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Research Article

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Homogenization in BV of a model for layered composites in finite crystal plasticity

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Abstract: In this work, we study the effective behavior of a two-dimensional variational model within finite crystal plasticity for high-contrast bilayered composites. Precisely, we consider materials arranged into periodically alternating thin horizontal strips of an elastically rigid component and a softer one with one active slip system. The energies arising from these modeling assumptions are of integral form, featuring linear growth and non-convex differential constraints. We approach this non-standard homogenization problem via Gamma-convergence. A crucial first step in the asymptotic analysis is the characterization of rigidity properties of limits of admissible deformations in the space BV of functions of bounded variation. In particular, we prove that, under suitable assumptions, the two-dimensional body may split horizontally into finitely many pieces, each of which undergoes shear deformation and global rotation. This allows us to identify a potential candidate for the homogenized limit energy, which we show to be a lower bound on the Gamma-limit. In the framework of non-simple materials, we present a complete Gamma-convergence result, including an explicit homogenization formula, for a regularized model with an anisotropic penalization in the layer direction.

Keywords: Homogenization, Γ-convergence, linear growth, composites, finite crystal plasticity, non-simple materials

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1 Introduction

Metamaterials are artificially engineered composites whose heterogeneities are optimized to improve structural performances. Due to their special mechanical properties, arising as a result of complex microstructures, metamaterials play a key role in industrial applications and are an increasingly active field of research. Two natural questions when dealing with composite materials are how the effective material response is influenced by the geometric distribution of its components, and how the mechanical properties of the components impact the overall macroscopic behavior of the metamaterial.

In what follows, we investigate these questions for a special class of metamaterials with two characteristic features that are of relevance in a number of applications: (i) the material consists of two components arranged in a highly anisotropic way into periodically alternating layers, and (ii) the (elasto)plastic properties of the two components exhibit strong differences, in the sense that one is rigid, while the other one is con-

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siderably softer, allowing for large (elasto)plastic deformations. The analysis of variational models for such layered high-contrast materials was initiated in [12]. There, the authors derive a macroscopic description for a two-dimensional model in the context of geometrically nonlinear but rigid elasticity, assuming that the softer component can be deformed along a single active slip system with linear self-hardening.

These results have been extended to general dimensions, to energy densities with p-growth ($1 < p < +\infty$), and to the case with non-trivial elastic energies, which allows treating very stiff (but not necessarily rigid) layers, see [11, 13].

In this paper, we carry the ideas of [12] forward to a model for plastic composites without linear hardening, in the spirit of [18]. This change turns the variational problem in [12], having quadratic growth (cf. also [15, 16]), into one with energy densities that grow merely linearly.

The main novelty lies in the fact that the homogenization analysis must be performed in the class BV of functions of bounded variation (see [2]) to account for concentration phenomena. This gives rise to conceptual mathematical difficulties: on the one hand, the standard convolution techniques commonly used for density arguments in BV or SBV cannot be directly applied because they do not preserve the intrinsic constraints of the problem; on the other hand, constraint-preserving approximations in this weaker setting of BV are rather challenging, as one needs to simultaneously regularize the absolutely continuous part of the distributional derivative of the functions and accommodate their jump sets.

To state our results precisely, we first introduce the relevant model with its main modeling hypotheses. Throughout the article, we analyze two versions of the model, namely with and without regularization.

Let e_1 and e_2 be the standard unit vectors in $\mathbb{R}^2$, and let $x = (x_1, x_2)$ denote a generic point in $\mathbb{R}^2$. Unless specified otherwise, $\Omega \subset \mathbb{R}^2$ is an x_1-connected, bounded domain with Lipschitz boundary, that is, an open set whose slices in the x_1-direction are (possibly empty) open intervals (see Section 2.4 for the precise definition). For such a domain Ω, we set

$$a_\Omega := \inf_{x\in\Omega} x_2 \quad \text{and} \quad b_\Omega := \sup_{x\in\Omega} x_2, \tag{1.1}$$

as well as

$$c_\Omega := \inf_{x\in\Omega} x_1 \quad \text{and} \quad d_\Omega := \sup_{x\in\Omega} x_1. \tag{1.2}$$

Assume that Ω is the reference configuration of a body with heterogeneities in the form of periodically alternating thin horizontal layers. To describe the bilayered structure mathematically, consider the periodicity cell $Y := [0, 1)^2$, which we subdivide into $Y = Y_{\text{soft}} \cup Y_{\text{rig}}$ with $Y_{\text{soft}} := [0, 1) \times [0, \lambda)$ for $\lambda \in (0, 1)$ and $Y_{\text{rig}} := Y \setminus Y_{\text{soft}}$. All sets are extended by periodicity to $\mathbb{R}^2$. The (small) parameter $\varepsilon > 0$ describes the thickness of a pair (one rigid, one softer) of fine layers, and can be viewed as the intrinsic length scale of the system. The collections of all rigid and soft layers in Ω can be expressed as $\varepsilon Y_{\text{rig}} \cap \Omega$ and $\varepsilon Y_{\text{soft}} \cap \Omega$, respectively. For an illustration of the geometrical assumptions, see Figure 1.

Following the classical theory of elastoplasticity at finite strains (see, e.g., [31] for an overview), we assume that the gradient of any deformation $u : \Omega \to \mathbb{R}^2$ decomposes into the product of an elastic strain, F_{el}, and a plastic one, F_{pl}. In the literature, different models of finite plasticity have been proposed (see, e.g., [3, 22, 29, 30, 37]), as well as alternative descriptions via the theory of structured deformations (see [6, 9, 10, 24] and the references therein). Here, we adopt the classical model by Lee on finite crystal plasticity introduced in [33–35], according to which the deformation gradients satisfy

$$\nabla u = F_{\text{el}} F_{\text{pl}}. \tag{1.3}$$

In addition, we suppose that the elastic behavior of the body is purely rigid, meaning that

$$F_{\text{el}} \in SO(2) \quad \text{almost everywhere in } \Omega, \tag{1.4}$$

and that the plastic part satisfies

$$F_{\text{pl}} = \mathbb{I} + \gamma s \otimes m, \tag{1.5}$$

where $s \in \mathbb{R}^2$ with $|s| = 1$ is the slip direction of the slip system, $m = s^\perp$ is the normal to the slip plane, and the map γ measures the amount of slip. Denoting by $\mathcal{M}_s$ the set

$$\mathcal{M}_s := \{F \in \mathbb{R}^{2\times 2} : \det F = 1 \text{ and } |Fs| = 1\},$$

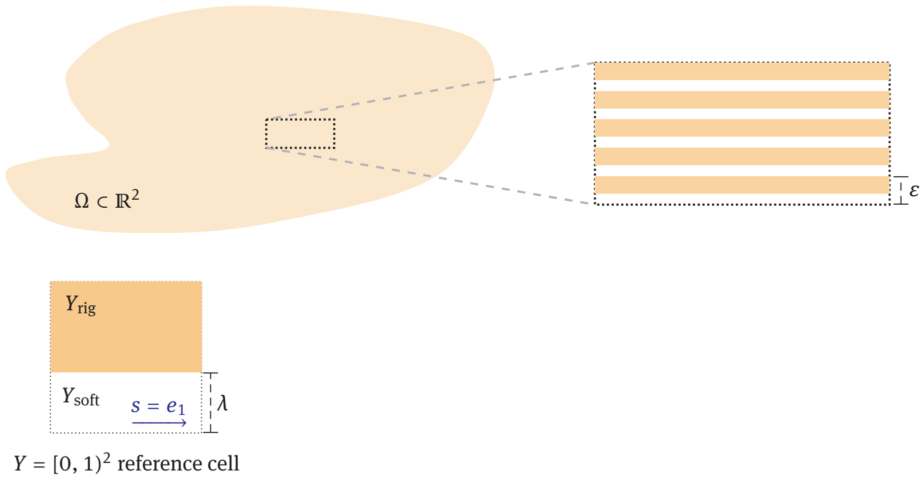


Figure 1: A bilayered x_1-connected domain Ω.

the multiplicative decomposition (1.3) (under assumptions (1.4) and (1.5)) is equivalent to $\nabla u \in \mathcal{M}_s$ almost everywhere in Ω. Whereas the material is free to glide along the slip system in the softer phase, it is required that γ vanishes on the layers consisting of a rigid material, i.e., $\gamma = 0$ in $\varepsilon Y_{\mathrm{rig}} \cap \Omega$.

