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Reasoning about knowledge and conditional probability [☆]

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ABSTRACT

We present a proof-theoretical and model-theoretical approach to reasoning about knowledge and conditional probability. We extend both the language of epistemic logic and the language of linear weight formulas, allowing statements like "Agent Ag knows that the probability of A given B is at least a half". We present both a propositional and a first-order version of the logic. We provide sound and complete axiomatizations for both logics and we prove decidability in the propositional case.

1. Introduction

Epistemic logics are formal models designed in order to reason about the knowledge of agents and their knowledge of each other's knowledge. During the last couple of decades, they have found applications in various fields such as game theory, the analysis of multi-agent systems in computer science and artificial intelligence, and for analyzing the behavior and interaction of agents in a distributed system [7,10,28]. In parallel, uncertain reasoning has emerged as one of the main fields in artificial intelligence, with many different tools developed for representing and reasoning with uncertain knowledge. A particular line of research concerns the formalization in terms of logic, and the questions of providing an axiomatization and decision procedure for *probabilistic logic* attracted the attention of researchers and triggered investigation about formal systems for probabilistic reasoning [2,17,9,11,16,26].

Fagin and Halpern [8] emphasized the need for combining the fields of epistemic and probabilistic logics for many application areas, and in particular in distributed systems applications, when one wants to analyze randomized or probabilistic programs.[1] They developed a joint propositional framework for reasoning about knowledge and probability, proposed a complete axiomatization and investigated decidability of the framework. Using the approach from the seminal paper by Fagin, Halpern and Meggido [9], they extended the propositional epistemic language with formulas which express linear combinations of probabilities, called *linear weight formulas*, i.e., the formulas of the form $a_1 w(\alpha_1) + \cdots + a_k w(\alpha_k) \geq r$, where a_j's and r are rational numbers. They proposed a finitary axiomatization and proved weak completeness, using a small model theorem. It is worth mentioning that a richer propositional

☆ This paper is revised and extended version of the conference paper [6] presented at 17th Edition of the European Conference on Logics in Artificial Intelligence (JELIA 2021), in which we presented a propositional epistemic logic with conditional probability operators. In this work we also develop a first order extension of that logic. We have opted to start by presenting the first order version of the logic and proving its completeness in detail, since that proof can be easily reduced to the proof of our propositional version of the logic (first presented at JELIA 2021). In addition, we incorporated the so called *consistency condition*, which was not present on our preliminary work, and which forbids an agent to place positive probabilities to the events she knows to be false.

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[1] More recently, epistemic probabilistic logics are also applied for and characterization of agents' knowledge when running a blockchain protocol [23] (following [15]) and in robotic applications when the sensors are assumed to be noisy [5].

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probabilistic language, obtained by adding multiplication to the syntax (so called *polynomial weight formulas*) is also proposed in [9] in order to allow representation of conditional probabilities.

In this paper, we extend the logic from [8] by also allowing formulas that can represent conditional probability. Thus, our language contains both knowledge operators K_i (one for each agent i) and conditional probability formulas of the form $a_1\mathbf{w}_i(\alpha_1,\beta_1)+\cdots+a_k\mathbf{w}_i(\alpha_k,\beta_k)\geq r$. The expressions of the form $\mathbf{w}_i(\alpha,\beta)$ represent conditional probabilities that agent i places on events according to Kolmogorov definition: $P(A|B)=\frac{P(A\cap B)}{P(B)}$ if $P(B)>0$, while $P(A|B)$ is undefined when $P(B)=0$.

One of our main objectives is to provide a framework in which we can also model situations that require an underlying first-order language. The need for such frameworks is recognized both in case of epistemic logic and probability logic. Indeed, Wolter [28] and Bacchus [3] independently advocated use of a first-order language as necessary whenever an application domain is either infinite, or finite, but whose size (and its bound) is not known in advance, which is a frequent case in the field of Knowledge Representation. For that reason we allow predicates and quantification over variables in our language. The corresponding semantics extends the semantics from [8], and it consists of enriched Kripke models, with a probability measure assigned to every agent in each world, where each world also carries a first order structure. We denote that logic by CKL^{fo}.[2] It should be mentioned that first-order probability logics are already studied in the field of Artificial Intelligence from two perspectives: one that puts a probability on the domain, and is appropriate for representation of statistical information, and the one that we follow - that puts a probability on possible worlds and can be used to model degrees of belief [13,4,20]. An epistemic extension of first order probability logic (in which conditional probabilities cannot be neither expressed nor compared) is introduced in [27].

Our main result is a strongly complete axiomatization of the logic. It is known that even in probabilistic fragment of such framework there are theoretical difficulties when one considers first-order models with possibly infinite domains. Namely, it is shown by Abadi and Halpern [1] that the set of valid first-order probability formulas is not recursively enumerable, so that there is no complete recursive axiomatization. In order to overcome that problem, we present an infinitary inference system.

From the technical point of view, we obtain completeness combining and extending the approaches from [8] and [27]. On one hand, most of our axioms (including those for reasoning about linear inequalities) extend those from [8]. On the other hand, because of the problem of axiomatizing conditional probabilities using linear weight formulas with the approach from [9,8] (mentioned below in this paragraph), we modify the Henkin-style proof strategy from [27] and obtain strong version of completeness ("every consistent set of sentences has a model") using infinitary inference rules. The premises and conclusions of those rules are in the form of k-nested implications. This form of infinitary rules is a technical solution already used in probabilistic, epistemic and temporal logics for obtaining various *strong necessitation* results [21,24,27].

For practical purposes, we also propose a propositional variant of the logic, denoted by CKL, for which we prove both completeness and decidability result. The completeness proof is a straightforward simplification of the corresponding proof for CKL^{fo}. The main advantage of CKL is that the satisfiability problem is decidable (while CKL^{fo} is trivially undecidable as it extends classical first-order logic). We prove decidability combining the method of filtration [18] and a reduction to a finite set of systems of inequalities. While we have already emphasized impossibility of (even weak) completeness of a finitary axiomatization of our first-order logic, one can pose the question can we develop a finitary system for CKL which would be weakly complete (strong completeness of a finitary system is impossible due to the noncompactness phenomena for probability logics, see [17]). We do not know a finitary axiomatization for this rich language. Moreover, even for propositional logics which need to express conditional probabilities only (i.e., without knowledge operators), the task of developing a finitary system turned out to be very hard to accomplish. Fagin, Halpern and Meggido [9] faced problems when they tried to represent conditional probabilities by adding multiplication to the syntax of linear weight formulas, and they needed to introduce a first-order quantification over coefficients of their probability polynomial terms in order to obtain completeness. The only finitary axiomatization of conditional probabilities (in propositional setting) we are aware of is the fuzzy approach of Marchioni and Godo [22], who consider the probability of a conditional event of the form "α given β" as the truth-value of the fuzzy proposition $P(\alpha|\beta)$ which is read as "$P(\alpha|\beta)$ is probable."

The structure of this paper is as follows. In Section 2 we present the syntax and semantics of our first order logic CKL^{fo} in detail. In Section 3 we propose an axiomatization for CKL^{fo} and we prove its soundness. In Section 4 we prove that the axiomatization is strongly complete with respect to the proposed semantics. Propositional restriction CKL we present in Section 5. We show that CKL is decidable, and we propose strong complete axiomatization for the logic. We conclude in Section 6.

2. The logic CKL^{fo}: syntax and semantics

In this section we introduce the set of formulas of our logic CKL^{fo}, provide a class of measurable models in which those formulas are evaluated, and define the satisfiability relation.

2.1. Syntax

Let $\mathbf{A}$ be a finite set of agents. As we emphasized in Introduction, here we want to extend the language of the logic from [8], which contains the formulas of the form $\mathbf{w}_i(\alpha)\geq r$ with intended meaning "according to the agent i, the formula α holds with the probability at least r". Our language $\mathcal{L}$ allows formulas of the form $\mathbf{w}_i(\alpha,\beta)\geq r$ which we read as "according to the agent i, conditional probability of α based on β is at least r". Actually, as in [9], we extend language further. If $\alpha_1,\beta_1,\ldots,\alpha_n,\beta_n$ are formulas

[2] Here, C stands for "Conditional probability" and K stands for "Knowledge", while L denotes "Logic".

then $a_1\mathbf{w}_i(\alpha_1,\beta_1)+\cdots+a_n\mathbf{w}_i(\alpha_n,\beta_n)\geq r$ is also a formula of the language, where a_k's and r are arbitrary rational numbers, for every $i\in\mathbf{A}$. The language $\mathcal{L}$ further contains:

- a countable set of variables $Var=\{x,y,z,\dots\}$,
- universal quantifier $\forall$, and classical propositional connectives,
- for every integer $k\geq 0$, denumerably many function symbols $F_0^k, F_1^k,\dots$ of arity k,
- for every integer $k\geq 1$, denumerably many relation symbols $P_0^k, P_1^k,\dots$ of arity k,
- a list of unary knowledge operators K_i, one for every $i\in\mathbf{A}$.

The function symbols of arity 0 are called constant symbols. Terms (denoted by $t_1,t_2,\dots$) and atomic formulas (formulas of the form $P_j^k(t_1,\dots,t_k)$) are defined as usual, as well as the notion of a term that is free for a variable.

Let Q denote the set of all rational numbers and let $[0,1]_Q$ denote the set $[0,1]\cap Q$.

Definition 1 *(Formula)*. The set For of all formulas of the logic is defined as follows:

$$\alpha ::= P_j^k(t_1,\dots,t_k) \mid K_i\alpha \mid \sum_{l=1}^{k} a_l\mathbf{w}_i(\alpha,\alpha)\geq r \mid \alpha\wedge\alpha \mid \neg\alpha \mid (\forall x)\alpha$$

where $i\in\mathbf{A}$, $k,j\in\mathbb{N}$, $k\geq 1$ and $a_1,\dots,a_k,r\in Q$.

The meaning of the formula $K_i\alpha$ is "agent i knows α", while the expression $\mathbf{w}_i(\alpha,\beta)$ denotes conditional probability of α given β, according to the agent i.

An expression of the form $a_1\mathbf{w}_i(\alpha_1,\alpha_1')+\cdots+a_k\mathbf{w}_i(\alpha_k,\alpha_k')$ is called *probabilistic term*. Following [8], we do not allow appearance of multiple agents inside of a probabilistic term. We denote probabilistic terms with f_i, g_i and h_i.

We use a number of abbreviations through this paper. The propositional connectives, $\vee$, $\rightarrow$ and $\leftrightarrow$, are introduced as usual. We define $\top$ to be an abbreviation for the formula $\alpha\vee\neg\alpha$, while $\bot$ is $\neg\top$. We also use abbreviations to define other types of inequalities, for example: $f_i\geq g_i$ as an abbreviation for $f_i-g_i\geq 0$, $f_i\leq g_i$ is $g_i\geq f_i$, $f_i=g_i$ is $(f_i\geq g_i)\wedge(f_i\leq g_i)$ and $f_i>g_i$ is $(f_i\geq g_i)\wedge\neg(f_i=g_i)$.

A formula without free variables is called a *sentence*. A set of sentences we call *theory*. The set of all sentences of the logic we denote by $Sent$.

Example 1. The sentence "The agent j knows that according to the agent i, the probability that all birds from the flock take off if one of them takes off, is 0.99" can be formalized by the formula

$$K_j\mathbf{w}_i((\forall x)(\mathit{InFlock}(x)\rightarrow \mathit{TakesOff}(x)),(\exists x)(\mathit{InFlock}(x)\wedge \mathit{TakesOff}(x)))=0.99$$

2.2. Semantics

Now we introduce the semantics of our logic. We define structure as an extension of the first order possible-words for epistemic logics with probabilistic spaces.

