

On high-order schemes for tempered fractional partial differential equations

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ABSTRACT

In this paper, we propose third-order semi-discretized schemes in space based on the tempered weighted and shifted Grunwald difference (tempered-WSGD) operators for the tempered fractional diffusion equation. We also show stability and convergence analysis for the fully discrete scheme based a Crank–Nicolson scheme in time. A third-order scheme for the tempered Black–Scholes equation is also proposed and tested numerically. Some numerical experiments are carried out to confirm accuracy and effectiveness of these proposed methods.

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1. Introduction

Fractional calculus including fractional integration and fractional differentiation is often regarded as an extension of traditional integer calculus theory. It was proposed in a letter from Leibniz to L'Hôpital, in which the one-half order derivative was discussed and remarked [22]. For quite a long time, the theory of fractional derivatives developed slowly as a purely theoretical field of mathematics with little use in science and engineering. It was not until the 1970s that Mandelbrot [27] put forward the fractal theory, used Riemann–Liouville fractional calculus to study Brownian motion, and the fractional differential equations became particularly noticeable. Fractional derivatives provide an accurate description of memory properties and hereditary effects of various materials and processes. The fractional derivatives in time are concerned, for example, with particle sticking and trapping, while the spatial versions can be used to model long particle jumps. Nowadays differential equations with fractional operators are often encountered, in a variety of science and engineering fields, such as in physics [6,15,16,18,25,39,40], finance [12,21,24,41], hydrology [2,4,5,14,36], and biology [3,20,26]. Mathematically, fractional calculus concerns time-space coupled operators.

Since the 1990s, application of fractional derivatives in anomalous diffusion models has caught great interest (for an extensive review, we refer to [30]). Anomalous diffusion provides an instrument to describe the transmission process in complex heterogeneous systems. These models depend on fractional derivatives to express the observed sub-linear or super-linear growth of the variance of the variable of interest. The former relates to sub-diffusion and the latter to super-diffusion. Sub-diffusion is often characterized by temporally non-local transport (i.e., memory effects) with a fractional time derivative. Super-diffusion is characterized by spatially non-local transport (i.e., large displacements) and modeled by a fractional diffusion term. Lévy flight models are mathematically used to describe the super-diffusion whose jumps have infinite moments

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in complex systems. However, there are upper bounds on the displacements that a particle can follow in most practical applications. To overcome this discrepancy, different approaches have been adopted. Mantegna and Stanley [28] proposed a truncated Lévy flight, thereby eliminating arbitrarily large steps of the Lévy flight. Sokolov et al. [37] put forward two other modifications by adding a higher-order power-law factor, and Chechkin et al. [10] added a nonlinear friction term to achieve finite second moments. Rosiński [34] provided a different approach by exponentially tempering the probability of large jumps. Following the same method, the waiting time between particle displacements can also be tempered (see, for example, [29]), while tempering the displacement and waiting time gives a space-time tempered fractional diffusion equation yielding convergent moments in both space and time (see, for example, [17]).

The *tempered* fractional operators have been introduced to describe probability density functions for the positions of particles by exponentially tempering the probability of large jumps of a Lévy flight. We first introduce the well-known Riemann-Liouville fractional derivatives defined as follows.

Definition 1.1. (See, for example, in [35].) For $\alpha \in (n-1, n)$, let $u(x)$ be $(n-1)$ -times continuously differentiable on an interval (a, b) and its n -th derivative be integrable on any subinterval of $[a, b]$. Then, the *left* Riemann-Liouville fractional derivative of order α is defined as

$${}_a D_x^\alpha u(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x \frac{u(\xi)}{(x-\xi)^{\alpha-n+1}} d\xi, \quad (1.1)$$

and the *right* Riemann-Liouville fractional derivative of order α is defined as

$${}_x D_b^\alpha u(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^b \frac{u(\xi)}{(\xi-x)^{\alpha-n+1}} d\xi, \quad (1.2)$$

where Γ represents the gamma function.

The Riemann-Liouville *tempered* fractional derivatives were defined and employed as follows.

Definition 1.2. (See, for example, in [23].) For $\alpha \in (n-1, n)$, let $u(x)$ be $(n-1)$ -th continuously differentiable on (a, b) with its n -times derivative integrable on any subinterval of $[a, b]$, $\lambda \geq 0$. Then, the *left* Riemann-Liouville *tempered* fractional derivative of order α is defined as

$${}_a D_x^{\alpha, \lambda} u(x) = (e^{-\lambda x} {}_a D_x^\alpha e^{\lambda x}) u(x) = \frac{e^{-\lambda x}}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x \frac{e^{\lambda \xi} u(\xi)}{(x-\xi)^{\alpha-n+1}} d\xi; \quad (1.3)$$

the *right* Riemann-Liouville *tempered* fractional derivative of order α is defined as

$${}_x D_b^{\alpha, \lambda} u(x) = (e^{\lambda x} {}_x D_b^\alpha e^{-\lambda x}) u(x) = \frac{(-1)^n e^{\lambda x}}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^b \frac{e^{-\lambda \xi} u(\xi)}{(\xi-x)^{\alpha-n+1}} d\xi. \quad (1.4)$$

The Riemann-Liouville tempered fractional derivatives ${}_a D_x^{\alpha, \lambda} u(x)$ and ${}_x D_b^{\alpha, \lambda} u(x)$ can be transformed into the Riemann-Liouville tempered fractional derivatives ${}_a D_x^\alpha u(x)$ and ${}_x D_b^\alpha u(x)$ when $\lambda = 0$. Variants of the left and right Riemann-Liouville tempered fractional derivatives are defined in [1,7] as follows

$${}_a \mathbf{D}_x^{\alpha, \lambda} u(x) = \begin{cases} {}_a D_x^{\alpha, \lambda} u(x) - \lambda^\alpha u(x), & 0 < \alpha < 1 \\ {}_a D_x^{\alpha, \lambda} u(x) - \alpha \lambda^{\alpha-1} \partial_x u(x) - \lambda^\alpha u(x), & 1 < \alpha < 2; \end{cases} \quad (1.5)$$

and

$${}_x \mathbf{D}_b^{\alpha, \lambda} u(x) = \begin{cases} {}_x D_b^{\alpha, \lambda} u(x) - \lambda^\alpha u(x), & 0 < \alpha < 1 \\ {}_x D_b^{\alpha, \lambda} u(x) - \alpha \lambda^{\alpha-1} \partial_x u(x) - \lambda^\alpha u(x), & 1 < \alpha < 2. \end{cases} \quad (1.6)$$

The *tempered* derivatives have been applied in physics [1,17,23], finance [8,42] and also in ground water hydrology [29]. Exponentially tempered Lévy processes lead to a tempered fractional diffusion equation in the fluid limit (see, for example, [1,23]). Cartea and Del-Castillo-Negrete [8] used Lévy processes to model the dynamics of financial assets and presented a finite difference scheme to numerically price exotic options. Zhang etc. [42] provided a second-order discretization for the tempered fractional Black-Scholes equation with stability and convergence results. Meerschaert et al. [29] proposed a novel

tempered anomalous diffusion model to capture the pre-asymptotic behavior of passive tracers in heterogeneous aquifers. Haneter and Piret [17] presented a high-order numerical discretization of the space-time fractional diffusion equation with exponential tempering in both space and time.

There are quite a few numerical approximations for tempered fractional diffusion available in the literature already. Existing numerical techniques include finite elements ([13]), finite differences ([1,11,23]), and also spectral methods ([17, 19,43]). Deng and Zhang [13] provided Galerkin and Petrov–alerkin finite element methods for the tempered PDEs and proved convergence, numerical stability, and a series of variational equalities. Baeumera and Meerschaert [1] developed a finite difference and particle tracking method to simulate the tempered anomalous diffusion. Huang etc. [19] focused on both Petrov–Galerkin and spectral collocation methods for two types of substantial fractional differential equations. Li and Deng [23] constructed higher-order discretizations, by weighted and shifted Grünwald type approximations to tempered fractional derivatives. They also presented stability and convergence properties of the second-order scheme for the tempered fractional diffusion equation. Based on the methods in [23], we construct the third-order accurate scheme for the fractional diffusion equation, which can be proved to be unconditional stable for a range of α -values and stability can be numerically confirmed when parameter α lies outside this interval. The convergence analysis is also given for the third-order numerical scheme. Besides, we propose a third-order accurate scheme for the tempered fractional Black-Scholes equations and test the numerical scheme.

This paper is organized as follows. In Section 2, we give discretization details for the tempered fractional derivatives. In Section 3, we present the third-order schemes for the fractional diffusion equation and carry out the corresponding stability and error analysis. Section 4 contains some numerical results to confirm the accuracy and efficiency of the proposed discretization methods. We also consider the tempered fractional Black-Scholes equation and provide numerical experiments to confirm the scheme's accuracy. Finally, a short summary is made in the last section.

2. Definitions and discretization of the tempered fractional derivatives

In this section, we define tempered fractional derivatives and the approximation accuracy of the shifted Grünwald type difference operator for the Riemann–Liouville tempered fractional derivatives. We denote ${}_a\mathcal{D}_x^{\alpha,\lambda}u(x) = {}_aD_x^{\alpha,\lambda}u(x) - \lambda^\alpha u(x)$ and ${}_x\mathcal{D}_b^{\alpha,\lambda}u(x) = {}_xD_b^{\alpha,\lambda}u(x) - \lambda^\alpha u(x)$, where ‘a’ and ‘b’ can be extended to $-\infty$ and ∞ , respectively.

Lemma 2.1. From [23]. Let $u(x) \in L^1(\Omega)$, $-\infty D_x^{\alpha,\lambda}u$ and its Fourier transform belong to $L^1(\Omega)$; $p \in \mathbb{R}$, $h > 0$, $\lambda \geq 0$ and $n - 1 < \alpha \leq n$. Define the shifted Grünwald type difference operator, as

$$A_{h,p}^{\alpha,\lambda}u(x) := \frac{1}{h^\alpha} \sum_{k=0}^{\infty} \omega_k^{(\alpha)} e^{-(k-p)h\lambda} u(x - (k-p)h) - \frac{1}{h^\alpha} \left(e^{ph\lambda} (1 - e^{-h\lambda})^\alpha \right) u(x), \quad (2.1)$$

then

$$A_{h,p}^{\alpha,\lambda}u(x) = {}_{-\infty}\mathcal{D}_x^{\alpha,\lambda}u + O(h), \quad (2.2)$$

where $\omega_k^{(\alpha)} = (-1)^k C_k^\alpha$, $k \geq 0$ denote the normalized Grünwald weights from [31]. Let ${}_x\mathcal{D}_{+\infty}^{\alpha,\lambda}u$ and its Fourier transform belong to $L^1(\Omega)$, then we define

$$B_{h,p}^{\alpha,\lambda}u(x) := \frac{1}{h^\alpha} \sum_{k=0}^{\infty} \omega_k^{(\alpha)} e^{-(k-p)h\lambda} u(x + (k-p)h) - \frac{1}{h^\alpha} \left(e^{ph\lambda} (1 - e^{-h\lambda})^\alpha \right) u(x), \quad (2.3)$$

so that

$$B_{h,p}^{\alpha,\lambda}u(x) = {}_x\mathcal{D}_\infty^{\alpha,\lambda}u(x) + O(h). \quad (2.4)$$

To develop and analyze the higher-order discretization schemes, we repeat here a highly relevant theorem from [23].

Theorem 2.2. From [23]. Let $u(x) \in L^1(\mathbb{R})$, $-\infty\mathcal{D}_x^{\alpha+l,\lambda}u(x)$ and its Fourier transform belong to $L^1(\mathbb{R})$. We define the left tempered-WSGD operator by

$${}_L\mathcal{D}_{h,p_1,p_2,\dots,p_m}^{\alpha,\gamma_1,\gamma_2,\dots,\gamma_m} = \sum_{j=1}^m \gamma_j A_{h,p_j}^{\alpha,\lambda}u(x), \quad (2.5)$$

where p_j , $\gamma_j \in \mathbb{R}$ (and they are determined by the Equations (2.9)–(2.12) to follow). Then, for any integer $m \geq l$, there exists an operator such that

$${}_L\mathcal{D}_{h,p_1,p_2,\dots,p_m}^{\alpha,\gamma_1,\gamma_2,\dots,\gamma_m} = {}_{-\infty}\mathfrak{D}_x^{\alpha,\lambda}u(x) + O(h^l), \quad (2.6)$$

uniformly for $x \in \mathbb{R}$.

Let $u(x) \in L^1(\mathbb{R})$, ${}_x\mathfrak{D}_b^{\alpha,\lambda}u(x)$ and its Fourier transform belong to $L^1(\mathbb{R})$; and define the right tempered-WSGD operator by

$${}_R\mathcal{D}_{h,p_1,p_2,\dots,p_m}^{\alpha,\gamma_1,\gamma_2,\dots,\gamma_m} = \sum_{j=1}^m \gamma_j B_{h,p_j}^{\alpha,\lambda}u(x). \quad (2.7)$$

Then, for any integer $m \geq l$, there exists an operator such that

$${}_R\mathcal{D}_{h,p_1,p_2,\dots,p_m}^{\alpha,\gamma_1,\gamma_2,\dots,\gamma_m} = {}_x\mathfrak{D}_{\infty}^{\alpha,\lambda}u(x) + O(h^l), \quad (2.8)$$

uniformly for $x \in \mathbb{R}$.