Collecting the previous modeling assumptions, we define, for $\varepsilon > 0$, the class $\mathcal{A}_\varepsilon$ of *admissible layered deformations* by

$$\begin{aligned}\mathcal{A}_\varepsilon &:= \{u \in W^{1,1}(\Omega;\mathbb{R}^2) : \nabla u \in \mathcal{M}_s \text{ a.e. in } \Omega,\ \nabla u \in \mathrm{SO}(2) \text{ a.e. in } \varepsilon Y_{\mathrm{rig}} \cap \Omega\}\\ &= \{u \in W^{1,1}(\Omega;\mathbb{R}^2) : \nabla u = R(\mathbb{I} + \gamma s \otimes m) \text{ a.e. in } \Omega,\ R \in L^\infty(\Omega;\mathrm{SO}(2)) \text{ and } \gamma \in L^1(\Omega)\\ &\qquad \text{with } \gamma = 0 \text{ a.e. in } \varepsilon Y_{\mathrm{rig}} \cap \Omega\}. \end{aligned} \tag{1.6}$$

The elastoplastic energy of a deformation $u \in L^1_0(\Omega;\mathbb{R}^2) := \{u \in L^1(\Omega;\mathbb{R}^2) : \int_\Omega u\,dx = 0\}$, given by

$$E_\varepsilon(u) = \begin{cases} \int_\Omega |\gamma|\,dx & \text{for } u \in \mathcal{A}_\varepsilon,\\ \infty & \text{otherwise in } L^1_0(\Omega;\mathbb{R}^2), \end{cases} \tag{1.7}$$

represents the internal energy contribution of the system during a single incremental step in a time-discrete variational description. This way of modeling excludes preexistent plastic distortions, and can be considered a reasonable assumption for the first time step of a deformation process. The elastoplastic energy can be complemented with terms modeling the work done by external body or surface forces.

The limit behavior of sequences $(u_\varepsilon)_\varepsilon$ of low energy states for $(E_\varepsilon)_\varepsilon$ gives information about the macroscopic material response of the layered composites. In the following, we focus the analysis of this asymptotic behavior on the $s = e_1$ case, when the slip direction is parallel to the orientation of the layers, cf. also Figure 1. Note that different slip directions can be treated similarly, but the arguments are technically more involved. In fact, for $s \notin \{e_1, e_2\}$, small-scale laminate microstructures on the softer layers need to be taken into account, which requires an extra relaxation step. We refer to [18] for the relaxation mechanism and to [12] for the strategy of how to apply it to layered structures.

An important first step towards identifying the limit behavior of the energies $(E_\varepsilon)_\varepsilon$ (in the sense of Γ-convergence) is the proof of a general statement of asymptotic rigidity for layered structures in the context of functions of bounded variation. The following result characterizes the weak* limits in BV of deformations whose gradients coincide pointwise with rotations on the rigid layers of the material. Note that no additional constraints are imposed on the softer components at this point.

Theorem 1.1 (Asymptotic rigidity of layered structures in BV). *Let $\Omega \subset \mathbb{R}^2$ be an x_1-connected domain, and assume that $(u_\varepsilon)_\varepsilon \subset W^{1,1}(\Omega;\mathbb{R}^2)$ is a sequence satisfying*

$$\nabla u_\varepsilon \in \mathrm{SO}(2) \quad \textit{a.e. in } \varepsilon Y_{\mathrm{rig}} \cap \Omega \textit{ for all } \varepsilon, \tag{1.8}$$

and that $u_\varepsilon \overset{*}{\rightharpoonup} u$ *in* $\mathrm{BV}(\Omega;\mathbb{R}^2)$ *for some* $u \in \mathrm{BV}(\Omega;\mathbb{R}^2)$ *as* $\varepsilon \to 0$. *Then,*

$$u(x) = R(x_2)x + \psi(x_2) \quad \text{for } \mathcal{L}^2\text{-a.e. } x \in \Omega, \tag{1.9}$$

where $R \in \mathrm{BV}(a_\Omega, b_\Omega; \mathrm{SO}(2))$ *and* $\psi \in \mathrm{BV}(a_\Omega, b_\Omega;\mathbb{R}^2)$ *(cf.* (1.1)*).*

Conversely, any function $u \in \mathrm{BV}(\Omega;\mathbb{R}^2)$ *as in* (1.9) *can be attained as weak*-limit in* $\mathrm{BV}(\Omega;\mathbb{R}^2)$ *of a sequence* $(u_\varepsilon)_\varepsilon \subset W^{1,1}(\Omega;\mathbb{R}^2)$ *satisfying* (1.8).

To prove the first part of Theorem 1.1, we adapt the arguments in [12] to the BV-setting. The second assertion follows from a tailored one-dimensional density result in BV, which involves approximating functions that are constant on the rigid layers (see Lemma 3.3 below). Up to minor adaptations, analogous statements hold in higher dimensions. We refer to Remark 3.4 for the specific assumptions on the geometry of the set Ω under which a higher-dimensional counterpart of Theorem 1.1 can be proved.

In view of Remark 5.1 below, a natural potential candidate for the limiting behavior of $(E_\varepsilon)_\varepsilon$ in the sense of Γ-convergence (see [8, 20] for an introduction, as well as the references therein) is the functional $E : L^1_0(\Omega;\mathbb{R}^2) \to [0,\infty]$, given by

$$E(u) = \begin{cases} \displaystyle\int_\Omega |\psi' \cdot Re_1|\,\mathrm{d}x + |D^s u|(\Omega) & \text{if } u \in \mathcal{A}, \\ \infty & \text{otherwise,} \end{cases} \tag{1.10}$$

where

$$\begin{aligned} \mathcal{A} := \{u \in \mathrm{BV}(\Omega;\mathbb{R}^2) : u(x) = R(x_2)x + \psi(x_2) \text{ for a.e. } x \in \Omega \text{ with } R \in \mathrm{BV}(a_\Omega, b_\Omega; \mathrm{SO}(2)),\\ \psi \in \mathrm{BV}(a_\Omega, b_\Omega;\mathbb{R}^2), \text{ and } \det \nabla u = 1 \text{ a.e. in } \Omega\}, \end{aligned} \tag{1.11}$$

and $|D^s u|$, along with other basic properties of BV functions, is introduced in Section 2.2.

The next theorem states that E provides indeed a lower bound for our homogenization problem.

Theorem 1.2 (Lower bound on the Γ-limit of $(E_\varepsilon)_\varepsilon$). *Let* $\Omega \subset \mathbb{R}^2$ *be an* x_1*-connected domain, and let* E_ε *and* E *be the functionals introduced in* (1.7) *and* (1.10), *respectively. Then, every sequence* $(u_\varepsilon)_\varepsilon \subset L^1_0(\Omega;\mathbb{R}^2)$ *with uniformly bounded energies,* $\sup_\varepsilon E_\varepsilon(u_\varepsilon) < \infty$, *has a subsequence that converges weakly* in* $\mathrm{BV}(\Omega;\mathbb{R}^2)$ *to some* $u \in \mathcal{A} \cap L^1_0(\Omega;\mathbb{R}^2)$. *Additionally,*

$$\Gamma(L^1)\text{-}\liminf_{\varepsilon\to 0} E_\varepsilon \geq E. \tag{1.12}$$

The proof of the first assertion is given in Proposition 4.3. It relies on Theorem 1.1 in combination with a technical argument about the weak continuity properties of Jacobian determinants (see Lemma 4.2). In Section 5 and in the Appendix, we exhibit two different proofs of (1.12): A first one relying on the properties of the admissible layered deformations, and an alternative one exploiting a Reshetnyak's lower semicontinuity theorem (see, e.g., [2, Theorem 2.38]). The identification of E as the Γ-limit of the sequence $(E_\varepsilon)_\varepsilon$, though, remains an open problem. Indeed, verifying the optimality of the lower bound in Theorem 1.2 is rather challenging, as it requires to approximate elements of $\mathcal{A}$ by means of sequences in $\mathcal{A}_\varepsilon$ at least in the sense of the strict convergence in BV. We refer to Remark 5.2 for a detailed discussion of the main difficulties. Even if the requirement on the convergence of the energies is dropped, recovering the jumps of maps in the effective domain of E under consideration of the non-standard differential inclusions in $\mathcal{A}_\varepsilon$ is by itself another challenging problem. Solving this problem requires delicate geometrical constructions, which are currently not available for all elements in $\mathcal{A}$.

Yet, there are two subclasses of physically relevant deformations in $\mathcal{A}$ for which we can find suitable approximations by sequences of admissible layered deformations. The precise statement is given in Theorem 1.3 below.

The first of these two subclasses is $\mathcal{A} \cap \mathrm{SBV}_<(\Omega;\mathbb{R}^2)$ (we refer to Section 2.3 for the definition of $\mathrm{SBV}_<(\Omega;\mathbb{R}^2)$) whose jump sets are given by a union of finitely many lines. Heuristically, this subclass describes deformations that break Ω horizontally into a finite number of pieces, which may get sheared and rotated individually.

