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# Critical time-dependent branching process modelling epidemic spreading with containment measures\*

Hanlin Sun<sup>1</sup>, Ivan Kryven<sup>2</sup> and Ginestra Bianconi<sup>1,3,\*\*</sup> 

<sup>1</sup> School of Mathematical Sciences, Queen Mary University of London, London, E1 4NS, United Kingdom

<sup>2</sup> Mathematical Institute, Utrecht University, PO Box 80010, 3508TA Utrecht, The Netherlands

<sup>3</sup> The Alan Turing Institute, The British Library, United Kingdom

E-mail: [ginestra.bianconi@qmul.ac.uk](mailto:ginestra.bianconi@qmul.ac.uk)

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## Abstract

During the COVID pandemic, periods of exponential growth of the disease have been mitigated by containment measures that in different occasions have resulted in a power-law growth of the number of cases. The first observation of such behaviour has been obtained from 2020 late spring data coming from China by Ziff and Ziff in reference Ziff and Ziff (2020 Fractal kinetics of COVID-19 pandemic MedRxiv). After this important observation the power-law scaling (albeit with different exponents) has also been observed in other countries during periods of containment of the spread. Early interpretations of these results suggest that this phenomenon might be due to spatial effects of the spread. Here we show that temporal modulations of infectivity of individuals due to containment measures can also cause power-law growth of the number of cases over time. To this end we propose a stochastic well-mixed susceptible-infected-removed model of epidemic spreading in presence of containment measures resulting in a time dependent infectivity and we explore the statistical properties of the resulting branching process at criticality. We show

\*This work celebrates the 70th birthday of our dear friend and colleague Bob Ziff.

\*\* Author to whom any correspondence should be addressed.



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that at criticality it is possible to observe power-law growth of the number of cases with exponents ranging between one and two. Our asymptotic analytical results are confirmed by extensive Monte Carlo simulations. Although these results do not exclude that spatial effects might be important in modulating the power-law growth of the number of cases at criticality, this work shows that even well-mixed populations may already feature non trivial power-law exponents at criticality.

Keywords: epidemic spreading, branching process, criticality, scaling laws

(Some figures may appear in colour only in the online journal)

## 1. Introduction

Exponential growth of the number of cases is typically observed at the onset of an epidemic when the dynamics is in the supercritical regime. The COVID data has also supported this claim and at the beginning of the current pandemic the scientific community has extensively confirmed exponential growth of the number of cases in different countries. However Ziff and Ziff in reference [1] were the first to detect a power-law growth in the number of cases starting from data coming from the late spring of 2020 in China when the epidemic was suppressed by containment measures. Later on the power-law growth of the number of cases has been recorded in data coming from other countries [2, 3]. Interestingly these results have been obtained in cases of successful containment of the epidemic spreading after the implementation of efficient containment measures [4], such as contact tracing (automatic and not), social distancing, testing and or other policies aimed at isolating timely infectious individuals and at reducing their reproductive number.

An important question that arises is: what is the mechanism responsible for the power-law scaling of the number of cases? Is this a phenomenon caused by the spatial distribution of the cases? Is it the sign that the system is reaching a critical behaviour consistent with a  $R_0 = 1$ ? Or can it be a combination of the latter two effects? If not, is this the effect of the containment measures?

During the current pandemic there has been a surge in research on epidemic spreading. Many works have discussed the challenges of epidemic spreading modelling [5, 6], a number of works have addressed outstanding theoretical problems that the current pandemic has highlighted [7–13] and a vast attention has been devoted to extract information from epidemic data [4, 14–16]. Additionally scientific research has informed policy makers [17, 18] establishing the role that containment measures such as social distancing, or contact tracing [19–24] have in mitigating the epidemic spread. Among the theoretical results we mention possible explanation of the power-law scaling have been proposed including interpretation of the power-law scaling as a signature of criticality [7, 8], as an effect of the inhomogeneous network of contact [13] or as due to the fractal spatial distribution of the spread [1].

Here we consider a very stylized theoretical model in a well-mixed population that is simple enough to be analytically solvable neglecting many detailed aspects of the realistic epidemic spreading model, yet capturing important statistical aspects that go beyond the simplest branching process. We show that a power-law growth of the number of cases can be observed when the epidemic process reaches criticality due to containment measures that allow for a temporal modulation of the infectivity of infectious individuals. In particular, while the susceptible-infected-removed (SIR) model at criticality predicts a power-law growth of the number of infected individuals with a power-law exponent equal to two, here we show that containment

measures can be responsible for modulating the power-law exponent between one and two. In order to demonstrate this modulation of the dynamical critical exponent we propose a discrete time epidemic model based on a branching process in which an infected seed individual can infect a different number of individuals at each time during seed's infectious period. This branching process is characterized by the distribution  $D(t)$  of the duration of the infectious period of each infected individual and the function  $m(t')$  indicating the expected number of individuals infected by an infectious individual after time  $t'$  from contracting an infection. This model is chosen to capture a temporal modulation of the infectivity of the infectious individual and clearly differs from the age-dependent branching process [25–27] where each infected individual gives rise to new infected individuals at a single time, even if this time is chosen randomly. We characterise the critical properties of the proposed branching process as a function of  $D(t)$  and  $m(t)$ , derive the critical indices of the dynamics and compare the results with extensive Monte Carlo simulations. As expected, this analysis reveals that stochastic effects play a key role in determining these exponents, which may strongly deviate from the exponents in deterministic approaches [7]. Moreover, these results show that time-dependent modulation of the infectivity can be responsible for a modulation of the power-law exponent determining the power-law growth of the number of cases in time. We note that these results do not exclude *a priori* that spatial effects might also be important elements determining the power-law increase in the number of cases. In particular, hierarchical and hyperbolic networks describing nested communities of people during lockdown can be responsible for a broadening of the critical region in which one can observe the power-law critical behaviour [28] similarly to what happens for percolation on the same type of networks [29–33].

## 2. Epidemic spreading with containment measures

### 2.1. The major properties of the SIR model

The SIR model is a well-known model of epidemic spreading in which individuals can be in one of three possible states: (1) susceptible individuals can get infected when in contact with an infectious individual, (2) infected individuals can spread the infection to susceptible individual upon contact with infectivity rate  $\lambda$ , and (3) removed or recovered individuals cannot spread the infection anymore. This model is known to display three dynamical regimes depending on the value of infectivity: for  $\lambda > \lambda_c$  the epidemic is in the supercritical regime, when the epidemic affects a positive fraction of the population; (b) for  $\lambda < \lambda_c$  the subcritical regime is observed, when the epidemic dies out before spreading in the population, and (c) for  $\lambda = \lambda_c$  the epidemic is in the critical regime, when the epidemic affects a sublinear fraction of all individuals. Here,  $\lambda_c$  indicates the so-called *epidemic threshold*. However, it has to be noted that in hyperbolic and hierarchical structures the critical region may stretch out for a finite range of values of the infectivity [28], which corresponds to the fact that in these networks one can observe two percolation thresholds [29–33]. When the onset of the epidemic is started from a single infected individual, the latter three dynamical regions are characterised by different dynamical properties: the supercritical region is characterised by an exponential increase of the number of infected individuals, the critical regime—by a power-law increase with exponent 2, while the subcritical regime — by finite size stochastic fluctuations.

### 2.2. Introducing a time dependent infectivity

In a typical SIR epidemic model it is assumed that infectivity  $\lambda$  does not change with time as long as an infected individual is contagious. In other words, the total number of secondary infections is proportional to the time an individual was infectious. Moreover, it is assumed

that each infected individual is removed from the population with a probability that does not depend on time.

Here we consider a model in which each infected individual has a reproductive number that depends on the time elapsed since his/her infection. Hence we consider time-dependent infectivity by substituting

$$\lambda \rightarrow \lambda F(t), \tag{1}$$

where  $F(t)$  is a decreasing function of  $t$ , indicating the time elapsed since the infection of the infectious individual. We additionally assume that the probability that an infectious individual recovers is also time-dependent. This model can be considered as the stochastic model underlying the deterministic dynamics proposed in reference [7]. The decay of the effective infectivity can be due to different causes, including asymptomatic onset, early testing policies, and containment measures enforced once the infection becomes symptomatic, i.e. the transmission time. In the supercritical regime this model can be treated using a deterministic approach, which predicts an exponential increase in the number of infected individuals at the onset of the epidemics. In order to perform the asymptotic analysis of this process we consider the scenario of an infinite population.