Let p_j, γ_j be real-valued. For $l = 2$, p_j, γ_j should satisfy the following conditions

$$\begin{cases} \sum_{j=1}^m \gamma_j = 1, \\ \sum_{j=1}^m \gamma_j \left[p_j - \frac{\alpha}{2} \right] = 0. \end{cases} \quad (2.9)$$

For $l = 3$, p_j, γ_j should satisfy the following conditions

$$\begin{cases} \sum_{j=1}^m \gamma_j = 1, \\ \sum_{j=1}^m \gamma_j \left[p_j - \frac{\alpha}{2} \right] = 0, \\ \sum_{j=1}^m \gamma_j \left[\frac{p_j^2}{2} - \frac{\alpha p_j}{2} + \frac{\alpha}{6} + \frac{\alpha(\alpha-1)}{8} \right] = 0. \end{cases} \quad (2.10)$$

For $l = 4$, p_j, γ_j should satisfy

$$\begin{cases} \sum_{j=1}^m \gamma_j = 1, \\ \sum_{j=1}^m \gamma_j \left[p_j - \frac{\alpha}{2} \right] = 0, \\ \sum_{j=1}^m \gamma_j \left[\frac{p_j^2}{2} - \frac{\alpha p_j}{2} + \frac{\alpha}{6} + \frac{\alpha(\alpha-1)}{8} \right] = 0, \\ \sum_{j=1}^m \gamma_j \left[\frac{p_j^3}{6} - \frac{\alpha p_j^2}{4} + \left(\frac{\alpha}{6} + \frac{\alpha(\alpha-1)}{8} \right) p_j - \frac{\alpha}{24} - \frac{\alpha(\alpha-1)}{12} - \frac{\alpha(\alpha-1)(\alpha-2)}{48} \right] = 0, \end{cases} \quad (2.11)$$

while for $l = 5$, p_j, γ_j should satisfy

$$\left\{ \begin{array}{l} \sum_{j=1}^m \gamma_j = 1, \\ \sum_{j=1}^m \gamma_j \left[p_j - \frac{\alpha}{2} \right] = 0, \\ \sum_{j=1}^m \gamma_j \left[\frac{p_j^2}{2} - \frac{\alpha p_j}{2} + \frac{\alpha}{6} + \frac{\alpha(\alpha-1)}{8} \right] = 0, \\ \sum_{j=1}^m \gamma_j \left[\frac{p_j^3}{6} - \frac{\alpha p_j^2}{4} + \left(\frac{\alpha}{6} + \frac{\alpha(\alpha-1)}{8} \right) p_j - \frac{\alpha}{24} - \frac{\alpha(\alpha-1)}{12} - \frac{\alpha(\alpha-1)(\alpha-2)}{48} \right] = 0, \\ \sum_{j=1}^m \gamma_j \left[\frac{p_j^4}{24} - \frac{\alpha p_j^3}{4} + \frac{1}{2} \left(\frac{\alpha}{6} + \frac{\alpha(\alpha-1)}{8} \right) p_j^2 - \left(\frac{\alpha}{24} - \frac{\alpha(\alpha-1)}{12} - \frac{\alpha(\alpha-1)(\alpha-2)}{48} \right) p_j \right. \\ \left. + \frac{\alpha}{120} + \frac{5\alpha(\alpha-1)}{144} + \frac{\alpha(\alpha-1)(\alpha-2)}{48} + \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{384} \right] = 0. \end{array} \right. \quad (2.12)$$

Remark 2.1. Considering a well-defined function $u(x)$ on the bounded interval $[a, b]$, then the function $u(x)$ can be zero extended for $x < a$ or $x > b$ so that $u(x) \in L^1(\mathbb{R})$, ${}_a\mathcal{D}_x^{\alpha+l, \lambda} u(x)$ and its Fourier transform belong to $L^1(\mathbb{R})$. The α -th order left and right Riemann-Liouville tempered fractional derivatives of $u(x)$ at point x can then be approximated by the tempered-WSGD operators

$$\begin{aligned} {}_a\mathcal{D}_x^{\alpha, \lambda} u(x) &= \sum_{j=1}^m \frac{\gamma_j}{h^\alpha} \left(\sum_{k=0}^{\lceil \frac{x-a}{h} \rceil + p_j} \omega_k^{(\alpha)} u(x - (k - p_j)h) - e^{p_j h \lambda} (1 - e^{-h \lambda})^\alpha u(x) \right) \\ &\quad + O(h^l), \\ {}_x\mathcal{D}_b^{\alpha, \lambda} u(x) &= \sum_{j=1}^m \frac{\gamma_j}{h^\alpha} \left(\sum_{k=0}^{\lceil \frac{b-x}{h} \rceil + p_j} \omega_k^{(\alpha)} u(x + (k - p_j)h) - e^{p_j h \lambda} (1 - e^{-h \lambda})^\alpha u(x) \right) \\ &\quad + O(h^l), \end{aligned} \quad (2.13)$$

where the weight parameters γ_j are determined by the above linear algebraic systems given in Theorem 2.2.

Take $p_1 = 1$, $p_2 = 0$, and $p_3 = -1$ for the second-order scheme, with (2.9). The parameters γ_j ($j = 1, 2, 3$) then satisfy the linear system as follows,

$$\left\{ \begin{array}{l} \gamma_1 = \frac{\alpha}{2} + \gamma_3, \\ \gamma_2 = \frac{2-\alpha}{2} - 2\gamma_3. \end{array} \right. \quad (2.14)$$

The second-order operators are given by

$$\begin{aligned} {}_L\mathcal{D}_{h,1,0,-1}^{\alpha, \gamma_1, \gamma_2, \gamma_3} u(x) &= \sum_{j=1}^3 \frac{\gamma_j}{h^\alpha} \left(\sum_{k=0}^{\lceil \frac{x-a}{h} \rceil + p_j} \omega_k^{(\alpha)} u(x - (k - p_j)h) - e^{p_j h \lambda} (1 - e^{-h \lambda})^\alpha u(x) \right) \\ &= \frac{1}{h^\alpha} \left(\sum_{k=0}^{\lceil \frac{x-a}{h} \rceil + 1} g_{k, \lambda}^{(2, \alpha)} u(x - (k - 1)h) - \left(\gamma_1 e^{h \lambda} + \gamma_2 + \gamma_3 e^{-h \lambda} \right) (1 - e^{-h \lambda})^\alpha u(x) \right), \end{aligned} \quad (2.15)$$

and

$$\begin{aligned}
{}_R\mathcal{D}_{h,1,0,-1}^{\alpha,\gamma_1,\gamma_2,\gamma_3}u(x_j) &= \sum_{j=1}^3 \frac{\gamma_j}{h^\alpha} \left(\sum_{k=0}^{\left[\frac{b-x}{h}\right]+p_j} \omega_k^{(\alpha)} u(x + (k-p_j)h) - e^{p_j h \lambda} (1 - e^{-h\lambda})^\alpha u(x) \right) \\
&= \frac{1}{h^\alpha} \left(\sum_{k=0}^{\left[\frac{b-x}{h}\right]+1} g_{k,\lambda}^{(2,\alpha)} u(x + (k-1)h) - \left(\gamma_1 e^{h\lambda} + \gamma_2 + \gamma_3 e^{-h\lambda} \right) (1 - e^{-h\lambda})^\alpha u(x) \right).
\end{aligned} \tag{2.16}$$

The weights are given by

$$\begin{aligned}
g_{0,\lambda}^{(2,\alpha)} &= \gamma_1 \omega_0^{(\alpha)} e^{h\lambda}, \quad g_{1,\lambda}^{(2,\alpha)} = \gamma_1 \omega_1^{(\alpha)} + \gamma_2 \omega_0^{(\alpha)}, \\
g_{k,\lambda}^{(2,\alpha)} &= \left(\gamma_1 \omega_k^{(\alpha)} + \gamma_2 \omega_{k-1}^{(\alpha)} + \gamma_3 \omega_{k-2}^{(\alpha)} \right) e^{-(k-1)h\lambda}, \quad k \geq 2.
\end{aligned} \tag{2.17}$$

The third-order scheme is obtained by the following result with γ_j ($j = 1, 2, \dots, m$), satisfying Equation (2.10) in Theorem 2.2. Let $m = 4$, $p_1 = 1$, $p_2 = 0$, $p_3 = -1$, $p_4 = -2$. We have,

$$\begin{cases} \gamma_1 = \frac{\alpha^2}{8} + \frac{5}{24}\alpha - \gamma_4, \\ \gamma_2 = -\frac{\alpha^2}{4} + \frac{1}{12}\alpha + 1 + 3\gamma_4, \\ \gamma_3 = \frac{\alpha^2}{8} - \frac{7}{24}\alpha - 3\gamma_4. \end{cases} \tag{2.18}$$

The third-order operators are given by

$$\begin{aligned}
{}_L\mathcal{D}_{h,1,0,-1,-2}^{\alpha,\gamma_1,\gamma_2,\gamma_3,\gamma_4}u(x) &= \sum_{j=1}^4 \frac{\gamma_j}{h^\alpha} \left(\sum_{k=0}^{\left[\frac{x-a}{h}\right]+p_j} \omega_k^{(\alpha)} u(x - (k-p_j)h) - e^{p_j h \lambda} (1 - e^{-h\lambda})^\alpha u(x) \right) \\
&= \frac{1}{h^\alpha} \left(\sum_{k=0}^{\left[\frac{x-a}{h}\right]+1} g_{k,\lambda}^{(3,\alpha)} u(x - (k-1)h) - \left(\gamma_1 e^{h\lambda} + \gamma_2 + \gamma_3 e^{-h\lambda} + \gamma_4 e^{-2h\lambda} \right) (1 - e^{-h\lambda})^\alpha u(x) \right),
\end{aligned} \tag{2.19}$$

and

$$\begin{aligned}
{}_R\mathcal{D}_{h,1,0,-1,-2}^{\alpha,\gamma_1,\gamma_2,\gamma_3,\gamma_4}u(x) &= \sum_{j=1}^4 \frac{\gamma_j}{h^\alpha} \left(\sum_{k=0}^{\left[\frac{b-x}{h}\right]+p_j} \omega_k^{(\alpha)} u(x - (k-p_j)h) - e^{p_j h \lambda} (1 - e^{-h\lambda})^\alpha u(x) \right) \\
&= \frac{1}{h^\alpha} \left(\sum_{k=0}^{\left[\frac{b-x}{h}\right]+1} g_{k,\lambda}^{(3,\alpha)} u(x + (k-1)h) - \left(\gamma_1 e^{h\lambda} + \gamma_2 + \gamma_3 e^{-h\lambda} + \gamma_4 e^{-2h\lambda} \right) (1 - e^{-h\lambda})^\alpha u(x) \right).
\end{aligned} \tag{2.20}$$

The weights are found to be

$$\begin{aligned}
g_{0,\lambda}^{(3,\alpha)} &= \gamma_1 \omega_0^{(\alpha)} e^{h\lambda}, \quad g_{1,\lambda}^{(3,\alpha)} = \gamma_1 \omega_1^{(\alpha)} + \gamma_2 \omega_0^{(\alpha)}, \\
g_{2,\lambda}^{(3,\alpha)} &= \left(\gamma_1 \omega_2^{(\alpha)} + \gamma_2 \omega_1^{(\alpha)} + \gamma_3 \omega_0^{(\alpha)} \right) e^{-h\lambda}, \\
g_{k,\lambda}^{(3,\alpha)} &= \left(\gamma_1 \omega_k^{(\alpha)} + \gamma_2 \omega_{k-1}^{(\alpha)} + \gamma_3 \omega_{k-2}^{(\alpha)} + \gamma_4 \omega_{k-3}^{(\alpha)} \right) e^{-(k-1)h\lambda}, \quad k \geq 3.
\end{aligned} \tag{2.21}$$

We will discretize and analyze this latter scheme in the section to follow.

3. Numerical schemes for the tempered fractional diffusion equation

Here we construct a high-order scheme based on the tempered-WSGD operators in space and the Crank–Nicolson scheme in time for the tempered fractional diffusion equation. Then we establish the stability and convergence for the third-order scheme.

Consider the following tempered fractional diffusion equation, from [23],

$$\begin{cases} \frac{\partial u(x, t)}{\partial t} = c_l \cdot ({}_a\mathcal{D}_x^{\alpha, \lambda} u(x, t)) + c_r \cdot ({}_x\mathcal{D}_b^{\alpha, \lambda} u(x, t)) + p(x, t), \\ (x, t) \in (a, b) \times (0, T) \\ u(a, t) = \Phi_l(t), u(b, t) = \Phi_r(t), t \in (0, T) \\ u(x, T) = S(x), x \in (a, b), \end{cases} \quad (3.1)$$

where $\alpha \in (1, 2)$, c_l and c_r are constants in front of the fractional derivatives with $c_l + c_r \neq 0$, that usually control the bias of the diffusion. And if $c_l \neq 0$, then $\Phi_l(t) \equiv 0$; if $c_r \neq 0$, then $\Phi_r(t) \equiv 0$.

Let $t_j = j\tau$ ($0 \leq t_j \leq T$, $j = 0, \dots, N$) and $x_i = a + ih$ ($a \leq x_i \leq b$, $i = 0, \dots, M$), where $\tau = T/N$ and $h = (b - a)/M$. Using the k -th order tempered-WSGD operators ${}_L\mathcal{D}_{h,k}^{\alpha, \lambda_1} = {}_L\mathcal{D}_{h,p_1,p_2,\dots,p_m}^{\alpha, \gamma_1, \gamma_2, \dots, \gamma_m}$ and ${}_R\mathcal{D}_{h,k}^{\alpha, \lambda_2} = {}_R\mathcal{D}_{h,p_1,p_2,\dots,p_m}^{\alpha, \gamma_1, \gamma_2, \dots, \gamma_m}$ for the tempered fractional derivatives and Crank-Nicolson time discretization, the numerical scheme for (3.1) reads

$$\frac{u_i^{j+1} - u_i^j}{\tau} = c_l \left({}_L\mathcal{D}_{h,k}^{\alpha, \lambda} u_i^{j+\frac{1}{2}} \right) + c_r \left({}_R\mathcal{D}_{h,k}^{\alpha, \lambda} u_i^{j+\frac{1}{2}} \right) + p_i^{j+\frac{1}{2}} + O(\tau^2 + h^k), \quad (3.2)$$

where u_i^j represents the solution of (3.1) at the point (x_i, t_j) , and $p_i^{j+\frac{1}{2}} = p(x_i, t_{j+\frac{1}{2}})$.