Definition 2 *(CKL^{fo}-structure)*. An CKL^{fo}-structure is a tuple $(W,D,I,\mathcal{K},Prob)$ where:

1. W is a non-empty set of objects called *worlds*.
2. D is non-empty domain for every $w\in W$,
3. I assigns an interpretation $I(w)$ to every $w\in W$ such that for all i and k:
   - $I(w)(F_i^k)$ is a function from D^k to D,
   - for every $w'\in W$, $I(w)(F_i^k)=I(w')(F_i^k)$,
   - $I(w)(P_i^k)$ is a subset of D^k,
4. $\mathcal{K}=\{\mathcal{K}_i \mid i\in\mathbf{A}\}$ is a set of binary equivalence relations on W. We denote $\mathcal{K}_i(w)=\{w' \mid (w',w)\in\mathcal{K}_i\}$, and we write $w\mathcal{K}_iw'$ if $w'\in\mathcal{K}_i(w)$,
5. $Prob$ assigns to every $i\in\mathbf{A}$ and $w\in W$ a probability space $Prob(i,w)=(W_i(w),H_i(w),\mu_i(w))$, where
   - $W_i(w)$ is a non-empty subset of $\mathcal{K}_i(w)$,
   - $H_i(w)$ is an algebra of subsets of $W_i(w)$, i.e. a set such that
     (a) $W_i(w)\in H_i(w)$,
     (b) if $A\in H_i(w)$, then $W_i(w)\setminus A\in H_i(w)$, and
     (c) if $A,B\in H_i(w)$, then $A\cup B\in H_i(w)$.
   - $\mu_i(w): H_i(w)\longrightarrow[0,1]$ is a finitely additive measure, i.e.,
     (a) $\mu_i(w)(W_i(w))=1$,
     (b) $\mu_i(w)(A\cup B)=\mu_i(w)(A)+\mu_i(w)(B)$, whenever $A\cap B=\emptyset$.

The elements of $H_i(w)$ are called measurable sets.

In this definition we use two assumptions that are fairly standard for the first order modal logics. First assumption is that the domain is fixed in a model, it means domain is the same in all the worlds of a considered model. The second assumption is that the terms are *rigid* which means that for every model their meanings are the same in all the worlds of a considered model. These assumptions give us validity of all fist-order axioms (see Section 3).

The notion of *variable valuation* is defined in usual way. Let $M = (W, D, I, Prob)$ be an CKL^{fo}-structure. A variable valuation v assigns some element of the corresponding domain to every variable x, i.e., $v(x) \in D$. If $d \in D$ and v is a valuation, then $v[d/x]$ is a valuation same as v except that $v[d/x](x) = d$.

Then we can define the *value* of terms. The value of a term t, denoted by $I(w)(t)_v$ is:

- if t is a variable x, then $I(w)(x)_v = v(x)$, and
- if $t = F_i^m(t_1, \dots, t_m)$, then $I(w)(t)_v = I(w)(F_i^m)(I(w)(t_1)_v, \dots, I(w)(t_m)_v)$.

Now, we define *satisfiability* of formulas in the worlds of introduced models.

Definition 3. The truth value of a formula α in a world $w \in W$ of a CKL^{fo}-structure $M = (W, D, I, \mathcal{K}, Prob)$, under a valuation v (denoted by $(M, w, v) \models \alpha$) is:

- $(M, w, v) \models P_i^m(t_1, \dots, t_m)$ iff $(I(w)(t_1)_v, \dots, I(w)(t_m)_v) \in I(w)(P_i^m)$,
- $(M, w, v) \models K_i\alpha$ iff $(M, w', v) \models \alpha$ for all $w' \in \mathcal{K}_i(w)$,
- $(M, w, v) \models \sum_{k=1}^n a_k \mathbf{w}_i(\alpha_k, \beta_k) \geq r$ if $\mu_i(w)(\{w' \in W_i(w) \mid (M, w', v) \models \beta_k\}) > 0$ for every $k \in \{1, \dots, n\}$ and $\sum_{k=1}^n a_k \mu_i(w)(\{w' \in W_i(w) \mid (M, w', v) \models \alpha_k\} | \{w' \in W_i(w) \mid (M, w', v) \models \beta_k\}) \geq r$,
- $(M, w, v) \models \neg\alpha$ iff $(M, w, v) \not\models \alpha$
- $(M, w, v) \models \alpha \wedge \beta$ iff $(M, w, v) \models \alpha$ and $(M, w, v) \models \beta$
- $(M, w, v) \models (\forall x)\alpha$, iff for every $d \in D$, $(M, w, v[d/x]) \models \alpha$

From the previous definition we can see that the satisfiability relation is not in general defined for all formulas. Because of that we will consider only CKL^{fo}-measurable structures: an CKL^{fo}-structure M is measurable if for every formula α, every valuation v and every world w from M, the set

$$[\alpha]_{i,w}^v = \{u \in W_i(w) \mid (M, u, v) \models \alpha\}$$

belongs to $H_i(w)$. If α is a sentence, we omit the superscript v in $[\alpha]_{i,w}^v$. We denote the set of all measurable CKL^{fo}-structure with $\mathrm{CKL}^{\mathrm{fo}}_{\mathrm{Meas}}$.

We say that a formula α holds in a world w from an CKL^{fo}-structure $M = (W, D, I, Prob)$ (denoted by $(M, w) \models \alpha$) if for every valuation v, $(M, w, v) \models \alpha$. If $d \in D$, we will use $(M, w) \models \alpha(d)$ to denote that $(M, w, v[d/x]) \models \alpha$, for every valuation v. A formula is valid in an CKL^{fo}-structure $M = (W, D, I, Prob)$ (denoted by $M \models \alpha$), if it is satisfied in every world w from W. A formula α is valid if for every CKL^{fo}-structure M, $M \models \alpha$. A sentence α is satisfiable if there is a world w in an CKL^{fo}-structure M such that $(M, w) \models \alpha$. A set T of sentences is satisfiable if there is a world w in an CKL^{fo}-structure M such that $(M, w) \models \alpha$ for every $\alpha \in T$. We also say that (M, w) is a *model* (or *pointed model*) of T.

At the end of this section, we emphasize that the non-compactness property, which is a common characteristic of many real valued probability logics [26], also holds for CKL^{fo}.

Example 2. Consider the set

$$T = \{\neg\mathbf{w}_i(\alpha, \beta) = 0\} \cup \{\mathbf{w}_i(\alpha, \beta) < \frac{1}{n} \mid n \textit{ is a positive integer}\}.$$

Every finite subset of T is satisfiable, but the set T itself is not. Therefore, the compactness theorem ("if every finite subset of T is satisfiable, then T is satisfiable") does not hold for the logic CKL^{fo}.

3. Aximatization Ax(CKL^{fo})

In this section we present an axiomatization of our logic, which we denote by Ax(CKL^{fo}). The axiom system Ax(CKL^{fo}) contains twenty axiom schemes and six inference rules. But first we need to define one useful notion. In the following definition, for any tuple of objects $\mathbf{Y} = (Y_1, \dots, Y_m)$, let $rest(\mathbf{Y}) = (Y_2, \dots, Y_m)$, i.e., the operator $rest$ removes the first coordinate from the tuple.

In this paper we will employ the notion of k-nested implications.

Definition 4 *(k-nested implication)*. For every formula $\alpha \in For$, a non-negative integer $k \in \mathbb{N}$, a sequence of formulas $\Theta = (\theta_0, \dots, \theta_k)$ and a sequence $\mathbf{X} = (X_1, \dots, X_k)$ of knowledge operators from $\{K_i \mid i \in \mathbf{A}\}$, we define a k-nested implication formula $\Phi_{k,\Theta,\mathbf{X}}(\alpha)$ recursively as follows:

$$\Phi_{0,\theta_0,\emptyset}(\alpha) = \theta_0 \to \alpha$$

$$\Phi_{k,\Theta,\mathbf{X}}(\alpha) = \theta_k \to X_k \Phi_{k-1,rest(\Theta),rest(\mathbf{X})}(\alpha).$$

For example, if $\mathbf{X} = (K_a, K_b, K_c)$, $a, b, c \in \mathbf{A}$, then

$$\Phi_{3,\Theta,\mathbf{X}}(\alpha) = \theta_3 \to K_c(\theta_2 \to K_b(\theta_1 \to K_a(\theta_0 \to \alpha))).$$

Formulas of this form will be used to formulate infinitary inference rules in the axiomatization, and allow us to apply those rules not only to the outermost operators, but also to the operators inside formulas. This form is necessary for the inductive proofs of Deduction theorem (Theorem 2) and Strong necessitation theorem (Theorem 3). We further discuss the form of k-nested implications in Remark 1.

Our axiomatization contains the following axiom schemes and inference rules.

Axioms and rules for classical first-order reasoning

(A1) All instances of classical propositional tautologies.
(A2) $(\forall x)(\alpha \to \beta) \to (\alpha \to (\forall x)\beta)$ where x is not free in α
(A3) $(\forall x)(\alpha(x)) \to \alpha(t)$, where $\alpha(t)$ is obtained by substituting all free occurrences of x in $\alpha(x)$ by the term t which is free for x in $\alpha(x)$
(R1) From $\{\alpha, \alpha \to \beta\}$ infer β.
(R2) From α infer $(\forall x)\alpha$.

Axioms and rules for reasoning about knowledge

(A4) $\forall x K_i \alpha(x) \to K_i \forall x \alpha(x)$ (Barcan formula)
(A5) $(K_i\alpha \wedge K_i(\alpha \to \beta)) \to K_i\beta$, for every $i \in G$
(A6) $K_i\alpha \to \alpha$,
(A7) $K_i\alpha \to K_iK_i\alpha$,
(A8) $\neg K_i\alpha \to K_i\neg K_i\alpha$,
(R3) From α infer $K_i\alpha$.

Axioms for reasoning about linear inequalities

(A9) $(\sum_{l=1}^{k} a_l \mathbf{w}_i(\alpha_l, \alpha_l') \leq r \wedge \mathbf{w}_i(\alpha_{k+1}', \top) > 0) \leftrightarrow (\sum_{l=1}^{k} a_l \mathbf{w}_i(\alpha_l, \alpha_l') + 0\mathbf{w}_i(\alpha_{k+1}, \alpha_{k+1}') \leq r)$
(A10) $(\sum_{l=1}^{k} a_l \mathbf{w}_i(\alpha_l, \alpha_l') \leq r) \to (\sum_{l=1}^{k} a_{j_l} \mathbf{w}_i(\alpha_{j_l}, \alpha_{j_l}') \leq r)$ where $j_1, \dots, j_k$ is a permutation of $1, \dots, k$.
(A11) $(\sum_{l=1}^{k} a_l \mathbf{w}_i(\alpha_l, \alpha_l') \leq r) \wedge (\sum_{l=1}^{k} a_l' \mathbf{w}_i(\alpha_l, \alpha_l') \leq r') \to (\sum_{l=1}^{k} (a_l + a_l')\mathbf{w}_i(\alpha_l, \alpha_l') \leq r + r')$
(A12) $(\sum_{l=1}^{k} a_l \mathbf{w}_i(\alpha_l, \alpha_l') \leq r) \leftrightarrow (\sum_{l=1}^{k} d a_l \mathbf{w}_i(\alpha_l, \alpha_l') \leq dr)$ where $d > 0$.
(A13) $\bigwedge_{i=0}^{n} \mathbf{w}_i(\alpha_i', \top) > 0 \to (\sum_{l=1}^{k} a_l \mathbf{w}_i(\alpha_l, \alpha_l') \leq r) \vee (\sum_{l=1}^{k} a_l \mathbf{w}_i(\alpha_l, \alpha_l') \geq r)$
(A14) $(f_i \geq r) \to (f_i > r')$ for $r > r'$

Axioms and rule for reasoning about probabilities

(A15) $\mathbf{w}_i(\alpha, \top) \geq 0$
(A16) $\mathbf{w}_i(\alpha \wedge \beta, \top) + \mathbf{w}_i(\alpha \wedge \neg\beta, \top) = \mathbf{w}_i(\alpha, \top)$
(A17) $\mathbf{w}_i(\alpha, \top) = \mathbf{w}_i(\beta, \top)$ if $\alpha \leftrightarrow \beta$ is an instance of propositional tautology
(A18) $\sum_{j=1}^{n} a_j \mathbf{w}_i(\alpha_j, \beta_j) \geq r \to \mathbf{w}_i(\beta_j, \top) > 0$ for every $j \in \{1, \dots, n\}$
(A19) $(\mathbf{w}_i(\beta, \top) \geq s \wedge \mathbf{w}_i(\alpha, \beta) \geq r) \to \mathbf{w}_i(\alpha \wedge \beta, \top) \geq s \cdot r$
(R4) From the set of premises $\{\Phi_{k,\Theta,\mathbf{X}}(f_i \geq r - \frac{1}{l}) \mid l \in \mathbb{N}\}$ infer $\Phi_{k,\Theta,\mathbf{X}}(f_i \geq r)$
(R5) From the set of premises $\{\Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_i(\beta, \top) > 0)\} \cup \{\Phi_{k,\Theta,\mathbf{X}}((\mathbf{w}_i(\beta, \top) \geq s \to \mathbf{w}_i(\alpha \wedge \beta, \top) \geq r \cdot s) \mid s \in [0,1]_Q\}$ infer $\Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_i(\alpha, \beta) \geq r)$

Axiom for consistency condition

(A20) $K_i\alpha \to \mathbf{w}_i(\alpha, \top) \geq 1$.