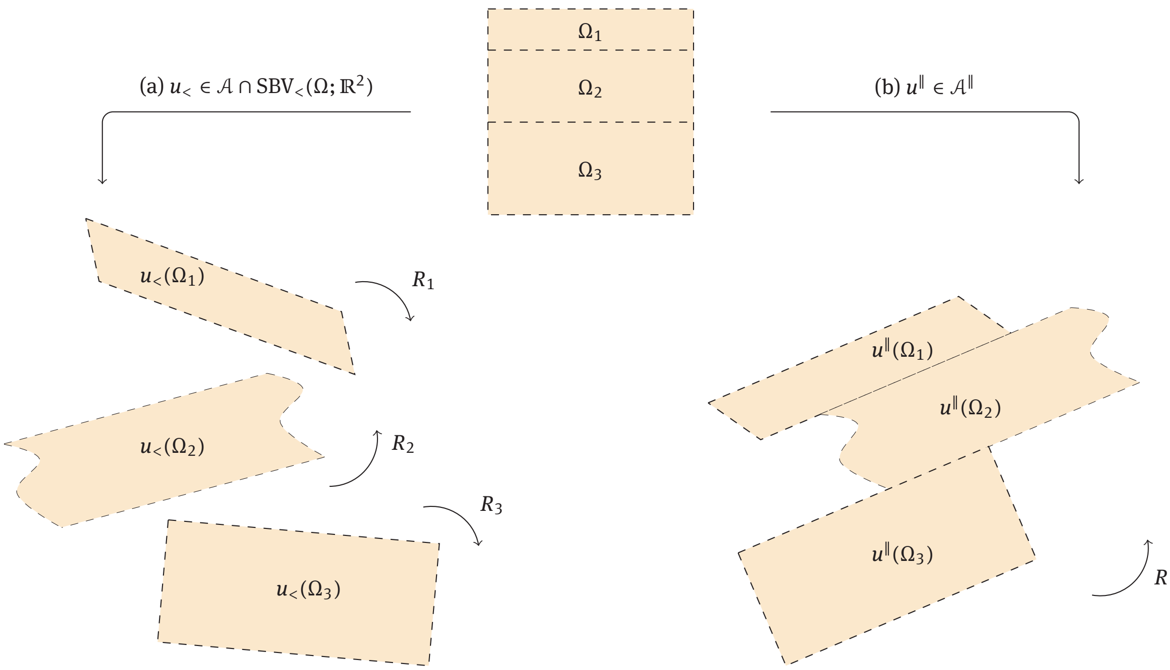


Figure 2: A typical deformation of a reference configuration $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$ through maps in (a) $\mathcal{A} \cap \mathrm{SBV}_<(\Omega; \mathbb{R}^2)$ and (b) $\mathcal{A}^\parallel$.

The second subclass is

$$\mathcal{A}^\parallel := \{u \in \mathrm{BV}(\Omega; \mathbb{R}^2) : u(x) = Rx + \vartheta(x_2)Re_1 + c \text{ for a.e. } x \in \Omega \text{ with } R \in \mathrm{SO}(2),\ \vartheta \in \mathrm{BV}(a_\Omega, b_\Omega), \text{ and } c \in \mathbb{R}^2\}. \tag{1.13}$$

In comparison with $\mathcal{A}$, functions in $\mathcal{A}^\parallel$ satisfy two additional constraints, namely the fact that the rotation R is constant and that the jumps of functions in $\mathcal{A}^\parallel$ are parallel to Re_1. With the notation $\mathcal{A}^\parallel$, we intend to highlight the second feature. The intuition behind maps in $\mathcal{A}^\parallel$ are non-trivial macroscopic deformations that (up to a global rotation) may make the material break along finite or infinitely many horizontal lines, induce sliding of the pieces relative to each other, and cause horizontal shearing within each individual piece. For an illustration of the two subclasses, see Figure 2.

Theorem 1.3 (Approximation of maps in $(\mathcal{A} \cap \mathrm{SBV}_<) \cup \mathcal{A}^\parallel$). *Let $\Omega \subset \mathbb{R}^2$ be an x_1-connected domain and let $u \in (\mathcal{A} \cap \mathrm{SBV}_<(\Omega; \mathbb{R}^2)) \cup \mathcal{A}^\parallel$. Then, there exists a sequence $(u_\varepsilon)_\varepsilon \subset W^{1,1}(\Omega; \mathbb{R}^2)$ such that $u_\varepsilon \in \mathcal{A}_\varepsilon$ for every ε, and $u_\varepsilon \overset{*}{\rightharpoonup} u$ in $\mathrm{BV}(\Omega; \mathbb{R}^2)$.*

As a first step towards proving Theorem 1.3, we establish an admissible piecewise affine approximation for limiting deformations with a single jump line (see Lemma 4.5). The construction relies on the characterization of rank-one connections in $\mathcal{M}_{e_1}$ proved in [12, Lemma 3.1], with transition lines stretching over the full width of Ω to avoid triple junctions (see Remark 4.6). In Propositions 4.7 and 4.9, we extend the arguments to $\mathcal{A} \cap \mathrm{SBV}_<(\Omega; \mathbb{R}^2)$ and $\mathcal{A}^\parallel$, respectively.

Problems in finite crystal plasticity without additional regularizations are generally known to be challenging because of the oscillations of minimizing sequences arising as a byproduct of relaxation mechanisms in the slip systems. This phenomenon is one of the main reasons why a full relaxation theory in finite crystal plasticity is still missing (see [17, Remark 3.2]). In our setting, it hampers the full characterization of weak limits of sequences with uniformly bounded energies. The observation that regularizations can help overcome the above compensated-compactness issue (see also Remark 6.2) motivates the introduction of a penalized version of our problem. After a higher-order penalization of the energy in the layer direction, we obtain the following Γ-convergence result. The attained limit deformations are given by the class $\mathcal{A}^\parallel$.

Theorem 1.4 (Γ-convergence of the regularized energies). *Let $\Omega \subset \mathbb{R}^2$ be an x_1-connected domain and $\mathcal{A}_\varepsilon$ the set introduced in (1.6). Fix $p > 2$ and $\delta > 0$. For each $\varepsilon > 0$, let $E_\varepsilon^\delta : L_0^1(\Omega; \mathbb{R}^2) \to [0, \infty]$ be the functional* *defined by*

$$E_\varepsilon^\delta(u) := \begin{cases} \int_\Omega |\gamma|\,\mathrm{d}x + \delta\|\partial_1 u\|^p_{W^{1,p}(\Omega;\mathbb{R}^2)} & \text{for } u \in \mathcal{A}_\varepsilon, \\ \infty & \text{otherwise.} \end{cases} \tag{1.14}$$

Then, the family $(E_\varepsilon^\delta)_\varepsilon$ Γ*-converges with respect to the strong* L^1*-topology to the functional* $E^\delta : L^1_0(\Omega;\mathbb{R}^2) \to [0,\infty]$ *given by*

$$E^\delta(u) := \begin{cases} \int_\Omega |\vartheta'(x_2)|\,\mathrm{d}x + |D^s u|(\Omega) + \delta|\Omega| & \text{for } u \in \mathcal{A}^\parallel, \\ \infty & \text{otherwise,} \end{cases}$$

where ϑ' *denotes the approximate differential of* ϑ *(cf. Section 2.2).*

The penalization in (1.14) can be viewed in the spirit of non-simple materials [39, 40]. Working with stored energy densities that depend on the Hessian of the deformations has proved successful in overcoming lack of compactness in a variety of applications; see, e.g., [5, 21, 27, 36, 38]. Very recently, there has been an effort towards weakening higher-order regularizations: It is shown in [7] that the full norm of the Hessian can be replaced by a control of its minors (gradient polyconvexity) in the context of locking materials; for solid-solid phase transitions, an anisotropic second-order penalization is considered in [23]. Along these lines, we introduce the regularized energies in (1.13) that penalize the variation of deformations only in the layer direction. This is enough to deduce that the limiting rotation (as $\varepsilon \to 0$) is global and that it determines the direction of the limiting jump. In Section 6 and in the Appendix, we provide two alternative proofs of this result: A first one relying on Alberti's rank-one theorem (see Section 2.2) in combination with the approximation result in Theorem 1.3, and a second one based on separate regularizations of the regular and the singular part of the limiting maps, and inspired by [19, Lemma 3.2].

This paper is organized as follows. In Section 2.1, we collect a few preliminaries, including some background on (special) functions of bounded variation. Section 3 is devoted to the analysis of asymptotic rigidity for layered structures in the setting of BV-functions. A characterization of limits of admissible layered deformations is provided in Section 4. Eventually, Sections 5 and 6 contain the proof of a lower bound for the homogenization problem without regularization (Theorem 1.2) and the full Γ-convergence analysis of the regularized problem (Theorem 1.4), respectively.

2 Preliminaries

2.1 Notation

In this section, unless mentioned otherwise, Ω is a bounded domain in $\mathbb{R}^N$ with $N \in \mathbb{N}$. Throughout the rest of the paper, we assume mostly that $N = 2$.