Then, we obtain,

$$\begin{aligned} u_i^{j+1} - \frac{\tau}{2} \left[c_l \left({}_L\mathcal{D}_{h,k}^{\alpha, \lambda} u_i^{j+1} \right) + c_r \left({}_R\mathcal{D}_{h,k}^{\alpha, \lambda} u_i^{j+1} \right) \right] \\ = u_i^j + \frac{\tau}{2} \left[c_l \left({}_L\mathcal{D}_{h,k}^{\alpha, \lambda} u_i^j \right) + c_r \left({}_R\mathcal{D}_{h,k}^{\alpha, \lambda} u_i^j \right) \right] + \tau p_i^{j+\frac{1}{2}} + O(\tau^3 + \tau h^k). \end{aligned} \quad (3.3)$$

We denote by U_i^j the solution of the numerical scheme for (3.1) at point (x_i, t_j) and $P_i^{j+\frac{1}{2}} = \frac{1}{2}(p_i^j + p_i^{j+1})$. The numerical scheme can now be written as

$$\begin{aligned} U_i^{j+1} - \frac{\tau}{2} \left[c_l \cdot \left({}_L\mathcal{D}_{h,k}^{\alpha, \lambda} U_i^{j+1} \right) + c_r \cdot \left({}_R\mathcal{D}_{h,k}^{\alpha, \lambda} U_i^{j+1} \right) \right] \\ = U_i^j + \frac{\tau}{2} \left[c_l \cdot \left({}_L\mathcal{D}_{h,k}^{\alpha, \lambda} U_i^j \right) + c_r \cdot \left({}_R\mathcal{D}_{h,k}^{\alpha, \lambda} U_i^j \right) \right] + \tau P_i^{j+\frac{1}{2}}. \end{aligned} \quad (3.4)$$

Denote $U^n = (U_1^n, U_2^n, \dots, U_{M-1}^n)^T$, $\phi^{(k,m)}(\lambda) = \sum_{j=1}^m \gamma_j e^{p_j h \lambda} (1 - e^{-h\lambda})^\alpha$, and

$$B_{k,\lambda} = \begin{pmatrix} g_{1,\lambda}^{(k,\alpha)} - \phi^{(k,m)}(\lambda) & g_{0,\lambda}^{(k,\alpha)} & & & \\ g_{2,\lambda}^{(k,\alpha)} & g_{1,\lambda}^{(k,\alpha)} - \phi^{(k,m)}(\lambda) & g_{0,\lambda}^{(k,\alpha)} & & \\ \vdots & g_{2,\lambda}^{(k,\alpha)} & g_{1,\lambda}^{(k,\alpha)} - \phi^{(k,m)}(\lambda) & \ddots & \\ g_{n-2,\lambda}^{(k,\alpha)} & \dots & \ddots & \ddots & g_{0,\lambda}^{(k,\alpha)} \\ g_{n-1,\lambda}^{(k,\alpha)} & g_{n-2,\lambda}^{(k,\alpha)} & \dots & g_{2,\lambda}^{(k,\alpha)} & g_{1,\lambda}^{(k,\alpha)} - \phi^{(k,m)}(\lambda) \end{pmatrix}. \quad (3.5)$$

The matrix form for (3.4) can be written as follows

$$\left(I - \frac{\tau}{2h^\alpha} (c_l B_{k,\lambda} + c_r B_{k,\lambda}^T) \right) U^{j+1} = \left(I + \frac{\tau}{2h^\alpha} (c_l B_{k,\lambda} + c_r B_{k,\lambda}^T) \right) U^j + \tau \hat{P}_i^{j+\frac{1}{2}}, \quad (3.6)$$

where

$$\hat{P}^{n+\frac{1}{2}} = \begin{pmatrix} P_1^{n+\frac{1}{2}} \\ P_2^{n+\frac{1}{2}} \\ \vdots \\ P_{M-2}^{n+\frac{1}{2}} \\ P_{M-1}^{n+\frac{1}{2}} \end{pmatrix} + \frac{1}{2h^\alpha} \begin{pmatrix} c_l g_{2,\lambda}^{(k,\alpha)} + c_r g_{0,\lambda}^{(k,\alpha)} \\ c_l g_{3,\lambda}^{(k,\alpha)} \\ \vdots \\ c_l g_{M-1,\lambda}^{(k,\alpha)} \\ c_l g_{M,\lambda}^{(k,\alpha)} \end{pmatrix} (U_0^n + U_0^{n+1}) + \frac{1}{2h^\alpha} \begin{pmatrix} c_r g_{M,\lambda}^{(k,\alpha)} \\ c_r g_{M-1,\lambda}^{(k,\alpha)} \\ \vdots \\ c_r g_{3,\lambda}^{(k,\alpha)} \\ c_r g_{2,\lambda}^{(k,\alpha)} + c_l g_{0,\lambda}^{(k,\alpha)} \end{pmatrix} (U_M^n + U_M^{n+1}). \quad (3.7)$$

3.1. The second-order numerical scheme

In this subsection, we detail the second-order scheme for the tempered fractional diffusion equation. Using the tempered-WSGD operators, ${}_L\mathcal{D}_{h,2}^{\alpha,\lambda_1} = {}_L\mathcal{D}_{h,-1,0,1}^{\alpha,\gamma_1,\gamma_2,\gamma_3}$ and ${}_R\mathcal{D}_{h,2}^{\alpha,\lambda_2} = {}_R\mathcal{D}_{h,-1,0,1}^{\alpha,\gamma_1,\gamma_2,\gamma_3}$, for the tempered fractional derivatives and the Crank-Nicolson discretization in time, we find the following second-order scheme for (3.1),

$$\begin{aligned} u_i^{j+1} - \frac{\tau}{2} \left[c_l \left({}_L\mathcal{D}_{h,2}^{\alpha,\lambda} u_i^{j+1} \right) + c_r \left({}_R\mathcal{D}_{h,2}^{\alpha,\lambda} u_i^{j+1} \right) \right] \\ = u_i^j + \frac{\tau}{2} \left[c_l \left({}_L\mathcal{D}_{h,2}^{\alpha,\lambda} u_i^j \right) + c_r \left({}_R\mathcal{D}_{h,2}^{\alpha,\lambda} u_i^j \right) \right] + \tau p_i^{j+\frac{1}{2}} + O(\tau^3 + \tau h^2). \end{aligned} \quad (3.8)$$

The numerical scheme can be written as,

$$\begin{aligned} U_i^{j+1} - \frac{\tau}{2} \left[c_l \left({}_L\mathcal{D}_{h,2}^{\alpha,\lambda} U_i^{j+1} \right) + c_r \left({}_R\mathcal{D}_{h,2}^{\alpha,\lambda} U_i^{j+1} \right) \right] \\ = U_i^j + \frac{\tau}{2} \left[c_l \left({}_L\mathcal{D}_{h,2}^{\alpha,\lambda} U_i^j \right) + c_r \left({}_R\mathcal{D}_{h,2}^{\alpha,\lambda} U_i^j \right) \right] + \tau P_i^{j+\frac{1}{2}}, \end{aligned} \quad (3.9)$$

and the following matrix form results,

$$\left(I - \frac{\tau}{2h^\alpha} (c_l B_\lambda + c_r B_\lambda^T) \right) U^{j+1} = \left(I + \frac{\tau}{2h^\alpha} (c_l B_\lambda + c_r B_\lambda^T) \right) U^j + \tau \hat{P}_i^{j+\frac{1}{2}}, \quad (3.10)$$

where $B_\lambda = B_{2,\lambda}$, as defined in (3.5).

The stability and convergence for the second-order scheme have already been presented in [23] in the following theorem, based on the lemma below.

Lemma 3.1. From [23]. For $1 < \alpha < 2$ and $\lambda \geq 0$, if

$$\max \left\{ \frac{(2-\alpha)(\alpha^2 + \alpha - 8)}{2(\alpha^2 + 3\alpha + 2)}, \frac{(1-\alpha)(\alpha^2 + 2\alpha)}{2(\alpha^2 + 3\alpha + 4)} \right\} < \gamma_3 < \frac{(2-\alpha)(\alpha^2 + 2\alpha - 3)}{2(\alpha^2 + 3\alpha + 2)},$$

then the weights coefficients $\omega_k^{(\alpha)}$ and $g_{k,\lambda}^{(2,\alpha)}$ satisfy

1. $\omega_0^{(\alpha)} = 1, \omega_1^{(\alpha)} = -\alpha, 0 \leq \dots \leq \omega_3^{(\alpha)} \leq \omega_2^{(\alpha)} \leq 1, \sum_{k=0}^{\infty} \omega_k^{(\alpha)} = 0$.
2. $g_{1,\lambda}^{(2,\alpha)} \leq 0, g_{0,\lambda}^{(2,\alpha)} + g_{2,\lambda}^{(2,\alpha)} \geq 0, g_{k,\lambda}^{(2,\alpha)} \geq 0 (k \geq 3)$.

Theorem 3.2. From [23]. For $1 < \alpha < 2$, and $\lambda_1, \lambda_2 \geq 0$, if $a_1 < \gamma_3 < a_2$, the numerical scheme (3.6) is stable for

$$a_1 = \max \left\{ \frac{(2-\alpha)(\alpha^2 + \alpha - 8)}{2(\alpha^2 + 3\alpha + 2)}, \frac{(1-\alpha)(\alpha^2 + 2\alpha)}{2(\alpha^2 + 3\alpha + 4)} \right\}$$

and

$$a_2 = \frac{(2-\alpha)(\alpha^2 + 2\alpha - 3)}{2(\alpha^2 + 3\alpha + 2)}.$$

Denoting $e_i^j = u_i^j - U_i^j$, $i = 1, 2, \dots, M-1$ and $E^j = (e_1^j, e_2^j, \dots, e_{M-1}^j)^T$, $j = 1, 2, \dots, N$, moreover, it is found that

$$\|E^j\|_h \leq c(\tau^2 + h^2), \quad 1 \leq j \leq N-1. \quad (3.11)$$

3.2. The third-order numerical scheme

In this subsection, we consider third-order accurate scheme for the tempered fractional diffusion equation. Denote

$$g_{k,\lambda} = g_{k,\lambda}^{(3,\alpha)}, \quad \phi(\lambda) = \phi^{(3,4)}(\lambda) = \left(\gamma_1 e^{h\lambda} + \gamma_2 + \gamma_3 e^{-h\lambda} + \gamma_4 e^{-2h\lambda} \right) (1 - e^{-h\lambda})^\alpha.$$

Using the tempered-WSGD operators, ${}_L\mathcal{D}_{h,3}^{\alpha,\lambda} = {}_L\mathcal{D}_{h,-1,0,1,2}^{\alpha,\gamma_1,\gamma_2,\dots,\gamma_4}$ and ${}_R\mathcal{D}_{h,3}^{\alpha,\lambda} = {}_R\mathcal{D}_{h,-1,0,1,2}^{\alpha,\gamma_1,\gamma_2,\dots,\gamma_4}$, for the tempered fractional derivatives and the Crank-Nicolson time discretization, we find the following discretization for (3.1),

$$\begin{aligned} u_i^{j+1} - \frac{\tau}{2} \left[c_l \left({}_L\mathcal{D}_{h,3}^{\alpha,\lambda} u_i^{j+1} \right) + c_r \left({}_R\mathcal{D}_{h,3}^{\alpha,\lambda} u_i^{j+1} \right) \right] \\ = u_i^j + \frac{\tau}{2} \left[c_l \left({}_L\mathcal{D}_{h,3}^{\alpha,\lambda} u_i^j \right) + c_r \left({}_R\mathcal{D}_{h,3}^{\alpha,\lambda} u_i^j \right) \right] + \tau p_i^{j+\frac{1}{2}} + O(\tau^3 + \tau h^3). \end{aligned} \quad (3.12)$$

This numerical scheme is then written as,

$$\begin{aligned} U_i^{j+1} - \frac{\tau}{2} \left[c_l \left({}_L\mathcal{D}_{h,3}^{\alpha,\lambda} U_i^{j+1} \right) + c_r \left({}_R\mathcal{D}_{h,3}^{\alpha,\lambda} U_i^{j+1} \right) \right] \\ = U_i^j + \frac{\tau}{2} \left[c_l \left({}_L\mathcal{D}_{h,3}^{\alpha,\lambda} U_i^j \right) + c_r \left({}_R\mathcal{D}_{h,3}^{\alpha,\lambda} U_i^j \right) \right] + \tau P_i^{j+\frac{1}{2}}. \end{aligned} \quad (3.13)$$

The matrix form looks as follows,

$$\left(I - \frac{\tau}{2h^\alpha} (c_l B_\lambda + c_r B_\lambda^T) \right) U^{j+1} = \left(I + \frac{\tau}{2h^\alpha} (c_l B_\lambda + c_r B_\lambda^T) \right) U^j + \tau \hat{P}_i^{j+\frac{1}{2}}, \quad (3.14)$$

where again, $B_\lambda = B_{3,\lambda}$, as defined in (3.5).

We will now analyze the stability and convergence of this third-order scheme. First of all, we introduce the following lemmas.

Lemma 3.3. See, for example, [33]. A real-valued matrix A of order n is positive definite if and only if its symmetric part, $H = \frac{A+A^T}{2}$, is positive definite; H is positive definite if and only if the eigenvalues of H are positive. For any eigenvalue μ of A , we have

$$\mu_{\min}(H) \leq \operatorname{Re}(\mu(A)) \leq \mu_{\max}(H), \quad (3.15)$$

where $\operatorname{Re}(\mu(A))$ represents the real part of μ , and $\mu_{\min}(H)$ and $\mu_{\max}(H)$ are the minimum and maximum eigenvalues of H , respectively.

To obtain the stability, we introduce the definition of the Toeplitz matrix T_n and its generating function f .