According to the type of reasoning, our axiomatization is divided into five groups. The first group tells us that classical first order logic is a sublogic of our logic. Recall that we use fixed domain CKL^{fo}-measurable models with rigid terms, which is similar to the objectual interpretation for first order modal logics [12]. If we reject the assumption that the terms are rigid, then the standard first order axiom A3 is not sound. The second group present standard axiomatization for epistemic logic. Axiom A4 is a variant of the well-known axiom for modal logics, called Barcan formula. It is proved that Barcan formula holds in the class of all first-order fixed domain modal models [19]. The axioms A9-A17 are adapted from axiom system from [8] to our approach to conditional probabilities. The axioms A18 and A19, together with the rule R5 properly capture the third condition of Definition 3. The rule R3 is Necessitation rule for the knowledge operator. In this logic we do not need Probabilistic Necessitation (*From* α *infer* $\mathbf{w}_i(\alpha, \top) \geq 1$) because it is derivable from R3 and A20. The rules R4 and R5 are infinitary inference rules. R4 is a variant of so called Archimedean rule, whose role is to prevent nonstandard values. Intuitively, it says that is the value of a term is infinitely close to r, then it must be equal to r. The necessity of employing such rules comes from the non-compactness phenomena (Example 2). Actually, it is known that in a real-valued probabilistic logic there exist unsatisfiable, but finitely satisfiable, sets of formulas. As pointed out in [17], one consequence of that fact is that any finitary axiomatization would not be strongly complete.

Let us now define some basic notions of proof theory.

Definition 5 *(Theorem, proof)*. A formula α is a *theorem*, denoted by $\vdash_{Ax(\mathrm{CKL}^{fo})} \alpha$, if there is a sequence of formulas $\alpha_0, \alpha_1, \ldots, \alpha_\lambda$ (λ is finite or countable ordinal), such that $\alpha_\lambda = \alpha$ and every α_i, $i < \lambda$, is an axiom, or it is derived from the preceding formulas by an inference rule.

A formula α is deducible from a set $T \subseteq For$ ($T \vdash_{Ax(\mathrm{CKL}^{fo})} \alpha$) if there is a sequence of formulas $\alpha_0, \alpha_1, \ldots, \alpha_\lambda$ (λ is finite or countable ordinal), such that $\alpha_\lambda = \alpha$ and every α_i is an axiom or a formula from T, or it is derived from the preceding formulas by an inference rule R1, R2, R4 or R5. The sequence $\alpha_0, \alpha_1, \ldots, \alpha$ is a *proof* of α from T.

We write $\vdash$ instead of $\vdash_{Ax(\mathrm{CKL}^{fo})}$ when it is clear from context.

Definition 6 *(Consistency)*. A set of formulas T is *inconsistent* if $T \vdash \bot$, otherwise it is *consistent*.

T is a *maximal consistent set* (mcs) of formulas if it is consistent and every proper superset of T is inconsistent.

A maximal consistent set T is *saturated* if $\neg(\forall x)\alpha(x) \in T$, then for some term t, $\neg\alpha(t) \in T$.

At the end of this section, we show that the axiom system $Ax(\mathrm{CKL}^{fo})$ is sound.

Theorem 1 *(Soundness). The axiomatization $Ax(\mathrm{CKL}^{fo})$ is sound with respect to the class of structures $\mathrm{CKL}^{\mathrm{fo}}_{\mathrm{Meas}}$.*

Proof. Let M be a $\mathrm{CKL}^{\mathrm{fo}}_{\mathrm{Meas}}$-structure, and w a world in M. Here, we will show only the cases for $A3$, $A4$, $R5$ and $A20$.

A3 Let v be any valuation such that $(M, w, v) \models (\forall x)\alpha$. By Definition 3 we have $(M, w, v[d/x]) \models \alpha(x)$, for all $d \in D$. Let $I(w)(t)_v = d'$, $d' \in D$ and we know that $v[d'/x](x) = d'$. Now, from the fact that $(M, w, v[d'/x]) \models \alpha(x)$ we have $(M, w, v) \models \alpha(t)$.

A4 Suppose that v is any valuation such that $(M, w, v) \models \forall x K_i \alpha(x)$. Similarly as before we have that for every $d \in D$, $(M, w, v[d/x]) \models K_i\alpha(x)$. Therefore, for every d and $w' \in \mathcal{K}_i(w)$, it is $(M, w', v[d/x]) \models \alpha(x)$. So, for every $w' \in \mathcal{K}_i(w)$ we have $(M, w', v) \models (\forall x)\alpha(x)$. Therefore $(M, w, v) \models K_i(\forall x)\alpha(x)$.

R5 We show the soundness of the rule R5 by induction on k. Let M be a model, w a world of M, and v a valuation. We show that if $(M, w, v) \models \Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_i(\beta, \top) > 0)$ and $(M, w, v) \models \Phi_{k,\Theta,\mathbf{X}}((\mathbf{w}_i(\beta, \top) \geq s \to \mathbf{w}_i(\alpha \wedge \beta, \top) \geq r \cdot s)$ for every $s \in [0,1]_Q$ then $(M, w, v) \models \Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_i(\alpha, \beta) \geq r)$.

[$k = 0$ *step*] It follows by the properties of real numbers.

[*Inductive hypothesis*] It holds for some k.

[$k+1$ *step*] Let $(M, w, v) \models \Phi_{k+1,\Theta,\mathbf{X}}((\mathbf{w}_i(\beta, \top) \geq s \to \mathbf{w}_i(\alpha \wedge \beta, \top) \geq r \cdot s)$ and $(M, w, v) \models \Phi_{k+1,\Theta,\mathbf{X}}(\mathbf{w}_i(\beta, \top) > 0)$, and let $X_{k+1} = K_i$, for some $i \in \mathcal{A}$. Therefore, $(M, w, v) \models \theta_{k+1} \to K_i\Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_i(\beta, \top) > 0)$ and $(M, w, v) \models \theta_{k+1} \to K_i\Phi_{k,\Theta,\mathbf{X}}((\mathbf{w}_i(\beta, \top) \geq s \to \mathbf{w}_i(\alpha \wedge \beta, \top) \geq r \cdot s)$, for every $s \in [0,1]_Q$.

Assume that $(M, w, v) \models \theta_{k+1}$, otherwise is trivial. Then $(M, w, v) \models K_i\Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_i(\beta, \top) > 0)$ and $(M, w, v) \models K_i\Phi_{k,\Theta,\mathbf{X}}((\mathbf{w}_i(\beta, \top) \geq s \to \mathbf{w}_i(\alpha \wedge \beta, \top) \geq r \cdot s)$, for every $s \in [0,1]_Q$. Now, for every $w' \in \mathcal{K}_i$ we have $(M, w', v) \models \Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_i(\beta, \top) > 0)$ and $(M, w', v) \models \Phi_{k,\Theta,\mathbf{X}}((\mathbf{w}_i(\beta, \top) \geq s \to \mathbf{w}_i(\alpha \wedge \beta, \top) \geq r \cdot s)$, for every $s \in [0,1]_Q$. By Inductive step, it is $(M, w', v) \models \Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_i(\alpha, \beta) \geq r)$, for every $w' \in \mathcal{K}_i(w)$. Then by the definition of satisfiability, $(M, w, v) \models K_i\Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_i(\alpha, \beta) \geq r)$. Thus $(M, w, v) \models \theta_{k+1} \to K_i\Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_i(\alpha, \beta) \geq r)$, i.e., $(M, w, v) \models \Phi_{k+1,\Theta,\mathbf{X}}(\mathbf{w}_i(\alpha, \beta) \geq r)$.

A20 Let $(M, w, v) \models K_i\alpha$, then for all $w' \in \mathcal{K}_i(w)$ we have $(M, w', v) \models \alpha$. From the definition of our structures, we have that $W_i(w) \subset \mathcal{K}_i(w)$, therefore $(M, w', v) \models \alpha$ for all $w' \in W_i(w)$, i.e., $(M, w, v) \models \mathbf{w}_i(\alpha, \top) \geq 1$. □

4. Completeness

In this section we show that the axiomatization $Ax(\mathrm{CKL}^{fo})$ is strongly complete for the logic CKL^{fo}, i.e., we prove that every consistent set of formulas has a model. We start with the deduction theorem.

Theorem 2 *(Deduction theorem). Let T be a theory and α and β be two sentences. Then*

$$T \cup \{\alpha\} \vdash \beta \textit{ iff } T \vdash \alpha \to \beta.$$

Proof. The direction from right to left is trivial. For left to right we use the transfinite induction on the length of the inference. Here we will only consider the case when β is obtained by the rule R5, i.e. $\beta = \Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_i(\alpha', \beta') \geq r)$, where $\Theta = (\theta_0, \ldots, \theta_k)$. Then we have by the definition of $\Phi_{k,\Theta,\mathbf{X}}$:

$$T, \alpha \vdash \theta_k \to X_k\Phi_{k-1,rest(\Theta),rest(\mathbf{X})}(\mathbf{w}_i(\beta', \top) > 0),$$
$$T, \alpha \vdash \theta_k \to X_k\Phi_{k-1,rest(\Theta),rest(\mathbf{X})}(\mathbf{w}_i(\beta', \top) \geq s \to \mathbf{w}_i(\alpha' \wedge \beta', \top) \geq r \cdot s).$$

By Inductive hypothesis and the axiom $A1$ we obtain:
$T \vdash (\alpha \wedge \theta_k) \to X_k \Phi_{k-1,rest(\Theta),rest(\mathbf{X})}(\mathbf{w}_i(\beta', \top) > 0)$,
$T \vdash (\alpha \wedge \theta_k) \to X_k \Phi_{k-1,rest(\Theta),rest(\mathbf{X})}(\mathbf{w}_i(\beta', \top) \geq s \to \mathbf{w}_i(\alpha' \wedge \beta', \top) \geq r \cdot s)$. For $\overline{\Theta} = (\theta_0, \dots, \alpha \wedge \theta_k)$ we have
$T \vdash \Phi_{k,\overline{\Theta},\mathbf{X}}(\mathbf{w}_i(\beta', \top) > 0)$,
$T \vdash \Phi_{k,\overline{\Theta},\mathbf{X}}(\mathbf{w}_i(\beta', \top) \geq s \to \mathbf{w}_i(\alpha' \wedge \beta', \top) \geq r \cdot s)$.
By the rule $R5$ and the definition of $\Phi_{k,\Theta,\mathbf{X}}$ we obtain
$T \vdash \Phi_{k,\overline{\Theta},\mathbf{X}}(\mathbf{w}_i(\alpha', \beta') \geq r)$,
$T \vdash (\alpha \wedge \theta_k) \to X_k \Phi_{k-1,rest(\Theta),rest(\mathbf{X})}(\mathbf{w}_i(\alpha', \beta') \geq r)$,
$T \vdash \alpha \to (\theta_k \to X_k \Phi_{k-1,rest(\Theta),rest(\mathbf{X})}(\mathbf{w}_i(\alpha', \beta') \geq r)$,
$T \vdash \alpha \to \Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_i(\alpha', \beta') \geq r)$
$T \vdash \alpha \to \beta$. □

Remark 1. Let us observe that the proof of the previous theorem relies on the fact that all the infinitary rules of inference are given in the implicative form, i.e., all the formulas in the premises and the conclusion of a rule are implications with the same antecedent. Thus, if we want to employ a rule that intuitively says

(IR) From the set of formulas $\{\alpha_m \mid m \geq 0\}$ infer α,

we need the following implicative generalizations:

From the set of formulas $\{\beta \to \alpha_m \mid m \geq 0\}$ infer $\beta \to \alpha$,

one for each β. This is a standard technical solution in infinitary proof systems which ensures that Deduction theorem can be proven using the transfinite induction on the length of the inference.

Moreover, in presence of knowledge operator, we will have to prove strong version of Necessitation for knowledge operators, namely that $T \vdash \alpha$ implies $\{K_i\beta \mid \beta \in T\} \vdash K_i\alpha$. As demonstrated in the proof of our next result below, the transfinite induction on the length of the proof requires another type of generalizations of our inference rules (IR):

From the set of formulas $\{K_i(K_{i_n} K_{i_{n-1}} \dots K_{i_1} \alpha_m) \mid m \geq 0\}$ infer $K_i(K_{i_n} K_{i_{n-1}} \dots K_{i_1} \alpha)$,

one for each block of knowledge operators $K_{i_n} K_{i_{n-1}} \dots K_{i_1}$.

