

# Efficient uniform approximation using Random Vector Functional Link networks

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## Abstract

A Random Vector Functional Link (RVFL) network is a depth-2 neural network with random inner weights and biases. As only the outer weights of such an architecture need to be learned, the learning process boils down to a linear optimization task, allowing one to sidestep the pitfalls of nonconvex optimization problems. In this paper, we prove that an RVFL with ReLU activation functions can approximate Lipschitz continuous functions provided its hidden layer is exponentially wide in the input dimension. Although it has been established before that such approximation can be achieved in  $L_2$  sense, we prove it for  $L_\infty$  approximation error and Gaussian inner weights. To the best of our knowledge, our result is the first of this kind. We give a non-asymptotic lower bound for the number of hidden layer nodes, depending on, among other things, the Lipschitz constant of the target function, the desired accuracy, and the input dimension. Our method of proof is rooted in probability theory and harmonic analysis.

## 1 Introduction

In this paper, we examine the approximation capacity of the Random Vector Functional Link (RVFL) network. An RVFL is a depth-2 neural network with random inner weights and biases. More precisely, an RVFL is a random function  $N_n : \mathbb{R}^m \rightarrow \mathbb{R}$  of the form

$$N_n(x) = \sum_{j=1}^n a_j \rho(\langle \mathbf{w}_j, x \rangle + \mathfrak{b}_j),$$

where  $w_j$ 's and  $b_j$ 's are iid random variables,  $\rho : \mathbb{R} \rightarrow \mathbb{R}$  is the *activation function*, and  $a_j$ 's are real numbers, chosen or learned so as to have  $N_n$  be close to a target function  $f$ .

Despite the simplicity of their architecture, RVFL models found their applications in signal classification and regression problems [7], forecasting [12], time-series data prediction [3], and others; for an overview, see [8]. At the same time, theoretical foundation for RVFL networks is still lacking [10, §1]. This paper aims to remedy this discrepancy, bringing us one step closer to understanding more complicated architectures on neural networks, widely used in practical applications.

Since only the outer weights of RVFL architectures need to be optimized, in practice the learning process boils down to a linear optimization task. Indeed, given training data  $\{x_p\}_{p=1}^k$ , we aim to choose  $a_j$ 's so that

$$\begin{aligned} \begin{Bmatrix} f(x_1) \\ \vdots \\ f(x_k) \end{Bmatrix} &\approx \begin{Bmatrix} \sum_{j=1}^n a_j \rho(\langle w_j, x_1 \rangle + b_j) \\ \vdots \\ \sum_{j=1}^n a_j \rho(\langle w_j, x_k \rangle + b_j) \end{Bmatrix} = \\ &\begin{Bmatrix} \rho(\langle w_1, x_1 \rangle + b_1) & \cdots & \rho(\langle w_n, x_1 \rangle + b_n) \\ \vdots & \ddots & \vdots \\ \rho(\langle w_1, x_k \rangle + b_1) & \cdots & \rho(\langle w_n, x_k \rangle + b_n) \end{Bmatrix} \begin{Bmatrix} a_1 \\ \vdots \\ a_n \end{Bmatrix}. \end{aligned}$$

Owing to the fact that the learning process boils down to a linear optimization problem, training RVFL networks sidesteps the usual pitfalls of nonconvex optimization problems, such as slow convergence and getting stuck in local optima.

In order to compare our main result to the existing literature, it will be useful to first state an approximate version of our main theorem.

**Main Theorem (approximate).** *Let  $K$  be compact subset of  $\mathbb{R}^m$  with at least two elements and circumradius  $R$ . Let  $p$  be a circumcenter of  $K$  and  $f : K \rightarrow \mathbb{R}$  be  $\ell$ -Lipschitz with  $2\zeta = \max f + \min f$ . There exist outer weights such that the corresponding RVFL network  $N_n$  with  $n$  hidden-layer nodes, ReLU activation functions, Gaussian inner weights, and uniformly distributed biases satisfies*

$$\mathbb{P} \left\{ \max_{x \in K+p} |f(x-p) - \zeta - N_n(x)| > \varepsilon \right\} \leq \eta$$

for any  $\eta > 0$  if

$$n \gtrsim \frac{2e}{\pi} \ln(2/\eta) (2\ell R \sqrt{e}/\varepsilon)^{2m+6} \exp\left(2d(K) \ln 2 - 2^{1/3} a m^{1/3} + 3 \ln m\right)$$

for large  $m$  and small  $\varepsilon$  such that  $\varepsilon^{-1/m} \approx 1$ . Here  $a \approx -2.3381$  is the first negative zero of the Airy function  $\text{Ai}$  and  $d(K) \in [1, m]$  is given by Definition 2 below.

*Remark.* Although  $K$  is in fact allowed to be a low-dimensional compact manifold, the lower bound for  $n$  will remain exponential in the ambient dimension, meaning more refined methods will be required to capitalize on the low-dimensional structure insofar possible. Assuming  $K$  is a low-dimensional manifold and trying to get a bound that is exponential in the manifold dimension in place of the ambient dimension is a setting commonly adopted to bridge the aforementioned gap between theory and practice [10].

While many results like this exist in the literature, ours offers an improvement on all of those in at least one way. To the best of our knowledge, two of the papers most similar to ours are [10] and [6]; both of them also examined how many hidden-layer nodes are sufficient for an RVFL to be able to approximate (Lipschitz) continuous functions.

In [10], Needell et al. obtained a sufficient number of hidden-layer nodes depending *superexponentially* on the dimension. Unlike us, they use  $L_2$  approximation error and the support of their uniformly distributed weights (and biases) crucially depend(s) on the approximation error, making their distributions not very suitable for practical use.

Our Gaussian weights, which are more common in practice than uniform ones, do not have variances depending on the approximation error. Our result additionally improves on [10, Thm 4.1] in that we measure the error in the  $L_\infty$  norm and our bound is only exponential in the dimension. In fact, our paper is, to the best of our knowledge, the first one to study Gaussian weights and  $L_\infty$  approximation error.

Hsu et al. not only managed to obtain a sufficient number of hidden-layer nodes that depends exponentially on the dimension, but also a necessary number of hidden-layer nodes, albeit in the  $L_2$  norm [6]. However, unlike our Gaussian weights, which align with common practice, their weight distribution is supported on a discrete subset of the unit sphere, which may be too restrictive for practical use. Their proof method, whilst conceptually similar to ours, differs in many details, such as using Fourier series where we use the Fourier transform.