Definition 3.1. See, for example, [9] Let the Toeplitz matrix T_n be of the following form,

$$T_n = \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{2-n} & t_{1-n} \\ t_1 & t_0 & t_{-1} & \cdots & t_{2-n} \\ \vdots & t_1 & t_0 & \ddots & \vdots \\ t_{n-2} & \cdots & \ddots & \ddots & t_{-1} \\ t_{n-1} & t_{n-2} & \ddots & t_1 & t_0 \end{pmatrix}. \quad (3.16)$$

If the diagonals $\{t_k\}_{k=-n+1}^{n-1}$ are the Fourier coefficients of a function f , i.e.,

$$t_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx, \quad (3.17)$$

then function f is called the generating function of T_n .

Lemma 3.4. (Grenander-Szegö theorem [9].) For a Toeplitz matrix T_n , we denote by $\mu_{\min}(T_n)$ and $\mu_{\max}(T_n)$ the smallest and largest eigenvalues of T_n , respectively. If f is a 2π -periodic continuous real-valued function, defined on $[-\pi, \pi]$, then

$$f_{\min} \leq \mu_{\min}(T_n) \leq \mu_{\max}(T_n) \leq f_{\max},$$

where f_{\min} and f_{\max} denote the minimum and maximum values of $f(x)$. Moreover, if $f_{\min} < f_{\max}$, then all the eigenvalues of T_n satisfy

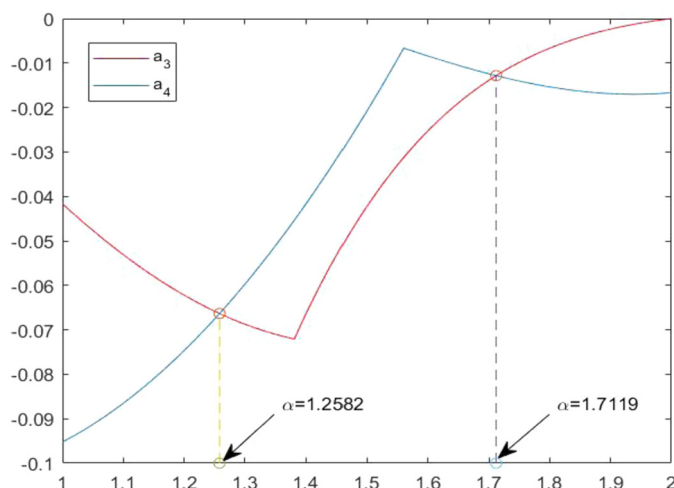
$$f_{\min} < \mu(T_n) < f_{\max},$$

for all $n > 0$; and furthermore if $f_{\min} \geq 0$, then T_n is positive definite.

Next, we define the function,

$$f_B(x) = \sum_{k=0}^{N-1} g_{k,\lambda} e^{i(k-1)x} - \phi(\lambda), \quad f_{B^T}(x) = \sum_{k=0}^{N-1} g_{k,\lambda} e^{-i(k-1)x} - \phi(\lambda), \quad (3.18)$$

and the generating function,

Fig. 1. a_3, a_4 with $\alpha \in (1, 2)$.

$$f(\alpha, \lambda; x) = \frac{f_B(x) + f_{B^T}(x)}{2}. \quad (3.19)$$

It's straightforward to get the following lemma by Lemmas 3.3–3.4.

Lemma 3.5. Let the matrices B_λ and B_λ^T be given via the numerical scheme (3.5). For $\lambda \geq 0$, $h > 0$ and $\alpha \in [1, 2]$, if we can find (analytically, or with the help of numerical techniques) values of γ_j for which the generating functions $f(\alpha, \lambda; x)$ of B_λ are negative, then the eigenvalues of the matrix B_λ are negative, too.

Corollary 3.6. If, for $1 < \alpha < 2$, the generating functions $f(\alpha, \lambda; x)$, given in (3.19), are negative, the numerical scheme (3.13) is stable.

Proof. Let $\mathcal{M} = \frac{\tau}{2h^\alpha} (c_l B_\lambda + c_r B_\lambda^T)$. We find,

$$\frac{\mathcal{M} + \mathcal{M}^T}{2} = \frac{\tau}{4h^\alpha} (c_l (B_\lambda + B_\lambda^T) + c_r (B_\lambda + B_\lambda^T)) \quad (3.20)$$

With $\mu(\mathcal{M})$ an eigenvalue of matrix \mathcal{M} , it follows that $\mu(\mathcal{M}) < 0$ when $f(\alpha, \lambda; x) < 0$ by Lemmas 3.3–3.4. Then $\frac{1+\mu(\mathcal{M})}{1-\mu(\mathcal{M})} < 1$ is an eigenvalue of matrix $|I - \mathcal{M}|^{-1} |I + \mathcal{M}|$. Hence, the numerical scheme (3.13) is stable. \square

For $\alpha \in (1, 2)$, we denote

$$a_3 = \max\left\{\frac{\frac{\alpha^5}{8} + \frac{7}{12}\alpha^4 - \frac{5}{8}\alpha^3 - \frac{49}{12}\alpha^2 + 3\alpha}{\alpha^3 + 6\alpha^2 + 11\alpha + 6}, \frac{\frac{\alpha^5}{8} + \frac{\alpha^4}{3} - \frac{67}{24}\alpha^3 - \frac{23}{6}\alpha^2 + \frac{175}{6}\alpha - 30}{\alpha^3 + 6\alpha^2 + 11\alpha + 6}\right\},$$

and

$$a_4 = \min\left\{\frac{\frac{1}{8}\alpha^4 + \frac{7}{12}\alpha^3 + \frac{1}{8}\alpha^2 - \frac{13}{6}\alpha}{\alpha^2 + 5\alpha + 8}, \frac{\frac{\alpha^5}{8} + \frac{11}{24}\alpha^4 - \frac{41}{24}\alpha^3 - \frac{107}{24}\alpha^2 + \frac{163}{12}\alpha - 8}{\alpha^3 + 6\alpha^2 + 11\alpha + 6}\right\}.$$

The impact of varying a_3 and a_4 is illustrated in Fig. 1. It can be seen that when $\alpha \in (1.26, 1.71)$, $a_3 < a_4$ and $(a_3, a_4) \neq \emptyset$.

For $\alpha \in (1.26, 1.71)$, we obtain the following result, similar to Lemma 3.1.

Theorem 3.7. For $\alpha \in (1.26, 1.71)$, $\lambda \geq 0$ and $a_3 \leq \gamma_4 \leq a_4$, then there exists $g_{1,\lambda} \leq 0$, $g_{0,\lambda} + g_{2,\lambda} \geq 0$, $g_{k,\lambda} \geq 0 (k \geq 3)$.

Proof. For the term $g_{0,\lambda} + g_{2,\lambda}$, we have

$$\begin{aligned}
 g_{0,\lambda} + g_{2,\lambda} &= \gamma_1 \omega_0^{(\alpha)} e^{h\lambda} + \left(\gamma_1 \omega_2^{(\alpha)} + \gamma_2 \omega_1^{(\alpha)} + \gamma_3 \omega_0^{(\alpha)} \right) e^{-h\lambda} \\
 &= \left(\frac{\alpha^2}{8} + \frac{5}{24} \alpha - \gamma_4 \right) e^{h\lambda} + \left(\frac{\alpha(\alpha-1)}{2} \left(\frac{\alpha^2}{8} + \frac{5}{24} \alpha - \gamma_4 \right) \right. \\
 &\quad \left. - \alpha \left(-\frac{\alpha^2}{4} + \frac{1}{12} \alpha + 1 + 3\gamma_4 \right) + \left(\frac{\alpha^2}{8} - \frac{7}{24} \alpha - 3\gamma_4 \right) \right) e^{-h\lambda} \\
 &\geq \left(\left(\frac{\alpha^2}{8} + \frac{5}{24} \alpha - \gamma_4 \right) + \frac{\alpha(\alpha-1)}{2} \left(\frac{\alpha^2}{8} + \frac{5}{24} \alpha - \gamma_4 \right) \right. \\
 &\quad \left. - \alpha \left(-\frac{\alpha^2}{4} + \frac{1}{12} \alpha + 1 + 3\gamma_4 \right) + \left(\frac{\alpha^2}{8} - \frac{7}{24} \alpha - 3\gamma_4 \right) \right) e^{-h\lambda} \\
 &= \left(\left(\frac{\alpha^4}{16} + \frac{7}{24} \alpha^3 + \frac{\alpha^2}{16} - \frac{13}{12} \alpha \right) - \left(\frac{\alpha^2}{2} + \frac{5}{2} \alpha + 4 \right) \gamma_4 \right) e^{-h\lambda}.
 \end{aligned} \tag{3.21}$$

If $\gamma_4 \leq \left(\frac{1}{8} \alpha^4 + \frac{7}{12} \alpha^3 + \frac{1}{8} \alpha^2 - \frac{13}{6} \alpha \right) / (\alpha^2 + 5\alpha + 8)$, it is immediate that $g_{0,\lambda} + g_{2,\lambda} \geq 0$.

For the term $g_{1,\lambda}$, it is found that

$$\begin{aligned}
 g_{1,\lambda} &= \gamma_1 \omega_1^{(\alpha)} + \gamma_2 \omega_0^{(\alpha)} = -\alpha \left(\frac{\alpha^2}{8} + \frac{5}{24} \alpha - \gamma_4 \right) + \left(-\frac{\alpha^2}{4} + \frac{1}{12} \alpha + 1 + 3\gamma_4 \right) \\
 &= -\frac{\alpha^3}{8} - \frac{11}{24} \alpha^2 + \frac{\alpha}{12} + 1 + (3 + \alpha) \gamma_4.
 \end{aligned} \tag{3.22}$$

When $\gamma_4 \leq \left(\frac{1}{8} \alpha^4 + \frac{7}{12} \alpha^3 + \frac{1}{8} \alpha^2 - \frac{13}{6} \alpha \right) / (\alpha^2 + 5\alpha + 8)$, $g_{1,\lambda} \leq 0$.

For the term $g_{3,\lambda}$, it follows that,

$$\begin{aligned}
 g_{3,\lambda} &= \left(\gamma_1 \omega_3^{(\alpha)} + \gamma_2 \omega_2^{(\alpha)} + \gamma_3 \omega_1^{(\alpha)} + \gamma_4 \omega_0^{(\alpha)} \right) e^{-2h\lambda} \\
 &= \left(\left(\frac{\alpha^2}{8} + \frac{5}{24} \alpha - \gamma_4 \right) \frac{\alpha(\alpha-1)(2-\alpha)}{6} + \left(-\frac{\alpha^2}{4} + \frac{\alpha}{12} + 1 + 3\gamma_4 \right) \frac{\alpha(\alpha-1)}{2} \right. \\
 &\quad \left. - \left(\frac{\alpha^2}{8} - \frac{7}{24} \alpha - 3\gamma_4 \right) \alpha + \gamma_4 \right) e^{-2h\lambda} \\
 &= \left(-\left(\frac{\alpha^5}{48} + \frac{7}{72} \alpha^4 - \frac{5}{48} \alpha^3 - \frac{49}{72} \alpha^2 + \frac{1}{2} \alpha \right) + \left(\frac{\alpha^3}{6} + \alpha^2 + \frac{11}{6} \alpha + 1 \right) \gamma_4 \right) e^{-2h\lambda}.
 \end{aligned} \tag{3.23}$$

If $\gamma_4 \geq \left(\frac{\alpha^5}{8} + \frac{7}{12} \alpha^4 - \frac{5}{8} \alpha^3 - \frac{49}{12} \alpha^2 + 3\alpha \right) / (\alpha^3 + 6\alpha^2 + 11\alpha + 6)$, we find $g_{3,\lambda} \geq 0$.

For the term $g_{4,\lambda}$, we derive

$$\begin{aligned}
 g_{4,\lambda} &= \left(\gamma_1 \omega_4^{(\alpha)} + \gamma_2 \omega_3^{(\alpha)} + \gamma_3 \omega_2^{(\alpha)} + \gamma_4 \omega_1^{(\alpha)} \right) e^{-3h\lambda} \\
 &= \left(\left(\frac{\alpha^2}{8} + \frac{5}{24} \alpha - \gamma_4 \right) \frac{(3-\alpha)(2-\alpha)(1-\alpha)}{24} + \left(-\frac{\alpha^2}{4} + \frac{\alpha}{12} + 1 + 3\gamma_4 \right) \frac{(2-\alpha)(1-\alpha)}{6} \right. \\
 &\quad \left. + \left(\frac{\alpha^2}{8} - \frac{7}{24} \alpha - 3\gamma_4 \right) \left(\frac{1-\alpha}{2} \right) + \gamma_4 \right) \omega_1^{(\alpha)} e^{-3h\lambda} \\
 &= -\frac{\alpha}{24} \left(-\left(\frac{\alpha^5}{8} + \frac{11}{24} \alpha^4 - \frac{41}{24} \alpha^3 - \frac{107}{24} \alpha^2 + \frac{163}{12} \alpha - 8 \right) + (\alpha^3 + 6\alpha^2 + 11\alpha + 6) \gamma_4 \right) e^{-3h\lambda}.
 \end{aligned} \tag{3.24}$$

If $\gamma_4 \leq \left(\frac{\alpha^5}{8} + \frac{11}{24} \alpha^4 - \frac{41}{24} \alpha^3 - \frac{107}{24} \alpha^2 + \frac{163}{12} \alpha - 8 \right) / (\alpha^3 + 6\alpha^2 + 11\alpha + 6)$, then it is straightforward to find $g_{4,\lambda} \geq 0$.

