y)$ up to isomorphism. By [Cox 1995, Corollary 4.7], $\mathbf{Aut}_Y$ is a k-linear algebraic group. Hence, $H^1_{\text{ét}}(k, \mathbf{Aut}_Y) \cong H^1_{\text{ét}}(k, \mathbf{Aut}_Y / \mathbf{Aut}_Y^0)$ by [Serre 2002, Corollary III.2.4.3]. By [Cox 1995, Corollary 4.7], there is a surjective homomorphism $\text{Aut}(\mathbb{Z}^n, \Sigma) \to \mathbf{Aut}_Y / \mathbf{Aut}_Y^0$, where $\text{Aut}(\mathbb{Z}^n, \Sigma)$ is the group of lattice automorphisms of $\mathbb{Z}^n$ that preserve the fan Σ. Therefore, there is a surjective map $H^1_{\text{ét}}(k, \text{Aut}(\mathbb{Z}^n, \Sigma)) \to H^1_{\text{ét}}(k, \mathbf{Aut}_Y / \mathbf{Aut}_Y^0)$ by [Serre 2002, Corollary III.2.4.2]. Moreover, to every element of $H^1_{\text{ét}}(k, \text{Aut}(\mathbb{Z}^n, \Sigma))$ corresponds an isomorphism class of normal k-varieties with a faithful action of a k-torus that has a dense orbit by [Elizondo et al. 2014, Theorems 3.2 and 3.4]. Hence, every k-form X of Y is a normal k-variety with a faithful action of a k-torus that has a dense orbit U. By [Voskresenskii 1982, Theorem 6], U is a principal homogeneous space under a torus T. Since tori are connected linear algebraic groups by [Borel 1991, p. 114], $H^1_{\text{ét}}(k, T) = 0$ by [Serre 2002, Corollary III.2.4.3]. Therefore, $U \cong T$ has a k-point (the unit element). Since tori are smooth, X has a smooth k-point. $\qquad\square$

Note that the assumption on the characteristic of k ensures that the automorphism group of the toric variety is smooth. The author does not know whether smoothness and the results in [Cox 1995, Corollary 4.7] hold in positive characteristics. The $\mathbb{Q}$-factoriality assumption can be removed using [Sancho de Salas et al. 2018, Theorem 7.8].

7B. *Intersections of low degree in products of weighted projective spaces.* We generalize [Lang 1952, Theorem 4] and [Kollár 1996, Theorem IV.6.7] to orbit complete intersections (in the sense of [Pieropan 2021]) in forms of products of weighted projective spaces over C_1 fields of arbitrary characteristic.

We first study the forms of products of projective spaces over C_1 fields.

Lemma 7.3. *Let k be a field of cohomological dimension ≤ 1. Let $\bar{k}$ be a separable closure of k. Let N, $n_1, \ldots, n_N$ and $m_1 < \cdots < m_N$ be positive integers. Then $H^1_{\text{ét}}(k, \mathbf{Aut}_{\prod_{i=1}^N \prod_{j=1}^{n_i} \mathbb{P}^{m_i}_k}) = \prod_{i=1}^N H^1_{\text{ét}}(k, S_{n_i})$.*

Proof. Since automorphisms preserve the effective cone in $\text{Pic}(Y_{\bar{k}})$ and the intersection product,

$$\text{Aut}\left(\prod_{i=1}^N \prod_{j=1}^{n_i} \mathbb{P}^{m_i}_k\right) = \prod_{i=1}^N \text{Aut}\left(\prod_{j=1}^{n_i} \mathbb{P}^{m_i}_k\right),$$

and for every $i \in \{1, \ldots, N\}$, there is an exact sequence of groups

$$1 \to \prod_{j=1}^{n_i} \text{Aut}(\mathbb{P}^{m_i}_k) \to \text{Aut}\left(\prod_{j=1}^{n_i} \mathbb{P}^{m_i}_k\right) \to S_{n_i} \to 1.$$

This yields exact sequences

$$\prod_{j=1}^{n_i} H^1_{\text{ét}}(k, \mathbf{Aut}_{\mathbb{P}^{m_i}_k}) \to H^1_{\text{ét}}(k, \mathbf{Aut}_{\prod_{j=1}^{n_i} \mathbb{P}^{m_i}_k}) \to H^1_{\text{ét}}(k, S_{n_i}).$$

for every $i \in \{1, \ldots, N\}$, where the second arrow is surjective by [Serre 2002, Corollary III.2.4.2], and $H^1_{\text{ét}}(k, \mathbf{Aut}_{\mathbb{P}^{m_i}_k}) = 0$ as $\text{Br}(k) = 0$ (see Remark 7.1). $\square$

Notation 7.4. For positive integers $a_1, \ldots, a_m$, we denote by $\mathbb{P}_k(a_1, \ldots, a_m)$ the $(m-1)$-dimensional weighted projective space over k with weights $a_1, \ldots, a_m$.

For all N-tuples of positive integers $\boldsymbol{n} = (n_1, \ldots, n_N)$ and $(m_1, \ldots, m_N)$ and every $\left(\sum_{i=1}^N m_i\right)$-tuple of positive integers $\boldsymbol{a} = (a_{i,l})_{1 \leq i \leq N, 1 \leq l \leq m_i}$, we denote by $\mathbb{P}(\boldsymbol{n}, \boldsymbol{a})$ a k-variety such that

$$\mathbb{P}(\boldsymbol{n}, \boldsymbol{a})_{\bar{k}} \cong \prod_{i=1}^N \prod_{j=1}^{n_i} \mathbb{P}_{\bar{k}}(a_{i,1}, \ldots, a_{i,m_i})$$

with a $\text{Gal}(\bar{k}/k)$-action by permutation of the factors with exactly N orbits given by

$$\prod_{j=1}^{n_i} \mathbb{P}_{\bar{k}}(a_{i,1}, \ldots, a_{i,m_i})_{\bar{k}} \quad \text{for } i \in \{1, \ldots, N\}.$$

That is, if we write $(x_{i,j,l})_{1 \leq i \leq N, 1 \leq j \leq n_i, 1 \leq l \leq m_i}$ for the coordinates (here $x_{i,j,1}, \ldots, x_{i,j,m_i}$ are the coordinates in the j-th factor $\mathbb{P}_{\bar{k}}(a_{i,1}, \ldots, a_{i,m_i})$ of the i-th orbit), then $\text{Gal}(\bar{k}/k)$ acts on the coordinates by permutations of the second index in a way that $\{x_{i,1,l}, \ldots, x_{i,n_i,l}\}$ forms an orbit for all $i \in \{i, \ldots, N\}$ and all $l \in \{1, \ldots, m_i\}$.

Remark 7.5. Lemma 7.3 together with [Serre 2002, Proposition III.1.3.5] shows that all forms of products of projective spaces over a field k of cohomological dimension ≤ 1 are isomorphic to $\mathbb{P}(\boldsymbol{n}, \boldsymbol{a})$ for some $\boldsymbol{n}$ and $\boldsymbol{a} = (1, \ldots, 1)$.

Proposition 7.6. *Let k be a C_1 field that admits normic forms of every degree. Let $\bar{k}$ be a separable closure of k. Let $\boldsymbol{n}$ and $\boldsymbol{a}$ be as in Notation 7.4. Let $X \subseteq \mathbb{P}(\boldsymbol{n}, \boldsymbol{a})$ be a subvariety such that $X_{\bar{k}}$ is an intersection of hypersurfaces $H_1, \ldots, H_s$ in $\mathbb{P}(\boldsymbol{n}, \boldsymbol{a})_{\bar{k}}$ of weighted multidegrees $(d_{i,j,t})_{1 \leq i \leq N, 1 \leq j \leq n_i}$ for $t \in \{1, \ldots, s\}$ such that $\sum_{t=1}^{s} d_{i,j,t} < \sum_{l=1}^{m_i} a_{i,l}$ for all $i \in \{1, \ldots, N\}$ and all $j \in \{1, \ldots, n_i\}$. Assume that for some $1 \leq s_1 < \cdots < s_r := s$ and $s_0 := 0$, the hypersurfaces $H_{s_{i-1}+1}, \ldots, H_{s_i}$ form an orbit under the $\mathrm{Gal}(\bar{k}/k)$-action on $\mathbb{P}(\boldsymbol{n}, \boldsymbol{a})_{\bar{k}}$ for all $i \in \{1, \ldots, r\}$. Then X has a k-rational point.*

Proof. Let $R = \bar{k}[x_{i,j,l}]_{1 \leq i \leq N, 1 \leq j \leq n_i, 1 \leq l \leq m_i}$ be the weighted coordinate ring of $\mathbb{P}(\boldsymbol{n}, \boldsymbol{a})_{\bar{k}}$ (here the variable $x_{i,j,l}$ corresponds to the l-th coordinate in the j-th factor $\mathbb{P}_{\bar{k}}(a_{i,1}, \ldots, a_{i,m_i})$ of the i-th orbit). Then the group $\mathrm{Gal}(\bar{k}/k)$ acts on the variables by permutations of the second index. Let $f_1, \ldots, f_s \in R$ that define $H_1, \ldots, H_s$, respectively, and such that $\{f_{s_{i-1}+1}, \ldots, f_{s_i}\}$ is an orbit under the $\mathrm{Gal}(\bar{k}/k)$-action on R for all $i \in \{1, \ldots, r\}$.