We would also like to draw attention to [1], which also concerns itself with RVFL networks, although without the RVFL moniker. Whereas the “corrective method” developed therein

is very different from our proof method, their use of spectral methods has been a great inspiration for this paper.

Lastly, many technical proof details for the lemmata below have been delegated to the appendix for readability's sake. We highly encourage the interested reader to check it out as well.

## Notation

- Square brackets may denote Iverson brackets
- Derivatives may be denoted by a dot atop the function
- $j_\nu$  is the first positive zero of the Bessel function  $J_\nu$
- $a \approx -2.3381$  symbolizes the first negative zero of the Airy function  $\text{Ai}$
- $\rho$  denotes the ReLU function
- $\text{sg}$  denotes the sign function
- Integrals without specified integration domains are understood to integrate over all of Euclidean space
- $\mathcal{F}\{\varphi\}(v) \equiv \int \varphi(u) \exp(-i\langle v, u \rangle) du$
- $\delta_X$  denotes the pdf of a random variable  $X$  (cf.,  $\delta_0$ )
- We write  $\varphi(\diamond)$  in lieu of the more common  $\varphi(\cdot)$
- $V_m$  denotes the volume of the  $m$ -dimensional unit ball
- Absolute value bars may denote either the  $m$ -dimensional Lebesgue measure or the  $\ell_2$  norm in any dimension
- $\|\diamond\|_K \equiv \max_K |\diamond|$

## 2 Bird's-eye overview

Our approximation procedure essentially comprises four steps. We first extend the  $\ell$ -Lipschitz target function  $f$ , which is only defined on some compactum  $K \subset \mathbb{R}^m$ , to a compactly supported  $\ell$ -Lipschitz function  $\tilde{f}$ . The next three steps constitute a chain of approximations:  $\tilde{f} \approx g \approx h \approx N_n$ .

First, we approximate the extension of the target function  $\tilde{f}$  with a “smoothed” version  $g$  obtained by convolving  $\tilde{f}$  with a specifically-constructed approximate delta function. We show that  $g$  can be viewed as an “infinite width” depth-2 neural network with Gaussian inner weights. The biases of  $g$ , however, are not random, and the activation function is a cosine function.

Secondly,  $g$  is approximated by  $h$ , which is an “infinite width” RVFL with ReLU activation functions and Gaussian inner weights, and can be viewed as the “infinite width limit” of the desired RVFL.

Lastly, we will use Hoeffding’s concentration inequality to approximate  $h$  by a finite width counterpart  $N_n$  with  $n$  hidden-layer nodes.

### 3 Main result and proof

Before we state our main theorem, the following definitions will prove useful.

**Definition 1.** Let  $K \subset \mathbb{R}^m$  be a compactum with at least two elements. We define its *circumradius* as

$$R = \min_p \max_{u \in K} |u - p|.$$

Note that  $R > 0$ . Any  $p \in \mathbb{R}^m$  achieving the minimum is called a *circumcenter*.

*Remark.* If  $q \notin K + \{x \in \mathbb{R}^m : |x| \leq \text{diam}(K)\}$ , then

$$\max_{u \in K} |u - q| > \text{diam}(K) \geq \min_{p \in K} \max_{u \in K} |u - p|,$$

so every compactum  $K$  with at least two elements has at least one circumcenter within  $K + \{x \in \mathbb{R}^m : |x| \leq \text{diam}(K)\}$ .

**Definition 2.** Let  $K \subset \mathbb{R}^m$  be a compactum with at least two elements and circumradius  $R$ . We denote

$$d(K) = \lg \frac{|K + \{x \in \mathbb{R}^m : |x| \leq R\}|}{|\{x \in \mathbb{R}^m : |x| \leq R\}|},$$

where  $\lg$  is the binary logarithm. Note that  $1 \leq d(K) \leq m$ , so that  $d(K)$  may be seen as some sort of unfamiliar notion of *effective dimension*.

Our main result is the following.

**Main Theorem.** Let  $K \subset \mathbb{R}^m$  be compact with at least two elements and circumradius  $R$ . Let  $p$  be a circumcenter of  $K$  and  $f : K \rightarrow \mathbb{R}$  be  $\ell$ -Lipschitz with  $\max f - \min f = 2M$  and  $\zeta = M + \min f$ . There exist outer weights such that the corresponding RVFL network  $N_n$  with  $n$  hidden-layer nodes, ReLU activation functions, inner weight distribution  $N(0, \sigma I_m)$  with  $\sigma > 0$ , and biases uniformly distributed on  $[-\sigma R\sqrt{m}, \sigma R\sqrt{m}]$  satisfies

$$\mathbb{P}\left\{\|f(\diamond - p) - \zeta - N_n\|_{K+p} > \varepsilon\right\} \leq \eta$$

for any  $\varepsilon, \eta > 0$  if

$$\begin{aligned} n \geq & \frac{1}{8\pi e} \ln(2/\eta)(1 + \vartheta)^2 \left\{1 + \frac{m+1}{m(m+2)}\right\}^{2(m+2)} \\ & \left(2 - 2^{1/3}am^{-2/3}\right)^4 \left(m^2 + 3m + 1\right)^{2+2/m} \\ & \exp\left(2d(K) \ln 2 - 2^{1/3}am^{1/3} - \ln m\right) \\ & (2\ell R\sqrt{e}/\varepsilon)^{2m+6+2/m}, \end{aligned}$$

where

$$\begin{aligned} \frac{1}{\vartheta} = & \left\{\frac{m^2 + 3m + 1}{\varepsilon} \times \frac{2\ell R}{\sqrt{\pi m}} V_m R^m\right\}^{1/m} \sqrt{\frac{em}{2\pi}} \\ & \left\{1 + \frac{m+1}{m(m+2)}\right\} \times \frac{\ell}{\varepsilon} \left(2 - 2^{1/3}am^{-2/3}\right). \end{aligned}$$

Note that  $1/\vartheta \approx 2\ell R e/\varepsilon$  for large  $m$  and small  $\varepsilon$  such that  $\varepsilon^{-1/m} \approx 1$  in light of the fact that

$$V_m^{1/m} \sim \sqrt{2\pi e/m},$$

which follows readily from applying Stirling's formula to  $V_m = \pi^{m/2}/\Gamma(m/2 + 1)$  [4, (5.19.4)].