For the term $g_{k,\lambda}$, $k \geq 5$, we finally find,

$$\begin{aligned} g_{k,\lambda} &= \left(\gamma_1 \omega_k^{(\alpha)} + \gamma_2 \omega_{k-1}^{(\alpha)} + \gamma_3 \omega_{k-2}^{(\alpha)} + \gamma_4 \omega_{k-3}^{(\alpha)} \right) e^{-(k-1)h\lambda} \\ &= \left(\left(\frac{\alpha^2}{8} + \frac{5}{24}\alpha - \gamma_4 \right) \left(\frac{k-1-\alpha}{k} \right) \left(\frac{k-2-\alpha}{k-1} \right) \left(\frac{k-3-\alpha}{k-2} \right) \right. \\ &\quad + \left(-\frac{\alpha^2}{4} + \frac{\alpha}{12} + 1 + 3\gamma_4 \right) \left(\frac{k-2-\alpha}{k-1} \right) \left(\frac{k-3-\alpha}{k-2} \right) \\ &\quad + \left(\frac{\alpha^2}{8} - \frac{7}{24}\alpha - 3\gamma_4 \right) \left(\frac{k-3-\alpha}{k-2} \right) + \gamma_4 \left. \right) \omega_{k-3}^{(\alpha)} e^{-(k-1)h\lambda} \\ &= \frac{\omega_{k-3}^{(\alpha)} e^{-(k-1)h\lambda}}{k(k-1)(k-2)} \left(k^3 - \left(\frac{\alpha^2}{2} + \frac{5}{2}\alpha + 5 \right) k^2 + \left(\frac{\alpha^4}{8} + \frac{13}{12}\alpha^3 + \frac{31}{8}\alpha^2 + \frac{83}{12}\alpha + 6 \right) k \right. \\ &\quad \left. - \left(\frac{\alpha^5}{8} + \frac{23}{24}\alpha^4 + \frac{21}{8}\alpha^3 + \frac{73}{24}\alpha^2 + \frac{5}{4}\alpha \right) + \left(\alpha^3 + 6\alpha^2 + 11\alpha + 6 \right) \gamma_4 \right). \end{aligned} \quad (3.25)$$

If,

$$\gamma_4 \geq \frac{-k^3 + \left(\frac{\alpha^2}{2} + \frac{5}{2}\alpha + 5 \right) k^2 - \left(\frac{\alpha^4}{8} + \frac{13}{12}\alpha^3 + \frac{31}{8}\alpha^2 + \frac{83}{12}\alpha + 6 \right) k + \left(\frac{\alpha^5}{8} + \frac{23}{24}\alpha^4 + \frac{21}{8}\alpha^3 + \frac{73}{24}\alpha^2 + \frac{5}{4}\alpha \right)}{\alpha^3 + 6\alpha^2 + 11\alpha + 6},$$

we can find $g_{k,\lambda} \geq 0$, as $\omega_{k-3}^{(\alpha)} > 0$, $k \geq 5$.

Next we consider

$$\frac{-k^3 + \left(\frac{\alpha^2}{2} + \frac{5}{2}\alpha + 5 \right) k^2 - \left(\frac{\alpha^4}{8} + \frac{13}{12}\alpha^3 + \frac{31}{8}\alpha^2 + \frac{83}{12}\alpha + 6 \right) k + \left(\frac{\alpha^5}{8} + \frac{23}{24}\alpha^4 + \frac{21}{8}\alpha^3 + \frac{73}{24}\alpha^2 + \frac{5}{4}\alpha \right)}{\alpha^3 + 6\alpha^2 + 11\alpha + 6},$$

and analyze the function,

$$\begin{aligned} g(x) &= -x^3 + \left(\frac{\alpha^2}{2} + \frac{5}{2}\alpha + 5 \right) x^2 - \left(\frac{\alpha^4}{8} + \frac{13}{12}\alpha^3 + \frac{31}{8}\alpha^2 + \frac{83}{12}\alpha + 6 \right) x \\ &\quad + \left(\frac{\alpha^5}{8} + \frac{23}{24}\alpha^4 + \frac{21}{8}\alpha^3 + \frac{73}{24}\alpha^2 + \frac{5}{4}\alpha \right). \end{aligned} \quad (3.26)$$

Function $g(x)$ is monotonically decreasing as the variable x ($x \geq 5$) for $1.26 < \alpha < 1.71$. When $\alpha^3 + 6\alpha^2 + 11\alpha + 6 \geq 0$, we have

$$\begin{aligned} \frac{g(x)}{\alpha^3 + 6\alpha^2 + 11\alpha + 6} &\leq \frac{g(5)}{\alpha^3 + 6\alpha^2 + 11\alpha + 6} \\ &= \frac{\frac{1}{8}\alpha^5 + \frac{1}{3}\alpha^4 - \frac{67}{24}\alpha^3 - \frac{23}{6}\alpha^2 + \frac{175}{6}\alpha - 30}{\alpha^3 + 6\alpha^2 + 11\alpha + 6}. \quad \square \end{aligned} \quad (3.27)$$

We then analyze the generating functions $f(\alpha, \lambda; x)$ of H given in (3.19).

Theorem 3.8. Let the matrices B_λ and B_λ^T be given by (3.5). For $\lambda \geq 0$, $h > 0$ and $\alpha \in (1.26, 1.71)$, $f(\alpha, \lambda; x)$ is the generating function of $H = \frac{B_\lambda + B_\lambda^T}{2}$, if $\gamma_4 \in (a_3, a_4)$, we have $f(\alpha, \lambda; x) < 0$ and B_λ is negative.

Proof. We consider the function $\phi(\lambda)$, and obtain,

$$\begin{aligned} \phi(\lambda) &= \left(\gamma_1 e^{h\lambda} + \gamma_2 + \gamma_3 e^{-h\lambda} + \gamma_4 e^{-2h\lambda} \right) (1 - e^{-h\lambda})^\alpha \\ &= \left(\left(\frac{\alpha^2}{8} + \frac{5}{24}\alpha - \gamma_4 \right) e^{h\lambda} + \left(-\frac{\alpha^2}{4} + \frac{1}{12}\alpha + 1 + 3\gamma_4 \right) + \left(\frac{\alpha^2}{8} - \frac{7}{24}\alpha - 3\gamma_1 \right) e^{-h\lambda} + \gamma_4 e^{-2h\lambda} \right) (1 - e^{-h\lambda})^\alpha \\ &= \left(\frac{\alpha^2}{8} (e^{h\lambda} + e^{-h\lambda} - 2) + \frac{\alpha}{24} (5e^{h\lambda} - 7e^{-h\lambda} + 2) + 1 + \gamma_4 (-e^{h\lambda} + 3 - 3e^{-h\lambda} + e^{-2h\lambda}) \right) (1 - e^{-h\lambda})^\alpha \\ &\geq \gamma_4 \left(-e^{h\lambda} + 3 - 3e^{-h\lambda} + e^{-2h\lambda} \right) (1 - e^{-h\lambda})^\alpha. \end{aligned} \quad (3.28)$$

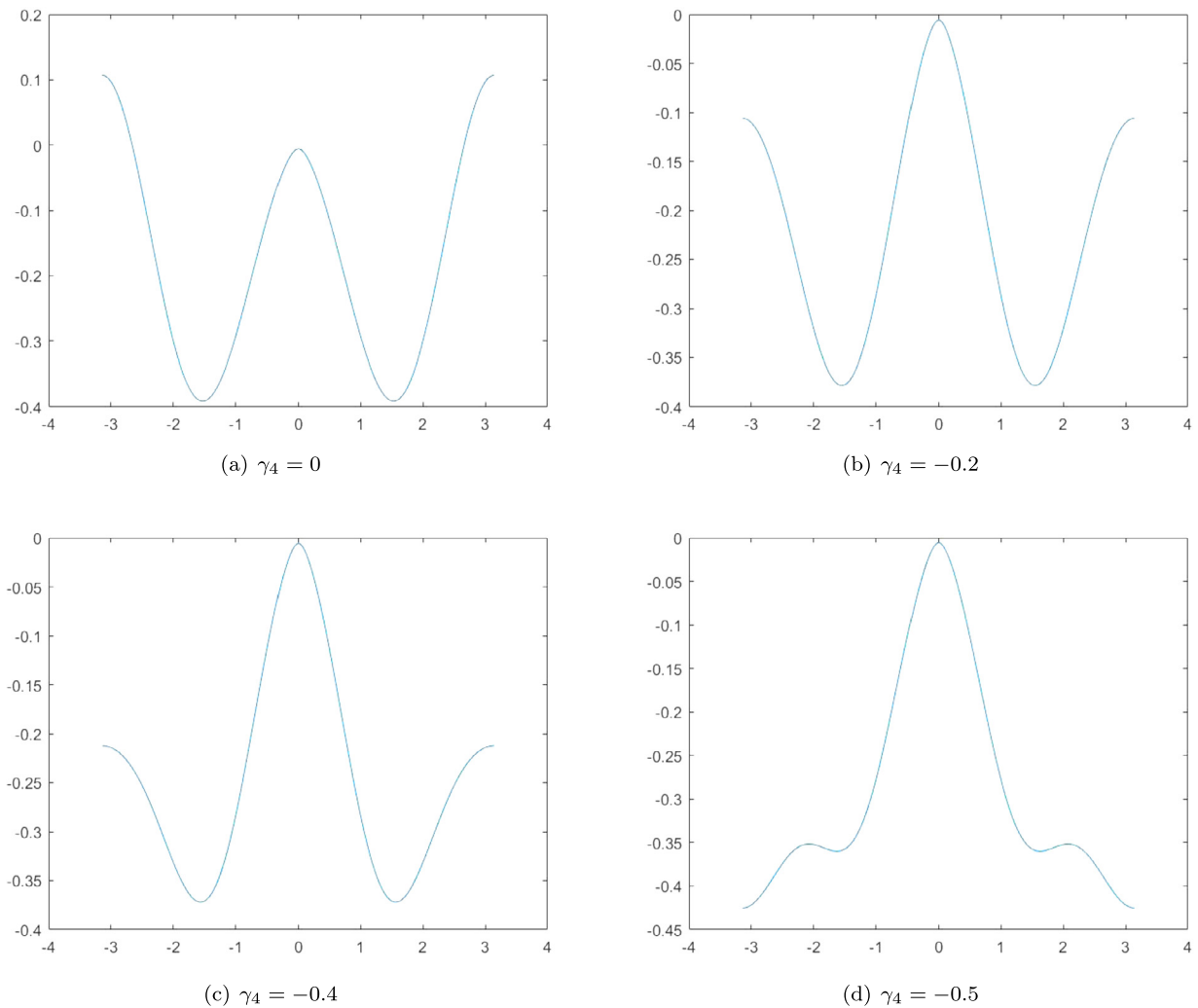


Fig. 2. Function $f(\alpha, \lambda; x)$ with $\alpha = 1.5$, $\lambda = 2$ and $N = 250$.

Denoting $h(x) = -e^x + 3 - 3e^{-x} + e^{-2x}$, we obtain $h(x) \leq h(0) = 0$, as $h(x)$ is monotonically decreasing when $x \geq 0$. It is found that $\phi(\lambda) > 0$ as $\gamma_4 < 0$.

By Theorem 3.7, we obtain

$$\begin{aligned}
 f(\alpha, \lambda; x) &= \sum_{k=0}^N g_{k,\lambda}^\alpha \cos(k-1)x - \phi(\lambda) \leq \sum_{k=0}^N g_{k,\lambda}^\alpha - \phi(\lambda) \\
 &= \sum_{k=0}^N g_{k,\lambda}^\alpha - \sum_{k=0}^{+\infty} g_{k,\lambda}^\alpha < 0.
 \end{aligned} \tag{3.29}$$

By Lemma 3.4, we see that B_λ is negative. \square

Remark 3.1. For any $\alpha \in (1.26, 1.71)$, if $\gamma_4 \in (a_3, a_4)$, the numerical scheme (3.13) will be unconditionally stable by Corollary 3.6 and Theorem 3.8. However, if $\gamma_4 \notin (a_3, a_4)$, we will also obtain a stable numerical scheme (3.13) whenever $f(\alpha, \lambda; x) < 0$. To show this numerically, we choose $\alpha = 1.5 \in (1.26, 1.71)$ with $\gamma_4 = 0, -0.2, -0.5 \notin (a_3, a_4)$ and $\gamma_4 = -0.4 \in (a_3, a_4)$ in Fig. 2. Clearly, we obtain a stable numerical scheme for a certain N , even if $\gamma_4 \notin (a_3, a_4)$, see Fig. 2. It is however nontrivial to find certain N -values when γ_4 is very different from (a_3, a_4) .

Remark 3.2. For any $\alpha \notin (1.26, 1.71)$, we obtain $g_{3,\lambda}^\alpha > 0$ and $g_{k,\lambda}^\alpha > 0$ ($k \geq 5$), if $\gamma_4 > a_3$, see Fig. 1. However, either the term $g_{0,\lambda}^\alpha + g_{2,\lambda}^\alpha$ or the term $g_{3,\lambda}^\alpha$ should be negative. It is difficult to find a range for γ_4 that makes $f(\alpha, \lambda; x)$ be negative.