This might seem incompatible with our requirement that the rules must formally be in an implicative form - in this case we simply use the implication with the antecedent $\top$ (see the proof of Theorem 3). The fact that our rules need to meet both requirements of implicative forms and nesting of knowledge operators leads to the form of k-nested implication as given by Definition 4.

Theorem 3 *(Strong necessitation). If T is a theory and $T \vdash \alpha$, then $K_j T \vdash K_j \alpha$, for all $j \in \mathbf{A}$, where $K_j T = \{K_j \alpha \mid \alpha \in T\}$.*

Proof. Let $T \vdash \alpha$. We will prove the theorem by using the transfinite induction on the length of the proof of $T \vdash \alpha$. Here we will only consider the application of the rule R5. Let α be the formula $\Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_i(\gamma, \beta) \geq r)$, with $\Theta = (\theta_0, \dots, \theta_k)$ and $\mathbf{X} = (X_1, \dots, X_k)$, which was obtained by the rule R5. Then we have

$T \vdash \Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_i(\beta, \top) > 0)$
$T \vdash \Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_i(\beta, \top) \geq s \to \mathbf{w}_i(\gamma \wedge \beta, \top) \geq r \cdot s)$ for all $s \in [0,1]_Q$
$K_j T \vdash K_j \Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_i(\beta, \top) > 0)$ by IH
$K_j T \vdash K_j \Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_i(\beta, \top) \geq s \to \mathbf{w}_i(\gamma \wedge \beta, \top) \geq r \cdot s)$ for all $s \in [0,1]_Q$, by IH
$K_j T \vdash \top \to K_j \Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_i(\beta, \top) > 0)$
$K_j T \vdash \top \to K_j \Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_i(\beta, \top) \geq s \to \mathbf{w}_i(\gamma \wedge \beta, \top) \geq r \cdot s)$ for all $s \in [0,1]_Q$
$K_j T \vdash \Phi_{k+1,\overline{\Theta},\overline{\mathbf{X}}}(\mathbf{w}_i(\beta, \top) > 0)$, $\overline{\Theta} = (\theta_0, \dots, \theta_k, \top), \overline{\mathbf{X}} = (X_1, \dots, X_k, K_j)$
$K_j T \vdash \Phi_{k+1,\overline{\Theta},\overline{\mathbf{X}}}(\mathbf{w}_i(\beta, \top) \geq s \to \mathbf{w}_i(\gamma \wedge \beta, \top) \geq r \cdot s)$ for all $s \in [0,1]_Q$,
$K_j T \vdash \Phi_{k+1,\overline{\Theta},\overline{\mathbf{X}}}(\mathbf{w}_i(\gamma, \beta) \geq r)$, by $R5$
$K_j T \vdash \top \to K_j \Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_i(\gamma, \beta) \geq r)$
$K_j T \vdash K_i \alpha$. □

Our next goal is to prove Lindenbaum's theorem. Before that, we will present several properties about maximal consistent sets with respect to our axiomatic system.

Lemma 1. *Let T be a maximal consistent set of formulas of our logic. Then, T satisfies the following properties:*

- *for every formula α, exactly one of α and $\neg\alpha$ is in T.*
- *T is deductively closed,*

- *$\alpha \wedge \beta \in T$ iff $\alpha \in T$ and $\beta \in T$.*
- *If $\{\alpha, \alpha \to \beta\} \subset T$, then $\beta \in T$,*
- *if $r = \sup\{r' \in [0,1]_Q \mid \mathbf{w}_i(\alpha, \top) \geq r' \in T\}$ and $r \in [0,1]_Q$, then $\mathbf{w}_i(\alpha, \top) \geq r \in T$.*

Proof. Let T be a maximal consistent set of formulas of our logic.

- If both formulas $\alpha, \neg\alpha \in T$, then T would be inconsistent. Suppose $\neg\alpha \notin T$. Since T is maximal consistent set, then $T \cup \{\neg\alpha\}$ is inconsistent. By the Theorem 2 we have $T \vdash \alpha$. Also, if $\alpha \notin T$, similarly we have $T \vdash \neg\alpha$, T is inconsistent.
- Suppose that T is not deductively closed, there is a formula α such that $T \vdash \alpha$ and $\alpha \notin T$. By the previous part we have $T \vdash \neg\alpha$, contradiction.
- Let $\alpha \wedge \beta \in T$, then $T \vdash \alpha \wedge \beta$, and also $T \vdash (\alpha \wedge \beta) \to \alpha$ and $T \vdash (\alpha \wedge \beta) \to \beta$. Using R1 and the previous step, we have $\alpha \in T$ and $\beta \in T$. Let now $\alpha \in T$ and $\beta \in T$. Then, $T \vdash \alpha$, $T \vdash \beta$ and $T \vdash \alpha \wedge \beta$. By second step of this Lemma we have $\alpha \wedge \beta \in T$.
- Let $\{\alpha, \alpha \to \beta\} \subseteq T$, then $T \vdash \alpha$ and $T \vdash \alpha \to \beta$. Similarly as before, we have $\beta \in T$.
- Let $r = \sup\{r' \in [0,1]_Q \mid \mathbf{w}_i(\alpha, \top) \geq r' \in T\}$, thus $T \vdash \mathbf{w}_i(\alpha, \top) \geq s$ for every $s < r$, $s \in [0,1]_Q$. By the rule R4 we have that $T \vdash \mathbf{w}_i(\alpha, \top) \geq r$. By the second step of this Lemma we have $\mathbf{w}_i(\alpha, \top) \geq r \in T$. □

Now we can prove that every consistent theory can be extended to a saturated theory in an extended language. This property, called Lindenbaum's theorem, is crucial for the proof of the Completeness theorem.

Theorem 4 *(Lindenbaum's theorem). Let T be a consistent theory in the language $\mathcal{L}$, and C an infinite enumerable set of new constant symbols such that $\mathcal{L} \cap C = \emptyset$. Then, T can be extended to a saturated theory $T^\star$ in the language $\mathcal{L}^\star = \mathcal{L} \cup C$.*

Proof. Let T be an arbitrary consistent theory in the language $\mathcal{L}$. Assume that $\{\gamma_i \mid i = 0, 1, 2, \dots\}$ is an enumeration of all sentences of CKL^{fo} logic. Let C be an infinite enumerable set of new constant symbols such that $\mathcal{L} \cap C = \emptyset$. We construct the set T^* recursively, in the following way:

1. $T_0 = T$.
2. If the formula γ_i is consistent with T_i, then $T_{i+1} = T_i \cup \{\gamma_i\}$.
3. If the formula γ_i is not consistent with T_i, then:
   (a) If $\gamma_i = (\forall x)\beta(x)$, then for some $c \in C$, which does not occur in any of the formulas from T_i, $T_{i+1} = T_i \cup \{\neg\gamma_i, \neg\beta(c)\}$, such that T_{i+1} is consistent.
   (b) If $\gamma_i = \Phi_{k,\Theta,\mathbf{X}}(f_i \geq r)$ and $f_i = \mathbf{w}_a(\alpha, \beta)$, then
   $$T_{i+1} = T_i \cup \left\{\neg\gamma_i, \neg\Phi_{k,\Theta,\mathbf{X}}(f_i \geq r - \frac{1}{m}), \gamma_i''\right\}$$
   where $\gamma_i'' = \neg\Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_a(\beta, \top) > 0)$, if $T_i \cup \{\neg\Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_a(\beta, \top) > 0\} \nvdash \bot$, otherwise $\gamma_i'' = \neg\Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_i(\beta, \top) \geq s \to \mathbf{w}_a(\alpha \wedge \beta, \top) \geq s \cdot r)$, for some $m \in \mathbb{N}$ and $s \in [0,1]_Q$ such that T_{i+1} is consistent.
   (c) If $\gamma_i = \Phi_{k,\Theta,\mathbf{X}}(f_i \geq r)$ and $f_i \neq \mathbf{w}_a(\alpha, \beta)$ then
   $$T_{i+1} = T_i \cup \left\{\neg\gamma_i, \neg\Phi_{k,\Theta,\mathbf{X}}(f_i \geq r - \frac{1}{m})\right\}$$
   for some $m \in \mathbb{N}$, such that T_{i+1} is consistent.
   (d) Otherwise, $T_{i+1} = T_i \cup \{\neg\gamma_i\}$.
4. $T^* = \bigcup_{n=0}^{\infty} T_n$.

First we will show that the set T^* is correctly defined, i.e., there exist the constant $c \in C$ from step (3a), the number $m \in \mathbb{N}$ exists from (3b) and (3c) and that the rational number s from the step (3b) of the construction exists.

First we consider the case (3a). It is clear that the formula $\neg(\forall x)\beta(x)$ can be added to T_i consistently. If there is some $c \in C$ such that $\neg\beta(c) \in T_i$, the proof is finished. If there is no such c, then observe that T_i is constructed by adding finitely many formulas to T, so there is a constant symbol $c \in C$ which does not appear in T_i. Let us show that we can choose that c in (3a). Assume that $T_i \cup \{\neg(\forall x)\beta(x), \neg\beta(c)\} \vdash \bot$, then by the Deduction theorem, we have

$$T_i, \neg(\forall x)\beta(x) \vdash \beta(c).$$

Note that c does not appear in $T_i \cup \{\neg(\forall x)\beta(x)\}$, so $T_i, \neg(\forall x)\beta(x) \vdash (\forall x)\beta(x)$, which is contradiction.

Now we consider the case (3b). Let us assume now that $T_i' = T_i \cup \{\Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_a(\alpha, \beta) \geq r)\}$ is inconsistent. From Deduction theorem we obtain

$$T_i \vdash \neg\Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_a(\alpha, \beta) \geq r).$$

Suppose that $T_i \cup \{\neg\Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_a(\alpha, \beta) \geq r - \frac{1}{m})\}$ inconsistent for every $m \in \mathbb{N}$. By Theorem 2, we have $T_i \vdash \Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_a(\alpha, \beta) \geq r - \frac{1}{m})$ for every $m \in \mathbb{N}$. Then by the rule R5 we have

$$T_i \vdash \Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_a(\alpha,\beta) \geq r).$$

Contradiction.

Now suppose that the set $T_i' \cup \{\neg\Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_a(\beta,\top) > 0)\}$ is inconsistent, and that the set $T_i' \cup \{\neg\Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_a(\beta,\top) \geq s \to \mathbf{w}_a(\alpha\wedge\beta,\top) \geq s\cdot r)\}$ is inconsistent for every s. By Theorem 2, we obtain that $T_i' \vdash \Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_a(\beta,\top) > 0)$ and $T_i' \vdash \Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_a(\beta,\top) \geq s \to \mathbf{w}_a(\alpha\wedge\beta,\top) \geq s\cdot r)$, for every s. By the rule $R5$ we have

$$T_i' \vdash \Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_a(\alpha,\beta) \geq r).$$

Contradiction. Step (3c) can be shown similarly.

Note that every T_i is consistent by the construction. This still doesn't imply consistency of $T^* = \bigcup_{n=0}^{\infty} T_n$, because of the presence of the infinitary rules. Therefore, we prove that $T^\star$ is deductively closed, using the induction on the length of proof. If the formula γ is an instance of some axiom, then $\gamma \in T^*$ by the construction of T^*. The proof is clear in the case of finitary rules. Here, we will only show that $T^\star$ is closed under the rule R5, since the cases when other infinitary rules are considered can be treated in a similar way.

Let us show that T^* is closed under the inference rule R5. Suppose $T^* \vdash \Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_a(\alpha,\beta) \geq r)$ was obtained by $R5$, where $\Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_a(\beta,\top) > 0) \in T^*$ and $\Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_a(\beta,\top) \geq s \to \mathbf{w}_a(\alpha\wedge\beta,\top) \geq s\cdot r) \in T^*$ for all $s \in [0,1]_Q$. Assume that $\Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_a(\alpha,\beta) \geq r) \notin T^*$. Let j be the positive integer such that $\gamma_j = \Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_a(\alpha,\beta) \geq r)$. Then, $T_j \cup \{\gamma_j\}$ is inconsistent, since otherwise $\Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_a(\alpha,\beta) \geq r) \in T_{j+1} \subset T^*$. By the step (3b) $\neg\Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_a(\beta,\top) > 0) \in T_{j+1}$ or there is $s' \in [0,1]_Q$ such that $\neg\Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_a(\beta,\top) \geq s' \to \mathbf{w}_a(\alpha\wedge\beta,\top) \geq s'r) \in T_{j+1}$. Suppose now that $\neg\Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_a(\beta,\top) > 0) \in T_{j+1}$ and from $\Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_a(\beta,\top) > 0) \in T^*$ there is nonegative integer k such that $\Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_a(\beta,\top) > 0) \in T_k$. Then $T_{\max\{k,j+1\}} \vdash \bot$, a contradiction.