Step 1: Assume that $f_1, \ldots, f_s$ are $\mathrm{Gal}(\bar{k}/k)$-invariant and $N = 1$. For each $l \in \{1, \ldots, m_1\}$, the $\bar{k}$-vector space $\sum_{1 \leq j \leq n_1} x_{1,j,l} \bar{k}$ is a $\mathrm{Gal}(\bar{k}/k)$-invariant subset of R, and therefore it has a basis $z_{1,l}, \ldots, z_{n_1,l}$ consisting of $\mathrm{Gal}(\bar{k}/k)$-invariant elements of degree $a_{1,l}$. Then a linear change of variables gives an isomorphism $\varphi : R \to \bar{k}[z_{j,l}]_{1 \leq j \leq n_1, 1 \leq l \leq m_1}$ of $\bar{k}$-algebras such that, for every $t \in \{1, \ldots, s\}$, $\varphi(f_t)$ is a weighted homogeneous polynomial of weighted degree $\sum_{j=1}^{n_1} d_{1,j,t}$ in the variables $z_{1,1}, \ldots, z_{n_1,m_1}$. Since $\sum_{1 \leq j \leq n_1, 1 \leq t \leq s} d_{1,j,t} < n_1 \sum_{l=1}^{m_1} a_{1,l}$, the system of equations $\varphi(f_1) = \cdots = \varphi(f_s) = 0$ has a solution $(\tilde{z}_{1,1}, \ldots, \tilde{z}_{n_1,m_1}) \in k^{n_1 m_1} \setminus \{0\}$ by [Kollár 1996, Theorem IV.6.7]. Therefore, the system $f_1 = \cdots = f_s = 0$ has a $\mathrm{Gal}(\bar{k}/k)$-invariant solution $(\tilde{x}_{1,j,l})_{1 \leq j \leq n_1, 1 \leq l \leq m_1} \in \bar{k}^{n_1 m_1}$ with $\tilde{x}_{1,\tilde{j},\tilde{l}} \neq 0$ for some $(\tilde{j}, \tilde{l}) \in \{1, \ldots, n_1\} \times \{1, \ldots, m_1\}$. Since $\{\tilde{x}_{1,1,\tilde{l}}, \ldots, \tilde{x}_{1,n_1,\tilde{l}}\}$ is an orbit under the $\mathrm{Gal}(\bar{k}/k)$-action, $\tilde{x}_{1,j,\tilde{l}} \neq 0$ for all $j \in \{1, \ldots, n_1\}$. Thus $(\tilde{x}_{1,j,l})_{1 \leq j \leq n_1, 1 \leq l \leq m_1}$ defines a $\mathrm{Gal}(\bar{k}/k)$-invariant point in $X_{\bar{k}}(\bar{k})$, and hence a k-rational point on X.

Step 2: Assume that $f_1, \ldots, f_s$ are $\mathrm{Gal}(\bar{k}/k)$-invariant and N is arbitrary. We proceed by induction on N. The case $N = 1$ is Step 1. Assume that $N > 1$. Let $F \subseteq \{f_1, \ldots, f_s\}$ be the set of elements f_t such that $d_{N,j,t} = 0$ for all $j \in \{1, \ldots, n_N\}$. We observe that F is invariant under the $\mathrm{Gal}(\bar{k}/k)$-action on R. If $F \neq \varnothing$, by the hypothesis of induction, the subvariety $V(F) \subseteq \prod_{i=1}^{N-1} \prod_{j=1}^{n_i} \mathbb{P}_{\bar{k}}(a_{i,1}, \ldots, a_{i,m_i})$ contains a $\mathrm{Gal}(\bar{k}/k)$-invariant point with coordinates $(\tilde{x}_{i,j,l})_{1 \leq i \leq N-1, 1 \leq j \leq n_i, 1 \leq l \leq m_i} \in \bar{k}^{\sum_{i=1}^{N-1} n_i m_i}$. If $F = \varnothing$, let

$$\tilde{x}_{i,j,l} := 1 \quad \text{for all } i \in \{1, \ldots, N-1\}, \ j \in \{1, \ldots, n_i\}, \ l \in \{1, \ldots, m_i\}.$$

Evaluating $f_1, \ldots, f_s$ in $x_{i,j,l} = \tilde{x}_{i,j,l}$ for all $i \in \{1, \ldots, N-1\}$, $j \in \{1, \ldots, n_i\}$, $l \in \{1, \ldots, m_i\}$ yields a system of $\mathrm{Gal}(\bar{k}/k)$-invariant forms $f_1', \ldots, f_s'$ in $\bar{k}[x_{N,j,l}]_{1 \leq j \leq n_N, 1 \leq l \leq m_N}$. By Step 1, the subvariety $V(f_1', \ldots, f_s') \subseteq \prod_{j=1}^{n_N} \mathbb{P}_{\bar{k}}(a_{N,1}, \ldots, a_{N,m_N})$ contains a $\mathrm{Gal}(\bar{k}/k)$-invariant point with coordinates $(\tilde{x}_{N,j,l})_{1 \leq j \leq n_N, 1 \leq l \leq m_N} \in \bar{k}^{n_N m_N}$. Then $(\tilde{x}_{i,j,l})_{1 \leq i \leq N, 1 \leq j \leq n_i, 1 \leq l \leq m_i}$ defines a $\mathrm{Gal}(\bar{k}/k)$-invariant point in $X_{\bar{k}}(\bar{k})$, hence, a k-point on X.

<u>Step 3</u>: No restrictions on $f_1, \ldots, f_s$. For $l = 1, \ldots, r$, let $h_l := \prod_{t=s_{l-1}+1}^{s_l} f_t$. Then $h_l \in R$ is a $\mathrm{Gal}(\bar{k}/k)$-invariant weighted multihomogeneous element of R of weighted degree $\left(\sum_{t=s_{l-1}+1}^{s_l} d_{i,j,t} \right)_{1 \leq i \leq N, 1 \leq j \leq n_i}$. The subvariety of $Y_{\bar{k}}$ defined by $h_1 = \cdots = h_r = 0$ has a $\mathrm{Gal}(\bar{k}/k)$-invariant $\bar{k}$-point $\tilde{x}$ by Step 2. The point $\tilde{x}$ belongs to at least one hypersurface in each orbit under the $\mathrm{Gal}(\bar{k}/k)$-action on $\{H_1, \ldots, H_s\}$, and hence it belongs to all. So $\tilde{x}$ defines a k-point on X. $\qquad\square$

7C. *Some cyclic coverings.* We study rational points over C_1 fields for cyclic coverings of complete intersections in forms of products of projective spaces that have Picard rank 1 over the base field. The assumption on the Picard rank is essential for the proof.

Proposition 7.7. *Let k be a C_1 field that admits normic forms of every degree. Let $\bar{k}$ be a separable closure of k. Let Y be a k-variety such that $Y_{\bar{k}} \cong \prod_{i=1}^{n} \mathbb{P}_{\bar{k}}^m$ and $\mathrm{rk\,Pic}(Y) = 1$. Let $V \subseteq Y$ be an intersection of r hypersurfaces of degrees $d_1(1, \ldots, 1), \ldots, d_r(1, \ldots, 1)$ in $\mathrm{Pic}(Y_{\bar{k}})$. Let $X \to V$ be a cyclic covering of degree d with reduced ramification divisor given by the restriction to V of a hypersurface of Y of degree $d_0(1, \ldots, 1)$ in $\mathrm{Pic}(Y_{\bar{k}})$. If $\sum_{i=0}^{r} d_i \leq m + d_0/d$, then $X(k) \neq \varnothing$.*

Proof. Let $R = \bar{k}[x_{1,0}, \ldots, x_{n,m}]$ be the coordinate ring of $Y_{\bar{k}}$. By Remark 7.5, we can assume that $\mathrm{Gal}(\bar{k}/k)$ acts by permutations of the first index. Since $\mathrm{rk\,Pic}(Y) = 1$, the set $\{x_{1,j}, \ldots, x_{n,j}\}$ is an orbit under the $\mathrm{Gal}(\bar{k}/k)$-action on R for all $j \in \{0, \ldots, m\}$. Let $f_0, \ldots, f_r \in R$ be $\mathrm{Gal}(\bar{k}/k)$-invariant homogeneous elements of degrees $d_0(1, \ldots, 1), \ldots, d_r(1, \ldots, 1)$ such that $f_1, \ldots, f_r$ define V and f_0 defines the ramification divisor Z.