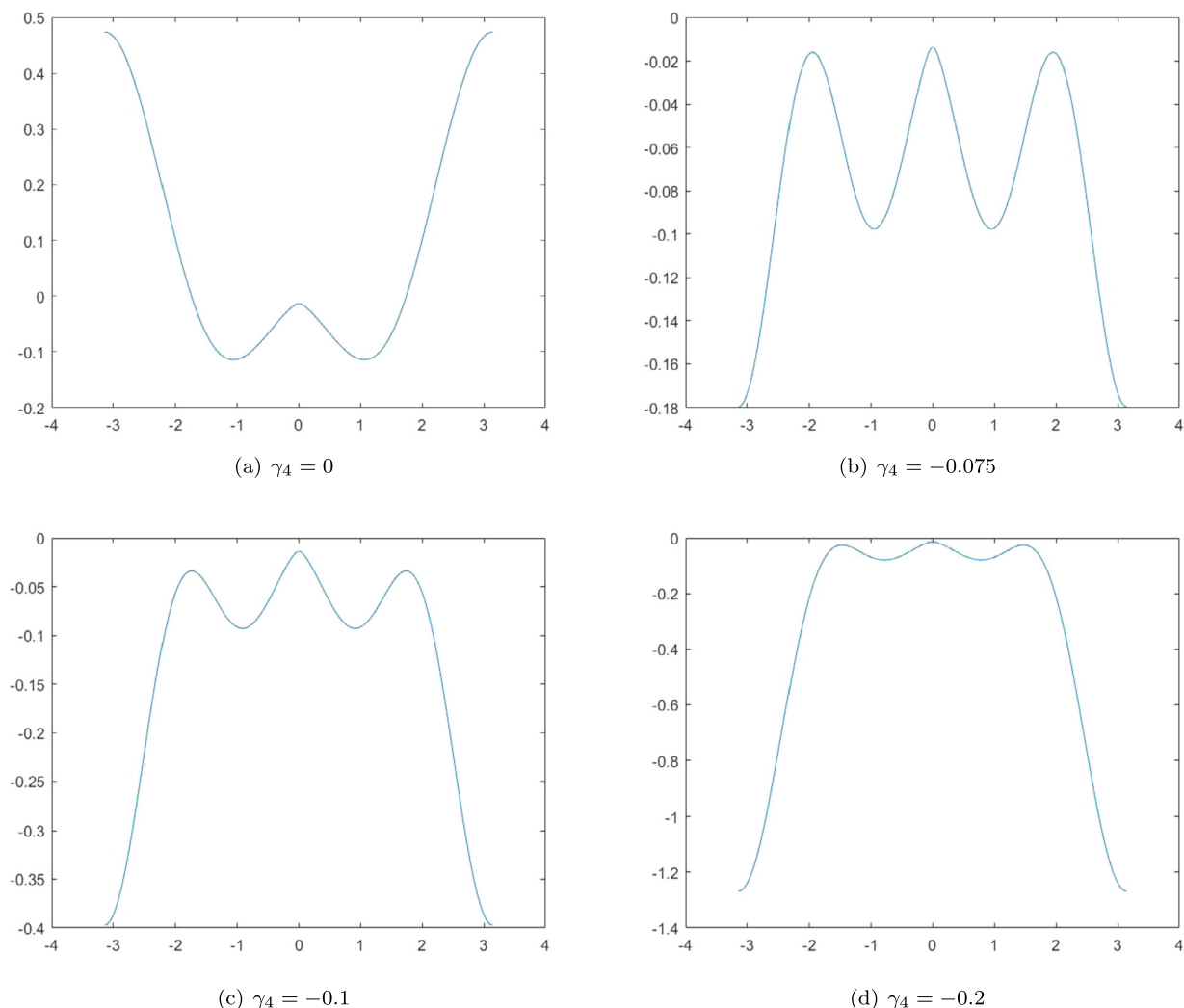


Fig. 3. Function $f(\alpha, \lambda; x)$ with $\alpha = 1.2$, $\lambda = 2$ and $N = 250$.

We will check numerically whether $f(\alpha, \lambda; x) < 0$ for different γ_4 with a fixed value of N in Figs. 3 and 4. Choosing $\alpha = 1.2 \in (1, 1.26)$ with $\gamma_4 = 0, -0.075, -0.1, -0.2$, see Fig. 3, and $\alpha = 1.8 \in (1.71, 2)$ with $\gamma_4 = 0, -0.1, -0.2, -0.5$, see Fig. 4, we find that γ_4 can be determined so that $f(\alpha, \lambda; x) < 0$, for certain N -values. We cannot define a valid range for N , but we can find the value of N to satisfy $f(\alpha, \lambda; x) < 0$ which is based on the γ_i -values from Figs. 3 and 4.

In fact, $\forall \alpha \in (1, 2)$, if a γ_i exists which satisfies $f(\alpha, \lambda; x) < 0$ for certain $N > 0$, the numerical scheme (3.13) will be stable. The numerical examples in Section 4 also showcase this.

Error estimates for the fully discrete scheme (3.13) are based on the following lemma.

Lemma 3.9. (Discrete Gronwall's inequation, see, for example, [32].) Assume that $\{k_n\}$ and $\{m_n\}$ are nonnegative sequences, and the sequence $\{\phi\}$ satisfies

$$\phi_0 \leq g_0, \quad \phi_n \leq g_0 + \sum_{l=0}^{n-1} m_l + \sum_{l=0}^{n-1} k_l \phi_l, \quad n \geq 1,$$

where $g_0 \leq 0$. Then, the sequence $\{\phi\}$ satisfies

$$\phi_n \leq \left(g_0 + \sum_{l=1}^{n-1} m_l \right) \exp \left(\sum_{l=1}^{n-1} k_l \right), \quad n \geq 1.$$

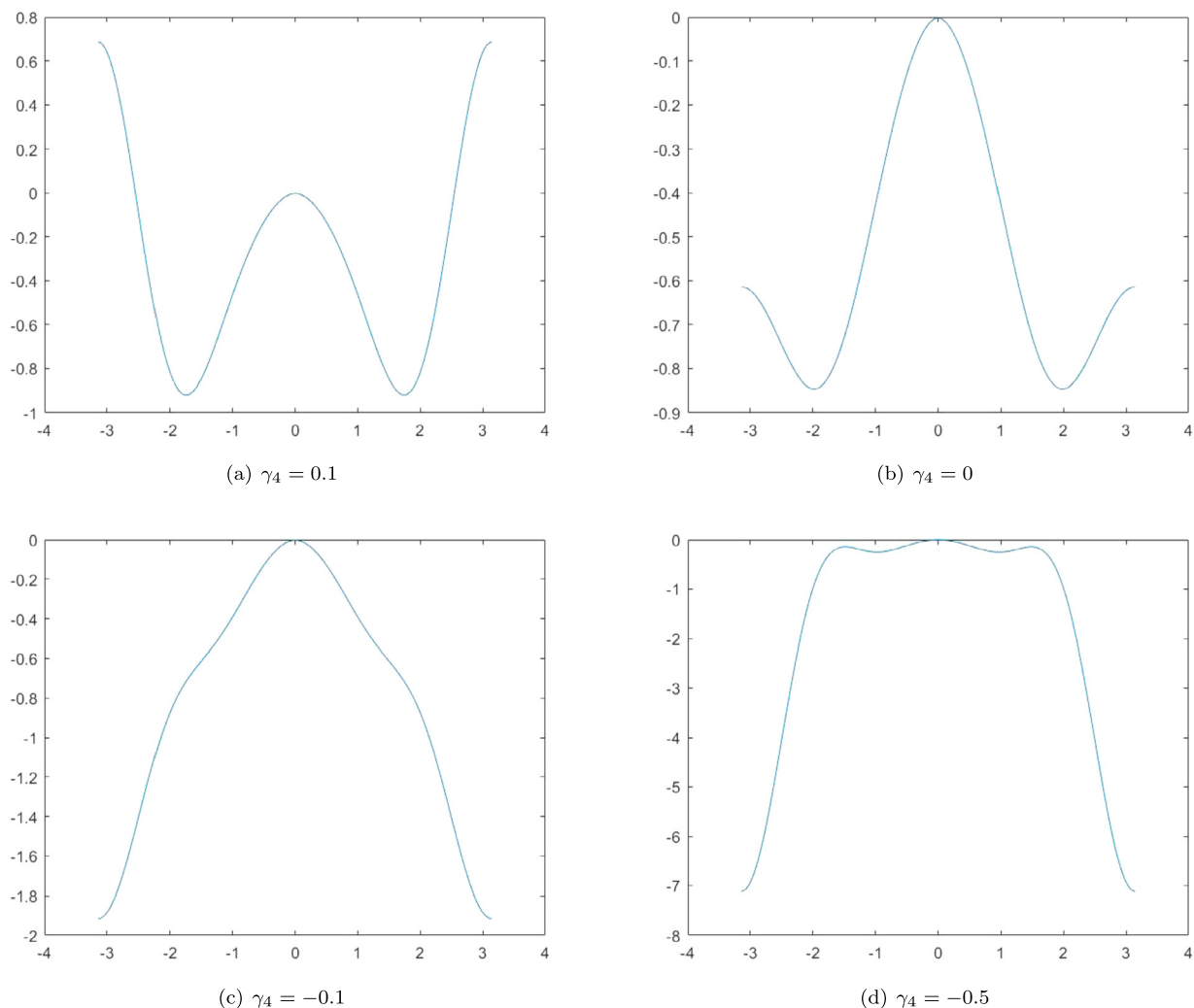


Fig. 4. Function $f(\alpha, \lambda; x)$ with $\alpha = 1.8$, $\lambda = 2$ and $N = 250$.

Theorem 3.10. Assuming function $u(x, t)$ is the solution of Equation (3.1) on a bounded interval $(a, b) \times (0, T)$, which can be zero extended for $x < a$ or $x > b$, so that $u \in L^1(0, T; \mathbb{R})$, ${}_a\mathfrak{D}_x^{\alpha+l, \lambda} u$ and its Fourier transform belong to $L^1(0, T; \mathbb{R})$. Let's denote by $e_i^j = u_i^j - U_i^j$, $i = 1, 2, \dots, M-1$, and $E^j = (e_1^j, e_2^j, \dots, e_{M-1}^j)^T$, $j = 1, 2, \dots, N$. With solutions u_i^j and U_i^j of Equations (3.1) and (3.13), respectively, we have, for $1 < \alpha < 2$, if $f(\alpha, \lambda_i; x) < 0$, $i = 1, 2$,

$$\|E^j\|_h \leq c(\tau^2 + h^3), \quad 1 \leq j \leq N-1. \quad (3.30)$$

Proof. Combining (3.12) and (3.13), gives us,

$$\begin{aligned} e_i^{j+1} - \frac{\tau}{2} \left[c_l \cdot {}_L\mathcal{D}_{h,3}^{\alpha, \lambda_1} e_i^{j+1} + c_r \cdot {}_R\mathcal{D}_{h,3}^{\alpha, \lambda_2} e_i^{j+1} \right] \\ = e_i^j + \frac{\tau}{2} \left[c_l \cdot {}_L\mathcal{D}_{h,3}^{\alpha, \lambda_1} e_i^j + c_r \cdot {}_R\mathcal{D}_{h,3}^{\alpha, \lambda_2} e_i^j \right] + \tau \rho_i^j, \end{aligned} \quad (3.31)$$

where $\rho_i^j = O(\tau^2 + h^3)$. In matrix form, this is given by

$$(I - \mathcal{M})E^{j+1} = (I + \mathcal{M})E^j + \tau \rho^j, \quad (3.32)$$

where $\rho^j = (\rho_1^j, \rho_2^j, \dots, \rho_{M-1}^j)^T$.

After multiplication on both sides of Equation (3.32) by $(E^{j+1} + E^j)^T$, we have

$$(E^{j+1} + E^j)^T (E^{j+1} - E^j) - (E^{j+1} + E^j)^T \mathcal{M} (E^{j+1} + E^j) = \tau (E^{j+1} + E^j)^T \rho^j. \quad (3.33)$$

By Theorem 3.6, it is known that \mathcal{M} is negative. So, we obtain,

$$(E^{j+1} + E^j)^T \mathcal{M}(E^{j+1} + E^j) < 0. \quad (3.34)$$

Hence, we obtain

$$\begin{aligned} (E^{j+1} + E^j)^T (E^{j+1} - E^j) &= \sum_{i=1}^{M-1} ((e_i^{j+1})^2 - (e_i^j)^2) \\ &\leq \tau (E^{j+1} + E^j)^T \rho^j = \tau \sum_{i=1}^{M-1} (e_i^{j+1} + e_i^j) \rho_i^j. \end{aligned} \quad (3.35)$$

Summing up for all $j \in [0, n-1]$, we conclude that

$$\begin{aligned} \sum_{i=1}^{M-1} (e_i^n)^2 &\leq \tau \sum_{j=0}^{n-1} \sum_{i=1}^{M-1} (e_i^{j+1} + e_i^j) \rho_i^j \\ &= \tau \sum_{j=1}^{n-1} \sum_{i=1}^{M-1} (\rho_i^j + \rho_i^{j-1}) e_i^j + \tau \sum_{i=1}^{M-1} e_i^n \rho_i^{n-1} \\ &\leq \frac{\tau}{2} \sum_{j=0}^{n-1} \sum_{i=1}^{M-1} (e_i^j)^2 + \frac{\tau}{2} \sum_{j=0}^{n-1} \sum_{i=1}^{M-1} (\rho_i^j + \rho_i^{j-1})^2 + \frac{1}{2} \sum_{i=1}^{M-1} (e_i^n)^2 + \frac{1}{2} \sum_{i=1}^{M-1} (\tau \rho_i^{n-1})^2. \end{aligned} \quad (3.36)$$

As $\rho_i^j = O(\tau^2 + h^3)$, we have the following result by the discrete Gronwall's inequality (3.9),

$$\|E^n\|^2 \leq \tau \sum_{j=1}^{n-1} \|E^j\|^2 + c(\tau^2 + h^3)^2 \leq e^T c(\tau^2 + h^3)^2 \leq C(\tau^2 + h^3)^2. \quad \square \quad (3.37)$$

4. Numerical examples

In this section, we present some numerical results for several experiments, on tempered fractional derivatives in Example 4.1, the tempered fractional diffusion equations in Example 4.2 and the tempered fractional Black–Scholes equation in Example 4.3–4.4, to verify the theoretical results.

4.1. The tempered fractional derivatives

In this subsection, we take the second-order operators (2.15)–(2.16) and third-order operators (2.19)–(2.20) for the left and right tempered fractional derivatives to test the accuracy of the tempered-WSGD operators.

Example 4.1. In the example, we choose $\alpha = 0.6$ and $\alpha = 1.6$, and consider different λ in the interval $[0, 1]$ for the left and right tempered fractional derivatives.

1. We analyze the schemes for the following left tempered fractional derivative,

$${}_0\mathfrak{D}_x^{\alpha, \lambda} (e^{-\lambda x} x^{3+\alpha}) = e^{-\lambda x} x^3 \left(\frac{\Gamma(4+\alpha)}{6} - \lambda^\alpha x^\alpha \right),$$

which is discretized by the second-order operator (2.15) and the third-order scheme (2.19), respectively. Tables 1 and 2 show the corresponding L^2 errors and the orders of accuracy for different λ -values, with $\alpha = 0.6$, and $\alpha = 1.6$, $\gamma_3 = 0.001$ for the second-order operator, and $\gamma_4 = 0.001$ for the third-order operator. The results confirm the desired accuracy.