Our model has discrete time. By taking the moment when an individual becomes infectious as a reference, we denote the time that has elapsed since this event as  $t = 1, 2, \dots$ . We then assume that at every time step  $t > 0$ , this individual recovers/is removed with probability  $q(t) = 1 - p(t)$  or remains to be infectious with probability  $p(t)$ . Therefore the probability that the infected individual is still infectious at  $t$  is given by

$$P(t) = \prod_{t'=1}^t p(t'). \tag{2}$$

Additionally, we assume that at time  $t$ , an individual transmits infection to  $z_t$  susceptible individuals. Here,  $z_t$  is a random number drawn from the Poisson distribution with mean  $\lambda m(t)$ , where  $m(t)$  is either a constant or a decreasing function of time. In expectation, an individual that recovers at time  $t$  has a cumulative number of transmissions given by

$$\lambda M(t) = \lambda \sum_{t'=1}^{t-1} m(t'), \tag{3}$$

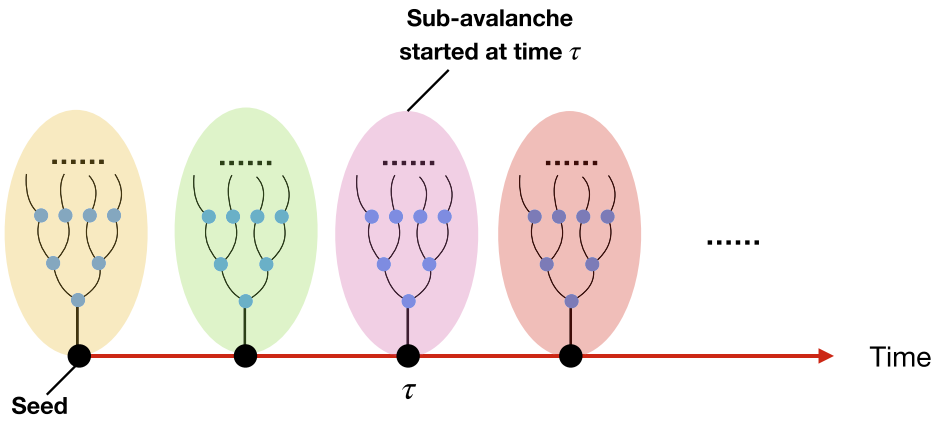
where we have assumed  $m(0) = 0$ .

In this stochastic model it is immediate to show that an infectious individual infects, in average,

$$\lambda F(t) = \lambda P(t)m(t) \tag{4}$$

other individuals after time  $t$ . It follows that  $F(t)$  acts as an overall dressing of the infectivity, capturing timely detection, tracking and isolation of the cases. Let us indicate with  $n(t)$  the average number of newly infected individuals at timestep  $t$ . Starting with a single individual infected at time  $t = 0$ , i.e.  $i(0) = 1$ , in average, the number of new infected individuals at time  $t$  reads

$$i(t) = \lambda \sum_{t'=1}^{t-1} F(t-t')i(t'). \tag{5}$$



**Figure 1.** A schematic figure of the branching processes. The seed starts a new sub-avalanche at each time step during its infectious period. The avalanche size  $n$  is given by summing up the size of all sub-avalanches and the seed itself. Note that each infected individual of any subavalanches will also produce a series of different subavalanches during each time step of its infectious period (not shown for simplifying the figure).

The expected number of newly removed individuals  $r(t)$  at time  $t$  is then

$$r(t) = \sum_{t'=1}^{t-1} \prod_{t''=t'+1}^{t-1} p(t)[1 - p(t'' - t')]i(t'). \tag{6}$$

The average number  $I(t)$  and  $R(t)$  of infected and removed individuals at time  $t$  is

$$I(t) = \sum_{t'=1}^t P(t - t')i(t'), \tag{7}$$

$$R(t) = \sum_{t'=1}^t [1 - P(t - t')]i(t'). \tag{8}$$

### 3. Time dependent branching process with containment measures

The model described in a previous section can be studied by considering a branching process. In this branching process the avalanche generated by a single node is due to the sum of sub-avalanches generated by each of the individuals infected by the seed node at any given time (see figure 1 for a schematic representation of this time-dependent branching process). Note that this branching process differs from the widely studied time-dependent branching process [25–27] because the infectious individual can infect new individuals at any time during his infectious period and not just at the end of its infectious period.

Let the durations of the infectious periods be distributed according to

$$D(t) = \left[ \prod_{t'=1}^{t-1} p(t') \right] [1 - p(t)]. \tag{9}$$

Moreover let  $\pi(n)$  be the distribution of the avalanche sizes started by a single infected individual. The branching process is described by the distribution  $\pi(n)$ , or equivalently, by its generating function  $H_1(x)$  defined as

$$H_1(x) = \sum_{n=1}^{\infty} \pi(n)x^n. \tag{10}$$

Assuming that  $t$  is the duration of the infectivity of the seed individual, and that at each time  $1 \leq t' < t$  the individual infects  $z_{t'}$  other individuals drawn from a Poisson distribution, i.e.  $z_{t'} \sim \text{Poisson}(\lambda m(t'))$  the size of the avalanche  $n$  generated by the seed individual is given by one plus the sum of the avalanches  $n'_j$  generated by each of the individuals  $j$  infected by the seed individual at time  $t'$ . Therefore the distribution  $\pi(n)$  can be expressed as

$$\pi(n) = \sum_{t=1}^{\infty} D(t) \sum_{\{z_{t'}\}} \sum_{\{n'_j\}} \left\{ \prod_{t'=1}^{t-1} \left[ P_{t'}(z_{t'}) \prod_{j=1}^{z_{t'}} \pi(n'_j) \right] \delta \left( n, \sum_{t'=1}^{t-1} \sum_{j=1}^{z_{t'}} n'_j + 1 \right) \right\}, \tag{11}$$

where  $P_t(z)$  is given by

$$P_t(z) = \frac{(\lambda m(t))^z}{z!} e^{-\lambda m(t)}. \tag{12}$$

The latter recursive equation can be rewritten by using generating function  $H_1(x)$  as

$$H_1(x) = x(F(H_1(x))), \quad \text{with} \quad F(x) = \sum_{t=1}^{\infty} D(t) \prod_{t'=1}^{t-1} G_{0,t'}(x), \tag{13}$$

where  $G_{0,t'}(x)$  is the generating function of  $P_{t'}(z)$

$$G_{0,t'}(x) = \sum_{z=0}^{\infty} P_{t'}(z)x^z = e^{\lambda m(t')(x-1)}, \tag{14}$$

where in the last expression we have used the explicit form of  $P_{t'}(z)$  given by equation (12). Therefore it follows that  $F(x)$  is given by

$$F(x) = \sum_{t=1}^{\infty} D(t)e^{\lambda M(t)(x-1)}, \tag{15}$$

where  $\lambda M(t)$ , indicating the expected total number of primary infected individuals, is given by equation (3). Summarizing, we conclude that the self-consistent equation for the generating function  $H_1(x)$  can be written as

$$H_1(x) = x(F(H_1(x))) = x \sum_{t=1}^{\infty} D(t)e^{\lambda M(t)(H_1(x)-1)}. \tag{16}$$

### 3.1. Relevant kernels

Let us consider different kernels for both  $D(t)$  and  $M(t)$ . The  $D(t)$  kernel that we will take under consideration are

(1) *The exponential kernel*

The exponential kernel is characterized by a  $p(t)$  equal to a constant

$$p(t) = a, \tag{17}$$

with  $0 < a < 1$ . Therefore we obtain

$$D(t) = a^{t-1}(1 - a). \tag{18}$$

(2) *The power-law kernel*

The power-law kernel is characterized by a  $p(t)$  given by

$$p(t) = 1 - \frac{\alpha - 1}{t + \alpha - 1}, \tag{19}$$

with  $\alpha > 1$  leading to the asymptotic scaling

$$D(t) = (\alpha - 1)\Gamma(\alpha) \frac{\Gamma(t)}{\Gamma(t + \alpha)} \simeq (\alpha - 1)\Gamma(\alpha)t^{-\alpha}, \tag{20}$$

where the last expression indicates the asymptotic scaling valid for  $t \gg 1$ .  
The  $M(t)$  kernels that we will consider are:

(A) *The linear kernel*

The linear  $M(t)$  kernel is characterized by a constant  $m(t)$ ,

$$m(t) = \bar{m}. \tag{21}$$

Therefore we obtain

$$M(t) = \sum_{t'=1}^{t-1} m(t') = \bar{m}(t - 1) \simeq \bar{m}t, \tag{22}$$

where the last expression refers to the asymptotic scaling valid for  $t \gg 1$ .