The Segre embedding $\alpha : Y_{\bar{k}} \subseteq \mathbb{P}_{\bar{k}}^{(m+1)^n - 1}$ is defined over k. Let $S = \bar{k}[z_{\underline{j}} : \underline{j} \in \{0, \ldots, m\}^n]$ be a coordinate ring of $\mathbb{P}_{\bar{k}}^{(m+1)^n - 1}$ such that α corresponds to the morphism $\varphi : S \to R$ that sends a coordinate $z_{(j_1, \ldots, j_n)}$ to $\prod_{i=1}^{n} x_{i,j_i}$. Let $I \subseteq S$ be the kernel of φ. Then S/I is isomorphic to a subring of R that contains $f_0, \ldots, f_r$. Let $\tilde{f}_0, \ldots, \tilde{f}_r$ be the corresponding elements in S/I. Then $X_{\bar{k}}$ is the subvariety of an $(m+1)^n$-dimensional weighted projective space $\mathbb{P}_{\bar{k}}(1, \ldots, 1, d_0/d)$ defined by the ideal $I + (\tilde{f}_1, \ldots, \tilde{f}_r, t^d - \tilde{f}_0) \subseteq S[t]$, where t is a new variable of degree d_0/d.

Let
$$\boldsymbol{n} = (n) \quad \text{and} \quad \boldsymbol{a} = (\underbrace{1, \ldots, 1}_{m+1}, d_0/d).$$

We consider the k-variety $\mathbb{P}(\boldsymbol{n}, \boldsymbol{a})$ with coordinates
$$((x_{1,0} : \cdots : x_{1,m} : t_1), \ldots, (x_{n,0} : \cdots : x_{n,m} : t_n))$$

on $\mathbb{P}(\boldsymbol{n}, \boldsymbol{a})_{\bar{k}}$ and a $\mathrm{Gal}(\bar{k}/k)$-action that is compatible with the $\mathrm{Gal}(\bar{k}/k)$-action on $Y_{\bar{k}}$ under the embedding that identifies $Y_{\bar{k}}$ with the subvariety of $\mathbb{P}(\boldsymbol{n}, \boldsymbol{a})_{\bar{k}}$ defined by $t_1 = \cdots = t_n = 0$. The coordinate ring of $\mathbb{P}(\boldsymbol{n}, \boldsymbol{a})_{\bar{k}}$ is $R' := R[t_1, \ldots, t_n]$, where $t_1, \ldots, t_n$ are new variables with $\deg t_i = (d_0/d) \deg x_{i,0}$ for all $i \in \{1, \ldots, n\}$. We observe that $f_1, \ldots, f_r, (t_1 \cdots t_n)^d - f_0$ define a $\mathrm{Gal}(\bar{k}/k)$-invariant subvariety of $\mathbb{P}(\boldsymbol{n}, \boldsymbol{a})_{\bar{k}}$, which has a $\mathrm{Gal}(\bar{k}/k)$-invariant $\bar{k}$-point $\tilde{x} = (\tilde{x}_{1,0} : \cdots : \tilde{x}_{n,m} : \tilde{t}_1 : \cdots : \tilde{t}_n)$ by Proposition 7.6. If $\tilde{x}_{i,j} \neq 0$ for some $(i, j) \in \{1, \ldots, n\} \times \{0, \ldots, m\}$, then $\tilde{x}_{1,j}, \ldots, \tilde{x}_{n,j} \neq 0$, as they form an orbit under the $\mathrm{Gal}(\bar{k}/k)$-action on $\bar{k}$. If $\tilde{x}_{1,0}, \ldots, \tilde{x}_{n,m} = 0$, then $t_1, \ldots, t_n \neq 0$, as $\tilde{x}$ is a point of $\prod_{j=1}^{n} \mathbb{P}_{\bar{k}}^{m+1}$.

Let $\tilde{t} := \tilde{t}_1 \cdots \tilde{t}_n$, and let $\tilde{z}_{\underline{j}}$ be the evaluation of $\varphi(z_{\underline{j}})$ at $(\tilde{x}_{1,0}, \ldots, \tilde{x}_{n,m})$ for all $\underline{j} \in \{0, \ldots, m\}^n$. The $\bar{k}$-point of $\mathbb{P}(1, \ldots, 1, d_0/d)$ with coordinates $(\{\tilde{z}_{\underline{j}}\}_{\underline{j} \in \{0,\ldots,m\}^n}, \tilde{t})$ is then a $\mathrm{Gal}(\bar{k}/k)$-point of $X_{\bar{k}}$. $\square$

7D. *A symmetric Cremona transformation.* In this section we consider the varieties described in [Pro 2013b, (1.2.3)] (see Section 5B). Some of the computations have been carried out using the web interface SageMathCell of [SageMath].

Notation 7.8. We denote the adjugate of an invertible matrix M by $\mathrm{Ad}(M) := \det(M) M^{-1}$. We recall that, given an automorphism σ of a field k, a morphism $\theta : V \to W$ of k-vector spaces is called σ-linear if $\theta(av) = \sigma(a)\theta(v)$ for all $a \in k$ and $v \in V$. We denote by $\sigma : \mathbb{P}^n(k) \to \mathbb{P}^n(k)$ the σ-linear automorphism that sends $(y_0 : \cdots : y_n)$ to $(\sigma^{-1}(y_0) : \cdots : \sigma^{-1}(y_n))$.

Let k be a field of characteristic 0. Let $S \subseteq \mathbb{P}^5_k$ be the image of the Veronese embedding $v : \mathbb{P}^2_k \to \mathbb{P}^5_k$,

$$v(x_0 : x_1 : x_2) = (x_0^2 : x_1^2 : x_2^2 : x_1 x_2 : x_0 x_2 : x_0 x_1).$$

Denote by $(y_0 : \cdots : y_5)$ the coordinates on $\mathbb{P}^5_k$. Then S is defined by the quadrics

$$A_0 := y_1 y_2 - y_3^2, \qquad A_1 := y_0 y_2 - y_4^2, \qquad A_2 := y_0 y_1 - y_5^2,$$
$$A_3 := y_4 y_5 - y_0 y_3, \quad A_4 := y_3 y_5 - y_1 y_4, \quad A_5 := y_3 y_4 - y_2 y_5,$$

and $A_0, \ldots, A_5$ is a basis of the linear system $\mathscr{S}$ of quadrics in $\mathbb{P}^5_k$ containing S.

Let S_3 be the k-vector space of symmetric 3×3 matrices. We consider the isomorphism $\mathbb{P}^5_k \to \mathbb{P}(S_3)$ induced by the isomorphism

$$\alpha : k^6 \to S_3, \quad (y_0, \ldots, y_5) \mapsto \begin{pmatrix} y_0 & y_5 & y_4 \\ y_5 & y_1 & y_3 \\ y_4 & y_3 & y_2 \end{pmatrix}.$$

Every automorphism of $\mathbb{P}^5_k$ that preserves S belongs to the image of the embedding $\mathrm{PGL}(3, k) \subseteq \mathrm{PGL}(6, k)$ induced by the group homomorphism $\varphi : \mathrm{GL}(3, k) \subseteq \mathrm{GL}(6, k)$ that sends a matrix $M \in \mathrm{GL}(3, k)$ to the element of $\mathrm{GL}(6, k)$ defined by $\varphi(M)(y) = \alpha^{-1}(M\alpha(y)M^t)$ for all $y \in k^6$.

Let $A : \mathbb{P}^5_k \dashrightarrow \mathbb{P}^5_k$ be the rational map defined by $(A_0, \ldots, A_5)$. It is a symmetric Cremona transformation. For every matrix $M \in \mathrm{GL}(3, k)$ the birational map $\varphi(M) \circ A$ is a Cremona transformation of $\mathbb{P}^5_k$ with inverse $\varphi(M^t) \circ A$.

Let $f : Y \to \mathbb{P}^5_k$ be the blowup with center S. Then f resolves any Cremona transformation $\mathbb{P}^5_k \dashrightarrow \mathbb{P}^5_k$ induced by the linear system $\mathscr{S}$ (see [Ein and Shepherd-Barron 1989], for example). Let H be the pullback of a hyperplane in Y and E the exceptional divisor of f. Then $|2H - E|$ is the linear system of strict transforms of elements of $\mathscr{S}$ under f.