Now suppose that

$$\neg\Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_a(\beta,\top) \geq s' \to \mathbf{w}_a(\alpha\wedge\beta,\top) \geq s'r) \in T_{j+1},$$

where $s' \in [0,1]_Q$. We have that $\Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_a(\beta,\top) \geq s \to \mathbf{w}_a(\alpha\wedge\beta,\top) \geq s\cdot r) \in T^*$ for all $s \in [0,1]_Q$, so we have $\Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_a(\beta,\top) \geq s' \to \mathbf{w}_a(\alpha\wedge\beta,\top) \geq s'r) \in T^*$. Then, there is nonegative integer k' such that

$$\Phi_{k,\Theta,\mathbf{X}}(\mathbf{w}_a(\mathbf{w}_a(\beta,\top) \geq s' \to \mathbf{w}_a(\alpha\wedge\beta,\top) \geq s'r) \in T_k'.$$

Then $T_{\max\{k',j+1\}} \vdash \bot$, a contradiction. Consequently, the set T^* is deductively closed.

From the fact that T^* is deductively closed we can prove that T^* is consistent. Indeed, if T^* is inconsistent, there is $\gamma' \in For$ such that $T^* \vdash \gamma' \wedge \neg\gamma'$. But then there is a nonnegative integer i such that $\gamma' \wedge \neg\gamma' \in T_i$, a contradiction.

Finally, the step (3a) of the construction guarantees that the theory $T^\star$ is saturated in the extended language $\mathcal{L}^\star$. □

Now we will construct a special Kripke structure using saturated theories. First, we need to introduce some notation. For a given set of formulas T and $i \in \mathbf{A}$, we define the set T/K_i as the set of all formulas α, such that $K_i\alpha$ belongs to T, i.e.,

$$T/K_i = \{\alpha \mid K_i\alpha \in T\}.$$

Definition 7 *(Canonical model).* The canonical model M_C is the tuple $(W, D, I, \mathcal{K}, Prob)$ where:

- $W = \{w \mid w \text{ is a saturated theory}\}$,
- D is the set of all variable-free terms;
- $I(w)$ is an interpretation such that:
  - For each function symbol F_j^k, $I(w)(F_j^k)$ is a function from D^k to D such that for all variable-free terms $t_1,\dots,t_k$, $I(w)(F_j^k) : (t_1,\dots,t_k) \to F_j^k(t_1,\dots,t_k)$,
  - For each relational symbol R_j^k, $I(w)(R_j^k) = \{(t_1,\dots,t_k) \mid t_1,\dots,t_k \textit{ are variable-free terms in } R_j^k(t_1,\dots,t_k) \in w\}$;
- $\mathcal{K} = \{\mathcal{K}_i \mid i \in \mathbf{A}\}$ where $\mathcal{K}_i = \{(u,w) \mid u/K_i \subset w\}$, and as before we denote $\mathcal{K}_i(w) = \{w' \mid (w',w) \in \mathcal{K}_i\}$;
- $Prob(i,w) = (W_i(w), H_i(w), \mu_i(w))$ such that:
  - $W_i(w) = \mathcal{K}_i(w)$,
  - $H_i(w) = \{[\![\alpha]\!]_{i,w} \mid \alpha \in Sent\}$ where $[\![\alpha]\!]_{i,w} = \{w' \in \mathcal{K}_i(w) \mid \alpha \in w'\}$,
  - $\mu_i(w) : H_i(w) \to [0,1]$ such that $\mu_i(w)([\![\alpha]\!]_{i,w}) = \sup\{r \in [0,1]_Q \mid \mathbf{w}_i(\alpha,\top) \geq r \in w\}$.

We will write $[\![\alpha]\!]$ instead of $[\![\alpha]\!]_{i,w}$ if i and w are clear from the context. Next we want to show that $M_C \in \mathrm{CKL}^{\mathrm{fo}}_{\mathrm{Meas}}$. We start by proving that M_C is an CKL^{fo}-structure.

Lemma 2. *The canonical model M_C is an CKL^{fo}-structure.*

Proof. Here we will show that $Prob(i,w)$ is a probability space, since the rest of the proof is trivial. First we show that $H_i(w)$ is an algebra of subsets of $W_i(w)$.

We know that $W_i(w) = \mathcal{K}_i(w)$. We have $[\![\alpha\vee\neg\alpha]\!]_{i,w} = \{w' \in \mathcal{K}_i(w) \mid \alpha\vee\neg\alpha \in w'\} = \mathcal{K}_i(w)$, so from the definition of $H_i(w)$ we have that $[\![\alpha\vee\neg\alpha]\!]_{i,w} = \mathcal{K}_i(w) \in H(w)$. Also, if $[\![\alpha]\!]_{i,w}, [\![\beta]\!]_{i,w} \in H_i(w)$ then $[\![\alpha]\!]_{i,w} \cup [\![\beta]\!]_{i,w} = (\{w' \in \mathcal{K}_i(w) \mid \alpha \in w'\}) \cup (\{w' \in \mathcal{K}_i(w) \mid \beta \in w'\}) = \{w' \in \mathcal{K}_i(w) \mid \alpha\vee\beta \in w'\} = [\![\alpha\vee\beta]\!]_{i,w} \in H_i(w)$. Similarly, $\mathcal{K}_i(w)/[\![\alpha]\!]_{i,w} = \mathcal{K}_i(w)/\{w' \in \mathcal{K}_i(w) \mid \alpha \in w'\}$, therefore

$\mathcal{K}_i(w)/[\![\alpha]\!]_{i,w} = \{w' \in \mathcal{K}_i(w) \mid \neg\alpha \in w'\} = [\![\neg\alpha]\!]_{i,w} \in H_i(w)$.

Next we show that $\mu(w)$ is correctly defined. First we show that if $[\![\alpha]\!]_{i,w} = [\![\beta]\!]_{i,w}$ then $\sup\{r \in [0,1]_Q \mid \mathbf{w}_i(\alpha,\top) \geq r \in w\} = \sup\{r \in [0,1]_Q \mid \mathbf{w}_i(\beta,\top) \in w\}$, in other words, $\mu_i(w)([\![\alpha]\!]_{i,w}) = \mu_i(w)([\![\beta]\!]_{i,w})$.

We prove that if $[\![\alpha]\!]_{i,w} \subseteq [\![\beta]\!]_{i,w}$ then $\mu(w)([\![\alpha]\!]_{i,w}) \leq \mu(w)([\![\beta]\!]_{i,w})$. From $[\![\alpha]\!]_{i,w} \subseteq [\![\beta]\!]_{i,w}$ we have that $w' \vdash \neg(\alpha \wedge \neg\beta)$ for every $w' \in \mathcal{K}_i(w)$. Using R3 and A20 we obtain $w' \vdash \mathbf{w}_i(\alpha \to \beta, \top) \geq 1$ for every $w' \in \mathcal{K}_i(w)$. Now, we must show that if $\mathbf{w}_i(\alpha,\top) \geq r \in w'$ then $\mathbf{w}_i(\beta,\top) \geq r \in w'$, for every $w' \in \mathcal{K}_i(w)$, i.e., $\mu_i(w)([\![\alpha]\!]_{i,w}) \leq \mu_i(w)([\![\beta]\!]_{i,w})$. It is sufficient to show that $w' \vdash \mathbf{w}_i(\alpha \to \beta,\top) \geq 1 \to (\mathbf{w}_i(\alpha,\top) \geq r \to \mathbf{w}_i(\beta,\top) \geq r)$ for all $w' \in \mathcal{K}_i(w)$. Suppose now that $w' \nvdash \mathbf{w}_i(\alpha \to \beta,\top) \geq 1 \to (\mathbf{w}_i(\alpha,\top) \geq r \to \mathbf{w}_i(\beta,\top) \geq r)$ for some $w' \in \mathcal{K}_i(w)$. Using Lemma 1 and A7 we can obtain $w' \vdash \mathbf{w}_i(\neg\alpha \vee \beta,\top) \geq 1 \wedge \mathbf{w}_i(\neg\alpha,\top) \leq 1 - r \wedge \mathbf{w}_i(\beta,\top) < r$.

Now we show that $w' \vdash (\mathbf{w}_i(\alpha,\top) \leq r \wedge \mathbf{w}_i(\beta,\top) < t) \to \mathbf{w}_i(\alpha \vee \beta,\top) < r + t$ for every $w' \in \mathcal{K}_i(w)$. Suppose that there is a world $w'' \vdash \mathbf{w}_i(\alpha,\top) \leq r \wedge \mathbf{w}_i(\beta,\top) < t \wedge \neg\mathbf{w}_i(\alpha \vee \beta,\top) < r + t$. By the axiom A11 and Lemma 1 it is

$$w'' \vdash \mathbf{w}_i(\alpha,\top) + \mathbf{w}_i(\beta,\top) < r + t$$

and $w'' \vdash \mathbf{w}_i(\neg(\alpha \vee \beta),\top) \leq 1 - r - t$. Using A16 twice we have $w'' \vdash \mathbf{w}_i(\neg\alpha\top) - \mathbf{w}_i(\beta,\top) + \mathbf{w}_i(\alpha \wedge \beta) \leq 1 - r - t$. It is easy to see (by A9, A11 and A14) $\vdash (\mathbf{w}_i(\alpha,\top) + \mathbf{w}_i(\beta,\top) \leq r) \to \mathbf{w}_i(\alpha,\top) \leq r$. Using that we obtain

$$w'' \vdash \mathbf{w}_i(\neg\alpha,\top) - \mathbf{w}_i(\beta,\top) \leq 1 - r - t.$$

Therefore, by A11, $w'' \vdash \mathbf{w}_i(\alpha,\top) + \mathbf{w}_i(\neg\alpha,\top) < 1$, a contradiction. Hence, it is

$$w' \vdash (\mathbf{w}_i(\alpha,\top) \leq r \wedge \mathbf{w}_i(\beta,\top) < t) \to \mathbf{w}_i(\alpha \vee \beta,\top) < r + t$$

for every $w' \in \mathcal{K}_i(w)$. So we have $w' \vdash (\mathbf{w}_i(\neg\alpha,\top) \leq 1 - r \wedge \mathbf{w}_i(\beta,\top) < r) \to \mathbf{w}_i(\neg\alpha \vee \beta,\top) < r$, i.e., $w'' \vdash \neg((\mathbf{w}_i(\alpha \to \beta),\top) \geq 1 \to (\mathbf{w}_i(\alpha,\top) \geq r \to \mathbf{w}_i(\beta,\top) \geq r)) \to \mathbf{w}_i(\neg\alpha \vee \beta,\top) \geq 1 \wedge \neg\mathbf{w}_i(\neg\alpha \vee \beta) \geq 1)$, a contradiction. Thus, $w' \vdash \mathbf{w}_i(\alpha \to \beta,\top) \geq 1 \to (\mathbf{w}_i(\alpha,\top) \geq r \to \mathbf{w}_i(\beta,\top) \geq r)$ for all $w' \in \mathcal{K}_i(w)$.

Therefore, $\mu(w)([\![\alpha]\!]_{i,w}) \leq \mu(w)([\![\beta]\!]_{i,w})$.

Next we show that $\mu(w)$ is a finitely additive probability measure.

Since $\mathbf{w}_i(\alpha,\top) \geq 0$ is an axiom, we have $\mu(w)([\![\alpha]\!]_{i,w}) \geq 0$.

To show that $\mu(w)([\![\alpha]\!]_{i,w} \cup [\![\beta]\!]_{i,w}) = \mu(w)([\![\alpha]\!]_{i,w}) + \mu(w)([\![\beta]\!]_{i,w})$ for all disjoint $[\![\alpha]\!]_{i,w}$ and $[\![\beta]\!]_{i,w}$ first we prove that $\mu(w)([\![\alpha]\!]_{i,w}) = 1 - \mu(w)([\![\neg\alpha]\!]_{i,w})$.