(B) *The power-law decaying kernel*

The power-law kernel is characterized by decaying  $m(t)$  given by

$$m(t) = \tilde{m}\eta \frac{1}{t^{1-\eta}}. \tag{23}$$

Therefore for large time limit,  $M(t)$  admits the power-law decay

$$M(t) = \tilde{m}\eta \sum_{t'=1}^{t-1} \frac{1}{t'^{1-\eta}} \simeq \tilde{m}t^\eta, \tag{24}$$

where the last expression indicates the asymptotic scaling of  $M(t)$  valid for  $t \gg 1$ .

In the following section we will characterize the critical behaviour of this branching process and its dependence on the different kernels that can be adopted for the functions  $D(t)$  and the function  $M(t)$ .



#### 4. Epidemic threshold of the considered epidemic spreading model

The time-dependent branching process with containment measures displays finite avalanches whose distribution is fully described by the self-consistent equation for its generating function  $H_1(x)$ , i.e. equation (16).

Depending on the value of the infectivity  $\lambda$  and the expected number  $\langle M \rangle = \sum_{t=1}^{\infty} D(t)M(t)$  of primary infections of the seed individual during the entire duration of its infective period we distinguish the three phases of the considered epidemic model.

- When  $\lambda \langle M \rangle < 1$ , we are in the *subcritical phase*. In this phase all avalanches of the branching process are finite, i.e.  $H_1(1) = 1$  and the expected size of the outbreak started from a single infected individual is given by:

$$\langle n \rangle = H'(1) = \frac{1}{1 - \lambda \langle M \rangle}. \tag{25}$$

- When  $\lambda \langle M \rangle > 1$ , we are in the *supercritical phase*. In this phase there is a positive probability  $S$  that the branching process does not stop, leading to finite avalanches only with probability  $H_1(1) = 1 - S$  where  $S \in (0, 1]$  is the unique solution of

$$S = 1 - F(1 - S). \tag{26}$$

- When  $\lambda \langle M \rangle = 1$  we are in the *critical phase* characterized by having  $F'(x) = 1$ , which corresponds to the *epidemic threshold*  $\lambda_c$  given by

$$\lambda_c = \frac{1}{\langle M \rangle} \tag{27}$$

which is greater than zero as long as  $\langle M \rangle$ , is finite. As  $\lambda \rightarrow \lambda_c^\pm$  the average size of the finite component diverges as

$$\langle n \rangle \propto \frac{1}{|\lambda - \lambda_c|^{\gamma, \gamma'}}, \tag{28}$$

with  $\gamma = 1$  and  $\gamma' = 1$  indicating the critical exponents for  $\lambda \rightarrow \lambda_c^-$  and for  $\lambda \rightarrow \lambda_c^+$  respectively.

Let us now establish the epidemic threshold  $\lambda_c$  for the different kernels taken in consideration. In particular we are interested in determining when the epidemic threshold is finite and greater than zero, and when it is zero. In fact a zero epidemic threshold implies that the epidemic will be always in the supercritical phase, i.e. for any arbitrarily small value of the infectivity  $\lambda$  the epidemic spreads over the population affecting an infinite number of individuals. Depending on the adopted kernels for  $D(t)$  and  $M(t)$  the epidemic threshold can be finite or zero:

(1A) *Exponential  $D(t)$  kernel and  $M(t) = \bar{m}(t - 1)$*

Let us consider the exponential kernel with  $D(t)$  given by equation (18) and assume  $M(t) = \bar{m}(t - 1)$ . The expected number of contacts  $\langle M \rangle$  of a random individual is given by

$$\langle M \rangle = \bar{m} \sum_{t \geq 1} D(t)(t - 1) = \bar{m}(1 - a) \sum_{t \geq 1} a^{t-1}(t - 1) = \bar{m} \frac{a}{1 - a} \tag{29}$$

The critical threshold  $\lambda_c$  is finite for every value of  $a \in (0, 1)$  with

$$\lambda_c = \frac{1}{\langle M \rangle} = \frac{1-a}{a\bar{m}}. \tag{30}$$

(2A) *Power-law  $D(t)$  kernel, and  $M(t) = \bar{m}(t - 1)$*

Let us consider the exponential kernel with  $D(t)$  given by equation (20) with  $\alpha > 1$  and the kernel  $M(t) = \bar{m}(t - 1)$ . For  $\alpha > 2$  the expected number of contacts of a random individual is finite and given by

$$\langle M \rangle = \bar{m} \sum_{t \geq 1} (\alpha - 1) \Gamma(\alpha) \frac{\Gamma(t)}{\Gamma(t + \alpha)} (t - 1) = \bar{m} \frac{1}{\alpha - 2}. \tag{31}$$

Therefore as long as  $\alpha > 2$  the epidemic threshold is finite and given by

$$\lambda_c = (\alpha - 2) \frac{1}{\bar{m}}. \tag{32}$$

However for  $\alpha \rightarrow 2$ ,  $\langle M \rangle$  diverges and the epidemic threshold  $\lambda_c$  vanishes, i.e.

$$\lambda_c = 0. \tag{33}$$

The epidemic threshold remains zero for all values of  $\alpha \in (1, 2]$ .

(1B) *Exponential  $D(t)$  kernel and power-law decaying  $M(t)$  kernel.*

Let us consider for  $D(t)$  the exponential kernel and for  $M(t)$  the power-law kernel with  $\eta \in (0, 1)$  (as  $\eta = 1$  reduces to the constant kernel). In this case the expected number of primary infections  $\langle M \rangle$  is finite and given by

$$\langle M \rangle = \sum_{t \geq 1} D(t)M(t) = \frac{\bar{m}}{\eta} \text{Li}_{1-\eta}(a). \tag{34}$$

Therefore, the critical threshold  $\lambda_c$  is finite and given by

$$\lambda_c = \frac{1}{\langle M \rangle} = \frac{\eta}{\bar{m} \text{Li}_{1-\eta}(a)}. \tag{35}$$

(2B) *Power-law  $D(t)$  kernel and power-law decaying  $M$  kernel.*

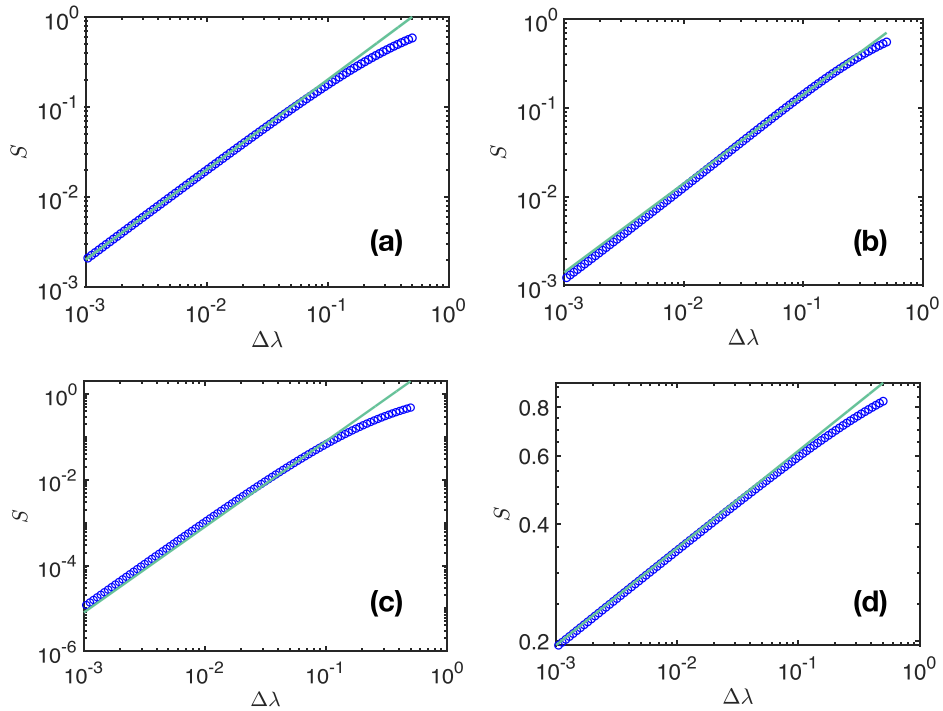
Let us consider for  $D(t)$  the power-law kernel and for  $M(t)$  the power-law kernel with  $\eta \in (0, 1)$  (as  $\eta = 1$  reduces to the constant kernel). In this case the expected number of primary infections  $\langle M \rangle$  can be expressed as

$$\langle M \rangle = \sum_{t \geq 1} D(t)M(t) = \frac{\bar{m}}{\eta} \Gamma(a) \sum_{t' \geq 1} \frac{1}{t'^{1-\eta}} \frac{\Gamma(t' + 1)}{\Gamma(t' + a)}. \tag{36}$$

Since asymptotically we have

$$\frac{1}{t'^{1-\eta}} \frac{\Gamma(t' + 1)}{\Gamma(t' + a)} \simeq \left( \frac{1}{t'} \right)^{a-\eta}, \tag{37}$$

we conclude that for  $\alpha \leq 1 + \eta$ ,  $\langle M \rangle$  diverges and therefore the epidemic threshold vanishes, i.e.  $\lambda_c = 0$ ; and that for  $\alpha > 1 + \eta$ ,  $\langle M \rangle$  converges and therefore the epidemic threshold is finite and non-zero  $\lambda_c > 0$ .