WLOG we henceforth suppose that  $p = 0$  and  $\zeta = 0$ , i.e., the image of  $f$  is  $[-M, M]$ . One can interpret  $\zeta$  as the bias of the outer layer of  $N_n$ , which would be convenient in practice. A good way to deal with  $p$  in practice would be approximating  $p$  as a part of preprocessing.

## Proof of the Main Theorem

Our first order of business is extending  $f$ . This would make it easier to construct a smooth approximation of  $f$ . Let

$$\tilde{f}(x) = \rho\left(|f(a)| - \ell|x - a|\right) \text{sg } f(a),$$

where  $a \in \operatorname{argmax}_{u \in K} \left(|f(u)| - \ell|x - u|\right)$ .

**Lemma 1.**  $\tilde{f}$  is a compactly supported  $\ell$ -Lipschitz extension of  $f$ .

*Proof.* To show that  $\tilde{f}$  extends  $f$  boils down to showing that  $x \in K \Rightarrow \tilde{f}(x) = f(x)$ . It suffices to show that  $x$  maximizes  $K \ni u \mapsto |f(u)| - \ell|x - u|$ , which is plain from the fact that

$$|f(u)| - |f(x)| \leq |f(x) - f(u)| \leq \ell|x - u|$$

by the reverse triangle inequality.

Because  $\operatorname{supp} \tilde{f} \subseteq K + \{x \in \mathbb{R}^m : |x| \leq M/\ell\}$ , its compact. Demonstrating the Lipschitz continuity is technical; as such, see the appendix.  $\square$

Henceforth,  $\tilde{K}$  shall denote the support of  $\tilde{f}$ .

Now that we have extended  $f$ , we can define  $g$  which we do as follows

$$g(x) = (2\pi)^{-m} \int F(v) \exp(i\langle v, x \rangle - |v|^2/2\lambda^2) \Psi(v/\lambda) dv,$$

where  $\lambda = \sigma\Lambda$  for a TBD  $\Lambda > 0$  and  $F = \mathcal{F}\{\tilde{f}\}$ . Moreover,  $\Psi(x) = (\omega * \omega)(x/\sqrt{m})$  with

$$\omega(x) \propto \left[|x| \leq \frac{1}{2}\right] J_\nu(2j_\nu|x|) / |x|^\nu,$$

where  $\nu = m/2 - 1$  and the proportionality constant is such that  $\psi = \mathcal{F}^{-1}\{\Psi\}$  is a pdf; see [5, §5] for more details. Note that  $t^{-\nu} J_\nu(t) \rightarrow 2^{-\nu} / \Gamma(\nu + 1)$  as  $t \rightarrow 0$  [4, (10.7.3)], whereupon  $\omega(0)$  is well-defined.

Upon recognizing  $g$  as an inverse Fourier transform, we see that  $g$  can be interpreted as  $\tilde{f}$  convolved with an approximate delta function, that is,  $g$  is a smoothed version of  $\tilde{f}$ , in light of the convolution theorem and the scaling property of the Fourier transform.

**Lemma 2.**  $\|\tilde{f} - g\|_\infty \leq \frac{\ell}{\lambda} \left(2 - 2^{1/3} a m^{-2/3}\right) \sqrt{m}$ .

*Proof.* Let  $Z \sim N(0, I_m)$ . Essentially, we first show that

$$g(x) = \int \tilde{f}(x - s/\lambda) (\delta_Z * \psi)(s) ds.$$

using standard integral manipulation. Since  $\psi$  is a pdf,

$$\begin{aligned} |\tilde{f}(x) - g(x)| &= \left| \tilde{f}(x) - \int \tilde{f}(x - s/\lambda) (\delta_Z * \psi)(s) ds \right| = \\ & \left| \int \tilde{f}(x) (\delta_Z * \psi)(s) ds - \int \tilde{f}(x - s/\lambda) (\delta_Z * \psi)(s) ds \right| \leq \\ & \int |\tilde{f}(x) - \tilde{f}(x - s/\lambda)| (\delta_Z * \psi)(s) ds \leq \\ & \frac{\ell}{\lambda} \int |s| (\delta_Z * \psi)(s) ds, \end{aligned}$$

so all that remains is to bound the integral. For the remaining details, see the appendix.  $\square$

Before we introduce  $h$ , it is useful to rewrite  $g$  in a yet different form. Since

$$\begin{aligned} (2\pi)^{-m} \int F(v) \exp(i\langle v, x \rangle - |v|^2/2\lambda^2) \Psi(v/\lambda) dv = \\ g(x) = \int \tilde{f}(x - s/\lambda) (\delta_Z * \psi)(s) ds \in \mathbb{R}, \end{aligned}$$

the inverse Fourier transform integral equals its own real part, that is,  $g(x)$  is equal to

$$\begin{aligned} (2\pi)^{-m} \int |F(v)| c(v, x) \exp(-|v|^2/2\lambda^2) \Psi(v/\lambda) dv = \\ (2\pi)^{-m} \lambda^m \int |F(\lambda w)| c(\lambda w, x) \exp(-|w|^2/2) \Psi(w) dw = \\ (2\pi)^{-m/2} \lambda^m \int |F(\lambda w)| c(\lambda w) \delta_Z(w) \Psi(w) dw = \\ (2\pi)^{-m/2} \lambda^m \mathbb{E} \left( |F(\lambda \mathbf{n})| \Psi(\mathbf{n}) c(\lambda \mathbf{n}, x) \right), \end{aligned}$$

where  $c(v, x) := \cos(\langle v, x \rangle + \arg F(v))$  and  $\mathbf{n} \sim N(0, I_m)$ . We now approximate the above expectation by

$$h(x) = \left\{ \frac{\lambda}{\sqrt{2\pi}} \right\}^m \mathbb{E} \left( |F(\lambda \mathbf{n})| \Psi(\mathbf{n}) \mathbb{1} \left[ |\mathbf{n}| > \vartheta \sqrt{m} \right] c(\lambda \mathbf{n}, x) \right).$$

**Lemma 3.**  $\|g - h\|_\infty \leq \frac{2\ell R}{\sqrt{\pi m}} V_m \left\{ \frac{R\vartheta\lambda}{\sqrt{2\pi/e}} \right\}^m$ .