Lemma 7.9. *Let L be a hyperplane in $\mathbb{P}^5_k$ such that $L \cap S$ spans L. Let Q be a quadric in $\mathscr{S}$. Let $X \subseteq Y$ be the strict transform of $L \cap Q$. Let σ be an automorphism of k. Then every σ-linear automorphism θ of X that induces a σ-linear isomorphism between the complete linear systems of $H|_X$, and $(2H - E)|_X$ is the restriction to X of a σ-linear automorphism θ_Y of Y that resolves a σ-linear Cremona transformation on $\mathbb{P}^5_k$ of the form $\sigma \circ \varphi(M) \circ A$ for some $M \in \mathrm{GL}(3, k)$. Moreover, if θ is an involution, so is θ_Y.*

Proof. We fix a basis $z_0, \ldots, z_5$ of the linear system $|H|$ such that z_5 defines the hyperplane L. Let $C \in \mathrm{GL}(6, k)$ such that the basis $y_0, \ldots, y_5$ of $|H|$ corresponding to the choice of coordinates on $\mathbb{P}^5_k$ can be written as $(y_0, \ldots, y_5) = C(z_0, \ldots, z_5)$. Let $B_0, \ldots, B_4, \widetilde{B}_0, \ldots, \widetilde{B}_4 \in \mathscr{S}$ such that $B_0|_Q, \ldots, B_4|_Q$ and $\widetilde{B}_0|_Q, \ldots, \widetilde{B}_4|_Q$ are the bases of $\mathscr{S}|_Q$ that are images of the basis $z_0|_L, \ldots, z_4|_L$ of $|H||_L$ under the isomorphisms induced by θ and θ^{-1}, respectively. Let $R, \widetilde{R} \in \mathrm{GL}(6, k)$ such that

$$\sigma(C)(B_0, \ldots, B_4, Q) = R(A_0, \ldots, A_5) \quad \text{and} \quad \sigma(C)(\widetilde{B}_0, \ldots, \widetilde{B}_4, Q) = \widetilde{R}(A_0, \ldots, A_5)$$

as bases of $\mathscr{S}$. Then θ induces via f a birational map $(\sigma \circ R \circ A)|_{Q \cap L} : Q \cap L \to Q \cap L$ with inverse $(\sigma \circ \widetilde{R} \circ A)|_{Q \cap L}$. Since $\sigma(S) = S$ by construction and $(\sigma \circ \widetilde{R} \circ A)|_{Q \cap L} = (A \circ R^{-1} \circ \sigma^{-1})|_{Q \cap L}$ is a birational map with exceptional locus $S \cap L = \sigma(R(S \cap L))$, we have $R(S \cap L) = S \cap \sigma^{-1}(L)$; that is, R induces an isomorphism of conics between $v^{-1}(S \cap L)$ and $v^{-1}(S \cap \sigma^{-1}(L))$. Hence there is a matrix $M \in \mathrm{GL}(3, k)$ such that $(v^{-1} \circ R \circ v)|_{v^{-1}(S \cap L)}$ is the restriction of the automorphism of $\mathbb{P}^2_k$ defined by M. Then $\varphi(M^{-1})|_{\sigma^{-1}(L)} = R^{-1}|_{\sigma^{-1}(L)}$ because they agree on $S \cap \sigma^{-1}(L)$ and $S \cap \sigma^{-1}(L)$ spans $\sigma^{-1}(L)$. So $(A \circ R^{-1} \circ \sigma^{-1})|_{Q \cap L} = (A \circ \varphi(M^{-1}) \circ \sigma^{-1})|_{Q \cap L}$, and the σ-linear Cremona transformations on $\mathbb{P}^5_k$ induced by R and $\varphi(M)$ have the same restriction to $Q \cap L$.

If $\theta \circ \theta$ is the identity on X, then the automorphism

$$\varphi(\sigma(M) \cdot \mathrm{Ad}(M)^t) = \varphi(\sigma(M)) \circ A \circ \varphi(M) \circ A$$

of $\mathbb{P}^5_k$ restricts to the identity on an open subset of $Q \cap L$, and hence on L. Then $\sigma(M) \cdot \mathrm{Ad}(M)^t$ is an automorphism of $\mathbb{P}^2_k$ that restricts to the identity on $v^{-1}(L \cap S)$. Since the conic $v^{-1}(L \cap S)$ spans $\mathbb{P}^2_k$, we conclude that $\sigma(M) \cdot \mathrm{Ad}(M)^t$ is the identity on $\mathbb{P}^2_k$, and so $\sigma \circ \varphi(M) \circ A$ is a symmetric σ-linear Cremona transformation. $\qquad \square$

Let

$$\beta : k^6 \to k^6, \quad (a_0, \ldots, a_5) \mapsto (a_0, a_1, a_2, 2a_3, 2a_4, 2a_5).$$

Let $\psi : \mathrm{GL}(3, k) \to \mathrm{GL}(6, k)$ be the group homomorphism defined by $\psi(M) = \beta \circ \varphi(\mathrm{Ad}(M)) \circ \beta^{-1}$. Let $M \in \mathrm{GL}(3, k)$ and $a = (a_0, \ldots, a_5) \in k^6$. Then

$$\sum_{i=0}^{5} a_i A_i(\varphi(M)(y)) = \sum_{i=0}^{5} \psi_i(M)(a) A_i(y) \tag{7-1}$$

for all $y \in k^6$, where $(\psi_0(M)(a), \ldots, \psi_5(M)(a)) = \psi(M)(a) = \varphi(\mathrm{Ad}(M)^t)^t(a)$. In particular,

$$A \circ \varphi(M) \circ A = \varphi(\mathrm{Ad}(M)^t) \tag{7-2}$$

as birational maps on $\mathbb{P}^5_k$.

The secant variety V of S in $\mathbb{P}^5_k$ is the cubic hypersurface defined by

$$y_0 y_1 y_2 + 2 y_3 y_4 y_5 - y_0 y_3^2 - y_1 y_4^2 - y_2 y_5^2 = 0.$$

The action of $\mathrm{PGL}(3, k)$ on $\mathbb{P}^5_k$ induced by φ has three orbits: S, $V \smallsetminus S$ and $\mathbb{P}^5_k \smallsetminus V$; see [Urech 2018, §3.1], for example. So the action of $\mathrm{PGL}(3, k)$ on $\mathbb{P}^5_k$ induced by ψ has three orbits: $\widetilde{\beta}(S)$, $\widetilde{\beta}(V)$ and $\mathbb{P}^5_k \smallsetminus \widetilde{\beta}(V)$,

where $\widetilde{\beta}$ is the automorphism of $\mathbb{P}^5_k$ induced by β. Now, $\widetilde{\beta}(S)$ is defined by $A_0 \circ \beta^{-1}, \ldots, A_5 \circ \beta^{-1}$, and $\widetilde{\beta}(V)$ is defined by $p = 0$, where

$$p = 4y_0 y_1 y_2 + y_3 y_4 y_5 - y_0 y_3^2 - y_1 y_4^2 - y_2 y_5^2.$$

Remark 7.10. If $a = (a_0, \ldots, a_5) \in k^6$ and $q_a := \sum_{i=0}^{5} a_i A_i$, the matrix J_a that defines the linear system

$$\frac{\partial q_a}{\partial y_0} = \cdots = \frac{\partial q_a}{\partial y_5} = 0 \tag{7-3}$$

satisfies $\det J_a = -p(a)^2$, and every 5×5 minor of J_a is divisible by $p(a)$. So, if nonempty, the linear subvariety of $\mathbb{P}^5_k$ defined by (7-3) has dimension ≥ 1, and hence it intersects the quadric $q_a = 0$. In particular, q_a is smooth if and only if $p(a) \neq 0$.

7D1. *Smooth case.*

Lemma 7.11. *Let k be an algebraically closed field. For $a_0, \ldots, a_3, b_0, b_1, b_2 \in k$, let*

$$g_3 := a_3 x^3 + a_2 x^2 + a_1 x + a_0, \quad g_2 := b_2 x^2 + b_1 x + b_0 \in k[x],$$

and

$$\delta(g_3, g_2) := b_2 \delta_1^2 - b_1 \delta_1 \delta_2 + b_0 \delta_2^2,$$

where

$$\delta_1 := a_3 b_0 b_1 - a_2 b_0 b_2 + a_0 b_2^2 \quad and \quad \delta_2 := a_3 b_1^2 - a_3 b_0 b_2 - a_2 b_1 b_2 + a_1 b_2^2.$$

If $\delta(g_3, g_2) \neq 0$, then $b_2 \neq 0$, and g_3 and g_2 have no common zero in k.

Proof. If $b_2 = 0$, then $\delta(g_3, g_2) = 0$. Assume now that $b_2 \neq 0$ and that g_3 and g_2 have a common zero $c \in k$. Then $c^2 = -(b_1/b_2)c - (b_0/b_2)$ and substituting it twice in g_3 gives $\delta_2 c + \delta_1 = 0$. If $\delta_2 = 0$, then $\delta_1 = 0$. If $\delta_2 \neq 0$, we substitute $c = -\delta_1/\delta_2$ in g_2 and obtain $\delta(g_3, g_2) = 0$. □

Lemma 7.12. *Let k be an algebraically closed field of characteristic 0. Let Q be a smooth quadric in the linear system $\mathscr{S}$. Let H be a hyperplane in $\mathbb{P}^5_k$. Then for a general quadric Q' in $\mathscr{S}$, the singular locus $\mathrm{Sing}(Q \cap Q')$ is contained in the union of three lines of $\mathbb{P}^5_k$, $\mathrm{Sing}(Q \cap Q') \cap S$ is nonempty of dimension 0, and $\mathrm{Sing}(Q \cap Q') \cap S \cap H = \varnothing$.*

Proof. By Remark 7.10, the open orbit under the action of $\mathrm{PGL}(3, k)$ on $\mathbb{P}^5(k)$ via ψ is the locus of $(a_0 : \cdots : a_5) \in \mathbb{P}^5(k)$ such that the quadric $\sum_{i=0}^{5} a_i A_i = 0$ is smooth. Hence, by (7-1), up to an automorphism of $\mathbb{P}^5_k$ that fixes S, we can assume that Q is defined by the quadratic form $q := A_0 + A_3$. Let $a := (1, 0, 0, 1, 0, 0)$. Let $q'(u) := \sum_{i=0}^{5} u_i A_i \in k[u_0, \ldots, u_5, y_0, \ldots, y_5]$, where $u := (u_0, \ldots, u_5)$. For every $b \in k^6$, let $Q'_b \subseteq \mathbb{P}^5_k$ be the quadric defined by the quadratic form $q'_b := q'(b_0, \ldots, b_5) \in k[y_0, \ldots, y_5]$.