We represent by $\mathcal{L}^N$ the N-dimensional Lebesgue measure and by $\mathcal{H}^{N-1}$ the $(N-1)$-dimensional Hausdorff measure. Whenever we write "a.e. in Ω", we mean "almost everywhere in Ω" with respect to $\mathcal{L}^N \lfloor \Omega$. To simplify the notation, we often omit the expression "a.e. in Ω" in mathematical relations involving Lebesgue measurable functions. Given a Lebesgue measurable set $B \subset \mathbb{R}^N$, we also use the shorter notation $|B| = \mathcal{L}^N(B)$ for the Lebesgue measure of B, while the characteristic function of B in $\mathbb{R}^N$ is denoted by $\mathbb{1}_B$ and takes values 0 and 1.

The set $\mathrm{SO}(N) := \{R \in \mathbb{R}^{N\times N} : RR^T = \mathbb{I},\ \det R = 1\}$, where $\mathbb{I}$ is the identity matrix in $\mathbb{R}^{N\times N}$, consists of all proper rotations. We recall that for $N = 2$, $R \in \mathrm{SO}(2)$ if and only if there is $\theta \in [-\pi, \pi)$ such that

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

For two vectors $a, b \in \mathbb{R}^d$, $a \otimes b := ab^T$ stands for their tensor product. If $a = (a_1, a_2)^T \in \mathbb{R}^2$, we set $a^\perp := (-a_2, a_1)^T$.

We use the standard notation for spaces of vector-valued functions; namely, $L^p_\mu(\Omega; \mathbb{R}^d)$ with $p \in [1, \infty]$ and a positive measure μ for L^p-spaces, $W^{1,p}(\Omega; \mathbb{R}^d)$ with $p \in [1, \infty]$ for Sobolev spaces, $C(\Omega; \mathbb{R}^d)$ for the space of continuous functions, $C^\infty(\Omega; \mathbb{R}^d)$ and $C^\infty_c(\Omega; \mathbb{R}^d)$ for the spaces of smooth functions without and with compact support, and $C^{0,\alpha}(\Omega; \mathbb{R}^d)$ with $\alpha \in [0, 1]$ for Hölder spaces. We denote by $C_0(\Omega; \mathbb{R}^d)$ the space of continuous functions that vanish on the boundary of Ω. Moreover, $\mathcal{M}(\Omega; \mathbb{R}^d)$ is the space of finite vector-valued Radon measures. In the case of scalar-valued functions and measures, we omit the codomain; for instance, we write $L^1(\Omega)$ instead of $L^1(\Omega; \mathbb{R})$.

The duality pairing between $C_0(\Omega; \mathbb{R}^d)$ and $\mathcal{M}(\Omega; \mathbb{R}^d)$ is represented by

$$\langle \mu, \zeta \rangle := \int_\Omega \zeta \, d\mu,$$

and $\mu \otimes \nu$ denotes the product measure of two measures μ and ν.

Throughout this manuscript, ε stands for a small (positive) parameter, and is usually thought of as taking values on a positive sequence converging to zero.

2.2 Functions of bounded variation

We adopt the standard notations for the space $\mathrm{BV}(\Omega; \mathbb{R}^d)$ of vector-valued functions of bounded variation, and refer the reader to [2] for a thorough treatment of this space. Here, we only recall some of its basic properties.

A function $u \in L^1(\Omega; \mathbb{R}^d)$ is called a *function of bounded variation*, written $u \in \mathrm{BV}(\Omega; \mathbb{R}^d)$, if its distributional derivative Du satisfies $Du \in \mathcal{M}(\Omega; \mathbb{R}^{d\times N})$. The space $\mathrm{BV}(\Omega; \mathbb{R}^d)$ is a Banach space when endowed with the norm $\|u\|_{\mathrm{BV}(\Omega;\mathbb{R}^d)} := \|u\|_{L^1(\Omega;\mathbb{R}^d)} + |Du|(\Omega)$, where $|Du| \in \mathcal{M}(\Omega)$ is the total variation of Du.

Let $D^a u$ and $D^s u$ denote the absolutely continuous and the singular part of the Radon–Nikodym decomposition of Du with respect to $\mathcal{L}^N \lfloor \Omega$, and let $D^j u$ and $D^c u$ be the jump and Cantor parts of Du. The following chain of equalities holds:

$$\begin{aligned} Du &= D^a u + D^s u = \nabla u \mathcal{L}^N \lfloor \Omega + D^s u = \nabla u \mathcal{L}^N \lfloor \Omega + D^j u + D^c u \\ &= \nabla u \mathcal{L}^N \lfloor \Omega + (u^+ - u^-) \otimes \nu_u \mathcal{H}^{N-1} \lfloor J_u + D^c u, \end{aligned} \tag{2.1}$$

where ∇u is the approximate differential of u (that is, the density of $D^a u$), u^+ and u^- are the approximate one-sided limits at the jump points, J_u is the jump set of u, and ν_u is the normal to J_u (cf. [2, Chapter 3]).

Following [2, p. 186], we can exploit the polar decomposition of a measure and the fact that all parts of the derivative of u in (2.1) are mutually singular to write $Du = g_u |Du|$ with a map $g_u \in L^1_{|Du|}(\Omega; \mathbb{R}^{d\times N})$ satisfying $|g_u| = 1$ for $|Du|$-a.e. $x \in \Omega$ and

$$D^a u = g_u |D^a u|, \quad D^s u = g_u |D^s u|, \quad D^j u = g_u |D^j u|, \quad D^c u = g_u |D^c u|.$$

Note that

$$g_u(x) = \frac{\nabla u(x)}{|\nabla u(x)|} \quad \text{for } \mathcal{L}^N\text{-a.e. } x \in \Omega \text{ such that } |\nabla u(x)| \neq 0,$$

$$g_u(x) = \frac{u(x^+) - u(x^-)}{|u(x^+) - u(x^-)|} \otimes \nu_u(x) \quad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in J_u, \tag{2.2}$$

$$g_u(x) = \bar{g}_u(x) \otimes n_u(x) \quad \text{for } |D^c u|\text{-a.e. } x \in \Omega \text{ with suitable Borel maps } \bar{g}_u : \Omega \to \mathbb{R}^d,\ n_u : \Omega \to \mathbb{R}^N. \tag{2.3}$$

The last equality relies on Alberti's rank-one theorem (see [1]).

Let $u \in \mathrm{BV}(\Omega; \mathbb{R}^d)$ and $(u_j)_{j\in\mathbb{N}} \subset \mathrm{BV}(\Omega; \mathbb{R}^d)$ be a sequence. One says that $(u_j)_{j\in\mathbb{N}}$ weakly* converges to u in $\mathrm{BV}(\Omega; \mathbb{R}^d)$, written $u_j \overset{*}{\rightharpoonup} u$ in $\mathrm{BV}(\Omega; \mathbb{R}^d)$, if $u_j \to u$ in $L^1(\Omega; \mathbb{R}^d)$ and $Du_j \overset{*}{\rightharpoonup} Du$ in $\mathcal{M}(\Omega; \mathbb{R}^{d\times N})$. The sequence $(u_j)_{j\in\mathbb{N}}$ is said to converge strictly to u in $\mathrm{BV}(\Omega; \mathbb{R}^d)$, written $u_j \overset{*}{\to} u$ in $\mathrm{BV}(\Omega; \mathbb{R}^d)$, if $u_j \to u$

in $L^1(\Omega;\mathbb{R}^d)$ and $|Du_j|(\Omega) \to |Du|(\Omega)$. We recall that strict convergence in $\mathrm{BV}(\Omega;\mathbb{R}^d)$ implies weak* convergence in $\mathrm{BV}(\Omega;\mathbb{R}^d)$. Moreover, from every bounded sequence in $\mathrm{BV}(\Omega;\mathbb{R}^d)$ one can extract a weakly* convergent subsequence (see [2, Theorem 3.23]).

In the one-dimensional setting, i.e., for $\varphi \in \mathrm{BV}(a,b;\mathbb{R}^d)$ with $\Omega = (a,b) \subset \mathbb{R}^N$ and $N = 1$, we write φ' in place of $\nabla\varphi$ to denote the approximate differential of φ. Accordingly, we use the notation $Du = \varphi'\mathcal{L}^1 + D^s\varphi$ for the decomposition of the distributional derivative of φ with respect to the Lebesgue measure.