**Figure 2.** The probability  $S$  that the branching process does not stop is displayed versus the deviation  $\Delta\lambda$  of  $\lambda$  from the criticality  $\lambda_c$ . The numerical solutions (blue circles) for exponential the  $D(t)$  kernel with  $a = 0.01$  (a), power-law  $D(t)$  kernel with exponent  $\alpha = 3.5$  (b),  $\alpha = 2.5$  (c) and  $\alpha = 1.2$  (d) are obtained by solving equation (26) numerically and considering always the  $M(t) = \bar{m}(t - 1)$  kernel. The predicted asymptotic scaling given by equation (38) using the analytically derived  $\beta$  exponents are shown as reference (green lines).

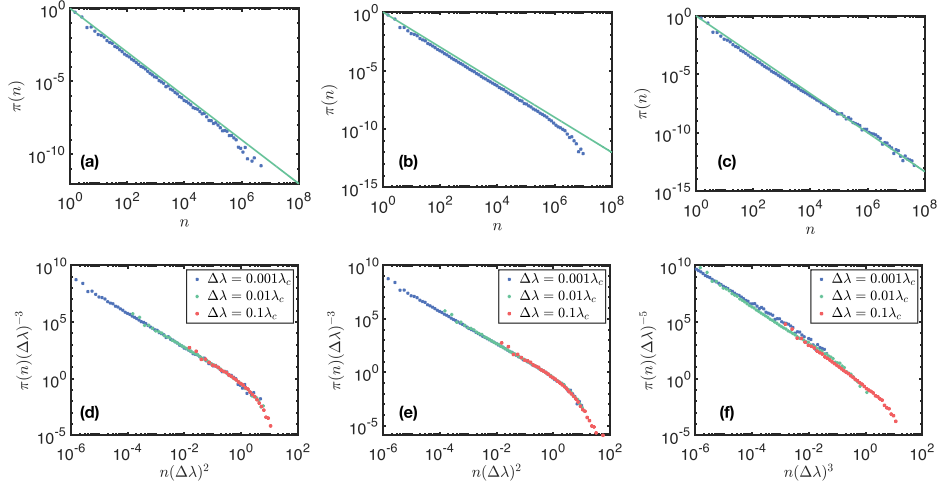
## 5. Critical indices associated to the size of the critical outbreak

### 5.1. Critical exponent $\beta$

The branching process undergoes a second order phase transition characterized by the order parameter  $S = 1 - H(1)$  indicating the probability of non-extinction of the branching process, with  $S$  satisfying equation (26). The critical exponent  $\beta$  characterizes the scaling of the probability  $S$  of observing an infinite avalanche, as a function of  $\lambda$  in the critical window  $0 < \lambda - \lambda_c \ll 1$ , in which  $S \ll 1$  can be approximated by

$$S \simeq A(\lambda - \lambda_c)^\beta, \tag{38}$$

with  $A$  being a positive constant, i.e.  $A > 0$ . This scaling can be predicted to hold in mean-field situations in which  $M(t)$  has finite moments, however when some moment diverges the scaling can acquire some logarithmic corrections as we will investigate in the following. Let us predict analytically the scaling of the exponent  $\beta$  of the studied epidemic model for the different kernels under consideration (see figure 2 for comparison of the analytical predictions and simulation results).



**Figure 3.** Monte Carlo simulations of the critical branching processes with exponential and power-law  $D(t)$  kernels. Panels (a)–(c) show the critical distribution of avalanche size  $\pi(n)$  corresponding to different  $D(t)$  kernels and panel (d)–(f) show the data collapse for distributions obtained away from criticality for the same  $D(t)$  kernels. Panels (a) and (d): exponential kernel with  $a = 0.01$ , panels (b) and (e) power-law kernel with  $\alpha = 3.5$ , panels (c) and (f) power-law kernel with  $\alpha = 2.5$ . The distributions are obtained from simulations of  $10^5$  realizations of the branching process with a linear  $M(t) = t$  kernel.

(1A & B) *Exponential  $D(t)$  kernel.* When the  $D(t)$  kernel is exponential, i.e., it is given by equation (18), independently on the choice of the kernel for  $M(t)$  we are in the mean-field regime where all the moments

$$\langle M^k \rangle = \sum_{t \geq 1} D(t) M^k(t), \tag{39}$$

are finite. In this regime, in order to find the critical exponent  $\beta$  we expand equation (26) up to the second order in  $S$ , obtaining

$$S \simeq 1 - \left[ F(1) - F'(1)S + \frac{1}{2}F''(1)S^2 \right], \tag{40}$$

where  $F(1) = 1$ ,  $F'(1) = \lambda \langle M \rangle$  and  $F''(1) = \lambda^2 \langle M^2 \rangle$ . For  $0 < \lambda - \lambda_c \ll 1$  we obtain that  $S$  scales according to equation (38) with the mean-field critical exponent  $\beta$  given by

$$\beta = 1, \tag{41}$$

and with  $A$  given by

$$A = 2 \frac{\langle M \rangle^3}{\langle M^2 \rangle}. \tag{42}$$

(2A) *Power-law  $D(t)$  kernel and linear  $M(t)$  kernel*

For the power-law  $D(t)$  kernel, the critical exponent  $\beta$  can deviate from the mean-field value  $\beta = 1$  and in general depends on the power-law exponent  $\alpha$ . Furthermore, for certain values of  $\alpha$  the scaling of  $S$  in equation (38) develops logarithmic corrections. Let us consider the linear

kernel  $M(t) = \bar{m}t$  and the power-law kernel  $D(t)$  with power-law exponent  $\alpha > 1$ . The critical index  $\beta$  will depend on the value of the power-law exponent  $\alpha$ .

- (a) For  $\alpha > 3$ , both  $\langle M \rangle$  and  $\langle M^2 \rangle$  are convergent, resulting in the mean-field critical exponent  $\beta$  given by

$$\beta = 1. \tag{43}$$

- (b) For  $\alpha = 3$  we observe logarithmic corrections to the critical scaling given by equation (38). Indeed by performing the asymptotic expansion of the self-consistent equation for  $S$  given by equation (26) for  $0 < S \ll 1$  we obtain

$$S \simeq \lambda \langle M \rangle S - S^2 \ln(1/S) \lambda^2 I_0, \tag{44}$$

where  $I_0$  is a constant. Since, according to equation (27) the epidemic threshold is given by  $\lambda_c = 1/\langle M \rangle$  we obtain that for  $0 < \lambda - \lambda_c \ll 1$ ,  $S$  follows the scaling

$$S \simeq A \frac{(\lambda - \lambda_c)^\beta}{\ln[1/(\lambda - \lambda_c)]} \tag{45}$$

with  $A$  indicating a constant, and  $\beta = 1$ .

- (c) For  $\alpha \in (2, 3)$ , the first moment  $\langle M \rangle$  is convergent, however the second moment  $\langle M^2 \rangle$  is divergent. We perform the asymptotic expansion of the self consistent equation for  $S$  (equation (26)) for  $0 < S \ll 1$  leading to

$$S \simeq \lambda \langle M \rangle S - S^{\alpha-1} \lambda^{\alpha-1} I_1 \tag{46}$$

where  $I_1$  is a finite constant. According to equation (27) we have  $\lambda_c \langle M \rangle = 1$ . Therefore we deduce that  $S$  scales follows the critical scaling given by equation (38) with the critical exponent  $\beta$  is given by

$$\beta = \frac{1}{\alpha - 2}. \tag{47}$$

- (d) For  $\alpha = 2$  we observe logarithmic corrections to the critical scaling given by equation (38). Indeed the asymptotic expansion of equation (26) for  $S \ll 1$  reads,

$$S \simeq cS \lambda \ln(1/S) I_2, \tag{48}$$

where  $I_2$  is a constant. By noticing that for  $\alpha = 2$  the epidemic threshold vanishes, i.e.  $\lambda_c = 0$  we deduce that close to criticality, for  $0 < \lambda \ll 1$  the order parameter  $S$  follows the scaling

$$S \simeq e^{-A/\lambda}, \tag{49}$$

where  $A$  is a constant.