Let $L \subseteq \mathbb{G}_{m,k} \times \mathbb{P}^5_k \times \mathbb{P}^5_k$ be the subvariety defined by

$$\frac{\partial(q - tq'(u))}{\partial y_0} = \cdots = \frac{\partial(q - tq'(u))}{\partial y_5} = 0, \tag{7-4}$$

where $(t, (u_0 : \cdots : u_5), (y_0 : \cdots : y_5))$ denotes the coordinates on $\mathbb{G}_{m,k} \times \mathbb{P}_k^5 \times \mathbb{P}_k^5$. Let $J(t, u) \in$ $\mathrm{Mat}_{6\times 6}(k[t, u_0, \ldots, u_5])$ be the matrix that defines (7-4) as a linear system in the variables $y_0, \ldots, y_5$ with coefficients in $k[t, u_0, \ldots, u_5]$. Then $\det(J(t, u)) = -p(a - tu)^2$ and every 5×5 minor of $J_{t,u}$ is divisible by $p(a - tu)$.

For every $b \in k^6$, the singular locus of $Q \cap Q_b'$ is contained in the subvariety of $\mathbb{P}_k^5$ where the Jacobian matrix of (q, q_b') has rank 1, that is, the union of the linear subspaces $L_{c,b}$, for $c \in k^\times$, where $L_{c,b} \subseteq \mathbb{P}_k^5$ is the subvariety defined by the linear system $J(c, b)(y_0, \ldots, y_5) = 0$. We observe that $L_{c,b} \neq \varnothing$ if and only if $p(a - cb) = 0$ and in that case $\dim L_{c,b} \geq 1$. Moreover, $\dim L_{c,b} \geq 2$ if and only if all the 4×4 minors of $J(c, b)$ vanish.

The 4×4 minor $m(t, u)$ of $J(t, u)$ obtained by deleting the 4-th and 6-th rows and the 4-th and 6-th columns is $-4u_1^2 t^3 \big(A_1(\beta^{-1}(u))t - u_2\big)$. If $b \in k^6$ satisfies

$$b_1 A_1(\beta^{-1}(b)) p\big(A_1(\beta^{-1}(b))a + b_2 b\big) \neq 0, \tag{7-5}$$

then $m(t, b)$ and $p(a - tb)$ have no common zeros in $k^\times$. Hence, for general $b \in k^6$, the singular locus of $Q \cap Q_b'$ is contained in a union of three lines of $\mathbb{P}_k^5$, as $p(a - tb) \in k[t]$ has degree ≤ 3.

The variety L is defined by equations

$$y_i = l_{t,u,i}(y_3, y_5), \quad i \in \{0, 1, 2, 4\}, \tag{7-6}$$

where $l_{t,u,i}$ are linear forms in $k(t, u_0, \ldots, u_5)[y_3, y_5]$ for $i \in \{0, 1, 2, 4\}$. If $b \in k^6$ satisfies $b_2 p(b_0 a - b) \neq 0$ and (7-5), then for every $c \in k^\times$ such that $L_{c,b} \neq \varnothing$, the line $L_{c,b}$ is defined by the linear equations $y_i = l_{c,b,i}(y_3, y_5)$, for $i \in \{0, 1, 2, 4\}$, obtained by evaluating (7-6) in $t = c$ and $u = b$. By substituting these equations into the equations that define S, we see that $L_{c,b} \cap S$ is the subvariety of $L_{c,b}$ defined by the quadratic equation

$$(1 - cb_0)y_3^2 + cb_4 y_3 y_5 - cb_2 y_5^2 = 0. \tag{7-7}$$

In particular, it is nonempty and of dimension 0.

Let $h_0, \ldots, h_5 \in k$ such that $h := \sum_{i=0}^5 h_i y_i$ is a linear form that defines H. Let $H_3(t, u), H_5(t, u) \in$ $k[t, u_0, \ldots, u_5]$ such that $H_3(t, u)y_3 + H_5(t, u)y_5 \in k(t, u_0, \ldots, u_5)[y_3, y_5]$ is the polynomial obtained by substituting (7-6) into $m(t, u)h$. If $b \in k^6$ satisfies (7-5), then $m(c, b) \neq 0$ for all $c \in k^\times$ such that $L_{c,b} \neq \varnothing$, so the intersection $L_{c,b} \cap H$ is the subvariety of $L_{c,b}$ defined by the equation $H_3(c, b)y_3 + H_5(c, b)y_5 = 0$.

We use Lemma 7.11 to show that for general $b \in k^6$ the polynomials $H_3(t, b), H_5(t, b)$ and $p(a - tb)$ have no common solutions, that is, there is no $c \in k^\times$ such that $L_{c,b} \neq \varnothing$ and $L_{c,b} \subseteq H$. Indeed, we observe that $H_5(t, u)/t \in k[t, u_0, \ldots, u_5]$ is a polynomial of degree 2 in t, and that $\delta(p(a - tu), H_5(t, u)/t)$ is a nonzero element in $k[u_0, \ldots, u_5]$ if $h_0 \neq 0$ or $h_1 \neq 0$ or $h_2 \neq 0$ or $h_4 \neq 0$ or $h_5 \neq 0$. If $h_0 = h_1 = h_2 = h_4 = h_5 = 0$, we observe that $H_3(t, u)/t \in k[t, u_0, \ldots, u_5]$ is a polynomial of degree 2 in t and $\delta(p(a - tu), H_3(t, u)/t)$ is a nonzero element in $k[u_0, \ldots, u_5]$, as $h_3 \neq 0$.

Then, for general $b \in k^6$ and for every $c \in k^\times$ such that $L_{c,b} \neq \varnothing$, the intersection $L_{c,b} \cap H$ consists of a point z, and $L_{c,b} \cap H \cap S$ is defined by evaluating (7-7) in z. Computations show that since $p(a - cb) = 0$,

if $m(c, b) \neq 0$, then $L_{c,b} \cap H \cap S$ is nonempty if and only if $C(c, b) = 0$, where

$$
\begin{aligned}
C(t, u) := &\, w_0^2 h_0^2 + w_1^2 h_1^2 + w_2^2 h_2^2 + w_1 w_2 h_3^2 + w_0 w_2 h_4^2 + w_0 w_1 h_5^2 + (w_5^2 - 2 w_0 w_1) h_0 h_1 \\
&+ (w_4^2 - 2 w_0 w_2) h_0 h_2 - (w_0 w_3 - w_4 w_5) h_0 h_3 + w_0 w_4 h_0 h_4 + w_0 w_5 h_0 h_5 \\
&+ (w_3^2 - 2 w_1 w_2) h_1 h_2 + w_1 w_3 h_1 h_3 - (w_1 w_4 - w_3 w_5) h_1 h_4 + w_1 w_5 h_1 h_5 + w_2 w_3 h_2 h_3 \\
&+ w_2 w_4 h_2 h_4 - (w_2 w_5 - w_3 w_4) h_2 h_5 + w_2 w_5 h_3 h_4 + w_1 w_4 h_3 h_5 + w_0 w_3 h_4 h_5 \quad \text{(7-8)}
\end{aligned}
$$

with $w_i := a_i - t u_i \in k[t, u_0, \dots, u_5]$ for all $i \in \{0, \dots, 5\}$. Since $\delta(p(a - tu), C(t, u))$ is a nonzero element in $k[u_0, \dots, u_5]$, we conclude by Lemma 7.11 that $L_{c,b} \cap H \cap S = \varnothing$ for general $b \in k^6$ and for every $c \in k^\times$ such that $L_{c,b} \neq \varnothing$. $\qquad\square$

Proposition 7.13. *Let X be a variety* [Pro 2013b, (1.2.3)] *defined over a field k of characteristic 0 and cohomological dimension ≤ 1. Then $X(k) \neq \varnothing$.*

Proof. By [Pro 2013b, pp. 426–427], there exists a quadratic extension k' of k such that $X_{k'}$ is a blowing-up of a smooth quadric $Q' \subseteq \mathbb{P}^4_{k'}$ along a twisted quartic curve Γ. Let $\pi : X_{k'} \to \mathbb{P}^4_{k'}$ be the induced morphism. We identify $\mathbb{P}^4_{k'}$ with a hyperplane L_1 of $\mathbb{P}^5_{k'}$ such that $L_1 \cap S = \Gamma$. Let $Q_1 \subseteq \mathbb{P}^5_{k'}$ be the unique quadric containing S such that $Q_1 \cap L_1 = Q'$. Then $X_{k'}$ embeds into $Y_{k'}$ as the strict transform of $Q_1 \cap L_1$ under f, and $\pi = f|_{X_{k'}}$. By Lemma 7.9, the action of $\mathrm{Gal}(k'/k)$ on $X_{k'}$ is induced by an action of $\mathrm{Gal}(k'/k)$ on $Y_{k'}$. Let σ be a generator of $\mathrm{Gal}(k'/k)$. Then $f(\sigma(f^{-1}(L_1))) = Q_1$.

