Teleportation of quantum coherence

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We investigate whether it is possible to teleport the coherence of an unknown quantum state from Alice to Bob by communicating a smaller number of classical bits in comparison to what is required for teleporting an unknown quantum state. We find that we cannot achieve perfect teleportation of coherence with one bit of classical communication for an arbitrary qubit. However, we find that if the qubit is partially known, i.e., chosen from the equatorial and polar circles of the Bloch sphere, then teleportation of coherence is possible with the transfer of one cbit of information when we have maximally entangled states as a shared resource. In the case of the resource being a nonmaximally entangled state, we can teleport the coherence with a certain probability of success. In a general teleportation protocol for coherence, we derive a compact formula for the final state at Bob's laboratory in terms of the composition of the completely positive maps corresponding to the shared resource state and a joint positive operator-valued measure (POVM) performed by Alice on her qubit and the unknown state. Using this formula, we show that teleportation of the coherence of a partially known state with real matrix elements is perfectly possible with the help of a maximally entangled state as a resource. Furthermore, we explore the teleportation of coherence with Werner states and show that even when Werner states become separable, the amount of teleported coherence is nonzero, implying the possibility of teleportation of coherence without entanglement.

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I. INTRODUCTION

Quantum coherence [1] and entanglement [2] are two central features in quantum theory that make the theory counterintuitive and can also be utilized as a resource to perform important information-theoretic and computational tasks. Motivated by this increasing importance, a general study from the perspective of resource theory is being formulated. Several measures based on resource-theoretic approaches are available to quantify the coherence and the entanglement present in a quantum system [1–19]. Quantum coherence can be interpreted in several ways, such as a measure of nonclassicality in physical systems, a measure of superposition, and a quantity capturing the wave aspect of the state vector [20–22]. It is also observed that perfect cloning and broadcasting of quantum coherence are not possible [23,24]. Coherence acts as a resource in tasks like quantum algorithms [25–27], biological processes [28,29], quantum metrology [30–34], reference-frame alignment [35], and thermodynamic tasks [36–40]. From the point of view of resource theory, coherence falls into two classes, namely, speakable coherence and unspeakable coherence [41]. The quantification of coherence involves a chosen basis. In situations where the labeling of the chosen basis is not important, the relevant notion of coherence is the speakable one, and in situations where the labeling or the identity of the basis elements matters, the relevant notion of coherence is the unspeakable one. The coherence involved in quantum metrology, reference-frame alignment, and thermodynamic tasks is unspeakable coherence, whereas the coherence involved in computational, cryptographic, and communication tasks is speakable coherence.

In the last two decades, several developments have been made in quantum-based technologies. These include quantum repeaters for communicating over large distances [42], quantum teleportation [43], and broadcasting of entanglement [44–47]. They were introduced in the process with the vision of having quantum networks [48–50]. In particular, "quantum teleportation" is one of the most fascinating discoveries of the 20th century [43]. The entangled states which are useful for teleportation are identified and detected in this process [51–54]. Research was also done to implement the teleportation process, if not perfectly, at least with a certain probability of success [55]. Recently, it was also utilized to understand phenomena like closed timelike curves [56]. It has been demonstrated in the laboratory with the help of various resources like photonic qubits, nuclear magnetic resonance [57], optical modes [58–66], atomic ensembles [67–70], trapped atoms [71–75], and solid-state systems [76–79]. Efficient teleportation was also achieved in terms of distance [80,81].

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In this paper, in general, we address the question of whether we can teleport the coherence of a quantum state with a smaller number of classical bits than what is required to teleport the state itself. It is already known that teleportation of an unknown state is possible with the transfer of two classical bits of information. The process of teleportation of a known state is known as remote state preparation, and it requires the transfer of one bit of information for the process to be (only for equatorial qubits) perfect [82]. It is obvious that once all the information about the state is recreated in a different location, the coherence of the state is also transferred. There are many instances where it has been seen that the coherence of the state is not all the information about the state [23]. So a natural question is, Will the transfer of coherence in either known or unknown cases require a smaller number of bits? We find in this paper that we are able to teleport the coherence of a partially known state with one bit of classical communication and with a maximally entangled resource when the qubits are taken from equatorial and polar circles. In addition, we show that if we start with a nonmaximally entangled state as a resource, then we can teleport coherence probabilistically. Here, we come up with a compact formula for the final state at Bob's laboratory. It is written in terms of the composition of the completely positive maps corresponding to the shared resource state and the joint positive operator-valued measure (POVM) performed by Alice on her qubit and the unknown particle. With this, we also determine the amount of coherence that is teleported when the shared resource state is maximally entangled mixed states and Werner states. We also discuss the general teleportation of coherence for an initial mixed state as an input state at Alice's laboratory. The results of the paper stand out from the perspective of transferring the coherence of a coherent state to a state where there is no coherence. This is important in a quantum network since coherence is a useful resource. In a quantum network, one node can teleport the coherence to another node where it is required. Here, we show that this is possible without actually transferring all of the information about the state. Interestingly, we are able to show this can be done much more cost-effectively with a smaller number of classical bits if the states are from the equatorial or polar circles. Our results are useful from the perspective of work extraction as well. Korzekwa *et al.* [19] showed the existence of thermal machines that can extract work from coherence arbitrarily well. These machines have to operate on only individual copies of a state. When Alice and Bob share a maximally entangled state, before the teleportation of coherence, the coherence on Bob's side is zero, and hence, no work extraction is possible. However, after the teleportation, Bob can use the teleported coherence for work extraction.

This paper is organized as follows: In Sec. II, we study the possibility of the teleportation of quantum coherence of a pure state with maximally as well as nonmaximally entangled resources using fewer cbits. In Sec. III, we investigate the same for an arbitrary mixed state with a maximally entangled mixed state and a Werner state as resources and demonstrate the possibility of teleporting some fraction of the initial coherence without entanglement. We conclude the paper in Sec. IV.

II. TELEPORTATION OF QUANTUM COHERENCE WITH PURE ENTANGLED STATES AS A RESOURCE

In this section, our aim is to address the question of whether we can perfectly teleport the coherence of an unknown state with a smaller number of classical bits than what is required for the teleportation of the state. Before we present our protocol, let us briefly summarize the notion of quantum coherence.