A function $\varphi \in \mathrm{BV}(a,b;\mathbb{R}^d)$ is called a jump or Cantor function if $D\varphi = D^j\varphi$ or $D\varphi = D^c\varphi$, respectively. We denote the sets of all jump and Cantor functions by $BV^j(a,b;\mathbb{R}^d)$ and $BV^c(a,b;\mathbb{R}^d)$, respectively. As shown in [2, Corollary 3.33], it is a special property of the one-dimensional setting that

$$\mathrm{BV}(a,b;\mathbb{R}^d) = W^{1,1}(a,b;\mathbb{R}^d) + BV^j(a,b;\mathbb{R}^d) + BV^c(a,b;\mathbb{R}^d). \tag{2.4}$$

In this paper, two-dimensional functions of the form

$$u(x) = R(x_2)x + \psi(x_2) \tag{2.5}$$

with $x = (x_1,x_2) \in Q := (c,d)\times(a,b) \subset \mathbb{R}^2$, where $R \in \mathrm{BV}(a,b;\mathrm{SO}(2))$ and $\psi \in \mathrm{BV}(a,b;\mathbb{R}^2)$, play a fundamental role. Maps u as in (2.5) satisfy $u \in \mathrm{BV}(\Omega;\mathbb{R}^2)$. Denoting by $D_1u := Du \otimes e_1$ and $D_2u := Du\otimes e_2$, the first and second columns of Du, respectively, we have for all $\zeta \in C_0(\Omega)$ that

$$\begin{aligned}\langle D_1u,\zeta\rangle &= \int_\Omega \zeta(x)R(x_2)e_1\,\mathrm{d}x_1\,\mathrm{d}x_2,\\ \langle D_2u,\zeta\rangle &= \int_\Omega \zeta(x)\big(R(x_2)e_2 + R'(x_2)x + \psi'(x_2)\big)\,\mathrm{d}x_1\,\mathrm{d}x_2 + \int_\Omega \zeta(x)x_1\,\mathrm{d}x_1\,\mathrm{d}D^sR(x_2)e_1\\ &\quad + \int_\Omega \zeta(x)x_2\,\mathrm{d}x_1\,\mathrm{d}D^sR(x_2)e_2 + \int_\Omega \zeta(x)\,\mathrm{d}x_1\,\mathrm{d}D^s\psi(x_2).\end{aligned}$$

Hence, $Du = D^au + D^su$ with

$$\begin{aligned}D^au &= (R + (R'x+\psi')\otimes e_2)\mathcal{L}^2\lfloor\Omega,\\ D^su &= \big((x^T\mathcal{L}^1\lfloor(c,d)\otimes D^sR^T)^T + \mathcal{L}^1\lfloor(c,d)\otimes D^s\psi\big)\otimes e_2,\end{aligned} \tag{2.6}$$

where $\mathcal{L}^1\lfloor(c,d)\otimes D^sR^T$ and $\mathcal{L}^1\lfloor(c,d)\otimes D^s\psi$ denote the restrictions to the Borel σ-algebra on $\Omega = Q$ of the product measures between $\mathcal{L}^1\lfloor(c,d)$ and D^sR^T and $D^s\psi$, respectively.

We observe further that there exists $\theta \in \mathrm{BV}(a,b;[-\pi,\pi])$ such that

$$R = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix} \quad\text{and}\quad R' = \theta'\begin{bmatrix}-\sin\theta & -\cos\theta\\ \cos\theta & -\sin\theta\end{bmatrix}, \tag{2.7}$$

where the representation of R' follows from the chain rule in BV; see, e.g., [2, Theorem 3.96].

2.3 Special functions of bounded variation

A function $u \in \mathrm{BV}(\Omega;\mathbb{R}^d)$ is said to be a *special function of bounded variation*, written $u \in \mathrm{SBV}(\Omega;\mathbb{R}^d)$, if the Cantor part of its distributional derivative satisfies

$$D^cu = 0.$$

In particular, it holds for every $u \in \mathrm{SBV}(\Omega;\mathbb{R}^d)$ that

$$Du = \nabla u\mathcal{L}^N\lfloor\Omega + (u^+ - u^-)\otimes\nu_u\mathcal{H}^{N-1}\lfloor J_u.$$

The space $\mathrm{SBV}(\Omega;\mathbb{R}^d)$ is a proper subspace of $\mathrm{BV}(\Omega;\mathbb{R}^d)$ (cf. [2, Corollary 4.3]).

Next, we recall the definition of the space $SBV_\infty(\Omega;\mathbb{R}^d)$ of special functions of bounded variation with bounded gradient and jump length, which is given by

$$SBV_\infty(\Omega;\mathbb{R}^d) := \{u \in SBV(\Omega;\mathbb{R}^d) : \nabla u \in L^\infty(\Omega;\mathbb{R}^{d\times N}) \text{ and } \mathcal{H}^{N-1}(J_u) < +\infty\}. \tag{2.8}$$

Dropping the L^∞-bound on the gradient in (2.8) gives a larger subspace of $SBV(\Omega;\mathbb{R}^d)$, which we call $SBV_<(\Omega;\mathbb{R}^d)$; precisely,

$$SBV_<(\Omega;\mathbb{R}^d) := \{u \in SBV(\Omega;\mathbb{R}^d) : \mathcal{H}^{N-1}(J_u) < +\infty\}.$$

By [14, Lemma 1.11], every element u in $SBV(\Omega;\mathbb{R}^d)$ satisfying $\nabla u = 0$ a.e. in Ω can be expressed with the help of a Caccioppoli partition $\{E_i\}_{i\in I}$ of Ω with index set $I \subset \mathbb{N}$ (see, e.g., [2, Definition 4.16]) as $u = \sum_{i\in I} a_i \mathbb{1}_{E_i}$ with suitable $a_i \in \mathbb{R}^d$.

Finally, we introduce the space

$$\begin{aligned} PC(a,b;\mathbb{R}^d) &= SBV_\infty(a,b;\mathbb{R}^d) \cap \{u \in BV(a,b;\mathbb{R}^d) : u' = 0\} \\ &= SBV_<(a,b;\mathbb{R}^d) \cap \{u \in BV(a,b;\mathbb{R}^d) : u' = 0\}, \end{aligned} \tag{2.9}$$

containing piecewise constant one-dimensional functions with values in $\mathbb{R}^d$.

2.4 Geometry of the domain

In this subsection, we specify our main assumptions on the geometry of Ω, which, as mentioned in the Introduction, will mostly be a bounded Lipschitz domain in $\mathbb{R}^2$. Let us first recall from [13, Section 3] the definitions of *locally one-dimensional* and *one-dimensional* functions.

Definition 2.1 (Locally one-dimensional functions in the e_2-direction). Let $\Omega \subset \mathbb{R}^2$ be an open set. A function $f : \Omega \to \mathbb{R}^d$ is *locally one-dimensional in the e_2-direction* if for every $x \in \Omega$, there exists an open cuboid $Q_x \subset \Omega$, containing x and with sides parallel to the standard coordinate axes, such that for all $y = (y_1, y_2)$ and all $z = (z_1, z_2) \in Q_x$,

$$f(y) = f(z) \quad \text{if } y_2 = z_2. \tag{2.10}$$

We say that f is *(globally) one-dimensional* in the e_2-direction if (2.10) holds for every $y, z \in \Omega$.

Analogous arguments to those in [13, Section 3] show that a function $f \in BV(\Omega;\mathbb{R}^d)$ satisfying $D_1 f = 0$ is locally one-dimensional in the e_2-direction. The following geometrical requirement is the counterpart of [13, Definitions 3.6 and 3.7] in our setting.

Definition 2.2 (x_1-connectedness). We say that an open set $\Omega \subset \mathbb{R}^2$ is *x_1-connected* if for every $t \in \mathbb{R}$, the set $\{x_2 = t\} \cap \Omega$ is a (possibly empty) interval.

In what follows, we always assume that the set $\Omega \subset \mathbb{R}^2$ is an x_1-connected domain. Under this geometrical assumption, the notions of locally and globally one-dimensional functions in the e_2-direction coincide. We refer to [13, Section 3] for an extended discussion on the topic, as well as for some explicit geometrical examples.

3 Asymptotic rigidity of layered structures in BV

In this section, we prove Theorem 1.1, which characterizes the asymptotic behavior of deformations of bilayered materials that correspond to rigid body motions on the stiff layers, but do not experience any further structural constraints on the softer layers. This qualitative result is not just limited to applications in crystal plasticity, but can be useful for a larger class of layered composites where fracture may occur.

We start by introducing some notation. Assume that $\Omega \subset \mathbb{R}^2$ is an x_1-connected domain. For $\varepsilon > 0$, let

$$\mathcal{B}_\varepsilon := \{u \in W^{1,1}(\Omega;\mathbb{R}^2) : \nabla u \in SO(2) \text{ in } \varepsilon Y_{\mathrm{rig}} \cap \Omega\} \tag{3.1}$$

represent the class of *layered deformations with rigid components,* and let

$$\mathcal{B}_0 := \{u \in \mathrm{BV}(\Omega;\mathbb{R}^2) : \text{there exists } (u_\varepsilon)_\varepsilon \subset W^{1,1}(\Omega;\mathbb{R}^2) \text{ with } u_\varepsilon \in \mathcal{B}_\varepsilon \text{ for all } \varepsilon \text{ such that } u_\varepsilon \overset{*}{\rightharpoonup} u \text{ in } \mathrm{BV}(\Omega;\mathbb{R}^2)\} \tag{3.2}$$

be the associated set of asymptotically attainable deformations.