2. We also consider the right tempered fractional derivative,

$$x\mathfrak{D}_1^{\alpha, \lambda} (e^{\lambda x} (1-x)^{3+\alpha}) = e^{\lambda x} (1-x)^3 \left(\frac{\Gamma(4+\alpha)}{6} - \lambda^\alpha (1-x)^\alpha \right),$$

computed by the second-order operator (2.16) and the third-order operators (2.20), respectively. Tables 3 and 4 show the L^2 errors and orders of accuracy for different λ -values, with $\gamma_3 = -0.001$ for the second-order and $\gamma_4 = -0.001$ for the third-order operators. The results clearly confirm the desired discretization accuracy.

Table 1
 L^2 errors and orders of accuracy for ${}_0\mathfrak{D}_x^{\alpha,\lambda}(e^{-\lambda x}x^{3+\alpha})$ by the second-order operator (2.15) for different λ with $\gamma_3 = 0.001$.

α	h	$\lambda = 0$		$\lambda = 1$		$\lambda = 3$	
		L^2 Error	Order	L^2 Error	Order	L^2 Error	Order
$\alpha = 0.6$	1/10	1.05e-02		4.87e-03		1.30e-03	
	1/20	2.58e-03	2.03	1.21e-03	2.01	3.25e-04	1.99
	1/40	6.39e-04	2.01	3.02e-04	2.00	8.15e-05	2.00
	1/80	1.59e-04	2.01	7.55e-05	2.00	2.04e-05	2.00
	1/160	3.96e-05	2.00	1.89e-05	2.00	5.11e-06	2.00
$\alpha = 1.6$	1/10	5.29e-02		2.50e-02		7.02e-03	
	1/20	1.32e-02	2.01	6.33e-03	1.98	1.83e-03	1.94
	1/40	3.29e-03	2.00	1.59e-03	1.99	4.68e-04	1.97
	1/80	8.21e-04	2.00	4.00e-04	1.99	1.18e-04	1.98
	1/160	2.05e-04	2.00	1.00e-04	2.00	2.97e-05	1.99

Table 2
 L^2 errors and orders of accuracy for ${}_0\mathfrak{D}_x^{\alpha,\lambda}(e^{-\lambda x}x^{3+\alpha})$ by the third-order operator (2.19) for different λ with $\gamma_4 = 0.001$.

α	h	$\lambda = 0$		$\lambda = 1$		$\lambda = 3$	
		L^2 Error	Order	L^2 Error	Order	L^2 Error	Order
$\alpha = 0.6$	1/10	6.62e-04		4.09e-04		2.20e-04	
	1/20	8.48e-05	2.96	5.38e-05	2.92	3.08e-05	2.84
	1/40	1.07e-05	2.98	6.91e-06	2.96	4.06e-06	2.92
	1/80	1.35e-06	2.99	8.75e-07	2.98	5.22e-07	2.96
	1/160	1.69e-07	3.00	1.10e-07	2.99	6.62e-08	2.98
$\alpha = 1.6$	1/10	5.79e-03		3.54e-03		1.84e-03	
	1/20	7.47e-04	2.95	4.73e-04	2.90	2.66e-04	2.79
	1/40	9.49e-05	2.98	6.12e-05	2.95	3.59e-05	2.89
	1/80	1.20e-05	2.99	7.78e-06	2.98	4.66e-06	2.95
	1/160	1.50e-06	2.99	9.82e-07	2.99	5.93e-07	2.97

Table 3
 L^2 errors and orders of accuracy for ${}_x\mathfrak{D}_1^{\alpha,\lambda}(e^{\lambda x}(1-x)^{3+\alpha})$ by the second-order operator (2.16) for different λ with $\gamma_3 = -0.001$.

α	h	$\lambda = 0$		$\lambda = 1$		$\lambda = 3$	
		L^2 Error	Order	L^2 Error	Order	L^2 Error	Order
$\alpha = 0.6$	1/10	8.94e-03		1.21e-02		2.57e-02	
	1/20	2.36e-03	1.92	3.13e-03	1.95	6.44e-03	2.00
	1/40	6.06e-04	1.96	7.94e-04	1.98	1.61e-03	2.00
	1/80	1.54e-04	1.98	2.00e-04	1.99	4.04e-04	2.00
	1/160	3.87e-05	1.99	5.03e-05	1.99	1.01e-04	2.00
$\alpha = 1.6$	1/10	4.49e-02		6.17e-02		1.40e-01	
	1/20	1.21e-02	1.90	1.63e-02	1.92	3.63e-02	1.94
	1/40	3.12e-03	1.95	4.19e-03	1.96	9.27e-03	1.97
	1/80	7.95e-04	1.97	1.06e-03	1.98	2.34e-03	1.98
	1/160	2.00e-04	1.99	2.67e-04	1.99	5.89e-04	1.99

Table 4
 L^2 errors and orders of accuracy for ${}_x\mathfrak{D}_1^{\alpha,\lambda}(e^{\lambda x}(1-x)^{3+\alpha})$ by the third-order operator (2.20) for different λ with $\gamma_4 = -0.001$.

α	h	$\lambda = 0$		$\lambda = 1$		$\lambda = 3$	
		L^2 Error	Order	L^2 Error	Order	L^2 Error	Order
$\alpha = 0.6$	1/10	6.53e-04		1.14e-03		4.63e-03	
	1/20	8.59e-05	2.93	1.51e-04	2.92	6.44e-04	2.85
	1/40	1.10e-05	2.96	1.95e-05	2.96	8.49e-05	2.92
	1/80	1.39e-06	2.98	2.47e-06	2.98	1.09e-05	2.96
	1/160	1.75e-07	2.99	3.11e-07	2.99	1.38e-06	2.98
$\alpha = 1.6$	1/10	5.58e-03		9.62e-03		3.77e-02	
	1/20	7.42e-04	2.91	1.30e-03	2.89	5.44e-03	2.79
	1/40	9.56e-05	2.96	1.69e-04	2.94	7.34e-04	2.89
	1/80	1.21e-05	2.98	2.15e-05	2.97	9.53e-05	2.94
	1/160	1.53e-06	2.99	2.72e-06	2.99	1.21e-05	2.97

Table 5

L^2 errors and orders of accuracy for (4.1) with the boundary conditions (4.2) and the initial value (4.3) by the second-order scheme (3.9).

h	$\gamma_3 = -0.01$		$\gamma_3 = 0$		$\gamma_3 = 0.01$	
	L^2 Error	Order	L^2 Error	Order	L^2 Error	Order
1/10	2.20e-04		2.30e-04		2.40e-04	
1/20	5.44e-05	2.02	5.73e-05	2.01	6.02e-05	2.00
1/40	1.33e-05	2.04	1.40e-05	2.03	1.48e-05	2.03
1/80	3.25e-06	2.03	3.45e-06	2.02	3.65e-06	2.02
1/160	8.06e-07	2.01	8.56e-07	2.01	9.05e-07	2.01

Table 6

L^2 errors and orders of accuracy for (4.1) with the boundary conditions (4.2) and the initial value (4.3) by the third-order scheme (3.13).

h	$\gamma_4 = -0.1$		$\gamma_4 = -0.2$		$\gamma_4 = -0.25$	
	L^2 Error	Order	L^2 Error	Order	L^2 Error	Order
1/10	7.55e-05		1.08e-04		1.25e-04	
1/20	9.44e-06	3.00	1.41e-05	2.94	1.66e-05	2.92
1/40	1.13e-06	3.07	1.71e-06	3.04	2.05e-06	3.02
1/80	1.37e-07	3.04	2.09e-07	3.04	2.51e-07	3.03
1/160	1.77e-08	2.96	2.61e-08	3.00	3.14e-08	3.00

4.2. The tempered fractional diffusion equation

In this subsection, we numerically test the accuracy of the second-order scheme (3.9) and the third-order scheme (3.13) for the tempered fractional diffusion equations.

Example 4.2. In this example, however, with $x \in [0, 1]$, we choose different α for three different tempered fractional diffusion equations. To ensure the stability, we take $\gamma_3 \in (a_1, a_2)$ for the second-order scheme (3.9), and $\gamma_4 \in (a_3, a_4)$ or γ_4 satisfying $f(\alpha, \lambda; x) < 0$ for the third-order scheme (3.13). The numerical results are shown in Table 5–10 which confirm the desired accuracy.

1. We consider the following tempered fractional diffusion equation with the left tempered fractional derivative,

$$\frac{\partial u(x, t)}{\partial t} = {}_0 D_x^{\alpha, \lambda} u(x, t) - e^{-\lambda x - t} \left(x^{3+\alpha} + \frac{\Gamma(4+\alpha)}{6} x^3 \right), \quad (4.1)$$

with the boundary conditions

$$u(0, t) = 0, \quad u(1, t) = e^{-\lambda - t}, \quad t \in [0, 1], \quad (4.2)$$

and the initial value

$$u(x, 0) = e^{-\lambda x} x^{1+\alpha}, \quad x \in [0, 1]. \quad (4.3)$$

The exact solution for (4.1) is given by $u(x, t) = e^{-\lambda x - t} x^{3+\alpha}$.

Let $\alpha = 1.2 \notin (1.26, 1.71)$ for the tempered fractional diffusion equation (4.1). For the second-order scheme (3.9), we choose $\gamma_3 \in (a_1, a_2)$ to ensure the stability. For the third-order scheme (3.13), we define γ_4 in such a way that $f(\alpha, \lambda; x) < 0$. Because of this latter choice, we expect a stable and accurate discretization, despite the fact that $\alpha \notin (1.26, 1.71)$. Take $\tau = h$, $\lambda = 4$. We construct the second-order scheme and the third-order scheme for this Equation (4.1). Tables 5 and 6 show L^2 errors and orders of accuracy for both schemes, confirming the desired accuracy, in both cases.

Remark 4.1. For the tempered fractional diffusion equation, we have employed the Crank-Nicolson discretization in the time-wise direction. The order of convergence in the time-wise direction is obviously second-order, which has been verified by the L^2 and L^∞ errors.

2. We also consider the following tempered fractional diffusion equation with the right tempered fractional derivative

$$\frac{\partial u(x, t)}{\partial t} = D_x^{\alpha, \lambda} u(x, t) - e^{\lambda x - t} \left((1-x)^{3+\alpha} + \frac{\Gamma(4+\alpha)}{6} (1-x)^3 \right), \quad (4.4)$$

with the boundary conditions

$$u(0, t) = e^{-t}, \quad u(1, t) = 0, \quad t \in [0, 1], \quad (4.5)$$

Table 7

L^2 errors and orders of accuracy for (4.4) with the boundary conditions (4.5) and the initial value (4.6) by the second-order scheme (3.9).

h	$\gamma_3 = -0.03$		$\gamma_3 = -0.02$		$\gamma_3 = -0.01$	
	L^2 Error	Order	L^2 Error	Order	L^2 Error	Order
1/10	1.41e-02		1.48e-02		1.55e-02	
1/20	3.51e-03	2.00	3.78e-03	1.97	4.04e-03	1.94
1/40	8.65e-04	2.02	9.44e-04	2.00	1.02e-03	1.98
1/80	2.14e-04	2.02	2.35e-04	2.00	2.57e-04	1.99
1/160	5.31e-05	2.01	5.87e-05	2.00	6.44e-05	2.00

Table 8

L^2 errors and orders of accuracy for (4.4) with the boundary conditions (4.5) and the initial value (4.6) by the third-order scheme (3.13).

h	$\gamma_4 = 0.01$		$\gamma_4 = 0.02$		$\gamma_4 = 0.03$	
	L^2 Error	Order	L^2 Error	Order	L^2 Error	Order
1/10	7.99e-03		7.62e-03		7.29e-03	
1/20	1.19e-03	2.75	1.11e-03	2.78	1.04e-03	2.81
1/40	1.61e-04	2.88	1.47e-04	2.91	1.35e-04	2.94
1/80	2.10e-05	2.94	1.90e-05	2.95	1.72e-05	2.98
1/160	2.72e-06	2.95	2.45e-06	2.96	2.19e-06	2.97

and the initial value

$$u(x, 0) = e^{\lambda x}(1-x)^{3+\alpha}, \quad x \in [0, 1]. \quad (4.6)$$

The exact solution for (4.4) is given by $u(x, t) = e^{\lambda x-t}(1-x)^{3+\alpha}$.

Let $\alpha = 1.8 \notin (1.26, 1.71)$ for the tempered fractional diffusion equation (4.4). We choose $\gamma_3 \in (a_1, a_2)$ for the second-order scheme (3.9), and define γ_4 satisfying $f(\alpha, \lambda; x) < 0$ for the third-order scheme (3.13). Take $\tau = h$, $\lambda = 4$. Tables 7 and 8 confirm the desired L^2 errors and orders of accuracy for both schemes.