- (e) For  $\alpha \in (1, 2)$ , both  $\langle M \rangle$  and  $\langle M^2 \rangle$  are divergent. In this case the asymptotic expansion of equation (26) determining the value of  $S$  reads

$$S \simeq cS^{\alpha-1} \lambda^{\alpha-1} I_3, \tag{50}$$

where  $I_3$  is a finite constant. Due to the diverging  $\langle M \rangle$  the epidemic threshold vanishes, i.e.  $\lambda_c = 0$ . Therefore,  $S$  scales as equation (38) with critical exponent  $\beta$  given by

$$\beta = \frac{\alpha - 1}{2 - \alpha}. \tag{51}$$

(2B) *Power-law  $D(t)$  kernel and power-law  $M(t)$  kernel*

Here we derive the critical exponent  $\beta$  for the branching process with the power-law  $M(t)$  kernel with  $\eta \in (0, 1)$  and the power-law  $D(t)$  kernel with power-law exponent  $\alpha > 1$ . Depending on the values of  $\eta$  and  $\alpha$  we can observe different critical exponents  $\beta$ . Note that in the limit in which  $\eta \rightarrow 1$  we recover the critical exponent  $\beta$  obtained in case (1B).

- (a) For  $(\alpha - 1)/\eta > 2$  both  $\langle M \rangle$  and  $\langle M^2 \rangle$  are convergent, therefore we can expand  $F(x)$  up to the second order. Inserting this expression into the equation (26) it is immediate to show that  $S$  follows the critical scaling given by equation (38) and that we recover the mean-field critical exponent  $\beta = 1$ .
- (b) For  $(\alpha - 1)/\eta = 2$  the first moment  $\langle M \rangle$  is convergent but the second moment  $\langle M^2 \rangle$  is diverging logarithmically. In this case we found logarithmic deviations from the scaling given by equation (38). Indeed the asymptotic expansion of equation (26) for  $0 < S \ll 1$  is given by

$$S \simeq \lambda \langle M \rangle S - \lambda^2 S^2 \ln(1/S) I'_0, \tag{52}$$

where  $I'_0$  is a constant. This asymptotic expansion, together with the expression of the epidemic threshold  $\lambda_c = 1/\langle M \rangle$ , leads to the critical scaling of  $S$ , valid for  $0 < \lambda - \lambda_c \ll 1$  given by

$$S \simeq A \frac{(\lambda - \lambda_c)}{\ln[1/(\lambda - \lambda_c)]}, \tag{53}$$

where  $A$  is a constant.

- (c) For  $(\alpha - 1)/\eta \in (1, 2)$ , the first moment  $\langle M \rangle$  is convergent while the second moment  $\langle M^2 \rangle$  diverges. The epidemic threshold  $\lambda_c$  is given by  $\lambda_c = 1/\langle M \rangle$  and the asymptotic expansion of equation (26) for  $0 < S \ll 1$  is given by

$$S \simeq \lambda \langle M \rangle S - \lambda^{\frac{\alpha-1}{\eta}} S^{\frac{\alpha-1}{\eta}} I'_1, \tag{54}$$

where  $I'_1$  is a constant. Therefore this asymptotic expansion leads to the critical scaling given by equation (38) with critical exponent

$$\beta = \frac{\eta}{\alpha - 1 - \eta}. \tag{55}$$

- (d) For  $(\alpha - 1)/\eta = 1$  both  $\langle M \rangle$  and  $\langle M^2 \rangle$  are diverging. The asymptotic expansion of equation (26) for  $0 < S \ll 1$  is given by

$$S \simeq \lambda S \ln(1/S) I'_2, \tag{56}$$

where  $I'_2$  is a constant. Given that the epidemic threshold in this case is vanishing  $\lambda_c = 0$  we get that close to criticality, for  $0 < \lambda \ll 1$   $S$  scales like

$$S \simeq e^{-A/\lambda}, \tag{57}$$

where  $A$  is a constant.

- (e) For  $(\alpha - 1)/\eta \in (0, 1)$ , both first moment  $\langle M \rangle$  and second moment  $\langle M^2 \rangle$  are diverging. In this case the epidemic threshold vanishes, i.e.  $\lambda_c = 0$ . The asymptotic expansion of equation (26) for  $0 < S \ll 1$  is given by

$$S \simeq \lambda^{\frac{\alpha-1}{\eta}} S^{\frac{\alpha-1}{\eta}} I'_3, \tag{58}$$

where  $I'_3$  is a constant. It follows that in this case, as long as  $0 < \lambda - \lambda_c \ll 1$  the order parameter  $S$  follows the critical scaling given by equation (38) with critical exponent  $\beta$  given by

$$\beta = \frac{\alpha - 1}{\eta + 1 - \alpha}. \tag{59}$$

### 5.2. Critical exponents $\tau$ and $\sigma$

At criticality the avalanche size distribution  $\pi(n)$  follows a power-law scaling with exponent  $\tau$  whose value depends on the statistical properties of the  $D(t)$  and the  $M(t)$  kernels. Close to criticality the avalanche size distribution  $\pi(n)$  acquires a cutoff determined by a scaling function  $\Phi(x)$ . Specifically for  $\lambda = \lambda_c + \Delta\lambda$ , the avalanche size distribution  $\pi(n)$  scales as

$$\pi(n) \simeq n^{-\tau} \Phi(n(\Delta\lambda)^\sigma), \tag{60}$$

where the function  $\Phi(x)$  approaches a constant value for  $x \rightarrow 0$  and decays to zero faster than any power for  $x \rightarrow \infty$ . In this section, we will derive the critical exponents  $\tau$  and  $\sigma$  for the different kernels under investigation starting from the self-consistent equation (16) for the generating function  $H_1(x)$ . We will show that the critical exponents will depend on the choice of the  $D(t)$  and the  $M(t)$  kernels. However, we notice here that the scaling relation [34]

$$\sigma(\tau - 1) = \beta, \tag{61}$$

relating the critical exponents  $\sigma, \tau$  to the critical exponent  $\beta$  will continue to be satisfied for every choice of the  $D(t)$  and  $M(t)$  kernels as long as the asymptotic expansion of  $F(x)$  for  $0 < 1 - x \ll 1$  does not have logarithmic corrections. In order to derive the value of the critical exponent  $\tau$  and  $\sigma$ , determining the scaling of  $\pi(n)$  according to equation (60), we first observe this scaling implies that the generating function  $H_1(x)$  defined as equation (10) for  $0 < 1 - x \ll 1$  scales as

$$H_1(x) \simeq 1 - (1 - x)^{\tau-1} h\left(\frac{1 - x}{(\Delta\lambda)^\sigma}\right). \tag{62}$$

where  $h(x)$  is a scaling function [35, 36]. By inserting this scaling relation into the self consistent equation for  $H_1(x)$  (equation (16)) which we rewrite here for convenience,

$$H_1(x) = xF(H_1(x)), \tag{63}$$

we will derive analytically the critical exponents  $\tau$  and  $\sigma$  for all the kernels under consideration (see figure 3 for comparison of the analytical predictions and the simulations results).

#### (1A & B) Exponential $D(t)$ kernel

With an exponential  $D(t)$  kernel, all the moments of  $M(t)$  are finite. Therefore we are in the mean-field regime, which is independent on the choice of  $M(t)$  kernel. We consider the self-consistent equation for  $H_1(x)$  given by equation (63) where we substitute the scaling of  $H_1(x)$  for  $0 < 1 - x \ll 1$  given by equation (62). In the case in which  $0 < 1 - x \ll 1$  we have  $0 < 1 - H_1(x) \ll 1$ , therefore in equation (63) we can substitute  $F(w)$  with this Taylor expansion around  $w = 1$  truncated at the second order. By putting  $1 - x = z(\Delta\lambda)^\sigma$  we get

$$F(H_1(x)) \simeq 1 - \lambda \langle M \rangle z^{\tau-1} (\Delta\lambda)^{\sigma(\tau-1)} h(z) + \frac{1}{2} \lambda^2 \langle M^2 \rangle z^{2(\tau-1)} (\Delta\lambda)^{2\sigma(\tau-1)} h^2(z). \tag{64}$$

Inserting this expression into equation (63) and using the explicit expression of the epidemic threshold  $\lambda_c = 1/\langle M \rangle$ , we get for  $0 < \lambda - \lambda_c \ll 1$

$$\langle M \rangle z^{\tau-1} (\Delta\lambda)^{\sigma(\tau-1)+1} h(z) + z(\Delta\lambda)^\sigma - cz^{2(\tau-1)} (\Delta\lambda)^{2\sigma(\tau-1)} h^2(z) = 0, \quad (65)$$

where  $c$  is a constant given by  $c = \lambda_c^2 \langle M^2 \rangle / 2$ . Imposing that all the terms in the above expansion are of the same order, i.e. putting

$$\sigma(\tau - 1) + 1 = \sigma = 2\sigma(\tau - 1), \quad (66)$$

we get the mean-field critical exponents

$$\tau = 3/2, \quad \sigma = 2. \quad (67)$$

(2A) Power-law  $D(t)$  kernel and linear  $M(t)$  kernel

When the  $D(t)$  kernel is power-law with power-law exponent  $\alpha > 1$ , the exponents  $\tau$  and  $\sigma$  depend on the value of  $\alpha$  and can deviate from the mean-field values. In the following we evaluate the exponents  $\tau$  and  $\sigma$  for values of the exponent  $\alpha$  that lead to an expansion of  $F(x)$  for  $0 < 1 - x \ll 1$  that does not have logarithmic corrections.