Let $r = \mu([\![\alpha]\!]_{i,w}) = \sup\{s \in [0,1]_Q \mid \mathbf{w}_i(\alpha,\top) \geq s \in w\}$. Suppose that $r = 1$ so $\mathbf{w}_i(\alpha,\top) \geq 1 \in w$. Thus, $\neg\mathbf{w}_i(\neg\alpha,\top) > 0 \in w$. If for some $r > 0$, $\mathbf{w}_i(\neg\alpha,\top) > r \in w$, it must be $\mathbf{w}_i(\neg\alpha,\top) > 0 \in w$ and it is a contradiction. It follows that $\mu(w)([\![\neg\alpha]\!]_{i,w}) = 0$. Suppose now that $r < 1$. Then, for every rational number $r' \in (r,1]_Q$,

$$\neg\mathbf{w}_i(\alpha,\top) \geq r' = \mathbf{w}_i(\alpha,\top) < r' \in w.$$

By Axiom A14 we get $\mathbf{w}_i(\alpha,\top) \leq r' \in w$ and $\mathbf{w}_i(\neg\alpha,\top) \geq 1 - r' \in w$. Also, if there is a rational number $r'' \in [0,r)_Q$ such that $\mathbf{w}_i(\neg\alpha,\top) \geq 1 - r'' \in w$ then $\neg\mathbf{w}_i(\alpha,\top) > r'' \in w$, a contradiction. Hence, $\sup\{r \in [0,1]_Q \mid \mathbf{w}_i(\neg\alpha,\top) \geq r \in w\} = 1 - \sup\{r \in [0,1]_Q \mid \mathbf{w}_i(\alpha,\top) \geq r \in w\}$, i.e., $\mu(w)([\![\alpha]\!]_{i,w}) = 1 - \mu(w)([\![\neg\alpha]\!]_{i,w})$.

Now, let $[\![\alpha]\!]_{i,w} \cap [\![\beta]\!]_{i,w} = \emptyset$, $\mu(w)([\![\alpha]\!]_{i,w}) = r$ and $\mu(w)([\![\beta]\!]_{i,w}) = s$. Since $[\![\beta]\!]_{i,w} \subseteq [\![\neg\alpha]\!]_{i,w}$, by the above steps we have $r + s \leq r + (1 - r) = 1$. Suppose that $r > 0$ and $s > 0$, then for every $r' \in [0,r]_Q$ and every $s' \in [0,s]_Q$ we have $\mathbf{w}_i(\alpha,\top) \geq r'$ and $\mathbf{w}_i(\beta,\top) \geq s'$ are in w. It follows, from the previous part, that $\mathbf{w}_i(\alpha \vee \beta,\top) \geq r' + s' \in w$. Hence,

$$r + s \leq t_0 = \sup\{t \in [0,1]_Q \mid \mathbf{w}_i(\alpha \vee \beta,\top) \geq t \in w\}.$$

If $r + s = 1$, then the statement trivially holds. Suppose that $r + s < 1$. If $r + s < t_0$ then for every $t' \in (r + s, t_0)_Q$ we have $\mathbf{w}_i(\alpha \vee \beta,\top) \geq t' \in w$. We can choose rational numbers $r'' > r$ and $s'' > s$ such that $\neg\mathbf{w}_i(\alpha,\top) \geq r'', \mathbf{w}_i(\alpha,\top) < r'' \in w$, $\neg\mathbf{w}_i(\beta,\top) \geq s'', \mathbf{w}_i(\beta,\top) < s'' \in w$ and $r'' + s'' = t' \leq 1$. Using A14 we have $\mathbf{w}_i(\alpha,\top) \leq r'' \in w$. As we proved before, we have $\mathbf{w}_i(\alpha \vee \beta,\top) < r'' + s'' \in w$, $\neg\mathbf{w}_i(\alpha \vee \beta,\top) \geq r'' + s'' \in w$ and $\neg\mathbf{w}_i(\alpha \vee \beta,\top) \geq t' \in w$, a contradiction. Hence $r + s = t_0$ and

$$\mu(w)([\![\alpha]\!]_{i,w} \cup [\![\beta]\!]_{i,w}) = \mu(w)([\![\alpha]\!]_{i,w}) + \mu(w)([\![\beta]\!]_{i,w}).$$

Finally, suppose that $r = 0$ or $s = 0$, then we can do the same as above with the only exception that $r' = 0$ or $s' = 0$. $\square$

Now we prove that the condition from the last item of Definition 7 naturally extends to all probabilistic terms.

Lemma 3. *Let w be a world in the canonical model M_C. If a probabilistic term $f_i = a_1\mathbf{w}_i(\alpha_1,\alpha_1') + \cdots + a_k\mathbf{w}_i(\alpha_k,\alpha_k')$ then $a_1\mu_i(w)([\![\alpha_1]\!] \mid [\![\alpha_1']\!]) + \cdots + a_k\mu_i(w)([\![\alpha_k]\!] \mid [\![\alpha_k']\!]) = \sup\{s \mid w \vdash f_i \geq s\}$.*

Proof. First we will show that

$$\mu_i(w)([\![\alpha]\!] \mid [\![\beta]\!]) = \sup\{r \in [0,1]_Q \mid \mathbf{w}_i(\alpha,\beta) \geq r \in w\}.$$

If $\mu_i(w)([\![\beta]\!]) = 0$ then $\mu_i(w)([\![\alpha]\!] \mid [\![\beta]\!])$ and $\sup\{r \in [0,1]_Q \mid \mathbf{w}_i(\alpha,\beta) \geq r \in w\}$ are not defined.

Suppose that $\mathbf{w}_i(\alpha,\beta) \geq r \in w$ and let $\{s_n \mid n \in \mathbb{N}\}$ be strictly increasing sequence of numbers from $[0,1]_Q$, such that $\lim_{n\to\infty} s_n = \mu_i(w)([\![\beta]\!])$. Let n be any number from $\mathbb{N}$. Then $w \vdash \mathbf{w}_i(\beta,\top) \geq s_n$. Using the assumption $\mathbf{w}_i(\alpha,\beta) \geq r \in w$, the axioms $A18$ and $A19$ and propositional reasoning, we obtain $w \vdash \mathbf{w}_i(\beta,\top) > 0$ and $w \vdash \mathbf{w}_i(\alpha\wedge\beta,\top) \geq r\cdot s_n$. Finally, by Definition 7 we have $\mu_i(w)([\![\beta]\!]) > 0$ and $\mu_i(w)([\![\alpha\wedge\beta]\!]) \geq \lim_{n\to\infty} rs_n = r\mu_i(w)([\![\beta]\!])$, i.e., $\mu_i(w)([\![\beta]\!]) > 0$ and $\mu_i(w)([\![\alpha]\!]|[\![\beta]\!]) \geq r$. We can conclude that

$$\mu_i(w)([\![\alpha]\!]|[\![\beta]\!]) \geq \sup\{r \in [0,1]_Q \mid \mathbf{w}_i(\alpha,\beta) \geq r \in w\}.$$

Let now $\mu_i(w)([\![\alpha]\!]|[\![\beta]\!]) \geq t$ and $\mu_i(w)([\![\beta]\!]) > 0$. We want to show that $w \vdash \mathbf{w}_i(\beta,\top) > 0$ and $w \vdash \mathbf{w}_i(\beta,\top) \geq s \to \mathbf{w}_i(\alpha\wedge\beta,\top) \geq ts$ for all $s \in [0,1]_Q$.

If $w \nvdash \mathbf{w}_i(\beta,\top) > 0$ then $w \vdash \mathbf{w}_i(\beta,\top) = 0$, i.e., $\mu_i(w)([\![\beta]\!]) = 0$, contradiction.

If $s > \mu_i(w)([\![\beta]\!])$, then $w \vdash \neg(\mathbf{w}_i(\beta,\top) \geq s)$, so $w \vdash \mathbf{w}_i(\beta,\top) \geq s \to \mathbf{w}_i(\alpha\wedge\beta,\top) \geq ts$. Let now $s \leq \mu_i(w)([\![\beta]\!])$, then $st \leq \mu_i(w)([\![\alpha\wedge\beta]\!])$, so $w \vdash \mathbf{w}_i(\alpha\wedge\beta,\top) \geq ts$. Now, we have that for every $s \in [0,1]_Q$, $w \vdash \mathbf{w}_i(\beta,\top) \geq s \to \mathbf{w}_i(\alpha\wedge\beta,\top) \geq ts$, by the rule R5 we get $w \vdash \mathbf{w}_i(\alpha,\beta) \geq t$. So

$$\mu_i(w)([\![\alpha]\!]|[\![\beta]\!]) \leq \sup\{r \in [0,1]_Q \mid \mathbf{w}_i(\alpha,\beta) \geq r \in w\}.$$

Let $f_i = a_1\mathbf{w}_i(\alpha_1,\alpha'_1) + \cdots + a_k\mathbf{w}_i(\alpha_k,\alpha'_k)$. Using the properties of supremum and A11 and A12 we obtain $a_1\mu_i(w)([\![\alpha_1]\!]|[\![\alpha'_1]\!]) + \cdots + a_k\mu_i(w)([\![\alpha_k]\!]|[\![\alpha'_k]\!]) = a_1 \sup\{s_1 \mid w \vdash \mathbf{w}_i(\alpha_1,\alpha'_1) \geq s_1\} + \cdots + a_k \sup\{s_k \mid w \vdash \mathbf{w}_i(\alpha_k,\alpha'_k) \geq s_k\} = \sup\{s \mid w \vdash f_i \geq s\}$. □

Next we prove that truth lemma holds for CKL^{fo}.

Lemma 4 *(Truth lemma). Let M_C be the canonical model and $\gamma \in For$. Then for every world w from M_C, $\gamma \in w$ iff $M_C, w \models \gamma$.*

Proof. We use induction on the complexity of the formula γ. If γ is an atomic formula, the statement follows from the construction of $I(w)$ in M_C. The cases when γ is a conjunction or a negation are straightforward.

Suppose $\gamma = K_i\beta$. Let $K_i\beta \in w$. Since $\beta \in w/K_i$, then $\beta \in w'$ for every w' such that $(w,w') \in \mathcal{K}_i$ (by the definition of $\mathcal{K}_i$). Therefore, $M_C, w \models \beta$ by induction hypothesis (β is subformula of $K_i\beta$), and then $M_C, w \models K_i\beta$.

Let now $M_C, w \models K_i\beta$. Assume the opposite, that $K_i\beta \notin w$. Then, $w/K_i \cup \{\neg\beta\}$ must be consistent. If it would not be consistent, then $w/K_i \vdash \beta$ by the Deduction theorem and $w \supset K_i(w/K_i) \vdash K_i\beta$ by Theorem 3, i.e., $K_i\beta \in w$, which is a contradiction. Therefore, $w/K_i \cup \{\neg\beta\}$ can be extended to a maximal consistent U, so $w\mathcal{K}_iU$. Since $\neg\beta \in U$, then $M_C, U \models \neg\beta$ by induction hypothesis, so we get the contradiction $M_C, w \not\models K_i\beta$.

Let $f_i = a_1\mathbf{w}_i(\alpha_1,\alpha'_1) + \cdots + a_k\mathbf{w}_i(\alpha_k,\alpha'_k)$. We suppose that $f_i \geq r \in w$, then $r \leq \sup\{s \mid w \vdash f_i \geq s\}$ and $\mathbf{w}_i(\alpha'_j,\top) > 0 \in w$ for every $j \in \{1,\ldots,k\}$. Then by Lemma 3, $M_C, w \models f_i \geq r$.

For the other direction, assume that $M_C, w \models f_i \geq r$. Suppose that $f_i \geq r \notin w$. Then we have $\mathbf{w}_i(\alpha'_j,\top) = 0 \in w$ for some $j \in \{1,\ldots,k\}$ or $f_i < r \in w$. If $\mathbf{w}_i(\alpha'_j,\top) = 0$ for some j then $M_C, w \not\models f_i \geq r$, a contradiction. Let $f_i < r \in w$, then, by the first part of this proof and abbreviations we conclude $M_C, w \models f_i < r$, a contradiction. □

Hence, we have shown that for every formula $\alpha \in For$, every agent $i \in \mathbb{A}$ and every world w from M_C the equality $[\alpha]_{i,M_C,w} = [\![\alpha]\!]$ holds, we have the following corollary.