We aim at proving that $\mathcal{B}_0$ coincides with the set of *asymptotically rigid deformations* given by

$$\mathcal{B} := \{u \in \mathrm{BV}(\Omega;\mathbb{R}^2) : u(x) = R(x_2)x + \psi(x_2) \text{ for a.e. } x \in \Omega \text{ with } R \in \mathrm{BV}(a_\Omega, b_\Omega; \mathrm{SO}(2)) \text{ and } \psi \in \mathrm{BV}(a_\Omega, b_\Omega;\mathbb{R}^2)\}, \tag{3.3}$$

cf. (1.1). This identity will be a consequence of Propositions 3.1 and 3.2 below.

Proposition 3.1 (Limiting behavior of maps in $\mathcal{B}_\varepsilon$). *Let* $\Omega = (0,1)\times(-1,1)$. *Then,*

$$\mathcal{B}_0 \subset \mathcal{B}, \tag{3.4}$$

where $\mathcal{B}_0$ *and* $\mathcal{B}$ *are the sets introduced in* (3.2) *and* (3.3)*, respectively.*

Proof. The proof is inspired by and generalizes ideas from [12, Proposition 2.1]. Let $u \in \mathcal{B}_0$. Then, there exists a sequence $(u_\varepsilon)_\varepsilon \subset W^{1,1}(\Omega;\mathbb{R}^2)$ satisfying $\nabla u_\varepsilon \in \mathrm{SO}(2)$ a.e. in $\varepsilon Y_{\mathrm{rig}} \cap \Omega$ for all ε, and $u_\varepsilon \overset{*}{\rightharpoonup} u$ in $\mathrm{BV}(\Omega;\mathbb{R}^2)$.

Fix $0 < \varepsilon < 1$, and let $I_\varepsilon := \{i \in \mathbb{Z} : (\mathbb{R}\times\varepsilon(i-1,i)) \cap \Omega \neq \emptyset\}$. For each $i \in I_\varepsilon$, we define a strip, P_ε^i, by setting

$$P_\varepsilon^i := (\mathbb{R}\times\varepsilon[i-1,i)) \cap \Omega.$$

Note that if $i \in \mathbb{Z}$ is such that $|i| > 1 + \lceil \frac{1}{\varepsilon} \rceil$, then $i \notin I_\varepsilon$. Moreover, defining $i_\varepsilon^+ := \max I_\varepsilon$ and $i_\varepsilon^- := \min I_\varepsilon$, then
(i) for $i_\varepsilon^- < i < i_\varepsilon^+$, P_ε^i is the union of two neighboring connected components of $\varepsilon Y_{\mathrm{rig}} \cap \Omega$ and $\varepsilon Y_{\mathrm{soft}} \cap \Omega$,
(ii) we may have $\varepsilon Y_{\mathrm{soft}} \cap P_\varepsilon^{i_\varepsilon^-} = \emptyset$ or $\varepsilon Y_{\mathrm{rig}} \cap P_\varepsilon^{i_\varepsilon^+} = \emptyset$.
From Reshetnyak's theorem, we infer that on each nonempty rigid layer $\varepsilon Y_{\mathrm{rig}} \cap P_\varepsilon^i$ with $i \in I_\varepsilon$, the gradient ∇u_ε is constant and coincides with a rotation $R_\varepsilon^i \in \mathrm{SO}(2)$. Moreover, there exists $b_\varepsilon^i \in \mathbb{R}^2$ such that $u_\varepsilon(x) = R_\varepsilon^i x + b_\varepsilon^i$ in $\varepsilon Y_{\mathrm{rig}} \cap P_\varepsilon^i$.

Using these rotations R_ε^i, we define a piecewise constant function, $\Sigma_\varepsilon : (-1,1) \to \mathbb{R}^{2\times 2}$, by setting

$$\Sigma_\varepsilon(t) = \sum_{i\in I_\varepsilon} R_\varepsilon^i \mathbb{1}_{\varepsilon[i-1,1)}(t) \quad \text{for } t \in (-1,1),$$

where $R_\varepsilon^{i_\varepsilon^+} := R_\varepsilon^{i_\varepsilon^+ - 1}$ if $\varepsilon Y_{\mathrm{rig}} \cap P_\varepsilon^{i_\varepsilon^+} = \emptyset$. We claim that there exist a subsequence of $(\Sigma_\varepsilon)_\varepsilon$, which we do not relabel, and a function $R \in \mathrm{BV}(-1,1;\mathrm{SO}(2))$ such that

$$\Sigma_\varepsilon \to R \quad \text{in } L^1(-1,1;\mathbb{R}^{2\times 2}). \tag{3.5}$$

To prove (3.5), we first observe that the total variation of the one-dimensional function Σ_ε coincides with its pointwise variation, and can be calculated to be

$$|D\Sigma_\varepsilon|(-1,1) = \sum_{i\in I_\varepsilon\setminus\{i_\varepsilon^-\}} |R_\varepsilon^i - R_\varepsilon^{i-1}| = \sqrt{2} \sum_{i\in I_\varepsilon\setminus\{i_\varepsilon^-\}} |R_\varepsilon^i e_1 - R_\varepsilon^{i-1} e_1|. \tag{3.6}$$

Next, we show that the right-hand side of (3.6) is uniformly bounded. By linear interpolation in the x_2-direction on the softer layers, it follows for all $i \in I_\varepsilon\setminus\{i_\varepsilon^-\}$ if $\varepsilon Y_{\mathrm{rig}} \cap P_\varepsilon^{i_\varepsilon^+} \neq \emptyset$ and $i \in I_\varepsilon\setminus\{i_\varepsilon^\pm\}$ if $\varepsilon Y_{\mathrm{rig}} \cap P_\varepsilon^{i_\varepsilon^+} = \emptyset$ that

$$\begin{aligned}\int_{\varepsilon Y_{\mathrm{soft}}\cap P_\varepsilon^i} |\nabla u_\varepsilon e_2|\,\mathrm{d}x &= \int_0^1 \int_{\varepsilon(i-1)}^{\varepsilon(i-1+\lambda)} |\partial_2 u_\varepsilon(x_1,x_2)|\,\mathrm{d}x_2\,\mathrm{d}x_1 \\ &\geq \int_0^1 |u_\varepsilon(x_1,\varepsilon(i-1+\lambda)) - u_\varepsilon(x_1,\varepsilon(i-1))|\,\mathrm{d}x_1 \\ &= \int_0^1 |(R_\varepsilon^i e_1 - R_\varepsilon^{i-1} e_1)x_1 + d_\varepsilon^i|\,\mathrm{d}x_1 \geq \frac{1}{4}|R_\varepsilon^i e_1 - R_\varepsilon^{i-1} e_1|,\end{aligned} \tag{3.7}$$

where $d_\varepsilon^i \in \mathbb{R}^2$. The first estimate is a consequence of Jensen's inequality, and optimization over translations yields the second one. To be more precise, the last estimate in (3.7) is based on the observation that for any given $a \in \mathbb{R}^2\backslash\{0\}$,

$$\min_{b\in\mathbb{R}^2}\int_0^1 |ta+b|\,\mathrm{d}t = \min_{\alpha,\beta\in\mathbb{R}}\int_0^1 |(t+\alpha)a+\beta a^\perp|\,\mathrm{d}t = |a|\min_{\alpha\in\mathbb{R}}\int_0^1 |t+\alpha|\,\mathrm{d}t = \frac{|a|}{4}.$$

From (3.6) and (3.7), since $(u_\varepsilon)_\varepsilon \subset W^{1,1}(\Omega;\mathbb{R}^2)$ as a weakly* converging sequence is uniformly bounded in $\mathrm{BV}(\Omega;\mathbb{R}^2)$, and recalling that $R_\varepsilon^{i_\varepsilon^+} = R_\varepsilon^{i_\varepsilon^+-1}$ if $\varepsilon Y_{\mathrm{rig}} \cap P_\varepsilon^{i_\varepsilon^+} = \emptyset$, we conclude that

$$|D\Sigma_\varepsilon|(-1,1) \le 4\sqrt{2}\int_\Omega |\nabla u_\varepsilon|\,\mathrm{d}x \le C. \tag{3.8}$$

The convergence in (3.5) follows now from the weak* relative compactness of bounded sequences in $\mathrm{BV}(-1,1;\mathbb{R}^{2\times2})$ (see Section 2.2), together with the fact that the set of integrable SO(2)-valued functions is closed with respect to strong L^1-convergence. The latter guarantees that the limit function $R \in \mathrm{BV}(-1,1;\mathbb{R}^{2\times2})$ takes values only in SO(2).