- (a) For  $\alpha > 3$ , both  $\langle M \rangle$  and  $\langle M^2 \rangle$  are finite. By expanding  $F(x)$  for  $0 < 1 - x \ll 1$  up to the second order we can reproduce the calculation performed for the exponential  $D(t)$  kernel. Therefore we recover the mean-field critical exponents

$$\tau = 3/2, \quad \sigma = 2. \quad (68)$$

- (b) For  $\alpha \in (2, 3)$ ,  $\langle M \rangle$  is convergent and  $\langle M^2 \rangle$  is divergent, while the epidemic threshold is finite and given by  $\lambda_c = 1/\langle M \rangle$ . We consider the asymptotic expansion of  $F(w)$  for  $w = H_1(x)$  and  $0 < 1 - x \ll 1$  given by

$$F(H_1(x)) \simeq 1 - z(\Delta\lambda)^{\sigma(\tau-1)} h(z) \lambda \langle M \rangle + \lambda^{\alpha-1} [(z(\Delta\lambda)^\sigma)^{\tau-1} h(z)]^{\alpha-1} I_1, \quad (69)$$

where  $I_1$  is a constant. By inserting this expression in the self consistent formula for  $H_1(x)$  given by equation (63) we get the leading terms

$$\langle M \rangle z^{\tau-1} (\Delta\lambda)^{\sigma(\tau-1)+1} h(z) + z(\Delta\lambda)^\sigma - \lambda^{\alpha-1} I_1 [(z(\Delta\lambda)^\sigma)^{\tau-1} h(z)]^{\alpha-1} = 0.$$

Imposing that all the terms in the above equation are of the same order,

$$\sigma(\tau - 1) + 1 = \sigma = \sigma(\tau - 1)(\alpha - 1), \quad (70)$$

we obtain the critical exponents

$$\tau = \frac{\alpha}{\alpha - 1}, \quad \sigma = \frac{\alpha - 1}{\alpha - 2}. \quad (71)$$

- (c) For  $\alpha \in (1, 2)$ , both  $\langle M \rangle$  and  $\langle M^2 \rangle$  are divergent, while the epidemic threshold is vanishing  $\lambda_c = 0$ . We proceed by considering the asymptotic expansion of  $F(w)$  for  $w = H_1(x)$  with  $0 < 1 - x \ll 1$ , with  $H_1(x)$  scaling according to equation (62), getting

$$F(H_1(x)) \simeq 1 - (\Delta\lambda)^{\alpha-1} [(z(\Delta\lambda)^\sigma)^{\tau-1} h(z)]^{\alpha-1} I_3, \quad (72)$$

where  $I_3$  is a constant. By inserting this expression in the self consistent formula for  $H_1(x)$  given by equation (63) we get the leading terms

$$\langle M \rangle z^{\tau-1} (\Delta\lambda)^{\sigma(\tau-1)} h(z) + z(\Delta\lambda)^\sigma - (\Delta\lambda)^{\alpha-1} [(z(\Delta\lambda)^\sigma)^{\tau-1} h(z)]^{\alpha-1} I_3 = 0.$$



By imposing that all the terms in the above equation are of the same order,

$$\sigma = \sigma(\tau - 1) = \sigma(\tau - 1)(\alpha - 1) + \alpha - 1, \tag{73}$$

we obtain the critical exponents

$$\tau = 2, \quad \sigma = \frac{\alpha - 1}{2 - \alpha}. \tag{74}$$

(2B) *Power-law  $D(t)$  kernel and power-law  $M(t)$  kernel*

(a) For  $(\alpha - 1)/\eta > 2$ , both  $\langle M \rangle$  and  $\langle M^2 \rangle$  are convergent. Thus we recover the mean-field exponents

$$\tau = 3/2, \quad \sigma = 2. \tag{75}$$

(b) For  $(\alpha - 1)/\eta \in (1, 2)$ ,  $\langle M \rangle$  is convergent,  $\langle M^2 \rangle$  is divergent and the epidemic threshold is finite and given by  $\lambda_c = 1/\langle M \rangle$ . We consider the asymptotic expansion of  $F(w)$  for  $w = H_1(x)$  and  $0 < 1 - x \ll 1$ :

$$F(H_1(x)) \simeq 1 - z^{\tau-1}(\Delta\lambda)^{\sigma(\tau-1)}h(z)\lambda \langle M \rangle + \lambda^{\frac{\alpha-1}{\eta}} [(x(\Delta\lambda)^\sigma)^{\tau-1}h(z)]^{\frac{\alpha-1}{\eta}} I'_1, \tag{76}$$

where  $I'_1$  is a constant. By inserting this expansion in the self-consistent equation (63) we find that the leading terms are given by

$$\langle M \rangle z^{\tau-1}(\Delta\lambda)^{\sigma(\tau-1)+1}h(z) + z(\Delta\lambda)^\sigma - \lambda_c^{\frac{\alpha-1}{\eta}} [(x(\Delta\lambda)^\sigma)^{\tau-1}h(z)]^{\frac{\alpha-1}{\eta}} I'_1 = 0. \tag{77}$$

By imposing that all these terms are of the same order, i.e. by imposing

$$\sigma(\tau - 1) + 1 = \sigma = \frac{\sigma(\tau - 1)(\alpha - 1)}{\eta}, \tag{78}$$

we obtain the critical exponents

$$\tau = \frac{\eta + \alpha - 1}{\alpha - 1}, \quad \sigma = \frac{\alpha - 1}{\alpha - 1 - \eta}. \tag{79}$$

(c) For  $(\alpha - 1)/\eta \in (0, 1)$ , both  $\langle M \rangle$  and  $\langle M^2 \rangle$  are divergent, and the epidemic threshold vanishes, i.e.  $\lambda_c = 0$ . By proceeding like the in three previous cases we consider the asymptotic expansion of  $F(w)$  for  $w = H_1(x)$  and  $0 < 1 - x \ll 1$ , given by

$$F(H_1(x)) \simeq 1 - (\Delta\lambda)^{\frac{\alpha-1}{\eta}} [(z(\Delta\lambda)^\sigma)^{\tau-1}h(z)]^{\frac{\alpha-1}{\eta}} I'_3, \tag{80}$$

where  $I'_3$  is a constant. By substituting this asymptotic expansion in the self consistent equation for  $H_1(x)$  we get to leading order,

$$\langle M \rangle z^{\tau-1}(\Delta\lambda)^{\sigma(\tau-1)}h(z) + z(\Delta\lambda)^\sigma - (\Delta\lambda)^{\frac{\alpha-1}{\eta}} [(z(\Delta\lambda)^\sigma)^{\tau-1}h(z)]^{\frac{\alpha-1}{\eta}} I'_3 = 0.$$