Let $\mathcal{H} = \mathbb{C}^d$ be the d-dimensional Hilbert space associated with the qudit of interest. Here, $S(\mathcal{H})$ is the set of positive trace-class linear operators, with the trace being 1 in Hilbert space $\mathcal{H}$. For a state $\rho \in S(\mathcal{H})$, the coherence of the state is given by $C_{l_1}(\rho) = \sum_{i\neq j}^{d} |\langle i|\rho|j\rangle|$, where $\{|i\rangle\}$ is an orthonormal basis of $\mathcal{H}$. It is clear from the definition that the l_1-norm of coherence is basis dependent. The states which are diagonal in the matrix representation with respect to this chosen basis are defined as incoherent states. Therefore, the incoherent states are of the form $\rho = \sum_{i=1}^{d} \lambda_i |i\rangle\langle i|$. Let us denote the set of all incoherent states by $I \subset S(\mathcal{H})$. Quantum operations on a physical system are mathematically represented by a completely positive and trace-preserving (CPTP) linear map $\Phi : B(\mathcal{H}) \to B(\mathcal{H})$, with $B(\mathcal{H})$ being the set of bounded linear operators on $\mathcal{H}$. It is well known that the action of a CPTP map Φ on a quantum state ρ can be represented as $\Phi(\rho) = \sum_i^n K_i \rho K_i^\dagger$, where $\sum_i^n K_i^\dagger K_i = \mathbb{I}$, with $\mathbb{I}$ being the identity operator on $\mathcal{H}$. This representation of a CPTP map is known as the operator-sum representation, and the linear operators $K_i : \mathcal{H} \to \mathcal{H}$ are called Kraus operators. Given a CPTP map, its Kraus representation is not unique. A quantum operation Φ is said to be incoherent if $K_i I K_i^\dagger \subset I$ for all $i = 1, \ldots, n$.

Now any measure of coherence C is defined to be a functional from the set of quantum states to the non-negative real numbers such that the following constraints are obeyed.

(i) $C(\rho) = 0$ for all $\rho \in I$; that is, for all incoherent states, the measure of coherence is zero.

(ii) The measure of coherence should not increase under incoherent operations. If Λ is an incoherent operation, then $C(\Lambda(\rho)) < C(\rho)$.

(iii) It is strongly monotonic: It does not increase under selective incoherent operations on average, i.e., $C(\sum_i p_i \rho_i) < C(\rho)$ for all $\{K_i\}$ with $\sum_i K_i^\dagger K_i = \mathbb{I}$ and $K_i I K_i^\dagger \subset I$, where $p_i = \mathrm{Tr}(K_i \rho K_i^\dagger)$ and $\rho_i = \frac{K_i \rho K_i^\dagger}{p_i}$.

Let us denote the system whose state is the unknown quantum state $|\psi\rangle$ as "system 1," one part of the shared state in Alice's laboratory as "system 2," and the other part on Bob's side as "system 3." We imagine that Alice and Bob share an entangled pair in the maximally entangled state $|\Phi^+\rangle_{23} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. An unknown state $\rho_1 = |\psi\rangle\langle\psi|$ is given to Alice, and it has an amount of coherence equal to $C(\rho_1)$. We consider a general scenario where Alice performs a joint measurement Π_i which is a POVM on the input particle and half of the entangled pair, and $\sum_i \Pi_i = \mathbb{I}$. Now, depending on the measurement outcomes, Bob's reduced state, which is un-normalized, can be written as

$$\rho_3^{(i)} = \mathrm{Tr}_{12}\big[\Pi_{12}^{(i)}(\rho_1 \otimes |\Phi^+\rangle_{23}\langle\Phi^+|)\Pi_{12}^{(i)}\big], \tag{1}$$

where we have adopted the notation that $\Pi_{12}^{(i)}$ means $\Pi_{12}^{(i)} \otimes \mathbb{I}$.

After Alice communicates the measurement outcome to Bob via a classical channel, Bob performs a local unitary, and

we expect the coherence of the state at Bob's location to have the same coherence as that of the input state, i.e., $C(\rho_3^{(i)}) = C(\rho)$. If this condition holds, then we say that teleportation is complete.

In this paper, we start with a two-level system as an input and an arbitrary two-qubit shared state as a resource. In particular, we also consider both maximally and nonmaximally entangled states as shared resources. For an unknown qubit on Alice's side, we question whether we can teleport the coherence of the qubit to Bob's side with one bit of classical communication or not. In each of these cases, we find that we cannot do it perfectly with one bit of classical communication universally. However, if we have a maximally entangled state as a resource and we have partial knowledge about the state of the input qubit, i.e., it is chosen from one of the two great circles (equatorial and the polar circle) on the Bloch sphere, coherence teleportation is possible perfectly with the use of one cbit.

A. Teleportation of coherence of a qubit with a maximally entangled state as a resource

The main idea is to teleport the coherence of an unknown state $|\psi\rangle$ from Alice's laboratory to Bob's laboratory, rather than the state itself. If, with the help of only one cbit, Alice can teleport a state to Bob's laboratory that has the same l_1-norm coherence as the unknown state, $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \in \mathbb{C}^2$, where α and β are complex numbers such that $|\alpha|^2 + |\beta|^2 = 1$, then the purpose is fulfilled.

The unknown quantum state $|\psi\rangle$ can be represented as

$$|\psi\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\phi}|1\rangle, \tag{2}$$

where $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \phi < 2\pi$ (the exact values of θ and ϕ are not known to us). The coherence of the state $|\psi\rangle$ in terms of these parameters with respect to the computational basis $\{|0\rangle, |1\rangle\}$ is $C(\psi) = 2|\alpha\beta| = \sin\theta$. Our aim is to have coherence $C(\psi) = \sin\theta$ in Bob's laboratory at the end of the protocol.

In the standard teleportation protocol, Alice and Bob share a maximally entangled state $|\Phi\rangle_{23} = |\Phi^+\rangle_{23} = \frac{1}{\sqrt{2}}(|00\rangle_{23} + |11\rangle_{23})$ as a resource. It is evident that if we teleport the unknown state itself, then the coherence also gets teleported. The l_1-norm coherence of all those states (before Bob's unitary operation) is $2\cos(\theta/2)\sin(\theta/2)$, which is the same as the coherence of the initial state. However, this still requires two bits of classical information.