Next, we show that there is $\psi \in \mathrm{BV}(-1,1;\mathbb{R}^2)$ such that

$$u(x) = R(x_2)x + \psi(x_2) \tag{3.9}$$

for a.e. $x \in \Omega$, which implies that $u \in \mathcal{B}$ and concludes the proof. To this end, we define auxiliary functions $\sigma_\varepsilon, b_\varepsilon \in L^\infty(\Omega;\mathbb{R}^2)$ for $\varepsilon > 0$ by setting

$$\sigma_\varepsilon(x) = \sum_{i\in I_\varepsilon}(R_\varepsilon^i x)\mathbb{1}_{P_\varepsilon^i}(x) \quad \text{and} \quad b_\varepsilon(x) = \sum_{i\in I_\varepsilon} b_\varepsilon^i \mathbb{1}_{P_\varepsilon^i}(x)$$

for $x \in \Omega$, where $R_\varepsilon^{i_\varepsilon^+} := R_\varepsilon^{i_\varepsilon^+-1}$ and $b_\varepsilon^{i_\varepsilon^+} := b_\varepsilon^{i_\varepsilon^+-1}$ if $\varepsilon Y_{\mathrm{rig}} \cap P_\varepsilon^{i_\varepsilon^+} = \emptyset$. Further, let $w_\varepsilon := \sigma_\varepsilon + b_\varepsilon$.

By Poincaré's inequality applied in the x_2-direction, we obtain

$$\begin{aligned}\int_\Omega |u_\varepsilon - w_\varepsilon|\,\mathrm{d}x &= \sum_{\substack{i\in I_\varepsilon\\ \varepsilon Y_{\mathrm{soft}}\cap P_\varepsilon^i\neq\emptyset}} \int_0^1 \int_{\max\{\varepsilon(i-1),-1\}}^{\min\{\varepsilon(i-1+\lambda),1\}} |u_\varepsilon - w_\varepsilon|\,\mathrm{d}x_2\,\mathrm{d}x_1\\ &\le \varepsilon\lambda \sum_{i\in I_\varepsilon}\int_{\varepsilon Y_{\mathrm{soft}}\cap P_\varepsilon^i} |\partial_2 u_\varepsilon - R_\varepsilon^i e_2|\,\mathrm{d}x \le \varepsilon\lambda(\|u_\varepsilon\|_{W^{1,1}(\Omega;\mathbb{R}^2)} + |\Omega|) \le C\varepsilon.\end{aligned}$$

Consequently,

$$w_\varepsilon \to u \quad \text{in } L^1(\Omega;\mathbb{R}^2). \tag{3.10}$$

Moreover, for $x \in \Omega$,

$$|\sigma_\varepsilon(x) - R(x_2)x| \le \Big|\sum_{i\in I_\varepsilon}(R_\varepsilon^i - R(x_2))\mathbb{1}_{P_\varepsilon^i}(x)\Big||x| \le \sqrt{2}|\Sigma_\varepsilon(x_2) - R(x_2)|,$$

which, together with (3.5), proves that

$$\sigma_\varepsilon \to \sigma \quad \text{in } L^1(\Omega;\mathbb{R}^2), \tag{3.11}$$

where $\sigma(x) := R(x_2)x \in \mathrm{BV}(\Omega;\mathbb{R}^2)$.

Finally, exploiting (3.10) and (3.11), we conclude that there exists $b \in \mathrm{BV}(\Omega;\mathbb{R}^2)$ such that $b_\varepsilon \to b$ in $L^1(\Omega;\mathbb{R}^2)$. In view of the one-dimensional character of the stripes P_ε^i, we infer that $\partial_1 b = 0$. Eventually, identifying b with a function $\psi \in \mathrm{BV}(-1,1;\mathbb{R}^2)$ yields (3.9). □

Next, we prove that the converse inclusion of (3.4) holds. In the following, let I_{rig} be the projection of Y_{rig} onto the second component; that is, I_{rig} corresponds to the 1-periodic extension of the interval $[\lambda, 1)$. Analogously, we write I_{soft} for the 1-periodic extension of $[0,\lambda)$.

Proposition 3.2 (Approximation of maps in $\mathcal{B}$). *Let* $\Omega = (0,1) \times (-1,1)$. *Then,*

$$\mathcal{B}_0 \supset \mathcal{B}. \tag{3.12}$$

Here, $\mathcal{B}_0$ *and* $\mathcal{B}$ *are the sets from* (3.2) *and* (3.3), *respectively.*

Proof. Let $u \in \mathcal{B}$, and let $R \in \mathrm{BV}(-1,1;\mathrm{SO}(2))$ and $\psi \in \mathrm{BV}(-1,1;\mathbb{R}^2)$ be such that

$$u(x) = R(x_2)x + \psi(x_2)$$

for a.e. $x \in \Omega$. Using Lemma 3.3 below, as well as the fact that strict convergence implies weak* convergence in BV, we construct sequences $(R_\varepsilon)_\varepsilon \subset W^{1,\infty}(-1,1;\mathrm{SO}(2))$ and $(\psi_\varepsilon)_\varepsilon \subset W^{1,\infty}(-1,1;\mathbb{R}^2)$ such that

$$R_\varepsilon' = 0 \quad \text{on } \varepsilon I_{\mathrm{rig}} \cap (-1,1) \quad \text{and} \quad \psi_\varepsilon' = 0 \quad \text{on } \varepsilon I_{\mathrm{rig}} \cap (-1,1), \tag{3.13}$$

$$R_\varepsilon \overset{*}{\rightharpoonup} R \quad \text{in } \mathrm{BV}(-1,1;\mathbb{R}^{2\times 2}) \quad \text{and} \quad \psi_\varepsilon \overset{*}{\rightharpoonup} \psi \quad \text{in } \mathrm{BV}(-1,1;\mathbb{R}^2). \tag{3.14}$$

Define $u_\varepsilon(x) := R_\varepsilon(x_2)x + \psi_\varepsilon(x_2)$ for $x \in \Omega$. Then, $u_\varepsilon \in W^{1,\infty}(\Omega;\mathbb{R}^2)$ for every ε, with

$$\nabla u_\varepsilon(x) = R_\varepsilon(x_2) + R_\varepsilon'(x_2)x \otimes e_2 + \psi_\varepsilon'(x_2) \otimes e_2$$

for a.e. $x \in \Omega$. In particular, $\nabla u_\varepsilon = R_\varepsilon \in \mathrm{SO}(2)$ a.e. in $\varepsilon Y_{\mathrm{rig}} \cap \Omega$ by (3.13); hence, $u_\varepsilon \in \mathcal{B}_\varepsilon$. Moreover, we have $\sup_\varepsilon \|\nabla u_\varepsilon\|_{L^1(\Omega;\mathbb{R}^{2\times 2})} < \infty$ and $u_\varepsilon \to u$ in $L^1(\Omega;\mathbb{R}^2)$ by (3.14), from which we conclude that $u_\varepsilon \overset{*}{\rightharpoonup} u$ in $\mathrm{BV}(\Omega;\mathbb{R}^2)$. This completes the proof. □

The next lemma states a one-dimensional approximation result of BV-maps by Lipschitz functions that are constant on $\varepsilon I_{\mathrm{rig}}$, which was an important ingredient in the previous proof.

Lemma 3.3 (One-dimensional approximation by maps constant on $\varepsilon I_{\mathrm{rig}}$). *Let* $I = (a,b) \subset \mathbb{R}$ *and* $w \in \mathrm{BV}(I;\mathbb{R}^d)$. *Then, there exists a sequence* $(w_\varepsilon)_\varepsilon \subset W^{1,\infty}(I;\mathbb{R}^d)$ *with the following three properties:*
(i) $w_\varepsilon \to w$ *in* $L^1(I;\mathbb{R}^d)$,
(ii) $\int_I |w_\varepsilon'|\,\mathrm{d}t \to |Dw|(I)$,
(iii) $w_\varepsilon' = 0$ *on* $\varepsilon I_{\mathrm{rig}} \cap I$.
If $R \in \mathrm{BV}(I;\mathrm{SO}(2))$, *then there exists a sequence* $(R_\varepsilon)_\varepsilon \subset W^{1,\infty}(I;\mathrm{SO}(2))$ *such that* $R_\varepsilon \overset{*}{\rightharpoonup} R$ *in* $\mathrm{BV}(I;\mathrm{SO}(2))$ *and* $R_\varepsilon' = 0$ *on* $\varepsilon I_{\mathrm{rig}} \cap I$.

Proof. Let $w \in \mathrm{BV}(I;\mathbb{R}^d)$. By [2, Theorem 3.9, Remark 3.22], w can be approximated by a sequence of smooth functions $(v_\delta)_\delta \subset C^\infty(\bar{I};\mathbb{R}^d)$ in the sense of strict convergence in BV; that is,

$$v_\delta \to w \quad \text{in } L^1(I;\mathbb{R}^d) \qquad \text{and} \qquad \int_I |v_\delta'|\,\mathrm{d}t \to |Dw|(I) \tag{3.15}$$

as $\delta \to 0$. To obtain property (iii), we will reparametrize v_δ so that it is *stopped* on the set $\varepsilon I_{\mathrm{rig}}$ and *accelerated* otherwise, and eventually apply a diagonalization argument.