*Proof.* Since  $\|\mathcal{F}\{\diamond\}\|_\infty \leq \|\diamond\|_1$ , and  $\psi$  is a pdf,

$$\begin{aligned} |g(x) - h(x)| &\leq \\ \left\{ \frac{\lambda}{\sqrt{2\pi}} \right\}^m \mathbb{E} &\left| |F(\lambda\mathbf{n})| \Psi(\mathbf{n}) \left[ |\mathbf{n}| \leq \vartheta\sqrt{m} \right] c(\lambda\mathbf{n}, x) \right| \leq \\ &(2\pi)^{-m/2} \lambda^m \|\tilde{f}\|_1 \mathbb{P}\{|\mathbf{n}| \leq \vartheta\sqrt{m}\}. \end{aligned}$$

The remaining details may be found in the appendix.  $\square$

In Section 2, we said that  $h$  would be an “infinite width” RVFL with ReLU activation functions and Gaussian inner weights. The following lemma corroborates this claim.

**Lemma 4.**  $h = \mathbb{E}(G(w, \mathfrak{b})\rho(\langle w, \diamond \rangle + \mathfrak{b}))$  on  $K$ , where

- $G(w, b) = -2\sigma R\sqrt{m}\Lambda^2(2\pi)^{-m/2}\lambda^m |F(\Lambda w)|\Psi(w/\sigma) \left[ |w| \geq \vartheta\sigma\sqrt{m} \right] \cos(\Lambda b - \arg F(\Lambda w))$ ;
- $\mathfrak{b} \sim \text{Unif}[-\sigma R\sqrt{m}, \sigma R\sqrt{m}]$ ;
- $w \sim N(0, \sigma I_m)$ .

*Proof.* The statement follows from straightforward manipulations of unwieldy integrals, that are crucially relying on the fact that the ReLU activation function satisfies “ $\ddot{\rho} = \delta_0$ ”. As always, the details can be found in the appendix.  $\square$

We now want to approximate our “infinite width” RVFL with a finite width one. Essentially, we are going to utilize Hoeffding’s inequality to approximate the expectation  $h$  by a sample mean.

There is a problem, however. Using Hoeffding’s inequality directly would yield an upper bound for

$$\mathbb{P}\left\{ \left| \frac{1}{n} \sum_{p=1}^n H_p(x) - \mathbb{E}(H(x)) \right| > t \right\},$$

where  $H(x) = G(w, \mathfrak{b})\rho(\langle w, x \rangle + \mathfrak{b})$  and  $H_1, \dots, H_n$  are iid copies of  $H$ . Instead, we need an upper bound for

$$\mathbb{P}\left\{\left\|\frac{1}{n}\sum_{p=1}^n H_p - \mathbb{E}(H)\right\|_K > t\right\}.$$

Arguably the cleanest solution is to find a measurable selector, that is, a random variable  $X_n$  such that

$$\left|\frac{1}{n}\sum_{p=1}^n H_p(X_n) - \mathbb{E}(H(X_n))\right| = \left\|\frac{1}{n}\sum_{p=1}^n H_p - \mathbb{E}(H)\right\|_K.$$

Such a measurable selector turns out to exist [2, thm 18.3] and can be used to deduce the following bound.

**Lemma 5.** *Let  $N_n = \frac{1}{n}\sum_{p=1}^n H_p$  and  $t > 0$ . Then*

$$\begin{aligned} & \mathbb{P}\left\{\|N_n - \mathbb{E}(H)\|_K > t\right\} \leq \\ & 2 \exp\left(-\frac{n}{2}\left(\frac{t}{2R^2\sqrt{m}(2\pi)^{-m/2}\lambda^{m+1}(1+1/\vartheta)\ell|\tilde{K}|}\right)^2\right). \end{aligned}$$

*Proof.* Technical; see the appendix. □

We are now ready to put Lemmas 2, 3, and 5 together to yield the desideratum. By setting

$$\begin{aligned} \alpha &= \frac{m(m+2)}{m^2+3m+1}; \\ \beta &= \frac{1}{m^2+3m+1}; \\ \Lambda &= \frac{1}{\sigma} \times \frac{\ell}{\alpha\varepsilon} \left(2 - 2^{1/3}am^{-2/3}\right) \sqrt{m}; \\ \frac{1}{\vartheta} &= \left\{\frac{1}{\beta\varepsilon} \times \frac{2\ell R}{\sqrt{\pi m}} V_m R^m\right\}^{1/m} \frac{\lambda}{\sqrt{2\pi/e}}, \end{aligned}$$

we obtain by design that  $\|f - g\|_\infty \leq \alpha\varepsilon$  and  $\|g - h\|_\infty \leq \beta\varepsilon$ . As such, letting  $\gamma =$

$1 - \alpha - \beta$  yields that

$$\begin{aligned}
& \mathbb{P}\left\{\|f - N_n\|_K > \varepsilon\right\} = \mathbb{P}\left\{\|\tilde{f} - N_n\|_K > \varepsilon\right\} \leq \\
& \mathbb{P}\left\{\|\tilde{f} - g\|_\infty + \|g - h\|_\infty + \|h - N_n\|_K > \varepsilon\right\} \leq \\
& \mathbb{P}\left\{\alpha\varepsilon + \beta\varepsilon + \|h - N_n\|_K > \varepsilon\right\} = \\
& \mathbb{P}\left\{\|h - N_n\|_K > \gamma\varepsilon\right\} \leq \\
& 2 \exp\left(-\frac{n}{2} \left(\frac{\gamma\varepsilon}{2R^2\sqrt{m}(2\pi)^{-m/2}\lambda^{m+1}(1+1/\vartheta)\ell|\tilde{K}|}\right)^2\right) \leq \eta,
\end{aligned}$$

upon plugging in the lower bound for  $n$  and doing a lot of simplifications, chief among them being

$$\begin{aligned}
|\tilde{K}| & \leq |K + \{x \in \mathbb{R}^m : |x| \leq R\}| \\
& = V_m R^m \frac{|K + \{x \in \mathbb{R}^m : |x| \leq R\}|}{|\{x \in \mathbb{R}^m : |x| \leq R\}|} \\
& = V_m R^m \exp(d(K) \ln 2).
\end{aligned}$$

It follows from the fact that  $\tilde{K} \subseteq K + \{x \in \mathbb{R}^m : |x| \leq M/\ell\}$  and  $M \leq \ell R$  (this is derived while proving Lemma 3 in the appendix). We also used the following inequalities.