However, the interesting question will be whether perfect teleportation of coherence can be done with a smaller number of cbits. The main idea is that instead of performing a Bell measurement, we perform a POVM measurement by adding different Bell projectors in various combinations. Given four Bell projectors, there are only three different ways we can add them to form a complete measurement. Let us consider the following situations in which the state shared between Alice and Bob is

$$|\Phi^+\rangle = \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \tag{3}$$

Case I. Let Alice and Bob share a maximally entangled state $|\Phi^+\rangle$. The combined state of the input and the shared state can be expressed as

$$|\psi\rangle \otimes |\Phi^+\rangle = \frac{1}{2}\sum_{i=0}^{3} |B_i\rangle \otimes u_i|\psi\rangle, \tag{4}$$

where $|B_i\rangle$ are the four mutually orthogonal Bell states and u_i are the local unitary operations $(I, \sigma_x, \sigma_y, \sigma_z)$.

Alice, instead of performing a Bell measurement, performs the following POVM on the input and half of the entangled pair:

$$\Pi_{12}^{(0)} = |\Phi^+\rangle\langle\Phi^+| + |\Psi^+\rangle\langle\Psi^+|, \tag{5}$$

$$\Pi_{12}^{(1)} = |\Phi^-\rangle\langle\Phi^-| + |\Psi^-\rangle\langle\Psi^-|, \tag{6}$$

where $|\Phi^\pm\rangle$ and $|\Psi^\pm\rangle$ are standard Bell states given by

$$|\Phi^-\rangle = \tfrac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \tag{7}$$

$$|\Psi^+\rangle = \tfrac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \tag{8}$$

$$|\Psi^-\rangle = \tfrac{1}{\sqrt{2}}(|01\rangle - |10\rangle). \tag{9}$$

After Alice performs the measurement given by the POVM element $\Pi_{12}^{(0)}$ she communicates her result to Bob. The state of the particle in Bob's laboratory is given by

$$\rho_3^{(0)} = \frac{1}{p_0}\mathrm{Tr}_{12}\big[\Pi_{12}^{(0)}(|\psi\rangle\langle\psi| \otimes |\Phi^+\rangle\langle\Phi^+|)\Pi_{12}^{(0)}\big],$$

where $p_0 = \mathrm{Tr}_{123}[\Pi_{12}^{(0)}(|\psi\rangle\langle\psi| \otimes |\Phi^+\rangle\langle\Phi^+|)\Pi_{12}^{(0)}]$. Therefore, Bob's state can be expressed as

$$\rho_3^{(0)} = \frac{\mathbb{I}}{2} + \mathrm{Re}(\alpha\beta^*)(|0\rangle\langle 1| + |1\rangle\langle 0|).$$

For the measurement with the POVM element $\Pi_{12}^{(1)}$, the state of the particle in Bob's laboratory is given by

$$\rho_3^{(1)} = \frac{1}{p_1}\mathrm{Tr}_{12}\big[\Pi_{12}^{(1)}(|\psi\rangle\langle\psi| \otimes |\Phi^+\rangle\langle\Phi^+|)\Pi_{12}^{(1)}\big],$$

where $p_1 = \mathrm{Tr}_{123}[\Pi_{12}^{(1)}(|\psi\rangle\langle\psi| \otimes |\Phi^+\rangle\langle\Phi^+|)\Pi_{12}^{(1)}]$. Therefore, Bob's state can be expressed as

$$\rho_3^{(1)} = \frac{\mathbb{I}}{2} - \mathrm{Re}(\alpha\beta^*)(|0\rangle\langle 1| + |1\rangle\langle 0|).$$

The coherence of the state at Bob's site for both cases is

$$C_{l1} = 2|\mathrm{Re}(\alpha\beta^*)| = \cos(\phi)\sin(\theta) \neq \sin(\theta).$$

Interestingly, this equality will hold when $\cos(\phi) = 1$. This condition actually tells us that if we have partial information about the state of the qubit, i.e., it comes from the equatorial circle, then the teleportation of the coherence is possible with the help of only one cbit.

Case II. In this case, Alice performs the following POVM on her part:

$$E_{12}^{(0)} = |\Phi^+\rangle\langle\Phi^+| + |\Psi^-\rangle\langle\Psi^-|,$$
$$E_{12}^{(1)} = |\Phi^-\rangle\langle\Phi^-| + |\Psi^+\rangle\langle\Psi^+|.$$

The state in Bob's part after Alice performs the measurement with POVM element $E_{12}^{(0)}$ and communicates the results to Bob is given by

$$\rho_3^{(0)} = \frac{1}{p_0}\mathrm{Tr}_{12}\big[E_{12}^{(0)}(|\psi\rangle\langle\psi| \otimes |\Phi^+\rangle\langle\Phi^+|)E_{12}^{(0)}\big],$$

where $p_0 = \mathrm{Tr}_{123}[E_{12}^{(0)}(|\psi\rangle\langle\psi| \otimes |\Phi^+\rangle\langle\Phi^+|)E_{12}^{(0)}]$. Therefore, Bob's state can be expressed as

$$\rho_3^{(0)} = \frac{\mathbb{I}}{2} + i\mathrm{Im}(\alpha\beta^*)(|0\rangle\langle 1| - |1\rangle\langle 0|).$$

The state in Bob's part after Alice performs the measurement with POVM element $E_{12}^{(1)}$ and communicates the results to Bob is given by

$$\rho_3^{(1)} = \frac{1}{p_1}\mathrm{Tr}_{12}\big[E_{12}^{(1)}(|\psi\rangle\langle\psi| \otimes |\Phi^+\rangle\langle\Phi^+|)E_{12}^{(1)}\big],$$

where $p_1 = \mathrm{Tr}_{123}[E_{12}^{(1)}(|\psi\rangle\langle\psi| \otimes |\Phi^+\rangle\langle\Phi^+|)E_{12}^{(1)}]$. Bob's state in this case can be expressed as

$$\rho_3^{(1)} = \frac{\mathbb{I}}{2} - i\mathrm{Im}(\alpha\beta^*)(|0\rangle\langle 1| - |1\rangle\langle 0|).$$

Now the l_1-norm coherence of these two states turns out to be

$$\begin{aligned} C_{l1}\big(\rho_3^{(0)}\big) = C_{l1}\big(\rho_3^{(1)}\big) &= 2|\mathrm{Im}(\alpha\beta^*)| \\ &= 2\sin\phi\cos(\theta/2)\sin(\theta/2) \neq 2\cos(\theta/2)\sin(\theta/2). \end{aligned}$$

Interestingly, the equality will hold when $\sin(\phi) = 1$. Like in the previous case, if the qubit is partially known, i.e., it comes from the polar circle, then the teleportation of coherence is perfectly possible with the use of only one cbit.