By construction, $v^{-1}(\Gamma)$ is a smooth conic, so $\Gamma(k')$ is nonempty and hence dense in Γ by Remark 7.1. Then the set of hyperplanes of $\mathbb{P}^5_{k'}$ that intersect $\Gamma(k')$ is dense in $\mathbb{P}(H^0(\mathbb{P}^5_{k'}, \mathcal{O}_{\mathbb{P}^5_{k'}}(1)))$. So we can choose a general hyperplane $L_2 \subseteq \mathbb{P}^5_{k'}$ that intersects $\Gamma(k')$ such that $Q_2 := f(\sigma(f^{-1}(L_2))$ is a general member of $\mathscr{S}$, and hence $Q_1 \cap Q_2$ is smooth outside the base locus S of $\mathscr{S}$ by Bertini [Kleiman 1998, Theorem (4.1)] and smooth at all k'-points in $L_1 \cap S$ by Lemma 7.12.

Let $x \in L_2 \cap \Gamma(k')$. Let Z be the strict transform of $Q_1 \cap Q_2$ under f. Then the morphism $f|_Z : Z \to Q_1 \cap Q_2$, which is the blowing-up of $Q_1 \cap Q_2$ at S, is an isomorphism around x because S is a smooth divisor in $Q_1 \cap Q_2$ and x is a smooth point of $Q_1 \cap Q_2$. We observe that $f(\sigma(Z)) = L_1 \cap L_2$. Let $L_3, L_4, L_5 \subseteq \mathbb{P}^5_{k'}$ be hyperplanes such that $L_1 \cap L_2 \cap L_3 \cap L_4 \cap L_5 = \{x\}$. Then

$$
f|_Z^{-1}(x) = Z \cap \sigma(Z) \cap \bigcap_{i=3}^{5} \left(f^{-1}(L_i) \cap \sigma(f^{-1}(L_i)) \right)
$$

consists of a $\mathrm{Gal}(k'/k)$-invariant point of $X_{k'}$. $\qquad\square$

7D2. *Singular case.*

Proposition 7.14. *Let X be a variety* [Pro 2013b, Theorem 6.6(iii)] *defined over a field k of characteristic 0 with $\mathrm{Br}(k) = 0$. Then X has a k-rational point.*

Proof. By [Pro 2013b, p. 432], there exists a quadratic extension k' of k, contained in an algebraic closure $\bar{k}$ of k, such that $X_{k'}$ is a blowing-up of a quadric cone $Q' \subseteq \mathbb{P}^4_{k'}$ with center a curve Γ such that Γ does not contain the vertex v of Q' and $\Gamma_{\bar{k}}$ is the union of two conics $\Gamma^{(1)}$ and $\Gamma^{(2)}$ that intersect each

other transversally. Let σ be the generator of $\mathrm{Gal}(k'/k)$, and let σ_X be the induced σ-linear automorphism of $X_{k'}$. By [Pro 2013b, p. 432], X contains at least two singular points. Since $f(\mathrm{Sing}(X_{\bar{k}})) \smallsetminus \Gamma = \{v\}$, the k'-point $\gamma := f\big(\sigma_X(f_1^{-1}(v))\big)$ belongs to $\mathrm{Sing}(\Gamma) = \Gamma^{(1)} \cap \Gamma^{(2)}$. Let ℓ be the strict transform of the line between v and γ. Since $\ell \cong \mathbb{P}^1_{k'}$, it suffices to show that ℓ is invariant under σ_X, because in that case ℓ contains a σ_X-invariant k'-point, as $\mathrm{Br}(k) = 0$.

Let $\pi : X_{k'} \to \mathbb{P}^4_{k'}$ be the morphism induced by the blowing-up. Let H' be the pullback of a hyperplane of $\mathbb{P}^4_{k'}$ under π. Let E' be the exceptional divisor of the blowing-up. Then $\sigma_X(H') = 2H' - E'$ in $\mathrm{Pic}(X_{k'})$ by [Pro 2013b, (5.2.1), Theorem 6.1, and p. 432]. Let $\mathcal{S}$ be the linear system of quadrics in $\mathbb{P}^4_{k'}$ that contain Γ. Since π is defined by the complete linear system $|H'|$, $|2H' - E'|$ is the linear system of strict transforms of elements of $\mathcal{S}|_{Q'}$. Thus $\dim \mathcal{S} - 1 = \dim \mathcal{S}|_{Q'} = \dim|2H' - E'| = \dim H' = 5$ as k'-vector spaces.

Let $P_i \subseteq \mathbb{P}^4_{\bar{k}}$ be the plane spanned by $\Gamma^{(i)}$ for $i \in \{1, 2\}$. We claim that $P_1 \cap P_2 = \{\gamma\}$. Clearly $P_1 \cap P_2$ contains γ because it contains the intersection $\Gamma^{(1)} \cap \Gamma^{(2)}$. If $P_1 = P_2$, then $\dim \mathcal{S} \geq 9$ as k'-vector space, which is a contradiction. If $P_1 \cap P_2$ has dimension 1, the linear system $\mathcal{S}'$ of quadrics containing $P_1 \cup P_2$ has dimension 6, and hence $\mathcal{S} = \mathcal{S}'$. So the base locus of $\mathcal{S}$ contains $(P_1 \cup P_2) \smallsetminus \Gamma$, contradicting the fact that $\sigma_X(|H'|) = |2H' - E'|$ is basepoint-free.

Now we show that X is a degeneration of a Fano threefold as in Proposition 7.13. Let k''/k' be a quadratic extension such that $\Gamma^{(1)}$ and $\Gamma^{(2)}$ are defined on k''. Let $u \in k''$ such that $k'' = k'(u)$ and $u^2 \in k'$. Without loss of generality, we can choose coordinates $z_0, \ldots, z_4$ on $\mathbb{P}^4_{k'}$ such that

$$\Gamma^{(1)} = \{z_3 - uz_2 = z_1 - uz_4 = z_0 z_2 - z_4^2\} \quad \text{and} \quad \Gamma^{(2)} = \{z_3 + uz_2 = z_1 + uz_4 = z_0 z_2 - z_4^2\}.$$

Let $w : \mathbb{P}^4_{k'} \subseteq \mathbb{P}^5_{k'}$ be the embedding given by $(z_0 : \cdots : z_4) \mapsto (z_0 : u^2 z_2 : z_2 : z_3 : z_4 : z_1)$. Then the image of w is the hyperplane $L = \{y_1 - u^2 y_2 = 0\}$. Also $w(\Gamma) = L \cap S$, and $v^{-1}(L \cap S)$ is the degenerate conic $\{x_1^2 - u^2 x_2^2 = 0\} \subseteq \mathbb{P}^2_{k'}$. Since $\dim \mathcal{S} = 6 = \dim \mathscr{S}$ as k'-vector spaces and $\mathscr{S}|_L = \mathcal{S}$, there exists a unique quadric Q in $\mathscr{S}$ such that $Q \cap L = w(Q')$. Then $X_{k'}$ embeds into $Y_{k'}$ as the strict transform of $Q \cap L$ under f, and $\pi = f|_{X_{k'}}$.

By Lemma 7.9, there is $M \in \mathrm{GL}(3, k')$ such that the σ-linear Cremona transformation $\sigma \circ \varphi(M) \circ A$ defines a σ-linear involution of Y that restricts to σ_X on $X_{k'}$. The quadric Q is the image of L under $\sigma \circ \varphi(M) \circ A$ and has equation $q = (0, 1, -\sigma(u)^2, 0, 0, 0) \cdot \varphi(M) \cdot (A_0, \ldots, A_5)$. Let

$$U := (Q \cap L) \smallsetminus \pi(E' \cup \sigma_X(E')).$$

Computations show that $\pi(\ell) \cap \pi(E' \cup \sigma_X(E')) = \{\gamma, v\}$, and $(\sigma \circ \varphi(M) \circ A)(\pi(\ell) \cap U) = \pi(\ell) \cap U$. Thus, ℓ is invariant under σ_X. $\qquad\square$

8. Terminal Fano varieties of large index

Proof of Theorem 1.7. We denote by $\bar{k}$ an algebraic closure of k. We recall that $r(X) \leq n + 1$ by a result of Shokurov (see [Iskovskikh and Prokhorov 1999, Corollary 2.1.13]).

Gorenstein terminal Fano varieties of index $r(X) \geq n$ are classified in [Fujita 1982, Theorem 0] (see also [Iskovskikh and Prokhorov 1999, Theorem 3.1.14]). If $r(X) = n+1$, then $X_{\bar{k}} \cong \mathbb{P}^n_{\bar{k}}$, and hence $X \cong \mathbb{P}^n_k$ by Remark 7.1. If $r(X) = n$, then the linear system $|-(1/n)K_X|$ embeds X as a quadric hypersurface in $\mathbb{P}^{n+1}_k$. Hence $X(k) \neq \varnothing$ by definition of C_1 field.