Corollary 1. *The canonical model M_C is an CKL^{fo}-measurable structure which is a model of every consistent set T.*

Theorem 5 *(Strong completeness of CKL^{fo}). A set of formulas T is consistent iff T is $\mathrm{CKL}^{\mathrm{fo}}_{\mathrm{Meas}}$-satisfiable.*

Proof. Note that the direction from right to left follows from Theorem 1. For the other direction, suppose that T is a consistent set of formulas. By Theorem 4, there is a saturated superset T^* of T. From the previous corollary we have that $M_C \in \mathrm{CKL}^{\mathrm{fo}}_{\mathrm{Meas}}$, so we only need to show that M_C is a model of T^*. By Lemma 4, if T is consistent set we know that T^* is a world in M_C, so we obtain $M_C, T^* \models T$. □

5. The logic CKL

In this section, we present the logic CKL which is the propositional restriction of the logic CKL^{fo}. We briefly present its syntax and semantic, and sketch the proof of completeness. We avoid repetition of some technical details that were already presented in detail in the first order case. We also show that the logic CKL is decidable.

Let $\mathcal{P} = \{p,q,r,\ldots\}$ be a denumerable set of propositional letters and as before let $\mathbf{A}$ be a finite set of agents.

The set For of all formulas of the logic is the smallest set such that:

- $\mathcal{P} \subset For$;
- If $\alpha \in For$ then $K_i\alpha \in For$.
- For any $i \in \mathbf{A}$ and $k \geq 1$, if $\alpha_1,\alpha'_1,\ldots,\alpha_k,\alpha'_k \in For$ and $a_1,\ldots,a_k,r \in \mathbb{Q}$, then $a_1\mathbf{w}_i(\alpha_1,\alpha'_1) + \cdots + a_k\mathbf{w}_i(\alpha_k,\alpha'_k) \geq r \in For$,
- If α and β are formulas then $\neg\alpha, \alpha\wedge\beta \in For$.

We define the semantics of the logic as follows:

Definition 8 *(CKL-structure).* A CKL-structure is a tuple $(W, \mathcal{K}, Prob, v)$ where:

1. W is a non-empty set of objects called *worlds*.
2. $v : W \times \mathcal{P} \to \{true, false\}$ assigns to each world $u \in W$ a two-valued evaluation $v(u, \cdot)$ of propositional letters,
3. $\mathcal{K} = \{\mathcal{K}_i \mid i \in \mathbf{A}\}$ is a set of binary equivalence relations on W. We denote $\mathcal{K}_i(u) = \{u' \mid (u', u) \in \mathcal{K}_i\}$, and write $u\mathcal{K}_i u'$ if $u' \in \mathcal{K}_i(u)$,
4. $Prob$ assigns to every $i \in \mathbf{A}$ and $u \in W$ a probability space $Prob(i, u) = (W_i(u), H_i(u), \mu_i(u))$, where
   - $W_i(u)$ is a non-empty subset of W,
   - $H_i(u)$ is an algebra of subsets of $W_i(u)$,
   - $\mu_i(u) : H_i(u) \longrightarrow [0, 1]$ is a finitely additive measure.

We define satisfiability similarly as in Definition 3 and, as before, we denote by $[\alpha]_{i,M,u}$ the set of all worlds from $W_i(u)$ in which α holds. Here will also consider only CKL-measurable structures.

5.1. *Decidability of* CKL

In this subsection we prove that the logic CKL is decidable. Recall the satisfiability problem: given an CKL-formula α, we want to determine if there exists a world w in an $\mathrm{CKL}_{\mathrm{Meas}}$-model M such that $M, w \vDash \alpha$. First, we show that an CKL-formula is satisfiable iff it is satisfiable in a measurable structure with a finite number of worlds.

For a formula α we denote $Subf(\alpha)$ the set of all subformulas of α.

Theorem 6. *If an* CKL*-formula α is satisfiable in a model $M \in \mathrm{CKL}_{\mathrm{Meas}}$, then it is satisfied in a model $M^* \in \mathrm{CKL}_{\mathrm{Meas}}$ with at most $2^{|Subf(\alpha)|}$ number of worlds.*

Proof. Let $M = (W, \mathcal{K}, Prob, v)$ be a CKL structure, with $Prob(i, u) = (W_i(u), H_i(u), \mu_i(u))$, and let s be a world from M such that $M, s \vDash \alpha$. Let $Subf(\alpha)$ be the set of all subformulas of α and $k = |Subf(\alpha)|$. By $\sim$ we denote the equivalence relation over $W \times W$, where $s \sim s'$ iff for every $\beta \in Subf(\alpha)$, $M, s \vDash \beta$ iff $M, s' \vDash \beta$. The quotient set $W_{/\sim}$ is finite and $|W_{/\sim}| \leq 2^{|Subf(\alpha)|}$. Now, for every class C_i we choose an element and denote it s_i^*. We consider the model $M^* = (W^*, \mathcal{K}^*, Prob^*, v^*)$, where:

- $W^* = \{s_i^* \mid C_i \in W_{/\sim}\}$,
- $\mathcal{K}^* = \{\mathcal{K}_a^* \mid a \in \mathbf{A}\}$ is a set of binary relations on W^* where $(s_i^*, s_j^*) \in \mathcal{K}_a^*$ iff for every $K_a\phi \in Subf(\alpha)$, $M, s_i^* \vDash K_a\phi$ iff $M, s_j^* \vDash K_a\phi$
- For every agent a and $s_i^* \in W^*$, $Prob^*(a, s_i^*) = (W_a^*(s_i^*), H_a^*(s_i^*), \mu_a^*(s_i^*))$ is defined as follows:
  - $W_a^*(s_i^*) = \{s_j^* \in W^* \mid (\exists u \in C_j) u \in W_a(s_i)\}$,
  - $H_a^*(s_i^*)$ is the power set of $W_a^*(s_i^*)$,
  - $\mu_a^*(s_i^*)(\{s_j^*\}) = \mu_a(s_i^*)(C_j(s_i^*))$, where $C_j(s_i^*) = C_j \cap W_a(s_i^*)$ and for any $D \in H_a^*(s_i^*)$, $\mu_a^*(s_i^*)(D) = \sum_{s_j^* \in D} \mu_a^*(s_i^*)(\{s_j^*\})$,
- $v^*(s_i, p) = v(s_i, p)$.

It is straightforward to show that $M^* \in \mathrm{CKL}_{\mathrm{Meas}}$. Now, we want to prove that for any $\beta \in Subf(\alpha)$, $M, s \vDash \beta$ iff $M^*, s_i^* \vDash \beta$ where s_i^* represents C_s in M^*. The proof is by induction on the complexity of the formulas. Let us briefly consider the cases when $\beta = K_a\alpha'$ and $\beta = \sum_{k=1}^n a_k \mathbf{w}_a(\alpha_k', \beta_k') \geq r$, while the proofs for the other cases are similar. In the case when $\beta = K_a\alpha'$ we have:

$M, s \vDash \beta$

iff $M, s_i^* \vDash \beta$ *(since* $s_i^* \in C_s$*)*

iff $M, u \vDash \alpha'$ *for all* $u \in K_a(s_i^*)$

iff $M^*, u^* \vDash \alpha'$ *for all* $u^* \in K_a^*(s_i^*)$ (IH)

iff $M^*, s_i^* \vDash \beta$.

For the case when $\beta = \sum_{k=1}^n a_k \mathbf{w}_a(\alpha_k', \beta_k') \geq r$ we have:

$M, s \vDash \beta$

iff $M, s_i^* \vDash \beta$

$$\textit{iff } \mu_a(s_i^*)([\beta_k']) > 0 \textit{ for all } k \in \{1, \dots, n\} \quad \textit{and} \quad \sum_{k=1}^n a_k \frac{\mu_a(s_i^*)([\alpha_k' \wedge \beta_k'])}{\mu_a(s_i^*)([\beta_k'])} \geq r$$

$$\textit{iff} \sum_{C_u(s_i^*) : M, u \vDash \beta_k'} \mu_a(s_i^*)(C_u(s_i^*)) > 0 \textit{ for all } k \in \{1, \dots, n\} \textit{ and}$$

$$\sum_{k=1}^{n} a_k \frac{\sum_{C_u(s_i^*):M,u\models\alpha_k'\wedge\beta_k'} \mu_a(s_i^*)(C_u(s_i^*))}{\sum_{C_u(s_i^*):M,v\models\beta_k'} \mu_a(s_i^*)(C_u(s_i^*))} \geq r$$

iff $\sum_{C_u(s_i^*):M^*,u\models\beta_k'} \mu_a^*(s_i^*)(\{u\}) > 0$ *for all* $k \in \{1,\dots,n\}$ *and*

$$\sum_{k=1}^{n} a_k \frac{\sum_{C_u(s_i^*):M^*,u\models\alpha_k'\wedge\beta_k'} \mu_a^*(s_i^*)(\{u\})}{\sum_{C_u(s_i^*):M^*,u\models\beta_k'} \mu_a^*(s_i^*)(\{u\})} \geq r$$

iff $\mu_a^*(s_i^*)([\beta_k']) > 0$ *for all* $k \in \{1,\dots,n\}$ *and* $\sum_{k=1}^{n} a_k \frac{\mu_a^*(s_i^*)([\alpha_k' \wedge \beta_k'])}{\mu_a^*(s_i^*)([\beta_k'])} \geq r$

iff $M^*, s_i^* \models \beta$ $\square$

Note that there are infinitely many finite models from $\mathrm{CKL}_{\mathrm{Meas}}$ with at most $2^{|Subf(\alpha)|}$ worlds, because there are infinitely many possibilities for real-valued probabilities. Thus, the previous theorem does not directly imply decidability. In order to show decidability, we use the previous theorem and we translate the problem of satisfiability of a formula to the problem of satisfiability of finite sets of equations and inequalities.

Theorem 7. *Satisfiability problem for* CKL *is decidable.*

Proof. Let α be an CKL-formula. We want to check whether there is a $\mathrm{CKL}_{\mathrm{Meas}}$-structure M and a world s from M such that $M, s \models \alpha$. Using the previous theorem, we will consider only the structures with l worlds, where $l \leq 2^{|Subf(\alpha)|}$.

The idea is to see is there any structure with at least l worlds whom we can join a valuation, a set of binary equivalence relations and finitely additive probabilities such that the formula α is satisfied in some world of the structure. For this we will use potential structures which we call pre-structures. In pre-structures we do not specify probability measures (in order to avoid infinitely many cases), but we want to specify enough information about measures from which we can determine satisfiability of all subformulas of α.

Let $Subf(\alpha)$ be the set of subformulas of α, let $\mathcal{P}^\alpha = \mathcal{P} \cap Subf(\alpha)$ and let $SubP(\alpha)$ be the set of all subformulas of α of the form $\sum_{k=1}^{n} a_k \mathbf{w}_i(\alpha_k, \beta_k) \geq r$. For every $l \leq 2^{|Subf(\alpha)|}$ we consider pre-structures $\overline{M} = (\overline{W}, \overline{\mathcal{K}}, \overline{S}, \overline{v})$ such that:

- $\overline{W}$ is a set of worlds such that $|\overline{W}| = l$
- $\overline{v} : \overline{W} \times \mathcal{P}^\alpha \to \{true, false\}$.
- $\overline{\mathcal{K}} = \{\overline{\mathcal{K}}_a \mid a \in \mathbf{A}\}$ on $\overline{W}$.
- $\overline{S} : \overline{W} \times SubP(\alpha) \to \{true, false\}$.

Note that for every number l we have finitely many possibilities for the choice of pre-structures, i.e., we have finite number of choices of valuation, binary equivalence relations and function $\overline{S}$. This pre-structure is not a CKL-structure, but we can check if a subformula of α holds in a world of a pre-structure $\overline{M}$ using the relation $\Vdash$, defined as follows:

1. If $\gamma \in \mathcal{P}^\alpha$ then $\overline{M}, s \Vdash \gamma$ iff $\overline{v}(s, \gamma) = true$,
2. $\overline{M}, s \Vdash \overline{K}_a\gamma$ iff $\overline{M}, s' \Vdash \gamma$ for all $w' \in \overline{\mathcal{K}}_a(s)$,
3. $\overline{M}, s \Vdash \sum_{k=1}^{n} a_k \mathbf{w}_a(\gamma_k, \beta_k) \geq r$ iff $\overline{S}(s, \sum_{k=1}^{n} a_k \mathbf{w}_a(\gamma_k, \beta_k) \geq r) = true$
4. $\overline{M}, s \Vdash \neg\gamma$ iff $\overline{M}, s \nVdash \gamma$,
5. $\overline{M}, s \Vdash \gamma \wedge \beta$ iff $\overline{M}, s \Vdash \gamma$ and $\overline{M}, s \Vdash \gamma$.