Case III. If Alice performs the POVM

$$\begin{aligned} F_{12}^{(0)} &= |\Phi^+\rangle\langle\Phi^+| + |\Phi^-\rangle\langle\Phi^-|, \\ F_{12}^{(1)} &= |\Psi^+\rangle\langle\Psi^+| + |\Psi^-\rangle\langle\Psi^-| \end{aligned}$$

on her part, the state in Bob's part after Alice performs the measurement with POVM element $F_{12}^{(0)}$ and communicates the results to Bob is given by

$$\rho_3^{(0)} = \frac{1}{p_0}\mathrm{Tr}_{12}\big[F_{12}^{(0)}(|\psi\rangle\langle\psi| \otimes |\Phi^+\rangle\langle\Phi^+|)F_{12}^{(0)}\big],$$

where $p_0 = \mathrm{Tr}_{123}[F_{12}^{(0)}(|\psi\rangle\langle\psi| \otimes |\Phi^+\rangle\langle\Phi^+|)F_{12}^{(0)}]$. Therefore, Bob's state can be expressed as

$$\rho_3^{(0)} = \frac{\mathbb{I}}{2}.$$

The state in Bob's part after Alice performs the measurement with POVM element $F_{12}^{(1)}$ and communicates the results to Bob is given by

$$\rho_3^{(1)} = \frac{1}{p_1}\mathrm{Tr}_{12}\big[F_{12}^{(1)}(|\psi\rangle\langle\psi| \otimes |\Phi^+\rangle\langle\Phi^+|)F_{12}^{(1)}\big],$$

where $p_1 = \mathrm{Tr}_{123}[F_{12}^{(1)}(|\psi\rangle\langle\psi| \otimes |\Phi^+\rangle\langle\Phi^+|)F_{12}^{(1)}]$. Therefore, Bob's state can be expressed as

$$\rho_3^{(1)} = \frac{\mathbb{I}}{2}.$$

So, in this case, with the help of one cbit, it is not possible to teleport the coherence of the qubit, even if it is partially known, because the final state has zero coherence. We show that perfect teleportation of coherence is *not* possible universally with a maximally entangled state as a resource and a POVM measurement on Alice's side with the transfer of one cbit of information. However, if the input qubit happens to be from the equatorial and polar circles, then we can perfectly teleport the coherence of the qubit with the help of one cbit of information.

When the partially known qubit is not from the equatorial and polar circles of the Bloch sphere, it is interesting to see whether Bob can increase the teleported coherence by applying any general unitary transformation on his part after receiving the cbit from Alice so that the coherence of his qubit becomes the same as the original coherence on Alice's side. Consider a general SU(2) matrix

$$U_{2\times 2} = \begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix},$$

where a and b are complex numbers such that $|a|^2 + |b|^2 = 1$.

In case I, after the application of U, the new state denoted by ρ_U will be

$$\rho_U = U\rho_3^{\Pi_0}U^\dagger = U\begin{bmatrix} 1/2 & \mathrm{Re}(\alpha\beta^*) \\ \mathrm{Re}(\alpha\beta^*) & 1/2 \end{bmatrix}U^\dagger.$$

The l_1-norm coherence turns out to be

$$\begin{aligned} C_{l1}(\rho_U) &= 2|\mathrm{Re}(\alpha\beta^*)||(a^2 - b^2)| \\ &= 2|\mathrm{Re}(\alpha\beta^*)|\sqrt{1 - [2\mathrm{Re}(ab^*)]^2} \neq 2|\alpha||\beta|. \end{aligned} \tag{10}$$

Note that the maximum possible value of the quantity $\sqrt{1 - [2\mathrm{Re}(ab^*)]^2}$ is 1. The right-hand side of Eq. (10) can, at most, be $2\mathrm{Re}(\alpha\beta^*)$, which is not equal to $2|\alpha||\beta|$, so the application of unitary operators on Bob's part does not give any significant advantage. Similarly, for the other cases, it can be shown that there is no *general* unitary operator that can take all of Bob's possible states to the state which will have the coherence of the original states. Hence, although perfect teleportation of quantum coherence is possible for specific classes of states (states from the equatorial and polar circles), universally, it is not possible even if Bob applies a unitary transform on his part.

B. Teleportation of coherence with the help of a nonmaximally entangled state

Here, we investigate whether we can teleport the coherence of an unknown state using a nonmaximally entangled state and by communicating one cbit of information. We will see that, indeed, it is possible with a certain probability of success if we have partial knowledge of the state.

Let the state shared between Alice and Bob be the following:

$$|\Phi\rangle_{AB} = \frac{1}{\sqrt{1 + |n|^2}}(|00\rangle + n|11\rangle), \tag{11}$$

where n is a complex number. Now let us consider the following four mutually orthonormal vectors, which form a basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$:

$$\begin{aligned} |\Phi_n^+\rangle &= \frac{1}{\sqrt{1 + |n|^2}}(|00\rangle + n|11\rangle), \\ |\Phi_n^-\rangle &= \frac{1}{\sqrt{1 + |n|^2}}(n^*|00\rangle - |11\rangle), \\ |\Psi_n^+\rangle &= \frac{1}{\sqrt{1 + |n|^2}}(|01\rangle + n^*|10\rangle), \\ |\Psi_n^-\rangle &= \frac{1}{\sqrt{1 + |n|^2}}(n|01\rangle - |10\rangle). \end{aligned}$$

Notice that these are nonmaximally entangled vectors in $\mathbb{C}^2 \otimes \mathbb{C}^2$.