Gorenstein terminal Fano varieties of index $r(X) = n - 1$ are del Pezzo varieties (see [Iskovskikh and Prokhorov 1999, Remarks 3.2.2]). Let X be a del Pezzo k-variety of dimension n. We prove that $X(k) \neq \varnothing$ by induction on n. If $n = 2$, then $X_{\bar{k}}$ is a smooth rational surface, and therefore $X(k) \neq \varnothing$ by [Colliot-Thélène 1987, Proposition 2]. Assume that $n > 2$. By [Iskovskikh and Prokhorov 1999, Proposition 3.2.3], the general element of the linear system $-(1/(n-1))K_X$ is a del Pezzo k-variety of dimension $n - 1$, so it has a k-rational point by the inductive hypothesis. Thus $X(k) \neq \varnothing$.

Non-Gorenstein terminal Fano varieties of index $r(X) > n - 2$ are classified in [Sano 1996] and have a rational point by Proposition 7.2 and [Kollár 1996, Theorem IV.6.7]. □

9. Terminal Fano threefolds

9A. *Gorenstein case.* We prove Theorem 1.8.

Proof of Theorem 1.8. We recall that the index $r(X)$ of a Gorenstein terminal Fano variety X is a positive integer. Terminal Fano threefolds of index ≥ 2 are covered by Theorem 1.7. Hence, we need to consider only the cases of index 1.

$\mathbb{Q}$-factorial Gorenstein terminal Fano threefolds of Picard rank 1 and geometric Picard rank > 1 are classified in [Pro 2013b, Theorem 1.2, §2, Theorem 6.6] (see Section 5B).

The varieties [Pro 2013b, (1.2.4), (1.2.7)] have index 2 by [Iskovskikh and Prokhorov 1999, Remarks (vi), p. 217].

The varieties [Pro 2013b, Case 1.2.1 a), p. 421 and (1.2.2), (1.2.8)] are k-models of complete intersections of divisors in products of projective spaces that have a k-rational point by [Pieropan 2021, Theorem 1.1] and Proposition 7.6 (see also [Pro 2013b, Remark 2.1]).

The variety [Pro 2013b, Case 1.2.1 b), p. 421] is a double cover $\pi : X \to V$ of a variety V that belongs to [Pro 2013b, (1.2.4)] with branch locus a member of $|-K_V|$. The morphism $\pi : X_{\bar{k}} \to \mathbb{P}^2_{\bar{k}} \times \mathbb{P}^2_{\bar{k}}$ is defined over k (see [Pro 2013b, proof of Theorem 6.6(ii)]) and the $\mathrm{Gal}(\bar{k}/k)$-action on $V_{\bar{k}}$ is induced by the one on $\mathbb{P}^2_{\bar{k}} \times \mathbb{P}^2_{\bar{k}}$. We observe that V is smooth by [Pro 2013b, Theorem 6.6 (i)] and $\mathrm{Pic}(V_{\bar{k}}) \cong \mathrm{Pic}(\mathbb{P}^2_{\bar{k}} \times \mathbb{P}^2_{\bar{k}})$ by [Hartshorne 1970, Corollary IV.3.3]. Then $\mathrm{rk}\,\mathrm{Pic}(\mathbb{P}^2_{\bar{k}} \times \mathbb{P}^2_{\bar{k}})^{\mathrm{Gal}(\bar{k}/k)} = \mathrm{rk}\,\mathrm{Pic}(V) = 1$. The variety $V_{\bar{k}}$ is a hypersurface of bidegree $(1, 1)$ in $\mathbb{P}^2_{\bar{k}} \times \mathbb{P}^2_{\bar{k}}$. Moreover, $-K_V = (2, 2)$ in $\mathrm{Pic}(V_{\bar{k}})$. Hence, the branch locus of π is defined by the restriction to V of a hypersurface of degree $(2, 2)$ in $\mathbb{P}^2_{\bar{k}} \times \mathbb{P}^2_{\bar{k}}$. Then $X(k) \neq \varnothing$ by Proposition 7.7.

The varieties [Pro 2013b, (1.2.3)] each have a k-rational point by Propositions 7.13 and 7.14.

The variety [Pro 2013b, (1.2.5)] is a k-model X of a double cover of $\mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}}$ with branch locus a member of $|-K_{\mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}}}|$ (which is a divisor of tridegree $(2, 2, 2)$). By [Pro 2013b, Lemma 4.4], the composition with the first projection $f_1 : X_{\bar{k}} \to \mathbb{P}^1_{\bar{k}}$ is a del Pezzo bundle with general fiber F_1.

Let $\{F_1, F_2, F_3\}$ be the $\mathrm{Gal}(\bar{k}/k)$-orbit of F_1 in $\mathrm{Pic}(X_{\bar{k}})$. Then there are two other del Pezzo bundles f_1 and f_2 conjugate to f_1. Then the product $f = f_1 \times f_2 \times f_3 : X_{\bar{k}} \to \mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}}$ is a finite map, and it is defined over k. Thus X is a double cover of a k-model Y of $\mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}}$. Since $\mathrm{rk}\,\mathrm{Pic}(X) = 1$, also $\mathrm{rk}\,\mathrm{Pic}(Y) = 1$ and hence $\mathrm{Pic}(Y)$ is the subgroup of $\mathrm{Pic}(\mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}})$ generated by $(1, 1, 1)$. Then X has a k-point by Proposition 7.7.

The variety [Pro 2013b, (1.2.6)] is a k-model X of an intersection of divisors of tridegrees $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$ in $\mathbb{P}^2_{\bar{k}} \times \mathbb{P}^2_{\bar{k}} \times \mathbb{P}^2_{\bar{k}}$. Let $\pi_1, \pi_2, \pi_3 : \mathbb{P}^2_{\bar{k}} \times \mathbb{P}^2_{\bar{k}} \times \mathbb{P}^2_{\bar{k}} \to \mathbb{P}^2_{\bar{k}} \times \mathbb{P}^2_{\bar{k}}$ be the three projections. The $\mathrm{Gal}(\bar{k}/k)$-action on $\mathbb{P}^2_{\bar{k}} \times \mathbb{P}^2_{\bar{k}} \times \mathbb{P}^2_{\bar{k}}$ is by permutation of the factors by Lemma 7.3, and $\mathrm{rk}\,\mathrm{Pic}(\mathbb{P}^2_{\bar{k}} \times \mathbb{P}^2_{\bar{k}} \times \mathbb{P}^2_{\bar{k}})^{\mathrm{Gal}(\bar{k}/k)} = 1$. So, for every $i, j \in \{1, 2, 3\}$, there exists $g_{i,j} \in \mathrm{Gal}(\bar{k}/k)$ such that $\pi_j = \pi_i \circ g_{i,j}$. Then $\pi_i(X) = \pi_i \circ g_{i,j}(X) = \pi_j(X)$ for all $i, j \in \{1, 2, 3\}$. By [Pro 2013b, Case 1.2.6, p. 422], $\pi_1(X_{\bar{k}}) \subseteq \mathbb{P}^2_{\bar{k}} \times \mathbb{P}^2_{\bar{k}}$ is a hypersurface of degree $(1, 1)$. So $X_{\bar{k}} = \pi_1^{-1}(\pi_1(X)) \cap \pi_2^{-1}(\pi_1(X)) \cap \pi_3^{-1}(\pi_1(X))$. Since $\pi_j^{-1}(\pi_1(X)) = g_{i,j}^{-1}(\pi_i^{-1}(\pi_1(X))$ for every $i, j \in \{1, 2, 3\}$, $X_{\bar{k}}$ is a complete intersections of hypersurfaces H_1, H_2, H_3 in $\mathbb{P}^2_{\bar{k}} \times \mathbb{P}^2_{\bar{k}} \times \mathbb{P}^2_{\bar{k}}$, of degrees $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$ respectively, such that $\{H_1, H_2, H_3\}$ is a $\mathrm{Gal}(\bar{k}/k)$-invariant set under the action of $\mathrm{Gal}(\bar{k}/k)$ over $X_{\bar{k}}$. Hence, $X(k) \neq \varnothing$ by Proposition 7.6.

Indecomposable Gorenstein terminal Fano threefolds of index 1 and geometric Picard rank 1 are classified in [Mukai 1995b, Theorems 1.10, 6.5] (see Sections 5A4 and 5A5). Let X be an indecomposable Gorenstein terminal Fano threefold of index 1 and geometric Picard rank 1. We use the notation introduced in Section 5A4.

If $g = 2$, then $X \subseteq \mathbb{P}(1, 1, 1, 1, 3)$ is a hypersurface of degree 6. Thus $X(k) \neq \varnothing$ by [Kollár 1996, Theorem IV.6.7].