- Since  $1 - x \leq e^{-x}$ , it follows that

$$\begin{aligned}
& \left(2 - 2^{-1/3} a m^{-2/3}\right)^{2m} = \\
& 4^m \left(1 - a(2m)^{-2/3}\right)^{2m} \leq \\
& 4^m \exp(-2^{1/3} a m^{1/3}).
\end{aligned}$$

- Since  $V_m = \pi^{m/2} / \Gamma(m/2 + 1)$  [4, (5.19.4)], inequality [4, (5.6.1)] yields that

$$V_m \leq \frac{1}{\sqrt{\pi m}} \left(\frac{2\pi e}{m}\right)^{m/2}.$$

Ergo,  $V_m^2 \leq \frac{1}{\pi m} \left(\frac{2\pi e}{m}\right)^m$  and  $V_m^{2/m} \leq 2\pi e/m$ .

Finally, note that  $N_n$  is indeed an RVFL with outer weights of the form  $G(w, \mathfrak{b})/n$ .

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## A Proofs of lemmata

Here you may find parts of proofs omitted in the main article. The numbering herein employed is the same as in the main article.

We remind the reader that  $K \subset \mathbb{R}^m$  is compact,  $f : K \rightarrow \mathbb{R}$  is  $\ell$ -Lipschitz, and

$$\begin{aligned}\tilde{f}(x) &= \rho\left(|f(a)| - \ell|x - a|\right) \operatorname{sg} f(a); \\ g(x) &= (2\pi)^{-m} \int F(v) \exp(i\langle v, x \rangle - |v|^2/2\lambda^2) \Psi(v/\lambda) dv; \\ h(x) &= (2\pi)^{-m/2} \lambda^m \mathbb{E}\left(|F(\lambda \mathbf{n})| \Psi(\mathbf{n}) \left[|\mathbf{n}| > \vartheta\right] c(\lambda \mathbf{n}, x)\right).\end{aligned}$$

Furthermore,  $H_1, \dots, H_p$  are iid copies of  $G(\mathfrak{w}, \mathfrak{t})\rho(\langle \mathfrak{w}, \diamond \rangle + \mathfrak{t})$  defined in Lemma 4 of the main paper.

**Lemma 1.**  *$\tilde{f}$  is a compactly supported  $\ell$ -Lipschitz extension of  $f$ .*

*Proof.* All that remains to be shown is that  $\tilde{f}$  is  $\ell$ -Lipschitz; the rest was shown in the main paper.

Let  $x_1, x_2 \in \mathbb{R}^m$ . Following [9] suppose WLOG that  $|\tilde{f}(x_1)| \geq |\tilde{f}(x_2)|$ . In other words,

$$\rho\left(|f(a_1)| - \ell|x_1 - a_1|\right) \geq \rho\left(|f(a_2)| - \ell|x_2 - a_2|\right).$$

If  $f(a_1)f(a_2) > 0$ , then

$$\begin{aligned}
|\tilde{f}(x_1) - \tilde{f}(x_2)| &= \left| \rho\left(|f(a_1)| - \ell|x_1 - a_1|\right) - \rho\left(|f(a_2)| - \ell|x_2 - a_2|\right) \right| \\
&= \rho\left(|f(a_1)| - \ell|x_1 - a_1|\right) - \rho\left(|f(a_2)| - \ell|x_2 - a_2|\right) \\
&\leq \rho\left(|f(a_1)| - \ell|x_1 - a_1|\right) - \rho\left(|f(a_1)| - \ell|x_2 - a_1|\right) \\
&\leq \left| \rho\left(|f(a_1)| - \ell|x_1 - a_1|\right) - \rho\left(|f(a_1)| - \ell|x_2 - a_1|\right) \right| \\
&\leq \left| \left(|f(a_1)| - \ell|x_1 - a_1|\right) - \left(|f(a_1)| - \ell|x_2 - a_1|\right) \right| \\
&= \ell \left| |x_2 - a_1| - |x_1 - a_1| \right| \leq \ell|x_1 - x_2|,
\end{aligned} \tag{1}$$

since  $\rho$  is evidently nondecreasing and 1-Lipschitz.

If  $f(a_1)f(a_2) \leq 0$ , however, we must distinguish three further cases.

If additionally  $|\tilde{f}(x_2)| > 0$ , i.e.,  $|f(a_2)| - \ell|x_2 - a_2| > 0$ , then

$$\begin{aligned}
|\tilde{f}(x_1) - \tilde{f}(x_2)| &= \rho\left(|f(a_1)| - \ell|x_1 - a_1|\right) + \rho\left(|f(a_2)| - \ell|x_2 - a_2|\right) \\
&= \left(|f(a_1)| - \ell|x_1 - a_1|\right) + \left(|f(a_2)| - \ell|x_2 - a_2|\right) \\
&= |f(a_1)| + |f(a_2)| - \ell|x_2 - a_2| - \ell|x_1 - a_1| \\
&= |f(a_1) - f(a_2)| - \ell|x_2 - a_2| - \ell|x_1 - a_1| \\
&\leq \ell|a_1 - a_2| - \ell|x_2 - a_2| - \ell|x_1 - a_1| \\
&\leq \ell|x_2 - a_1| - \ell|x_1 - a_1| \leq \ell|x_1 - x_2|.
\end{aligned}$$

If additionally  $|\tilde{f}(x_1)| > 0$  and  $|f(a_2)| - \ell|x_2 - a_2| \leq 0$ , that is,  $\tilde{f}(x_2)$  is zero, then

$$\begin{aligned}
|\tilde{f}(x_1) - \tilde{f}(x_2)| &= |\tilde{f}(x_1)| = |\tilde{f}(x_1)| - |\tilde{f}(x_2)| \\
&= \rho\left(|f(a_1)| - \ell|x_1 - a_1|\right) - \rho\left(|f(a_2)| - \ell|x_2 - a_2|\right),
\end{aligned}$$

which is (1). Lastly, if both  $\tilde{f}(x_1)$  and  $\tilde{f}(x_2)$  are zero, then  $|\tilde{f}(x_1) - \tilde{f}(x_2)| = 0$ , whence the desired Lipschitz continuity of  $\tilde{f}$  readily follows. □