Case I. Alice will perform the following POVM on her part:

$$e_{12}^{(0)} = |\Phi_n^+\rangle\langle\Phi_n^+| + |\Psi_n^-\rangle\langle\Psi_n^-|,$$
$$e_{12}^{(1)} = |\Phi_n^-\rangle\langle\Phi_n^-| + |\Psi_n^+\rangle\langle\Psi_n^+|.$$

The state in Bob's part after Alice performs the measurement with POVM element $e_{12}^{(0)}$ and communicates the results to Bob is given by

$$\rho_3^{(0)} = \frac{1}{p_0}\mathrm{Tr}_{12}\big[e_{12}^{(0)}(|\psi\rangle\langle\psi| \otimes |\Phi_n^+\rangle\langle\Phi_n^+|)e_{12}^{(0)}\big],$$

where $p_0 = \mathrm{Tr}_{123}[e_{12}^{(0)}(|\psi\rangle\langle\psi| \otimes |\Phi_n^+\rangle\langle\Phi_n^+|)e_{12}^{(0)}]$. Therefore, Bob's state can be expressed as

$$\begin{aligned}\rho_3^{(0)} = {} & \frac{1}{1+|n|^4}|0\rangle\langle 0| + \frac{|n|^4}{1+|n|^4}|1\rangle\langle 1| \\ & + \frac{|n|^2}{1+|n|^4}2i\mathrm{Im}(\alpha\beta^*)|0\rangle\langle 1| \\ & - \frac{|n|^2}{1+|n|^4}2i\mathrm{Im}(\alpha\beta^*)|1\rangle\langle 0|.\end{aligned}$$

The probability of success for the POVM $e_{12}^{(0)}$ to be clicked is given by $p_0 = \frac{1+|n|^4}{(1+|n|^2)^2}$.

The coherence of the state on Bob's side is given by

$$\begin{aligned}C_{l_1}(\rho_3^{(0)}) &= \frac{4|n|^2}{1+|n|^4}|\mathrm{Im}(\alpha\beta^*)| \\ &= \frac{2|n|^2}{1+|n|^4}2\sin\phi\cos(\theta/2)\sin(\theta/2) \\ &\neq 2\cos(\theta/2)\sin(\theta/2).\end{aligned}$$

Unlike the case discussed in the previous section, even if the state of the input qubit is partially known, i.e., it is from the polar circle of the Bloch sphere, the coherence is not perfectly teleported. It is $\frac{2|n|^2}{1+|n|^4}$ times the original coherence.

The state in Bob's part after Alice performs the measurement with POVM element $e_{12}^{(1)}$ and communicates the results to Bob is given by

$$\rho_3^{(1)} = \frac{1}{p_1}\mathrm{Tr}_{12}\big[e_{12}^{(1)}(|\psi\rangle\langle\psi| \otimes |\Phi_n^+\rangle\langle\Phi_n^+|)e_{12}^{(1)}\big],$$

where $p_1 = \mathrm{Tr}_{123}[e_{12}^{(1)}(|\psi\rangle\langle\psi| \otimes |\Phi_n^+\rangle\langle\Phi_n^+|)e_{12}^{(1)}]$. Therefore, Bob's state can be expressed as

$$\rho_3^{(1)} = \frac{\mathbb{I}}{2} + 2i\mathrm{Im}(\alpha\beta^*)|0\rangle\langle 1| - 2i\mathrm{Im}(\alpha\beta^*)|1\rangle\langle 0|.$$

This probability that the POVM $e_{12}^{(1)}$ will click is given by $p_1 = \frac{2|n|^2}{(1+|n|^2)^2}$.

The coherence of this state is given by

$$\begin{aligned}C_{l_1}\big(\rho_3^{(0)}\big) &= 2|\mathrm{Im}(\alpha\beta^*)| \\ &= 2\sin\phi\cos(\theta/2)\sin(\theta/2) \\ &\neq 2\cos(\theta/2)\sin(\theta/2).\end{aligned}$$

In this case if the partially known state happens to be from the polar circle of the Bloch sphere, then the coherence is perfectly teleported.

Case II. Alice performs the following POVM on her part:

$$\pi_{12}^{(0)} = |\Phi_n^+\rangle\langle\Phi_n^+| + |\Psi_n^+\rangle\langle\Psi_n^+|,$$
$$\pi_{12}^{(1)} = |\Phi_n^-\rangle\langle\Phi_n^-| + |\Psi_n^-\rangle\langle\Psi_n^-|.$$

The state in Bob's part after Alice performs the measurement with POVM element $\pi_{12}^{(0)}$ and communicates the results to Bob is given by

$$\rho_3^{(0)} = \frac{1}{p_0}\mathrm{Tr}_{12}\big[\pi_{12}^{(0)}(|\psi\rangle\langle\psi| \otimes |\Phi_n^+\rangle\langle\Phi_n^+|)\pi_{12}^{(0)}\big],$$

where $p_0 = \mathrm{Tr}_{123}[\pi_{12}^{(0)}(|\psi\rangle\langle\psi| \otimes |\Phi_n^+\rangle\langle\Phi_n^+|)\pi_{12}^{(0)}]$. Therefore, Bob's state can be expressed as

$$\begin{aligned}\rho_3^{(0)} = {} & \frac{1}{1+|n|^2}|0\rangle\langle 0| + \frac{|n|^2}{1+|n|^2}|1\rangle\langle 1| \\ & + \frac{|n|^2}{(1+|n|^2)(|\alpha|^2+|\beta|^2|n|^2)}2\mathrm{Re}(\alpha\beta^*)|0\rangle\langle 1| \\ & + \frac{|n|^2}{(1+|n|^2)(|\alpha|^2+|\beta|^2|n|^2)}2\mathrm{Re}(\alpha\beta^*)|1\rangle\langle 0|.\end{aligned}$$

The probability that the POVM $\pi_{12}^{(0)}$ will click is given by $p_0 = \frac{|\alpha|^2+|\beta|^2|n|^2}{(1+|n|^2)}$.