We will consider only those $\overline{M} = (\overline{W}, \overline{\mathcal{K}}, \overline{S}, \overline{v})$ such that $\overline{M}, s \Vdash \alpha$ for some world $s \in \overline{W}$. For each such $\overline{M}$ we want to check whether $\overline{M}$ can be extended to a structure, i.e., whether there is a measurable structure $M = (\overline{W}, \overline{\mathcal{K}}, Prob, v)$ such that $\overline{v}$ is a restriction of v and for every agent a and every $s \in \overline{W}$ and $\sum_{k=1}^{n} a_k \mathbf{w}_a(\gamma_k, \beta_k) \geq r \in SubP(\alpha)$ we have $M, s \models \sum_{k=1}^{n} a_k \mathbf{w}_a(\gamma_k, \beta_k) \geq r$ iff $\overline{S}(s, \sum_{k=1}^{n} a_k \mathbf{w}_a(\gamma_k, \beta_k) \geq r) = true$. It is straightforward to check that for such M we have $M, s \models \beta$ iff $\overline{M}, s \Vdash \beta$ holds for every $\beta \in Subf(\alpha)$. Since the way v extends $\overline{v}$ is irrelevant, it suffices to check whether $\overline{S}$ can be replaced with $Prob$ in some $\overline{M} = (\overline{W}, \overline{\mathcal{K}}, \overline{S}, \overline{v})$ such that $\overline{M}, s \Vdash \alpha$ for some world $s \in \overline{W}$. For that purpose, for each such $\overline{M}$ we consider specific equations and inequalities, that we describe below. We chose the variables of the form y_{a,s_i,s_j} which represent the values $\mu_a(s_i)(\{s_j\})$.

Now we state the equations and inequalities:

(1) $y_{a,s_i,s_j} \geq 0$, *for every world* s_j

(2) $\sum_{s_j \in \overline{M}} y_{a,s_i,s_j} = 1$

(3) $\sum_{s_j : \overline{M}, s_j \Vdash \beta_k} y_{a,s_i,s_j} > 0$ *for every* $k \in \{1,\dots,n\}$, *and*

$$\sum_{k=1}^{n}\Big(a_k \sum_{s_j:\overline{M},s_j\Vdash\beta_k\wedge\gamma_k} y_{a,s_i,s_j} \prod_{t\neq k,t=1}^{n} \sum_{s_j:\overline{M},s_j\Vdash\beta_t} y_{a,s_i,s_j}\Big) \geq$$

$$r\prod_{k=1}^{n} \sum_{s_j:\overline{M},s_j\Vdash\beta_k} y_{a,s_i,s_j}, \textit{ for every formula } \sum_{k=1}^{n} a_k \mathbf{w}_a(\gamma_k,\beta_k) \geq r$$

$$\textit{such that } \overline{S}(s_i, \sum_{k=1}^{n} a_k \mathbf{w}_a(\gamma_k,\beta_k) \geq r) = true$$

$$(4) \bigvee_{k=1}^{n}\Big(\sum_{s_j:\overline{M},s_j\Vdash\beta_k} y_{a,s_i,s_j} = 0\Big) \textit{ or}$$

$$\sum_{k=1}^{n}\Big(a_k \sum_{s_j:\overline{M},s_j\Vdash\beta_k\wedge\gamma_k} y_{a,s_i,s_j} \prod_{t\neq k,t=1}^{n} \sum_{s_j:\overline{M},s_j\Vdash\beta_t} y_{a,s_i,s_j}\Big) <$$

$$r\prod_{k=1}^{n} \sum_{s_j:\overline{M},s_j\Vdash\beta_k} y_{a,s_i,s_j}, \textit{ for every formula } \sum_{k=1}^{n} a_k \mathbf{w}_a(\gamma_k,\beta_k) \geq r$$

$$\textit{such that } \overline{S}(s_i, \sum_{k=1}^{n} a_k \mathbf{w}_a(\gamma_k,\beta_k) \geq r) = false$$

The inequality (1) above assures that all the probability measures are non-negative, and the equality (2) states that the probability of the set of all possible worlds has to be equal to 1. The equality (3) states that the probabilities of the sets of all evidences in a formula are greater than 0 and the linear combination of probabilities is greater than r, from the corresponding formula. It is easy to see that (3) corresponds to the third condition of the satisfiability relation from Definition 3, after we clean the denominators.

Let us show that in the case when $n = 2$. We use $\overline{[\beta]}$ for the set $\{s' \in \overline{W} \mid \overline{M}, s' \Vdash \beta\}$. From $\sum_{s_j:\overline{M},s_j\Vdash\beta_k} y_{a,s_i,s_j} > 0$ *for every* $k \in \{1,2\}$ we have that $\mu_a(s_i)(\overline{[\beta_k]}) > 0$ for all $k \in \{1,2\}$. From the

$$\sum_{k=1}^{2}\Big(a_k \sum_{s_j:\overline{M},s_j\Vdash\beta_k\wedge\gamma_k} y_{a,s_i,s_j} \prod_{t\neq k,t=1}^{2} \sum_{s_j:\overline{M},s_j\Vdash\beta_t} y_{a,s_i,s_j}\Big) \geq$$

$$r\prod_{k=1}^{2} \sum_{s_j:\overline{M},s_j\Vdash\beta_k} y_{a,s_i,s_j}$$

we obtain

$$\sum_{k=1}^{2}\Big(a_k\mu_a(s_i)(\overline{[\beta_k\wedge\gamma_k]}) \prod_{t\neq k,s=1}^{2} \mu_a(s_i)(\overline{[\beta_t]})\Big) \geq r\prod_{k=1}^{2}\mu_a(s_i)(\overline{[\beta_k]}),$$

i.e.,

$$\frac{a_1\mu_a(s_i)(\overline{[\beta_1\wedge\gamma_1]})\mu_a(s_i)(\overline{[\beta_2]}) + a_2\mu_a(s_i)(\overline{[\beta_2\wedge\gamma_2]})\mu_a(s_i)(\overline{[\beta_1]})}{\mu_a(s_i)(\overline{[\beta_1]})\mu_a(s_i)(\overline{[\beta_2]})} \geq r.$$

Then we have

$$\frac{a_1\mu_a(s_i)(\overline{[\beta_1\wedge\gamma_1]})}{\mu_a(s_i)(\overline{[\beta_1]})} + \frac{a_2\mu_a(s_i)(\overline{[\beta_2\wedge\gamma_2]})}{\mu_a(s_i)(\overline{[\beta_2]})} \geq r,$$

i.e.,

$$\sum_{k=1}^{2} a_k\mu_a(s_i)(\overline{[\gamma_k]}|\overline{[\beta_k]}) \geq r.$$

Similarly, (4), corresponds to the combination of the fourth and the third condition from Definition 3.

The equations and inequalities (1)-(4) form not one, but a number of finite systems of equations and inequalities. Note that adding (4) to any system Sys of equations and inequalities results with a disjunction of at least two different extensions of Sys. For the purpose of this proof, the fact that we always have finitely many systems is sufficient, and it is enough if one of the systems is solvable. Those systems are represented in the language of real closed fields, and it is well known that the theory of real closed fields is decidable. Since we have finitely many possibilities for the choice of l, and for every l finitely many possibilities for the choice of pre-structure, our logic CKL is decidable as well. □

5.2. Completeness of CKL

The axiomatic system for the logic CKL is very similar to the axiomatic system for the logic CKL^{fo}. We denote the system by $Ax(CKL)$.

Axiom schemes consist of the Axioms (5)–(20) from the logic CKL^{fo} and the following axiom: (a) all instances of the classical propositional tautologies, while Inference Rules consists of all the Inference Rules from the logic CKL^{fo} except R2. It is easy to see that classical propositional logic is sublogic of the CKL. Inference relation, consistent and maximally consistent sets are defined in the same way as for the logic CKL^{fo}.

It is clear that the axiomatic system is sound with respect to the class of $\mathrm{CKL}_{\mathrm{Meas}}$-models. Also, Deduction theorem and Strong necessitation hold for the logic. The proofs of those statements are straightforward adaptations of the proofs of corresponding results for CKL^{fo}.

In order to prove the strong completeness theorem for the logic CKL, we first need to adapt Theorem 4, since we now deal with a propositional language. We build maximal consistent set as in the proof of Theorem 4, except that we now omit the step 3(a). Using this theorem, we define canonical model.

Definition 9 *(Canonical model).* The canonical model $M_C = (W, \mathcal{K}, Prob, v)$ is defined as follows:

- $W = \{u \mid u \text{ is maximal consistent set}\}$,
- for every world u and every propositional letter $p \in \mathcal{P}$, $v(u, p) = true$ iff $p \in u$,
- $\mathcal{K} = \{\mathcal{K}_i \mid i \in \mathbf{A}\}$ where $\mathcal{K}_i = \{(u', u) \mid u'/K_i \subset u\}$,
- $Prob(i, u) = (W_i(u), H_i(u), \mu_i(u))$ such that:
  - $W_i(u) = \mathcal{K}_i(u)$,
  - $H_i(u) = \{\{u' \in \mathcal{K}_i(u) \mid \alpha \in u'\} \mid \alpha \in For\}$,
  - $\mu_i(u) : H_i(u) \to [0, 1]$ such that $\mu_i(u)(\{u' \in \mathcal{K}_i(u) \mid \alpha \in u'\}) = \sup\{r \in [0, 1]_Q \mid \mathbf{w}_i(\alpha, \top) \geq r \in u\}$.

We can prove that M_C is well defined and that it belongs to $\mathrm{CKL}_{\mathrm{Meas}}$ analogously as it is done before.

Theorem 8 *(Strong completeness of* CKL*). A set of formulas T is consistent iff T is* $\mathrm{CKL}_{\mathrm{Meas}}$*-satisfiable.*

The proof of this theorem is almost the same as the proof of the strong completeness theorem for a logic CKL^{fo}. The minor differences are caused by the fact that now we consider propositional formulas instead of classical first-order formulas (for example, in the proof of Truth lemma, the base step in the induction proof will consider propositional letters instead of atomic first-order formulas).

6. Conclusion

We have investigated extensions of epistemic logic that allows explicit reasoning about conditional probabilities. We have considered a first-order and a propositional case and we have been able to obtain strongly complete axiomatizations for both languages, combining the axiomatization approaches from two previously developed epistemic probabilistic logics [8] and [27]. In addition, we have obtained decidability result for the propositional version of the logic by combining the method of filtration and a reduction to a finite set of systems of inequalities. The novelty of our logic with respect to [8] and [27] does not lie in the epistemic part of the logic, but on expressiveness of the probabilistic part of the language. Indeed, while probabilities of formulas can be expressed in [27], they cannot be compared, and the conditional probabilities are not even expressible. Similarly, the logic from [8] cannot compare conditional probabilities, nor express first order statements. To the best of our knowledge, our work is the first axiomatization of an epistemic logic which can reason about conditional probabilities.

The semantic relationship between the modalities for knowledge and probability in CKL^{fo} is given by the condition $W_i(u) \subseteq \mathcal{K}_i(u)$, which forbids an agent to place positive probabilities to the events she knows to be false. The syntactical counterpart of that condition is given by the axiom A20. Fagin and Halpern [8] also considered several other modifications of their semantics, by posing relations between the indistinguishably relations and probability spaces, and which model some typical situations in the multi-agent systems. For example, if $Prob(i, w) = (W_i(w), H_i(w), \mu_i(w))$ and $w' \in W_i(w)$, they consider the assumption $Prob(i, w) = Prob(i, w')$, which they call *uniformity* and which divides the possible worlds to partitions which share the same probability distributions. Two other considered properties are *state determined property*, which requires that $Prob(i, w) = Prob(i, w')$ whenever $w' \in \mathcal{K}_i(w)$, and *objectivity*, which requires that $Prob(i, w) = Prob(j, w)$ for all i and j and for every $w \in W$. The paper [8] provides characterization of all those semantic assumptions in terms of corresponding axioms. Adding those axioms to our system would also make it complete for the considered semantics.

Finally, when we deal with epistemic logic we cannot ignore one of its main notions: common knowledge, which has been shown as crucial for many applications dealing with reaching agreements or coordinated actions [14]. Informally, a formula α is common knowledge of a group of agents exactly when everyone knows that everyone knows that everyone knows... that α holds. There are two similar proposals for a probabilistic variant of common knowledge [8,25], which assumes that coordinated actions hold with high probability. The first complete axiomatization that encompasses probabilistic common knowledge is proposed in [27]. It would be interesting to see whether there is a sensible generalization of that notion that uses conditional probabilities. We plan to investigate that topic in future work.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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