Imposing that all the terms of the above equation are of the same order, by putting

$$\sigma = \sigma(\tau - 1) = \frac{\sigma(\tau - 1)(\alpha - 1) + \alpha - 1}{\eta}, \tag{81}$$

we obtain the critical exponents

$$\tau = 2, \quad \sigma = \frac{\alpha - 1}{\eta - \alpha + 1}. \tag{82}$$

### 6. Distribution of the temporal duration of avalanches

In the previous section we have shown how the distribution of critical avalanche size depends on the kernel of the considered branching process modelling epidemics spreading with time-dependent containment measures. Here we show that instead the distribution of the avalanche duration is determined by critical exponents that are independent of the choice of the kernels under consideration. Let us define  $y$  as the critical exponent characterizing asymptotic scaling of the distribution  $P(T)$  of the duration  $T$  of critical avalanches

$$P(T) \simeq C' T^{-y}, \tag{83}$$

for  $T \gg 1$  where  $C'$  is a constant. The cumulative distribution of  $P(T)$  denoted by  $\hat{P}(T)$  indicates the probability that the avalanche has not stopped at time  $T$ , and scales for  $T \gg 1$  as

$$\hat{P}(T) \simeq C T^{-y+1}, \tag{84}$$

where  $C$  is a constant. We note that a critical avalanche started from a single initial seed is extinct at time  $T$  if each subavalanche generated by any of the offspring of the seed node is also extinct. Therefore, it is immediate to show that  $\hat{P}(T)$  satisfies

$$\begin{aligned} 1 - \hat{P}(T) &= \sum_{t \geq 1} D(t) \prod_{t'=1}^{t-1} \left\langle [1 - \hat{P}(T - t')]^{z_{t'}} \right\rangle_{z_{t'}} \\ &= \sum_{t \geq 1} D(t) \exp \left[ -\lambda \sum_{t'=1}^{t-1} m(t') \hat{P}(T - t') \right], \end{aligned} \tag{85}$$

where in the last expression we consider average over the Poisson distribution for  $z_{t'}$ . In order to determine the exponent  $y$ , we insert the critical scaling for  $\hat{P}(T)$  given by equation (84) into (85) and check that this equation is satisfied only for  $y = 2$  (which is the mean-field exponent) independently of the choice of the  $D(t)$  and  $M(t)$  kernels. To this end, let us take  $m(t) = \bar{m}$  corresponding to the linear  $M(t)$  kernel, and let us consider a generic  $D(t)$  kernel. By inserting the scaling function for  $\hat{P}(T)$  given by equation (84) with  $y = 2$  into the left-hand side of the self-consistent equation (85) we get, at the critical point  $\lambda = \lambda_c$ ,

$$\begin{aligned} \sum_{t \geq 1} D(t) \exp \left[ -\lambda_c \sum_{t'=1}^t m(t') \hat{P}_c(T - t') \right] \\ \simeq \sum_{t \geq 1} D(t) \exp \left[ -\lambda_c C \left( \phi^{(0)}(1 - T) - \psi^{(0)}(1 - T + t) \right) \right], \end{aligned} \tag{86}$$

where  $\psi^{(0)}(x)$  is the 0th PolyGamma function. We consider the expansion for  $T \gg t$ , getting

$$\phi^{(0)}(1 - T) - \psi^{(0)}(1 - T + t) = \frac{t}{T} + O(1/T^2). \tag{87}$$

Inserting this expansion in equation (86) we obtain to leading terms

$$\begin{aligned} \sum_{t \geq 1} D(t) \exp \left[ -\lambda_c \sum_{t'=1}^t m(t') \hat{P}_c(T - t') \right] &= \sum_{t \geq 1} D(t) \exp \left[ -\lambda_c C \bar{m} \frac{t}{T} \right] \\ &\simeq 1 - \frac{C}{T}, \end{aligned} \tag{88}$$

where in the last expression we have first expanded for  $T \gg 1$  and then we have used  $\lambda_c \langle M \rangle = 1$ . Therefore with this derivation we get that equation (85) is identically satisfied at criticality with the choice of  $\hat{P}(T)$  given by equation (84) as long as  $y = 2$ . By considering the power-law  $M(t)$  kernel it can be shown that the critical exponent  $y = 2$  is not modified. In fact, taking  $m(t) = (\bar{m}\eta)t^{\eta-1}$  with  $\eta \in (0, 1)$  we can evaluate the left-hand side of the self-consistent equation (85) for  $T \gg 1$  using continuous approximation to obtain:

$$\begin{aligned} & \sum_{t \geq 1} D(t) \exp \left[ -\lambda_c \sum_{t'=1}^t m(t') \hat{P}_c(T-t') \right] \\ & \simeq \sum_{t \geq 1} D(t) \exp \left[ -\lambda_c \int_0^t \frac{C\bar{m}\eta(t')^{\eta-1}}{T-t'} dt' \right] \\ & = \sum_{t \geq 1} D(t) \exp \left[ -\lambda_c C\bar{m}\eta T^{\eta-1} \mathcal{B}_{t/T}(\eta, 0) \right], \end{aligned} \tag{89}$$

where  $\mathcal{B}_{t/T}(\eta, 0)$  is the incomplete Beta function.

By further considering the expansion of the Beta function for  $T \gg 1$ , given by  $\mathcal{B}_x(\eta, 0) \simeq x^\eta$  we get

$$\begin{aligned} \sum_{t \geq 1} D(t) \exp \left[ -\lambda_c \bar{m}\eta C T^{\eta-1} \mathcal{B}_{t/T}(\eta, 0) \right] & \simeq \sum_{t \geq 1} D(t) \exp \left[ -\lambda_c \bar{m}\eta C \frac{t^\eta}{T} \right] \\ & \simeq \sum_{t \geq 1} D(t) \left( 1 - \lambda_c C \bar{m}\eta \frac{t^\eta}{T} \right) \\ & = 1 - \frac{C}{T}, \end{aligned} \tag{90}$$

where we have used  $\lambda_c \langle M \rangle = 1$  with  $\langle M \rangle$  given, in the continuous approximation, by

$$\langle M \rangle = \bar{m}\eta \langle t^\eta \rangle. \tag{91}$$

Therefore this derivation shows that also for the power-law  $M(t)$  kernel we get that equation (85) is identically satisfied at criticality provided  $y = 2$ .

In the sublinear regime, for  $0 < \Delta\lambda = \lambda_c - \lambda \ll 1$ , we can proceed in a similar manner as for the standard branching process [37, 38] and show that the power-law scaling of  $P(T)$  is modulated by a function of  $T(\Delta\lambda)^\epsilon$  with  $\epsilon = 1$  leading to the scaling

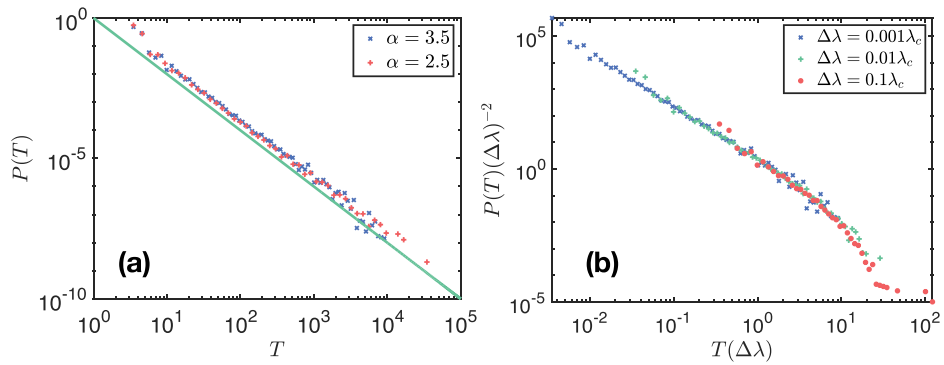
$$P(T) \simeq \frac{1}{T^2} \Psi(T(\Delta\lambda)), \tag{92}$$

where  $\Psi(x)$  converges to a constant for  $c \rightarrow 0$  and decays exponentially for  $x \rightarrow \infty$ . These predictions agree perfectly with the Monte Carlo simulations (see figure 4).

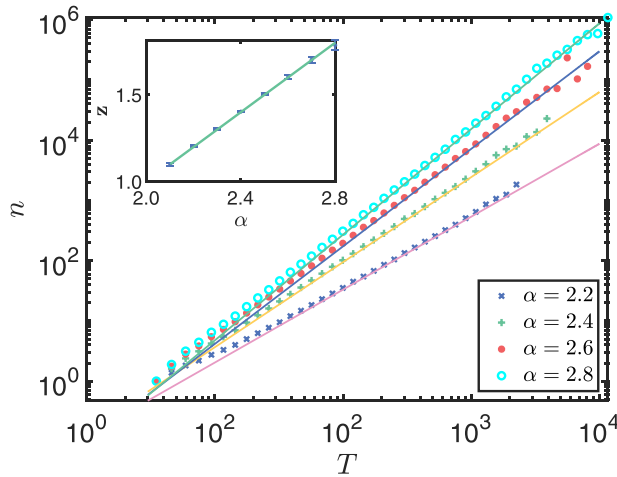
### 7. Dynamics of the critical branching process