If $g = 3$ and X is a quartic hypersurface in $\mathbb{P}^4_k$, then X has a k-rational point by definition of C_1 field. If $g = 3$ and X is a double cover of a quadric hypersurface Q in $\mathbb{P}^4_k$, then it is ramified along the restriction to Q of a hypersurface of degree 4 (see [Prokhorov 2004, (4.3.2)]). Therefore $X \subseteq \mathbb{P}(1, 1, 1, 1, 1, 2)$ is a complete intersection of two hypersurfaces of degrees 2 and 4, and $X(k) \neq \varnothing$ by [Kollár 1996, Theorem IV.6.7].

If $g \in \{4, 5\}$, the variety X has a k-rational point by [Lang 1952, Theorem 4].

If $g = 12$ and X is singular, then its singular locus consists of a k-point by [Prokhorov 2016, Theorem 1.3]. If $g = 12$ and X is smooth, then the Fano variety of conics of X is isomorphic to $\mathbb{P}^2_k$ by [Kuznetsov et al. 2018, Proposition B.4.1] and Remark 7.1. Hence, X contains smooth conics defined over k, which have k-points, again by Remark 7.1. $\qquad \square$

As recalled in Section 5A5, decomposable terminal Gorenstein Fano threefolds of index 1 and geometric Picard rank 1 are not completely classified. However, the cases with $g \in \{2, 3, 4, 5\}$ are completely classified in [Mukai 1995b, Theorem 6.5]. Moreover, if $g = 12$, the only decomposable cases we are interested in are classified in [Prokhorov 2016, Theorem 1.3]. These cases are covered by Theorem 1.8.

9B. *Non-Gorenstein case.* We start by considering non-Gorenstein terminal Fano threefolds of index 1 with only cyclic quotient singularities, which are classified in [Sano 1995].

Theorem 9.1. *Let X be a non-Gorenstein terminal Fano threefold of index 1 with only cyclic quotient singularities over a C_1 field k of characteristic 0 that admits normic forms of arbitrary degree. Then $X(k) \neq \varnothing$.*

Proof. Let $\bar{k}$ be an algebraic closure of k. By Lemma 5.2, the double covering in [Sano 1995, Theorem 1.1] is defined over k; we denote it by $Y \to X$. Moreover, Y is a smooth Fano threefold, and there are 14 possible cases for Y.

In the cases [Sano 1995, Theorem 1.1, nos. 8, 12, 14], the variety Y has index 2, and then $Y(k) \neq \varnothing$ by Theorem 1.7. In the remaining cases the variety Y has index 1. If Y has Picard rank 1 over k, then $Y(k) \neq \varnothing$ by Theorem 1.8. Therefore we can assume that Y has Picard rank ≥ 2 over k.

In the case [Sano 1995, Theorem 1.1, nos. 3, 7] (see also [Iskovskikh and Prokhorov 1999, §12.3, nos. 3, 10]), the variety Y is a blowup of a smooth Fano threefold of index 2 that has a k-point by Theorem 1.7. Then $Y(k) \neq \varnothing$ by [Nishimura 1955].

In the cases [Sano 1995, Theorem 1.1, nos. 4, 10, 13], by studying the automorphisms of $Y_{\bar{k}}$ as in [Bayle 1994, §§6.5.1, 6.6.1, 6.6.2], we conclude that Y is isomorphic to $C \times_k S_d$, where C is a smooth conic and S_d is a smooth del Pezzo surface of degree $d \in \{2, 4, 6\}$. Since $C(k) \neq \varnothing$ by Remark 7.1 and $S_d(k) \neq \varnothing$ by [Colliot-Thélène 1987, Proposition 2], we conclude that $Y(k) \neq \varnothing$.

In the case [Sano 1995, Theorem 1.1, no. 5] (see also [Iskovskikh and Prokhorov 1999, §12.4, no. 1]), the variety Y is a double covering of a k-form Z of $\mathbb{P}_k^1 \times \mathbb{P}_k^1 \times \mathbb{P}_k^1$ ramified along a divisor of tridegree $(2, 2, 2)$. Since $\operatorname{rk} \operatorname{Pic}(Y_{\bar{k}}) = 3$, we deduce that $\operatorname{rk} \operatorname{Pic}(Z) = \operatorname{rk} \operatorname{Pic}(Y) \geq 2$. By Lemma 7.3, this can only happen if Z is a product of a smooth conic C with a k-form of $\mathbb{P}_k^1 \times \mathbb{P}_k^1$. By [Pro 2013b, Lemma 4.4], the composition with the projection $Z \to C$ is a del Pezzo bundle structure on Y. By Remark 7.1, $C(k)$ is nonempty, and hence Zariski dense in C. Thus Y contains a smooth del Pezzo surface defined over k that has a k-point by [Colliot-Thélène 1987, Proposition 2].

In the case [Sano 1995, Theorem 1.1, no. 6], since $\operatorname{rk} \operatorname{Pic}(Y) = 2$, the two extremal contractions from [Iskovskikh and Prokhorov 1999, Case (C_1, C_1), p. 140] are defined over k. Hence there is a dominant morphism $Y \to \mathbb{P}_k^2$ whose general fiber is a smooth conic, and hence contains k-points by Remark 7.1.

In the case [Sano 1995, Theorem 1.1, no. 9], Y is a k-model of a complete intersection of three divisors of tridegree $(1, 1)$ in $\mathbb{P}_{\bar{k}}^3 \times \mathbb{P}_{\bar{k}}^3$, so Y is a complete intersection of three divisors of bidegree $(1, 1)$ in a k-form of $\mathbb{P}_k^3 \times \mathbb{P}_k^3$ by [Pieropan 2021, Theorem 1.1] and $Y(k) \neq \varnothing$ by Proposition 7.6.

In the case [Sano 1995, Theorem 1.1, no. 11], Y is a k-model of a divisor of multidegree $(1, 1, 1, 1)$ in $\mathbb{P}_{\bar{k}}^1 \times \mathbb{P}_{\bar{k}}^1 \times \mathbb{P}_{\bar{k}}^1 \times \mathbb{P}_{\bar{k}}^1$, so $Y(k) \neq \varnothing$ by Proposition 7.6. $\qquad\square$

The [Graded Ring Database] contains the list of possible baskets [Brown and Suzuki 2007, §2] of non-Gorenstein singularities for geometrically $\mathbb{Q}$-factorial non-Gorenstein terminal Fano threefolds of geometric Picard rank 1 over an algebraically closed field of characteristic 0. Such varieties X are studied in [Brown and Suzuki 2007] using the Fano index $f(X)$ defined in the introduction. The Fano index $f(X)$ divides $q(X)$ and they coincide if there is no torsion in the Weil divisor class group (see Prokhorov [2007, Corollary 2.3; 2010, Lemma 3.2]). In particular, $f(X) \in \{1, \ldots, 9, 11, 13, 17, 19\}$ by [Suzuki 2004].

Remark 9.2. The basket of singularities of a terminal Fano variety was introduced in [Reid 1987, (8.2), (10.2)]. It is a collection of quotient singularity germs. To each non-Gorenstein singular point of the variety there is an associated collection of quotient singularity germs. The disjoint union of such collections forms the basket of the variety. The collection of quotient singularity germs associated to a singular point is invariant under the automorphisms of the variety. Therefore, if a given quotient singularity germ appears only once in a basket (we say it has multiplicity one in the basket), then the corresponding singular point is a fixed point for all the automorphisms of the variety.

Theorem 9.3. *Let k be a field of characteristic 0 and $\bar{k}$ an algebraic closure. Let X be a non-Gorenstein terminal Fano threefold over k such that $X_{\bar{k}}$ is $\mathbb{Q}$-factorial and $\mathrm{rk}\,\mathrm{Pic}(X_{\bar{k}}) = 1$.*

(1) If $f(X_{\bar{k}}) \geq 6$, then $X(k) \neq \varnothing$.

(2) If $f(X_{\bar{k}}) = 5$ and k is C_1 and admits normic forms of arbitrary degree, then $X(k) \neq \varnothing$.

Proof. By inspection in the [Graded Ring Database, Fano 3-folds], if $f(X_{\bar{k}}) \geq 6$, each possible basket of singularities for X contains a quotient singularity germ that appears with multiplicity one. Hence, the corresponding singular point on $X(\bar{k})$ is invariant under the action of $\mathrm{Gal}(\bar{k}/k)$, by Remark 9.2. If $f(X_{\bar{k}}) = 5$, the same argument works for all possible baskets of singularities except one [Graded Ring Database, Fano 3-folds, ID 41439], which is realized by a hypersurface of degree 6 in $\mathbb{P}(1, 2, 2, 3, 3)$, and hence has a k-point by [Kollár 1996, Theorem IV.6.7]. □

The [Graded Ring Database] contains 1,847 cases with Fano index $f \in \{2, 3, 4\}$ and 52,646 cases with Fano index $f = 1$. The arguments used in Theorem 9.3 work for all except 9 cases if $f = 4$, for all except 13 cases if $f = 3$, for all except 109 cases if $f = 2$, and for at least 1,581 cases if $f = 1$ (the basket has cardinality 1 in 1,415 cases, and the variety is a Fano complete intersection of codimension at most 2 in a weighted projective space in 166 further cases).

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