**Lemma 2.**  $\|\tilde{f} - g\|_\infty \leq \frac{\ell}{\lambda} \left(2 - 2^{1/3}am^{-2/3}\right) \sqrt{m}$ .

*Proof.* Using various substitutions yields that

$$\begin{aligned}
& (2\pi)^{-m} \int F(v) \exp(i\langle v, x \rangle - |v|^2/2\lambda^2) \Psi(v/\lambda) dv = \\
& (2\pi)^{-m} \iint \tilde{f}(u) \exp(-i\langle v, u \rangle) \exp(i\langle v, x \rangle - |v|^2/2\lambda^2) \Psi(v/\lambda) du dv = \\
& (2\pi)^{-m} \iint \tilde{f}(u) \exp(i\langle v, x - u \rangle - |v|^2/2\lambda^2) \Psi(v/\lambda) dv du = \\
& (2\pi)^{-m} \int \tilde{f}(x - t) \int \exp(i\langle v, t \rangle - |v|^2/2\lambda^2) \Psi(v/\lambda) dv dt = \\
& (2\pi)^{-m} \int \tilde{f}(x - s/\lambda) \int \exp(i\langle w, s \rangle - |w|^2/2) \Psi(w) dw dt = \\
& \int \tilde{f}(x - s/\lambda) (\delta_Z * \psi)(s) ds,
\end{aligned}$$

where the second equality follows readily from Fubini's theorem because  $\tilde{f}$  and  $\Psi$  are compactly supported. In the main article it was shown that

$$\|\tilde{f} - g\|_\infty \leq \frac{\ell}{\lambda} \int |s| (\delta_Z * \psi)(s) ds,$$

so all that remains to be shown is that  $\int |s| (\delta_Z * \psi)(s) ds \leq (2 - 2^{1/3} a m^{-2/3}) \sqrt{m}$ .

Since  $\psi$  is a pdf,  $\psi = \delta_X$  for some random variable  $X$ . Thus,

$$\int |s| (\delta_Z * \psi)(s) ds = \int |s| (\delta_Z * \delta_X)(s) ds = \mathbb{E}|Z + X| \leq \mathbb{E}|Z| + \mathbb{E}|X|.$$

Since  $|Z|$  is chi distributed with  $m$  degrees of freedom, Wendel's inequality [13] yields that

$$\mathbb{E}|Z| = \frac{\Gamma((m+1)/2)}{\Gamma(m/2)} \sqrt{2} \leq \sqrt{m}.$$

By Jensen's inequality and [5, thm 5.1],

$$\mathbb{E}|X| \leq \sqrt{\int |x|^2 \psi(x) dx} = 2j_\nu / \sqrt{m}.$$

Indeed, as follows from the scaling property of the Fourier transform,

$$\begin{aligned} \int |x|^2 \mathcal{F}^{-1}\left\{(\omega * \omega)\left(\diamond/\sqrt{m}\right)\right\}(x) dx &= \\ m^{m/2} \int |x|^2 \mathcal{F}^{-1}\{\omega * \omega\}(x\sqrt{m}) dx &= \\ \frac{1}{m} \int |u|^2 \mathcal{F}^{-1}\{\omega * \omega\}(u) du &= 4j_\nu^2/m. \end{aligned}$$

We now claim that  $2j_\nu/\sqrt{m} < \sqrt{m} - 2^{1/3}am^{-1/6}$ . In [11], it has been proven that

$$j_\nu < \nu - a(\nu/2)^{1/3} + \frac{3}{20}a^2(\nu/2)^{-1/3}$$

for all  $\nu > 0$ . If  $m \geq 3$ , then  $\nu = m/2 - 1 > 0$ , so

$$\begin{aligned} 2j_\nu/\sqrt{m} &< \sqrt{m} - 2/\sqrt{m} - 2^{2/3}a(m/2 - 1)^{1/3}/\sqrt{m} + \frac{6}{20}a^2(\nu/2)^{-1/3}/\sqrt{m} \\ &< \sqrt{m} - 2^{2/3}a(m/2)^{1/3}/\sqrt{m} + (\frac{6}{20}a^2(\nu/2)^{-1/3} - 2)/\sqrt{m} \\ &= \sqrt{m} - 2^{1/3}am^{-1/6} + (\frac{6}{20}a^2(\nu/2)^{-1/3} - 2)/\sqrt{m}. \end{aligned}$$

Since  $(\nu/2)^{-1/3}$  is clearly decreasing in  $m$  and  $m = 3 \Rightarrow \frac{6}{20}a^2(\nu/2)^{-1/3} < 2$ , as can be verified numerically, the last term will always be negative and can thus be discarded to yield our claim for all  $m \geq 3$ . One can verify the cases  $m = 1, 2$  numerically. Putting everything together yields the desideratum.  $\square$

**Lemma 3.**  $\|g - h\|_\infty \leq \frac{2\ell R}{\sqrt{\pi m}} V_m \left\{ \frac{R\vartheta\lambda}{\sqrt{2\pi/e}} \right\}^m$ .

*Proof.* In the main article we has already shown that

$$\|g - h\|_\infty \leq (2\pi)^{-m/2} \lambda^m \|\tilde{f}\|_1 \mathbb{P}\left\{|\mathbf{n}| \leq \vartheta\sqrt{m}\right\},$$

so all that remains is to bound  $\|\tilde{f}\|_1$  and  $(2\pi)^{-m/2} \lambda^m \mathbb{P}\left\{|\mathbf{n}| \leq \vartheta\sqrt{m}\right\}$ .

Starting with the former, since  $|\tilde{f}(x)| = \rho(|f(a)| - \ell|x - a|)$ ,

$$\|\tilde{f}\|_1 = \int_{\bar{K}} |\tilde{f}| = \int_K |\tilde{f}| + \int_{\bar{K}-K} |\tilde{f}| \leq M|K| + \int_{|u| \leq M/\ell} (M - \ell|u|) du.$$

Now,

$$\int_{|u| \leq M/\ell} (M - \ell|u|) du = V_m \int_0^{M/\ell} (M - \ell y) y^{m-1} dy = \frac{V_m M^{m+1}}{\ell^m m(m+1)}.$$

In conjunction with the previous display, this yields that

$$\|\tilde{f}\|_1 \leq M \left( |K| + V_m \frac{(M/\ell)^m}{m(m+1)} \right) \leq M \left( V_m R^m + V_m (M/\ell)^m \right) \leq 2\ell R V_m R^m,$$

because  $M \leq \ell R$ . Indeed, since  $K$  has circumradius  $R$ , it follows that  $\text{diam}(K) \leq 2R$ , so  $2M = \max f - \min f \leq \ell \text{diam}(K) = 2\ell R$  because  $f$  is  $\ell$ -Lipschitz.