The coherence of the postmeasurement state on Bob's side is given by

$$C_{l_1}\big(\rho_3^{(0)}\big) = \frac{4|n|^2}{(1+|n|^2)(|\alpha|^2+|\beta|^2|n|^2)}|\mathrm{Re}(\alpha\beta^*)|.$$

Here, also unlike the case discussed in the previous section, even if the partially known state is from the equatorial circle of the Bloch sphere, the coherence is not perfectly teleported. It is $\frac{4|n|^2}{(1+|n|^2)(|\alpha|^2+|\beta|^2|n|^2)}$ times the original coherence.

The state in Bob's part after Alice performs the measurement with POVM element $\pi_{12}^{(1)}$ and communicates the results to Bob is given by

$$\rho_3^{(1)} = \frac{1}{p_1}\mathrm{Tr}_{12}\big[\pi_{12}^{(1)}(|\psi\rangle\langle\psi| \otimes |\Phi_n^+\rangle\langle\Phi_n^+|)\pi_{12}^{(1)}\big],$$

where $p_1 = \mathrm{Tr}_{123}[\pi_{12}^{(1)}(|\psi\rangle\langle\psi| \otimes |\Phi_n^+\rangle\langle\Phi_n^+|)\pi_{12}^{(1)}]$. Therefore, Bob's state can be expressed as

$$\begin{aligned}\rho_3^{(1)} = {} & \frac{1}{1+|n|^2}|0\rangle\langle 0| + \frac{|n|^2}{1+|n|^2}|1\rangle\langle 1| \\ & - \frac{|n|^2}{(1+|n|^2)(|\alpha|^2|n|^2+|\beta|^2)}2\mathrm{Re}(\alpha\beta^*)|0\rangle\langle 1| \\ & - \frac{|n|^2}{(1+|n|^2)(|\alpha|^2|n|^2+|\beta|^2)}2\mathrm{Re}(\alpha\beta^*)|1\rangle\langle 0|.\end{aligned}$$

The probability that the POVM $\pi_{12}^{(1)}$ will click is $p_1 = \frac{|\alpha|^2|n|^2+|\beta|^2}{(1+|n|^2)}$.

The coherence of this state is given by

$$C_{l_1}\big(\rho_3^{(1)}\big) = \frac{4|n|^2}{(1+|n|^2)(|\alpha|^2|n|^2+|\beta|^2)}|\mathrm{Re}(\alpha\beta^*)|.$$

Here, too, perfect teleportation of coherence is not possible, unlike in the case discussed in the previous section, even if the partially known state is from the equatorial circle of the Bloch sphere. It is $\frac{4|n|^2}{(1+|n|^2)(|\alpha|^2|n|^2+|\beta|^2)}$ times the original coherence.

Case III. In this case Alice performs the following POVM on her part:

$$f_{12}^{(0)} = |\Phi_n^+\rangle\langle\Phi_n^+| + |\Phi_n^-\rangle\langle\Phi_n^-|,$$
$$f_{12}^{(1)} = |\Psi_n^+\rangle\langle\Psi_n^+| + |\Psi_n^-\rangle\langle\Psi_n^-|.$$

The state in Bob's part after Alice performs the measurement with POVM element $f_{12}^{(0)}$ and communicates the results to Bob is given by

$$\rho_3^{(0)} = \frac{1}{p_0}\mathrm{Tr}_{12}\big[f_{12}^{(0)}(|\psi\rangle\langle\psi| \otimes |\Phi_n^+\rangle\langle\Phi_n^+|)f_{12}^{(0)}\big],$$

where $p_0 = \mathrm{Tr}_{123}[f_{12}^{(0)}(|\psi\rangle\langle\psi| \otimes |\Phi_n^+\rangle\langle\Phi_n^+|)f_{12}^{(0)}]$. Therefore, Bob's state can be expressed as

$$\rho_3^{(0)} = \frac{|\alpha|^2}{|\alpha|^2+|\beta|^2|n|^2}|0\rangle\langle 0| + \frac{|\beta|^2|n|^2}{|\alpha|^2+|\beta|^2|n|^2}|1\rangle\langle 1|.$$

The probability that the POVM $f_{12}^{(0)}$ will click is given by $p_0 = \frac{|\alpha|^2+|\beta|^2|n|^2}{1+|n|^2}$.

In this case, the coherence of this state is zero. This is not useful for the teleportation of coherence.

The state in Bob's part after Alice performs the measurement with POVM element $f_{12}^{(1)}$ and communicates the results to Bob is given by

$$\rho_3^{(1)} = \frac{1}{p_1}\mathrm{Tr}_{12}\big[f_{12}^{(1)}(|\psi\rangle\langle\psi| \otimes |\Phi_n^+\rangle\langle\Phi_n^+|)f_{12}^{(1)}\big],$$

where $p_1 = \mathrm{Tr}_{123}[f_{12}^{(1)}(|\psi\rangle\langle\psi| \otimes |\Phi_n^+\rangle\langle\Phi_n^+|)f_{12}^{(1)}]$. Therefore, Bob's state can be expressed as

$$\rho_3^{(1)} = \frac{|\beta|^2}{|\alpha|^2|n|^2+|\beta|^2}|0\rangle\langle 0| + \frac{|\alpha|^2|n|^2}{|\alpha|^2|n|^2+|\beta|^2}|1\rangle\langle 1|.$$

The probability that the POVM $f_{12}^{(1)}$ will click is $p_1 = \frac{|\alpha|^2|n|^2+|\beta|^2}{1+|n|^2}$.

We see that the coherence of the postmeasurement state is zero. So we observe that only in case I is perfect teleportation of coherence possible probabilistically, with the probability being $\frac{2|n|^2}{(1+|n|^2)^2}$ when the partially known states are from the polar circle of the Bloch sphere.

III. TELEPORTATION OF COHERENCE WITH AN ARBITRARY SHARED STATE AND ARBITRARY POVM MEASUREMENT

In this section, we will talk about the teleportation of the coherence of an arbitrary state (pure and mixed) with the help of an arbitrary shared state and by considering, in general, arbitrary POVM measurements. Consider two finite-dimensional Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ over the field of complex numbers, with the dimension of H_1 being n. Let us consider a linear map $\Phi : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$, where $B(\mathcal{H})$ is the vector space of bounded linear operators on $\mathcal{H}$. Let $\{e_{ij} = |i\rangle\langle j|\}_{i,j=1}^n$ be a complete set of matrix units for $B(\mathcal{H}_1)$. Then operator $\rho_\Phi = \sum_{i,j=1}^n e_{ij} \otimes \Phi(e_{ij}) \in B(\mathcal{H}_1) \otimes B(\mathcal{H}_2)$ is known as the Choi-Jamiołkowski-Kraus-Sudarshan (CJKS) matrix [83–87] corresponding to Φ. It can easily be verified that the map $\Phi \to \rho_\Phi$ is linear and bijective and is called the CJKS isomorphism.