At criticality, the avalanche size  $n$  is related to the duration of the avalanche by a power-law scaling determined by the critical dynamic exponents by  $z$ , i.e.,

$$n \propto T^z. \tag{93}$$



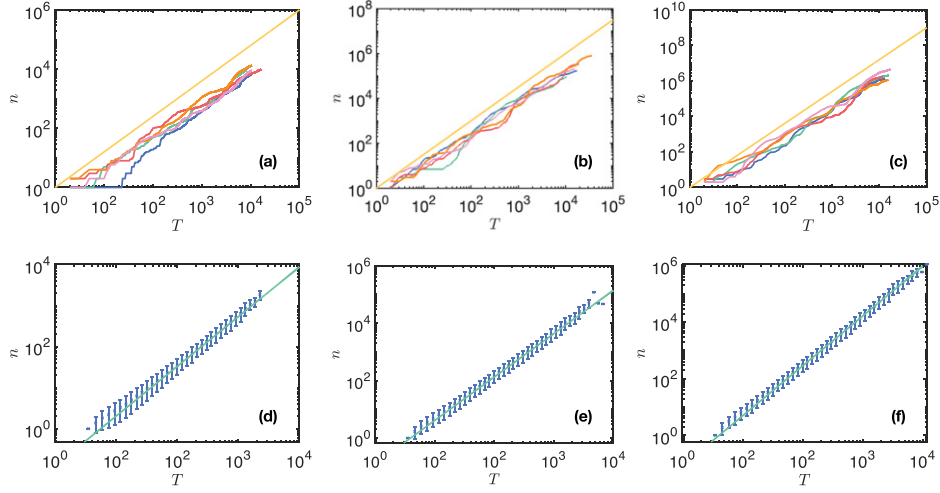
**Figure 4.** Monte Carlo simulations of time-duration distribution of critical branching processes with power-law  $D(t)$  kernel with  $\alpha = 3.5$  and  $\alpha = 2.5$ . The distributions of time-duration of avalanches (panel (a)) and the data collapse (panel (b)) with  $\alpha = 3.5$  are shown. The distributions are obtained from  $10^5$  samples of critical branching processes and  $M(t) = t$  is considered in the simulations. The distributions with different  $D(t)$  kernel give the same critical exponents.



**Figure 5.** The size of the avalanches  $n$  is shown versus the time-duration of the avalanches  $T$  for the critical branching process with  $D(t)$  power-law kernel with different power-law exponents  $\alpha$ . The  $M(t)$  kernel is linear. Symbols indicate numerical simulations, solid lines indicate power-law fit to the data. Inner panel: the fitted power-law exponent  $z$  (blue dots) is shown versus  $\alpha$  and compared with the theoretical expectation  $z = \alpha - 1$  (green line).

This power-law dependence of  $n$  with  $T$  is only observed exactly at criticality, for  $\lambda = \lambda_c$  while in the supercritical phase we have an exponential growth of the individual of an avalanche in time. The dynamical exponent  $z$  can be easily found once the exponents  $\tau$  and  $y$ , determining the critical scaling of  $\pi(n)$  and  $P(T)$ , are known. In fact  $z$  can be found by imposing that at criticality, i.e. for  $\lambda = \lambda_c$ ,

$$P(T)dT = \pi(n)dn, \tag{94}$$



**Figure 6.** The size of the avalanches  $n$  is shown versus the time-duration of the avalanches  $T$  for five single instance realizations of the critical branching process with  $D(t)$  power-law kernel and power-law exponents  $\alpha = 2.2$  (panel (a)),  $\alpha = 2.5$  (panel (b)) and  $\alpha = 2.8$  (panel (c)). In panels (a)–(c) the theoretical expectation is shown as a yellow line. In panel (d)–(f) we show the size of the avalanches  $n$  versus the time-duration of the avalanches  $T$  averaged over  $10^6$  realizations of the critical branching process with  $D(t)$  power-law kernel and power-law exponents  $\alpha = 2.2$  (panel (d)),  $\alpha = 2.5$  (panel (e)) and  $\alpha = 2.8$  (panel (f)) explicitly indicating the errorbars.

where  $n$  scales with  $T$  according to equation (93),  $\pi(n) \propto n^{-\tau}$  and  $P(T) \propto T^{-\gamma}$ . In this way, using the fact that  $y = 2$ , it is straightforward to show that the critical exponent  $z$  is given by

$$z = \frac{1}{\tau - 1}. \tag{95}$$

It follows that  $z$  depends of the choices of the  $D(t)$  and  $M(t)$  kernels.

**1A & 1B Exponential  $D(t)$  kernel.** In the case of the exponential  $D(t)$  kernel, we recover the mean-field exponents

$$\tau = 3/2, \quad z = 2, \tag{96}$$

both for the linear and the power-law  $M(t)$  kernel.

**2A & 2B power-law  $D(t)$  kernel.** In the case of power-law  $D(t)$  kernel the dynamical exponent  $z$  ranges between one and two, i.e.  $z \in [1, 2]$ . Let us treat the case of the linear  $M(t) \propto t$  kernel and the power-law  $M(t) \propto t^\eta$  together by taking  $\eta \in (0, 1]$  where for  $\eta = 1$  we recover the linear kernel. When neglecting the values of  $\alpha$  in which the expansion of  $F(x)$  around  $x = 1$  has logarithmic corrections, and considering the values of  $\tau$  derived in section 5.2, we see that the dynamical exponent  $z$  changes as a function of  $\alpha$  and  $\eta$  in the following way.

(a) For  $(\alpha - 1)/\eta > 2$  we recover the mean-field exponents

$$\tau = 3/2, \quad z = 2. \tag{97}$$

(b) For  $(\alpha - 1)/\eta \in (1, 2)$  we obtain

$$\tau = 1 + \frac{\eta}{\alpha - 1}, \quad z = \frac{\alpha - 1}{\eta}. \quad (98)$$

It follows then that in particular, for the linear kernel, i.e. for  $\eta = 1$  we obtain  $z = \alpha - 1$  which agrees with the numerical simulations (see figure 5 for comparison of the theoretical results with the average over simulations of the critical branching process and figure 6 for comparison of the theoretical results with simulations of single instances of the branching process).

(c) For  $(\alpha - 1)/\eta \in (0, 1)$  we obtain

$$\tau = 2, \quad z = 1. \quad (99)$$

As mentioned before these dynamical critical exponents agree with extensive Monte Carlo simulations, and display values that can be only obtained by taking into consideration the stochastic effects of the dynamics that play a crucial role of criticality. As a consequence it is possible to observe that the critical exponent  $z$  derived here deviates from the corresponding dynamical exponent that can be derived from the deterministic dynamics [7].

## 8. Conclusions

In this work we have studied a stochastic epidemic model with containment measures in which each infected individual is infectious for a time  $t$  with a given distribution  $D(t)$ . Additionally, during the infectious period an individual can infect a constant, or time-varying number of individuals resulting in a total number of secondary infections  $M(t)$  that either increases linearly or sublinearly with time. We have shown that depending of the choice of the  $D(t)$  and  $M(t)$  kernels, the critical behaviour of the branching process that captures this epidemic spreading model changes. In particular the critical index  $\tau$  that characterise the distribution of avalanche sizes depends on the choice of the kernels  $D(t)$  and  $M(t)$  and ranges in the interval between  $3/2$  and  $2$ , i.e.  $z \in [3/2, 2]$ . However, the critical exponent determining the avalanche duration appears to be universal and independent on the choice of the  $D(t)$  and  $M(t)$  kernels. Most relevantly, the study of this model allows us to derive the expression for the dynamical critical exponent  $z$  that determines the power-law growth of the number of infected individuals  $n$  and the avalanche duration of critical avalanches  $T$ , i.e.  $n \propto T^z$ . Interestingly, this critical exponent can be related to empirical observations on COVID data that starting from the work of Ziff and Ziff [1] have detected power-law increases of the number of cases in time [2–4].

We recover the classic results for the dynamical exponent  $z = 2$  in the standard branching process, and we predict that containment measures that have the effect of modulating the  $D(t)$  and the  $M(t)$  kernels can have the effect to modify the value of  $z$  allowing  $z \in [1, 2]$ . These theoretical results show that stochastic effects are important when determining the dynamical exponent  $z$ . Indeed, the exponent found in this paper improves on the deterministic treatment proposed in [7]. More importantly, the result presented in this work shows that the dynamical critical exponent  $z$  can be modulated by time-dependent containment measures in the range  $z \in [1, 2]$  which is consistent with some empirical observations made during few periods of strong mitigation of the COVID-19 pandemic observed in the last two years. We note however that this range does not include the value originally found by Ziff and Ziff in the first work [1] in which a power-law growth with exponent larger than two of the number of cases in time was reported. This implies that although containment measures that have the effect of modulating the  $D(t)$  and  $M(t)$  kernels can tune the value of the critical exponent  $z$ , other mechanisms

including for instance the role of a (hyperbolic) hierarchical, and nested spatial distribution of the spreading process might be also play a role in determining the actual value of  $z$  in real epidemics.

### Data availability statement

No new data were created or analysed in this study.

### ORCID iDs

Ginestra Bianconi  <https://orcid.org/0000-0002-3380-887X>

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