As for the latter,  $|\mathbf{n}|$  has a chi distribution with  $m$  degrees of freedom, so the cdf of  $|\mathbf{n}|$  may be expressed as  $P(m/2, \diamond^2/2)$  [4, §8.2(i)]. If  $x \geq 0$ , then [4, (8.6.3)]

$$P(a, x) = \frac{x^a}{\Gamma(a)} \int_0^\infty \exp(-at - xe^{-t}) dt \leq \frac{x^a}{\Gamma(a)} \int_0^\infty \exp(-at) dt = \frac{x^a}{a\Gamma(a)}.$$

Additionally  $a\Gamma(a) > (a/e)^a \sqrt{2\pi a}$  for all  $a > 0$  [4, (5.6.1)]. Ergo,

$$\begin{aligned} (2\pi)^{-m/2} \lambda^m \mathbb{P}\{|\mathbf{n}| \leq \vartheta \sqrt{m}\} &= (2\pi)^{-m/2} \lambda^m P(m/2, m\vartheta^2/2) \\ &< (2\pi)^{-m/2} \lambda^m \frac{1}{\sqrt{\pi m}} \left( \frac{m\vartheta^2/2}{m/2e} \right)^{m/2} \\ &= \frac{1}{\sqrt{\pi m}} \left( \frac{\vartheta \lambda}{\sqrt{2\pi/e}} \right)^m. \end{aligned}$$

Multiplying the obtained bounds readily yields the desideratum.  $\square$

**Lemma 4.**  $h = \mathbb{E}(G(\mathbf{w}, \mathfrak{t})\rho(\langle \mathbf{w}, \diamond \rangle + \mathfrak{t}))$  on  $K$ , where

- $G(w, b) = -2\sigma R \sqrt{m} \Lambda^2 (2\pi)^{-m/2} \lambda^m |F(\Lambda w)| \Psi(w/\sigma) \left[ |w| \geq \vartheta \sigma \sqrt{m} \right] \cos(\Lambda b - \arg F(\Lambda w));$
- $\mathfrak{t}$  being uniformly distributed on  $[-\sigma R \sqrt{m}, \sigma R \sqrt{m}]$ ;
- $\mathbf{w} \sim N(0, \sigma I_m)$ .

*Proof.* Letting  $N \sim N(0, \sigma I_m)$  and  $\varphi(v) = \arg F(v)$  allows us to write

$$\begin{aligned}
& (2\pi)^{-m/2} \lambda^m \mathbb{E} \left( |F(\lambda \mathbf{n})| \Psi(\mathbf{n}) \left[ |\mathbf{n}| \geq \vartheta \sqrt{m} \right] c(\mathbf{n}, x) \right) = \\
& (2\pi)^{-m/2} \lambda^m \int |F(\lambda u)| \Psi(u) \left[ |u| \geq \vartheta \sqrt{m} \right] c(\lambda u, x) \delta_Z(u) du = \\
& (2\pi)^{-m/2} \lambda^m \int |F(\Lambda w)| \Psi(w/\sigma) \left[ |w| \geq \vartheta \sigma \sqrt{m} \right] c(\Lambda w, x) \delta_N(w) dw = \\
& \int \left( (2\pi)^{-m/2} \lambda^m |F(\Lambda w)| \left[ |w| \geq \vartheta \sigma \sqrt{m} \right] \Psi(w/\sigma) \right) \\
& \cos(\varphi(\Lambda w) + \Lambda \langle w, x \rangle) \left[ |\langle w, x \rangle| \leq \sigma R \sqrt{m} \right] \delta_N(w) dw,
\end{aligned}$$

since  $|\langle w, x \rangle| \leq |w| \cdot |x|$  and the support of  $\Psi$  is  $\{w \in \mathbb{R}^m : |w| \leq \sqrt{m}\}$ .

Now, using the fundamental theorem of calculus and integration by parts,

$$\begin{aligned}
\cos(\varphi + \Lambda z) \left[ |z| \leq B \right] &= -\Lambda \int_{-\infty}^z \sin(\varphi + \Lambda y) \left[ |y| \leq B \right] dy \\
&= -\Lambda \int \sin(\varphi + \Lambda y) \left[ |y| \leq B \right] \rho(z - y) dy \\
&= -\Lambda^2 \int \cos(\varphi + \Lambda y) \left[ |y| \leq B \right] \rho(z - y) dy \\
&= -\Lambda^2 \int \cos(\varphi - \Lambda b) \rho(z + b) \left[ |b| \leq B \right] db.
\end{aligned}$$

Upon plugging back in, this yields that

$$h(x) = \iint G(w, b) \rho(\langle w, x \rangle + b) \frac{1}{2\sigma R \sqrt{m}} \left[ |b| \leq \sigma R \sqrt{m} \right] \delta_N(w) db dw,$$

which is the expanded form of desired expectation. □

**Lemma 5.** Let  $N_n = \frac{1}{n} \sum_{p=1}^n H_p$  and  $t > 0$ . Then

$$\begin{aligned}
& \mathbb{P} \left\{ \|N_n - \mathbb{E}(H)\|_K > t \right\} \leq \\
& 2 \exp \left( -\frac{n}{2} \left( \frac{t}{2R^2 \sqrt{m} (2\pi)^{-m/2} \lambda^{m+1} (1 + 1/\vartheta) \ell |\tilde{K}|} \right)^2 \right).
\end{aligned}$$

*Proof.* We first prove the existence of the measurable selector  $X_n$ .