Theorem 1. CJKS theorem on completely positive maps [83–87]. The CJKS matrix $\rho_\Phi = \sum_{i,j=1}^n e_{ij} \otimes \Phi(e_{ij}) \in B(\mathcal{H}_1) \otimes B(\mathcal{H}_2)$ is positive if and only if the map $\Phi : B(\mathcal{H}_1) \to B(\mathcal{H}_2)$ is completely positive.

From the Theorem 1. we conclude that for any state, τ and POVM E belonging to $B(\mathcal{H}_1) \otimes B(\mathcal{H}_2)$ can be written as follows for two completely positive maps T and Φ_E:

$$\tau = \sum_{i,j=1}^n e_{ij} \otimes T(e_{ij}), \tag{12}$$

$$E = \sum_{i,j=1}^n e_{ij} \otimes \Phi_E(e_{ij}). \tag{13}$$

In the following theorem, Φ_E^* denotes the completely positive map whose Kraus operators are the complex conjugate of that of the completely positive map Φ_E.

Theorem 2. Let Alice have the unknown state ρ whose coherence she needs to teleport to Bob. Let the shared state between Alice and Bob be τ. If Alice performs a general measurement with a POVM element E and communicates the result to Bob, then the state in Bob's laboratory is $\rho_B = T \circ \Phi_E^*(\rho)$, where T is the completely positive map corresponding to the shared state τ, Φ_E^* is the complex conjugate of the completely positive map corresponding to the POVM element E, and $\circ$ represents the composition of the two maps.

Proof. The reduced state at Bob's location is $\rho_B = \mathrm{Tr}_{12}[(\sqrt{E} \otimes \mathbb{I})(\rho \otimes \tau)(\sqrt{E} \otimes \mathbb{I})]$. Let A be any operator on the Hilbert space associated with Bob.

Now,

$$\begin{aligned}
\mathrm{Tr}(\rho_B A) &= \mathrm{Tr}\{\mathrm{Tr}_{12}[(\sqrt{E} \otimes \mathbb{I})(\rho \otimes \tau)(\sqrt{E} \otimes \mathbb{I})]A\} \\
&= \mathrm{Tr}[(\rho \otimes \tau)(E \otimes A)] \\
&= \sum_{i,j,k,l} \mathrm{Tr}\{[\rho \otimes e_{ij} \otimes T(e_{ij})](e_{kl} \otimes \Phi_E(e_{kl}) \otimes A)\} \\
&= \sum_{i,j,k,l} \mathrm{Tr}\{[\rho \otimes |i\rangle\langle j| \otimes T(|i\rangle\langle j|)][|k\rangle \\
&\quad \times \langle l| \otimes \Phi_E(|k\rangle\langle l|) \otimes A]\} \\
&= \sum_{i,j,k,l}\sum_p \langle l|\rho|k\rangle\langle j|M_p|k\rangle\langle l|M_p^\dagger|i\rangle \mathrm{Tr}[T(|i\rangle\langle j|)A] \\
&= \sum_{i,j,k,l}\sum_p \langle i|M_p^*|l\rangle\langle l|\rho|k\rangle\langle k|M_p^t|j\rangle \mathrm{Tr}[T(|i\rangle\langle j|)A] \\
&= \sum_{i,j}\langle i|\sum_p M_p^*\rho(M_p^*)^\dagger|j\rangle \mathrm{Tr}[T(|i\rangle\langle j|)A] \\
&= \sum_{i,j}\langle i|\Phi_E^*(\rho)|j\rangle \mathrm{Tr}[T(|i\rangle\langle j|)A] \\
&= \mathrm{Tr}[T(\Phi_E^*(\rho))A].
\end{aligned} \tag{14}$$

As A is arbitrary, we conclude that

$$\rho_B = T \circ \Phi_E^*(\rho). \tag{15}$$

A. Teleportation of coherence using a mixed entangled state as a resource

In this section, we study the teleportation of coherence using mixed entangled states as a resource state, unlike pure entangled states in the previous sections. Here, we consider two types of mixed entangled states as examples of resources that can be used. They are (i) maximally mixed entangled states and (ii) Werner states.

Maximally entangled mixed states. This class of two-qubit states appeared first in [88,89]. It has the maximum amount of entanglement because no global unitary transformation can increase the entanglement of formation or even the negativity of entanglement of the states in this class. Let Alice and Bob share a two-qubit maximally entangled mixed state given as follows:

$$\tau = p_1|\Psi^-\rangle\langle\Psi^-| + p_2|00\rangle\langle 00| + p_3|\Psi^+\rangle\langle\Psi^+| + p_4|11\rangle\langle 11|,$$

where p_1, p_2, p_3, and p_4 are probabilities in decreasing order, i.e., $p_1 \geqslant p_2 \geqslant p_3 \geqslant p_4$. Alice performs the following POVMs on the input and half of the entangled pair:

$$\Pi_{12}^{(0)} = |\Phi^+\rangle\langle\Phi^+| + |\Psi^+\rangle\langle\Psi^+|, \tag{16}$$

$$\Pi_{12}^{(1)} = |\Phi^-\rangle\langle\Phi^-| + |\Psi^-\rangle\langle\Psi^-|. \tag{17}$$

Let Alice perform the measurement given by the POVM element $\Pi_{12}^{(0)}$ and communicate her result to Bob. In this case, Φ_E^* and T are given by the following equations:

$$\begin{aligned}\Phi_E^*(A) &= A + \sigma_x A \sigma_x,\\ T(A) &= p_1\sigma_y A\sigma_y + p_2\langle 0|A|0\rangle|0\rangle\langle 0|\\ &\quad + p_3\sigma_x A\sigma_x + p_4\langle 1|A|1\rangle|1\rangle\langle 1|,\end{aligned}$$

where A belongs to the set of bounded linear operators on the Hilbert space associated with Bob. The state of the particle in Bob's laboratory is given by

$$\begin{aligned}\rho_B = T\Phi_E^*(\rho) = \frac{1}{1+p_1+p_3}(&p_1\sigma_z\rho\sigma_z + p_1\sigma_y\rho\sigma_y\\ &+ p_3\sigma_x\rho\sigma_x + p_3\rho + p_2|0\rangle\langle 0| + p_4|1\rangle\langle 1|).\end{aligned} \tag{18}$$