3. We also analyze the following tempered fractional advection-diffusion equation

$$\frac{\partial u(x, t)}{\partial t} = c_l ({}_0D_x^{\alpha, \lambda} u(x, t)) + c_r ({}_xD_1^{\alpha, \lambda} u(x, t)) + p(x, t), \quad (4.7)$$

with the boundary conditions,

$$u(0, t) = 0, \quad u(1, t) = 0, \quad t \in [0, 1], \quad (4.8)$$

and the initial value,

$$u(x, 0) = e^{-\lambda x}x^{1+\alpha}, \quad x \in [0, 1]. \quad (4.9)$$

With

$$p = -e^{-t}x^3(1-x)^3 - e^{-\lambda x-t} \left({}_0D_x^{\alpha} \left[e^{\lambda x}(x^3 - 3x^4 + 3x^5 - x^6) \right] \right) \\ - e^{\lambda x-t} \left({}_xD_1^{\alpha} \left[e^{-\lambda x}((1-x)^3 - 3(1-x)^4 + 3(1-x)^5 - (1-x)^6) \right] \right),$$

the exact solution for (4.7) reads $u(x, t) = e^{-t}x^3(1-x)^3$. To compute $p(x, t)$, we use the following formulae,

$${}_0D_x^{\alpha} (e^{\lambda x}x^m) = {}_0D_x^{\alpha} \left(\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} x^{n+m} \right) = \sum_{n=0}^{\infty} \frac{\lambda^n \Gamma(n+m+1)}{n! \Gamma(n+m-\alpha+1)} x^{n+m-\alpha}, \quad (4.10)$$

and

$${}_xD_1^{\alpha} (e^{\lambda x}(1-x)^m) = {}_xD_1^{\alpha} \left(\sum_{n=0}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} (1-x)^{n+m} \right) \\ = e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n \Gamma(n+m+1)}{n! \Gamma(n+m-\alpha+1)} (1-x)^{n+m-\alpha}. \quad (4.11)$$

Take $\alpha = 1.5 \in (1.26, 1.71)$ for the tempered fractional diffusion equation (4.7). We choose $\gamma_3 \in (a_1, a_2)$ for the second-order scheme (3.9), and $\gamma_4 = -0.04, -0.03, -0.02$ for the third-order scheme (3.13), where $\gamma_4 = -0.04, -0.03 \in (a_3, a_4)$ and $\gamma_4 = -0.02 \notin (a_3, a_4)$ but satisfies $f(\alpha, \lambda; x) < 0$. Let $\lambda = 0.5$, $c_r = c_l = 0.5$, and $\tau = 10^{-3}$. Tables 9 and 10 show the corresponding L^2 errors and the second and third orders of accuracy, respectively.

Table 9

L^2 errors and orders of accuracy for (4.7) with the boundary conditions (4.8) and the initial value (4.9) by the second-order scheme (3.9).

h	$\gamma_3 = -0.04$		$\gamma_3 = -0.03$		$\gamma_3 = -0.02$	
	L^2 Error	Order	L^2 Error	Order	L^2 Error	Order
$1/2^4$	4.15e-05		4.19e-05		4.25e-05	
$1/2^5$	9.55e-06	2.12	9.98e-06	2.07	1.04e-05	2.03
$1/2^6$	2.29e-06	2.06	2.44e-06	2.03	2.59e-06	2.01
$1/2^7$	5.62e-07	2.03	6.04e-07	2.01	6.46e-07	2.00
$1/2^8$	1.39e-07	2.01	1.50e-07	2.01	1.62e-07	2.00

Table 10

L^2 errors and orders of accuracy for (4.7) with the boundary conditions (4.8) and the initial value (4.9) by the third-order scheme (3.13).

h	$\gamma_4 = -0.04$		$\gamma_4 = -0.03$		$\gamma_4 = -0.02$	
	L^2 Error	Order	L^2 Error	Order	L^2 Error	Order
$1/2^4$	5.41e-05		5.33e-05		5.39e-05	
$1/2^5$	7.98e-06	2.76	7.68e-06	2.79	7.48e-06	2.85
$1/2^6$	1.12e-06	2.83	1.06e-06	2.85	1.02e-06	2.88
$1/2^7$	1.52e-07	2.88	1.43e-07	2.89	1.35e-07	2.91
$1/2^8$	2.02e-08	2.91	1.89e-08	2.92	1.78e-08	2.93

4.3. The tempered fractional Black–Scholes equations

In this subsection, we construct a third-order scheme for the tempered fractional Black–Scholes equation. These numerical results come without a proof of stability here. We experimentally show that the developed schemes are robust and accurate, also for a convection-(fractional) diffusion type equation.

Consider the fractional PDE,

$$\frac{\partial u(x, t)}{\partial t} + a \cdot \frac{\partial u(x, t)}{\partial x} + b \cdot B_d \mathfrak{D}_x^{\alpha, \lambda_1} u(x, t) + d \cdot x \mathfrak{D}_{B_u}^{\alpha, \lambda_2} u(x, t) = c \cdot u(x, t), \quad (4.12)$$

where $\alpha \in (1, 2)$, the parameters b , d , c , λ_1 and λ_2 are all non-negative. With different values for the parameters a , b , c , d , λ_1 and λ_2 , we find variations for Equation (4.12).

We consider problem (4.12) with a source term $p(x, t)$ added to test the numerical scheme in the following form,

$$\begin{cases} \frac{\partial u(x, t)}{\partial t} + a \frac{\partial u(x, t)}{\partial x} + b B_d \mathfrak{D}_x^{\alpha, \lambda_1} u(x, t) + d x \mathfrak{D}_{B_u}^{\alpha, \lambda_2} u(x, t) = c \cdot u(x, t) + p(x, t), \\ (x, t) \in (B_d, B_u) \times (0, T) \\ u(B_d, t) = 0, u(B_u, t) = 0, t \in (0, T) \\ u(x, T) = S(x), x \in (B_d, B_u), x \in (B_d, B_u). \end{cases} \quad (4.13)$$

Let $t_j = (N - j)\tau$, $0 \leq t_j \leq T$, $j = 0, \dots, N$ and $x_i = B_d + ih$, $B_d \leq x_i \leq B_u$, $I = 0, \dots, M$, where $\tau = T/N$ and $h = (B_u - B_d)/M$. Using the tempered-WSGD operators, ${}_L \mathcal{D}_h^{\alpha, \lambda_1} = {}_L \mathcal{D}_{h, -1, 0, \alpha-1, 1}^{\alpha, \gamma_1, \gamma_2, \dots, \gamma_4}$ and ${}_R \mathcal{D}_h^{\alpha, \lambda_2} = {}_R \mathcal{D}_{h, -1, 0, \alpha-1, 1}^{\alpha, \gamma_1, \gamma_2, \dots, \gamma_4}$, for the tempered fractional derivatives, and the forth-order scheme for the first-order space derivative, we get the following space discretization for (4.12)

$$\begin{aligned} \frac{\partial u_i}{\partial t} + a \frac{8(u_{i+1} - u_{i-1}) - (u_{i+2} - u_{i-2})}{12h} + b \left({}_L \mathcal{D}_h^{\alpha, \lambda_1} u_i \right) + d \left({}_R \mathcal{D}_h^{\alpha, \lambda_2} u_i \right) \\ = c \cdot u_i + p_i + O(h^3), \end{aligned} \quad (4.14)$$

where u_i is the solution of (4.13) when $x = x_i$, and $p_i = p(x_i, t)$.

For (4.14), the Crank-Nicolson time discretization reads

$$\begin{aligned} -\frac{u_i^{j+1} - u_i^j}{\tau} + a \frac{8(u_{i+1}^{j+\frac{1}{2}} - u_{i-1}^{j+\frac{1}{2}}) - (u_{i+2}^{j+\frac{1}{2}} - u_{i-2}^{j+\frac{1}{2}})}{12h} + b \left({}_L \mathcal{D}_h^{\alpha, \lambda_1} u_i^{j+\frac{1}{2}} \right) + d \left({}_R \mathcal{D}_h^{\alpha, \lambda_2} u_i^{j+\frac{1}{2}} \right) \\ = c \cdot u_i^{j+\frac{1}{2}} + p_i^{j+\frac{1}{2}} + O(\tau^2 + h^3), \end{aligned} \quad (4.15)$$

where u_i^j is the solution of (4.13) at the point (x_i, t_j) , and $p_i^{j+\frac{1}{2}} = p(x_i, t_{j+\frac{1}{2}})$.

Table 11
 L^2 and L^∞ errors and orders of accuracy for (4.17) with $\gamma_4 = -0.5$.

h	L^2 Error	Order	L^∞ Error	Order
$1/2^5$	7.95e-05		2.58e-04	
$1/2^6$	1.08e-05	2.88	3.99e-05	2.69
$1/2^7$	1.38e-06	2.97	5.62e-06	2.83
$1/2^8$	1.71e-07	3.02	7.54e-07	2.90
$1/2^9$	2.09e-08	3.03	9.81e-08	2.94

The numerical scheme can now be written as

$$\begin{aligned} (1 + \frac{\tau}{2}c)U_i^{j+1} - \frac{\tau}{2} \left[a \frac{8(U_{i+1}^{j+1} - U_{i-1}^{j+1}) - (U_{i+2}^{j+1} - U_{i-2}^{j+1})}{12h} + b_L \mathcal{D}_h^{\alpha, \lambda_1} U_i^{j+1} + d_R \mathcal{D}_h^{\alpha, \lambda_2} U_i^{j+1} \right] \\ = (1 - \frac{\tau}{2}c)U_i^j + \frac{\tau}{2} \left[a \frac{8(U_{i+1}^j - U_{i-1}^j) - (U_{i+2}^j - U_{i-2}^j)}{12h} + b_L \mathcal{D}_h^{\alpha, \lambda_1} U_i^j + d_R \mathcal{D}_h^{\alpha, \lambda_2} U_i^j \right] - P_i^{j+\frac{1}{2}}, \end{aligned} \quad (4.16)$$

where U_i^j is the solution of the numerical scheme for (4.13) at point (t_i, t_j) , and $P_i^{j+\frac{1}{2}} = \frac{1}{2}(p_i^j + p_i^{j+1})$.

Remark 4.2. If we have Dirichlet boundary conditions on the whole boundary, and convection in the equation of interest, then we typically exhibit a boundary layer. Boundary value problems involving fractional operators attract great interest (for example, see [38]). In the problem (4.13), we need the solution to be sufficiently smooth, to examine the accuracy of our proposed discretization. When the solution is locally (in space or in time) not sufficiently smooth, then locally the third order is not achieved. We here only use the following tempered fractional Black–Scholes equations with analytic solutions, to test our numerical schemes. Note that our analytic solutions are sufficiently smooth.

Example 4.3. We here consider the following tempered fractional equation,

$$\begin{cases} \frac{\partial u(x, t)}{\partial t} + a \frac{\partial u(x, t)}{\partial x} + b \left({}_0\mathcal{D}_x^{\alpha, \lambda_1} u(x, t) \right) = c \cdot u(x, t) + p(x, t), \\ (x, t) \in (0, 1) \times (0, T) \\ u(0, t) = 0, u(1, t) = 0, t \in (0, T) \\ u(x, T) = S(x), x \in (0, 1), \end{cases} \quad (4.17)$$

where

$$\begin{aligned} p(x, t) = e^{-\lambda x + (T-t)} \left\{ x^3(1-x)(-1 - a\lambda - b\lambda^\alpha - p) + a(3x^2 - 4x^3) \right. \\ \left. + b \left(\frac{\Gamma(4)}{\Gamma(4-\alpha)} x^{3-\alpha} - \frac{\Gamma(5)}{\Gamma(5-\alpha)} x^{4-\alpha} \right) \right\}. \end{aligned}$$

The exact solution of the above equation is $u = e^{T-t} x^3(1-x)^3$. Let the parameters $b = -\frac{1}{2}\sigma^\alpha \sec(\frac{\alpha\pi}{2})$, $a = r - b$ and $c = r$. Take $\alpha = 1.6$, $\lambda = 1$, $\sigma = 0.25$, $r = 0.05$, and $\tau = 10^{-4}$. Table 11 lists the L^2 and L^∞ errors and orders of accuracy for Equation (4.17), which confirm the desired accuracy with $\gamma_4 = -0.5$.

Example 4.4. We finally consider the following tempered fractional model

$$\begin{cases} \frac{\partial u(x, t)}{\partial t} + a \frac{\partial u(x, t)}{\partial x} + b \left({}_0\mathcal{D}_x^{\alpha, \lambda_1} u(x, t) \right) + d \left({}_x\mathcal{D}_1^{\alpha, \lambda_1} u(x, t) \right) = c \cdot u(x, t) + p(x, t), \\ (x, t) \in (0, 1) \times (0, T) \\ u(0, t) = 0, u(1, t) = 0, t \in (0, T) \\ u(x, T) = S(x), x \in (0, 1), \end{cases} \quad (4.18)$$

where

$$\begin{aligned} p = - (1 + a\lambda_1^\alpha + b\lambda_2^\alpha + p)u(x, t) + 3ae^{T-t}x^2(1-x)^2(1-2x) \\ + be^{-\lambda_1 x + (T-t)} \left({}_0\mathcal{D}_x^\alpha e^{\lambda_1 x} u(x, t) \right) + ce^{\lambda_2 x + (T-t)} \left({}_x\mathcal{D}_1^\alpha e^{-\lambda_2 x} u(x, t) \right). \end{aligned}$$

Table 12
 L^2 and L^∞ errors and orders of accuracy for (4.18) with γ_4 .

h	L^2 Error	Order	L^∞ Error	Order
$1/2^5$	2.61e-05		4.28e-05	
$1/2^6$	3.57e-06	2.87	5.52e-06	2.95
$1/2^7$	4.74e-07	2.91	7.08e-07	2.96
$1/2^8$	6.19e-08	2.94	9.06e-08	2.97
$1/2^9$	7.99e-09	2.95	1.16e-08	2.97

The exact solution of the above equation is given by $u = e^{-\lambda x + (T-t)x^3}(1-x)$.

Let the parameters $b = c = d = 1$ and $a = -0.5$. Take $\alpha = 1.8$, $\lambda_1 = 0.5$, $\lambda_2 = 1$, and $\tau = 10^{-4}$. Table 12 lists the L^2 and L^∞ errors and orders of accuracy for Equation (4.18), again confirming the desired accuracy with $\gamma_4 = 0$.

5. Conclusion

In this paper, we presented stability analysis and error estimates for numerical schemes for the tempered fractional diffusion equation. We focused on the third-order semi-discretized scheme in space and showed error analysis for these fully discrete scheme based on the Crank–Nicolson scheme in time. We also provided the third-order scheme for the tempered fractional Black–Scholes equation. Clearly, the stable numerical schemes proposed in this paper are computationally highly accurate and efficient.

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