To that end, let  $(\Omega, \Sigma)$  be the measurable space underlying  $\mathbb{P}$ . We define

$$\begin{aligned} \xi : \Omega \times K &\ni (\omega, x) \mapsto |N_n(x)(\omega) - \mathbb{E}(H(x))|; \\ \Xi(\omega) &= \operatorname{argmax}_{x \in K} \xi(\omega, x) = \left\{ x \in K : \xi(\omega, x) - \max_{u \in K} \xi(\omega, u) = 0 \right\} \subset K. \end{aligned}$$

Note that  $H(\diamond)(\omega)$  and  $\mathbb{E}(H(\diamond))$  are Lipschitz continuous for all  $\omega \in \Omega$ , and

$$\left\{ \max_{u \in K} \xi(\diamond, u) \leq c \right\} = \bigcap_{u \in K} \{ \xi(\diamond, u) \leq c \} = \bigcap_{q \in S} \{ \xi(\diamond, q) \leq c \},$$

where  $S$  is a countably dense subset of  $K$ , whence  $\max_{u \in K} \xi(\diamond, u)$  is measurable. Ergo,

$$\Omega \times K \ni (\omega, x) \mapsto \xi(\omega, x) - \max_{u \in K} \xi(\omega, u)$$

is a Carathéodory function [2, def. 4.50]. Since  $K$  is a compact subset of  $\mathbb{R}^m$ , it follows that  $\Xi$  is a (weakly) [2, lem. 18.2] measurable [2, cor. 18.8] correspondence [2, def. 17.1] with nonempty closed values from a measurable space into a Polish space. Thus  $\Xi$  admits a measurable selector [2, thm 18.13], that is, there exists a random variable

$$X_n : \Omega \ni \omega \mapsto X_n(\omega) \in \Xi(\omega)$$

such that  $|N_n(X_n) - \mathbb{E}(H(X_n))| = \|N_n - \mathbb{E}(H)\|_K$ .

In order to apply Hoeffding's inequality to

$$\mathbb{P}\left\{ \|N_n - \mathbb{E}(H)\|_K > t \right\} = \mathbb{P}\left\{ |N_n(X_n) - \mathbb{E}(H(X_n))| > t \right\},$$

we need to bound  $H(X_n)$  a.s., which we can do as follows.

$$\begin{aligned} |H(X_n)| &= |G(\mathbf{w}, \mathfrak{b})\rho(\langle \mathbf{w}, X_n \rangle + \mathfrak{b})| \leq \\ 2\sigma R\sqrt{m}\Lambda^2(2\pi)^{-m/2}\lambda^m |F(\Lambda\mathbf{w})| \cdot |\Psi(w/\sigma)| \left( \langle \mathbf{w}, X_n \rangle + \sigma R\sqrt{m} \right) \left[ |w| \geq \vartheta\sigma\sqrt{m} \right] &\leq \\ 2\sigma R\sqrt{m}\Lambda(2\pi)^{-m/2}\lambda^m \left( \Lambda |F(\Lambda\mathbf{w})| \right) R(1 + 1/\vartheta) |\mathbf{w}| &\leq \\ 2R^2\sqrt{m}(2\pi)^{-m/2}\lambda^{m+1}(1 + 1/\vartheta) \left\| |\Lambda\diamond| \cdot |F(\Lambda\diamond)| \right\|_\infty &= \\ 2R^2\sqrt{m}(2\pi)^{-m/2}\lambda^{m+1}(1 + 1/\vartheta) \left\| |\diamond| \cdot |F(\diamond)| \right\|_\infty, & \end{aligned}$$

because  $|\Psi| \leq 1$  and  $X_n(\omega) \in K$ . Now, using the fact that  $\|\mathcal{F}\{\diamond\}\|_\infty \leq \|\diamond\|_1$  and utilizing Minkowski's integral inequality twice yields that

$$\begin{aligned} \left\| |\diamond| \cdot |F(\diamond)| \right\|_\infty &= \left\| |\{\diamond_p F(\diamond)\}_{p=1}^m| \right\|_\infty \leq \left\| \left\{ \|\diamond_p F(\diamond)\|_\infty \right\}_{p=1}^m \right\| \\ &= \left\| \left\{ \|\mathcal{F}\{\partial_p \tilde{f}\}\|_\infty \right\}_{p=1}^m \right\| \leq \left\| \left\{ \|\partial_p \tilde{f}\|_1 \right\}_{p=1}^m \right\| \\ &\leq \left\| |\{\partial_p \tilde{f}\}_{p=1}^m| \right\|_1 = \|\nabla \tilde{f}\|_1 \leq \ell |\tilde{K}|, \end{aligned}$$

so  $H(X_n)$  is (a.s.) bounded by  $2R^2 \sqrt{m} (2\pi)^{-m/2} \lambda^{m+1} (1 + 1/\vartheta) \ell |\tilde{K}|$ . Here,  $|\{\diamond_p\}_{p=1}^m|$  denotes the Euclidean norm of the vector  $\{\diamond_p\}_{p=1}^m \in \mathbb{R}^m$ .

Above we implicitly used Rademacher's theorem to conclude that  $\nabla \tilde{f}$  exists a.e. We also used that  $\|\nabla \tilde{f}\|_\infty \leq \ell$ , which can be seen as follows.

It suffices to show that  $|\nabla \tilde{f}(x)| \leq \ell$  for all  $x \in \tilde{K}$  for which  $\nabla \tilde{f}(x)$  exists. Suppose  $x \in \tilde{K}$  is such that  $\nabla \tilde{f}(x)$  exists. If  $\nabla \tilde{f}(x) = 0$ , there is nothing to prove, so we may additionally assume that  $\nabla \tilde{f}(x) \neq 0$ . For almost every such  $x$  it holds that

$$\begin{aligned} \lim_{h \rightarrow 0} \left| \frac{|\tilde{f}(x + hu) - \tilde{f}(x)|}{h} - |\nabla \tilde{f}(x)| \right| &\leq \lim_{h \rightarrow 0} \left| \frac{\tilde{f}(x + hu) - \tilde{f}(x)}{h} - |\nabla \tilde{f}(x)| \right| \\ &= \lim_{h \rightarrow 0} \frac{|\tilde{f}(x + hu) - \tilde{f}(x) - \langle hu, \nabla \tilde{f}(x) \rangle|}{|hu|} = 0, \end{aligned}$$

where  $u = \frac{\nabla \tilde{f}(x)}{|\nabla \tilde{f}(x)|}$  and  $h > 0$ . As such,

$$|\nabla \tilde{f}(x)| = \lim_{h \rightarrow 0} \frac{|\tilde{f}(x + hu) - \tilde{f}(x)|}{h} \leq \ell,$$

because  $u$  is a unit vector.

The claim of the lemma then follows from directly applying Hoeffding's inequality.  $\square$

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