The coherence of this state is given by

$$C(\rho_B) = \frac{4|p_1 - p_3|}{1+p_1+p_3}|\mathrm{Re}(\rho_{01})|. \tag{19}$$

The concurrence of the above-mentioned maximally entangled mixed state is

$$\mathcal{C}(\tau) = \max\{0, p_1 - p_3 - \sqrt{p_2 p_4}\}. \tag{20}$$

Assuming $p_4 = 0$, the concurrence reduces to $\mathcal{C}(\tau) = p_1 - p_3$. From Eq. (18) we get

$$\begin{aligned}C(\rho_B) &= \frac{2\mathcal{C}(\tau)}{1+p_1+p_3}2|\mathrm{Re}(\rho_{01})| \leqslant \frac{2\mathcal{C}(\tau)}{1+p_1+p_3}C(\rho)\\ &\leqslant \frac{2\mathcal{C}(\tau)}{1+\mathcal{C}(\tau)}C(\rho).\end{aligned} \tag{21}$$

When POVM $\Pi_{12}^{(1)}$ clicks and Alice communicates the result to Bob, then the state in Bob's part is the same as Eq. (18), and hence, the relation between the coherence in Bob's part and the unknown state obeys Eq. (21).

Wener states. Now let's assume that Alice and Bob share the Warner state $\tau = p|\Psi^-\rangle\langle\Psi^-| + \frac{1-p}{4}\mathbb{I}$ with $0 \leqslant p \leqslant 1$ and Alice performs a measurement using the POVM described by Eqs. (16) and (17). If the POVM given by Eq. (16) clicks and Alice communicates the result to Bob, the state in Bob's part is given by

$$\rho_B = p\sigma_y\rho\sigma_y + p\sigma_z\rho\sigma_z + \frac{1-p}{2}\mathbb{I}. \tag{22}$$

The coherence of the above state with respect to the computational basis is as follows:

$$C(\rho_B) = \frac{2p}{1+p}|2\mathrm{Re}(\rho_{01})|. \tag{23}$$

If the POVM described by Eq. (17) clicks and Alice communicates the result to Bob, then the state in Bob's part is given by

$$\rho_B = p\sigma_x\rho\sigma_x + p\rho + \frac{1-p}{2}\mathbb{I}. \tag{24}$$

The coherence of the above state is the same as in Eq. (23).

It is a well-known fact that the Warner state is separable when $p \leqslant \frac{1}{3}$. From Eq. (23) we clearly see that the teleported coherence is nonzero for $0 < p \leqslant \frac{1}{3}$. Hence, we conclude that it is possible to teleport coherence without entanglement.

B. Teleportation of coherence of an unknown mixed state

Finally, we consider the case of teleportation of coherence of an unknown mixed state with a maximally entangled state as a resource and the POVM described by Eqs. (5) and (6). In this case the completely positive map T is the identity map, and the maps $\Phi^*_{\Pi_{12}^{(0)}}$ and $\Phi^*_{\Pi_{12}^{(1)}}$ are given by

$$\Phi^*_{\Pi_{12}^{(0)}}(\rho) = \rho + \sigma_x\rho\sigma_x, \tag{25}$$

$$\Phi^*_{\Pi_{12}^{(1)}}(\rho) = \sigma_z\rho\sigma_z + \sigma_y\rho\sigma_y. \tag{26}$$

So the state in Bob's part when the outcomes of the measurements of POVMs $\Pi_{12}^{(0)}$ and $\Pi_{12}^{(1)}$ are communicated to Bob is given by

$$\rho_B^{(0)} = \Phi^*_{\Pi_{12}^{(0)}}(\rho) = \rho + \sigma_x\rho\sigma_x, \tag{27}$$

$$\rho_B^{(1)} = \Phi^*_{\Pi_{12}^{(1)}}(\rho) = \sigma_z\rho\sigma_z + \sigma_y\rho\sigma_y. \tag{28}$$

In both cases, the coherence with respect to the computational basis turns out to be the same and is given by

$$C\big(\rho_B^{(0)}\big) = C(\rho_B^{(1)}) = |\rho_{01} + \rho_{10}| \tag{29}$$

$$\leqslant |\rho_{01}| + |\rho_{10}| \tag{30}$$

$$\leqslant C(\rho). \tag{31}$$

Clearly, the teleported coherence is less than or equal to the original coherence, even in the more general scenario in which the unknown state is a mixed state. However, if we have partial knowledge about the state, i.e., the matrix elements of the state are real, then the teleported coherence is equal to the original coherence.

IV. CONCLUSION

Because quantum coherence is a useful resource in various information processing tasks, it is of the utmost importance to transfer the coherence of a coherent state to a state that has no coherence. This is required in a quantum network in which one quantum processor can teleport the coherence to another quantum processor if there is a requirement. Therefore, it is important to know whether it is possible to teleport coherence without actually transferring all the information about the state. Interestingly, we were able to show that this can be done with a smaller number of cbits for states that are taken from the equatorial and polar circles.

In summary, we addressed the question of whether we can teleport the coherence of an unknown state with the transfer of a smaller number of classical bits. We showed that by transferring one cbit of information, we can teleport the coherence only for states which lie on the equatorial and polar circles. This is accomplished when we have a maximally entangled state as a resource. However, for a nonmaximally entangled state, we can teleport coherence if not deterministically, then with a certain probability of success. We also investigated the possibility of teleportation of coherence using two-qubit maximally entangled mixed states and Werner states as resources. We observed that even when the Werner state is separable, the teleported coherence is nonzero, which indicates that the teleportation of coherence is possible without entanglement. In a nutshell, we were able to provide a cost-effective way to teleport coherence as a resource to a place where there is a requirement for this resource. This can be done without actually transferring the state.

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