We start by introducing for every $\varepsilon > 0$ a Lipschitz function $\varphi_\varepsilon : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi_\varepsilon(t) := \begin{cases} \dfrac{1}{\lambda}(t - i\varepsilon) + i\varepsilon & \text{if } i\varepsilon \le t \le i\varepsilon + \lambda\varepsilon, \\ (i+1)\varepsilon & \text{if } i\varepsilon + \lambda\varepsilon \le t < \varepsilon(i+1), \end{cases}$$

for each $i \in \mathbb{Z}$ and $t \in \varepsilon[i, i+1)$. For all $t \in \mathbb{R}$, we have $t \le \varphi_\varepsilon(t) \le t + \varepsilon(1-\lambda)$ and $\varphi_\varepsilon'(t) = \psi(\frac{t}{\varepsilon})$, where ψ is the 1-periodic function such that $\psi(t) = \frac{1}{\lambda}$ if $0 \le t \le \lambda$, and $\psi(t) = 0$ if $\lambda < t < 1$. By the Riemann–Lebesgue lemma on weak convergence of periodically oscillating sequences, it follows that $\psi(\frac{\cdot}{\varepsilon}) \overset{*}{\rightharpoonup} 1$ in $L^\infty(\mathbb{R})$. Thus, $\varphi_\varepsilon \overset{*}{\rightharpoonup} \varphi$ in $W^{1,\infty}(\mathbb{R})$, where $\varphi(t) := t$. In particular, φ_ε converges uniformly to φ in $\mathbb{R}$.

Next, we define for $\varepsilon > 0$ sufficiently small a Lipschitz function $\tilde{\varphi}_\varepsilon : \bar{I} \to \bar{I}$ by setting

$$\tilde{\varphi}_\varepsilon(t) := \begin{cases} \varphi_\varepsilon(t) & \text{if } a \le t \le b_\varepsilon, \\ b & \text{if } b_\varepsilon \le t \le b, \end{cases}$$

where $b_\varepsilon \in (a, b]$ is such that $\varphi_\varepsilon(b_\varepsilon) = b$. Note that by the definition of φ_ε, there exists at least one such b_ε provided that $0 < \varepsilon < \frac{b-a}{1-\lambda}$. We claim that $b_\varepsilon \to b$ as $\varepsilon \to 0$. In fact, extracting a subsequence if necessary, we have $b_\varepsilon \to c$ for some $c \in [a, b]$. Then,

$$|b - c| = |\varphi_\varepsilon(b_\varepsilon) - \varphi(c)| \leq |\varphi_\varepsilon(b_\varepsilon) - \varphi_\varepsilon(c)| + |\varphi_\varepsilon(c) - \varphi(c)| \leq \tfrac{1}{\lambda}|b_\varepsilon - c| + |\varphi_\varepsilon(c) - \varphi(c)|,$$

from which we infer that $b = c$ by letting $\varepsilon \to 0$. Because the limit does not depend on the subsequence, the whole sequence $(b_\varepsilon)_\varepsilon$ converges to b. Consequently, we have $\tilde{\varphi}_\varepsilon(t) \to \varphi(t) = t$ for all $t \in \bar{I}$, and since also $\|\tilde{\varphi}_\varepsilon\|_{W^{1,\infty}(I)} = O(1)$ as $\varepsilon \to 0$, we deduce that

$$\tilde{\varphi}_\varepsilon \overset{*}{\rightharpoonup} \varphi \text{ in } W^{1,\infty}(I) \quad \text{and} \quad \|\tilde{\varphi}_\varepsilon - \varphi\|_{L^\infty(I)} \to 0. \tag{3.16}$$

Finally, we set $w_{\varepsilon,\delta} := v_\delta \circ \tilde{\varphi}_\varepsilon \in W^{1,\infty}(I; \mathbb{R}^d)$, and observe that

$$\|w_{\varepsilon,\delta} - w\|_{L^1(I;\mathbb{R}^d)} \leq \|v_\delta \circ \tilde{\varphi}_\varepsilon - v_\delta\|_{L^1(I;\mathbb{R}^d)} + \|v_\delta - w\|_{L^1(I;\mathbb{R}^d)} \quad \text{and} \quad \int_I |w'_{\varepsilon,\delta}| \,\mathrm{d}t = \int_I |v'_\delta \circ \tilde{\varphi}_\varepsilon| \, \tilde{\varphi}'_\varepsilon \,\mathrm{d}t.$$

Hence, by (3.15), (3.16), the boundedness of each v_δ and v'_δ, and a weak-strong convergence argument, it follows that

$$\lim_{\delta\to 0} \lim_{\varepsilon\to 0} \|w_{\varepsilon,\delta} - w\|_{L^1(I;\mathbb{R}^d)} = 0, \tag{3.17}$$

$$\lim_{\delta\to 0} \lim_{\varepsilon\to 0} \int_I |w'_{\varepsilon,\delta}| \,\mathrm{d}t = \lim_{\delta\to 0} \int_I |v'_\delta \circ \varphi| \, \varphi' \,\mathrm{d}t = \lim_{\delta\to 0} \int_I |v'_\delta| \,\mathrm{d}t = |Dw|(I). \tag{3.18}$$

In view of (3.17) and (3.18), we apply Attouch's diagonalization lemma [4] to find a sequence $(w_\varepsilon)_\varepsilon \subset W^{1,\infty}(I; \mathbb{R}^d)$ with $w_\varepsilon := w_{\varepsilon,\delta(\varepsilon)}$ satisfying (i) and (ii). We observe further that each w_ε satisfies (iii) by construction.

To conclude, we address the issue of constraint-preserving approximations for $R \in \mathrm{BV}(I; \mathrm{SO}(2))$. In this case, we define $\theta(t) := \arccos(R(t)e_1 \cdot e_1)$ for every $t \in I$. By the regularity of R, it follows directly that $\theta \in \mathrm{BV}(I)$. Applying to θ the construction described in the first part of the proof, we identify a sequence $(\theta_\varepsilon)_\varepsilon \subset W^{1,\infty}(I)$, satisfying
(i*) $\theta_\varepsilon \to \theta$ in $L^1(I)$,
(ii*) $\int_I |\theta'_\varepsilon| \,\mathrm{d}t \to |D\theta|(I)$,
(iii*) $\theta'_\varepsilon = 0$ on $\varepsilon I_{\mathrm{rig}} \cap I$.
For every $\varepsilon > 0$, we consider the map

$$R_\varepsilon := \begin{bmatrix} \cos(\theta_\varepsilon) & -\sin(\theta_\varepsilon) \\ \sin(\theta_\varepsilon) & \cos(\theta_\varepsilon) \end{bmatrix}.$$

The regularity of R_ε as well as the property that $R'_\varepsilon = 0$ on $\varepsilon I_{\mathrm{rig}} \cap I$ follow by the regularity of θ_ε and by (iii*). The weak* convergence in BV is a consequence of the bound $\int_I |R'_\varepsilon| \,\mathrm{d}t \leq \int_I |\theta'_\varepsilon| \,\mathrm{d}t$, and of the observation that $R_\varepsilon \to R$ strongly in $L^1(I; \mathrm{SO}(2))$ by (i*) and by the Lipschitz regularity of $\cos(\cdot)$ and $\sin(\cdot)$. □

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. In view of the discussion on locally and globally one-dimensional functions in Section 2.4, it suffices to prove the statement on rectangles with sides parallel to the axes. A simple modification of the proofs of Propositions 3.1 and 3.2 shows that these results hold for any such rectangle. Then, Theorem 1.1 follows by extension and exhaustion arguments in the spirit of [13, Lemma A.2]. □

Remark 3.4 (The higher-dimensional setting). We point out that the results of Theorem 1.1 continue to hold for domains $\Omega \subset \mathbb{R}^N$, $N \in \mathbb{N}$, satisfying the flatness and cross-connectedness assumptions in [13, Definitions 3.6 and 3.7]. We omit the proof here as it follows from that of Theorem 1.1 up to minor adaptations. Notice in particular that [12, Lemma A1] provides a higher-dimensional version of (3.7). Moreover, the one-dimensional approximation in Lemma 3.3 for $R \in \mathrm{BV}(I; \mathrm{SO}(N))$ can be proved by replacing the density argument leading to (3.15) by its analogue for BV-functions with values on manifolds, see [28, Theorem 1.2].

In fact, by [28, Theorem 1.2], we find a sequence $(S_\delta)_\delta \in C^\infty(I; \mathrm{SO}(N))$ satisfying (3.15) with $\mathcal{E}_{\mathrm{TV}}(R)$ in place of $|DR|(I)$, where $\mathcal{E}_{\mathrm{TV}}(R)$ stands for a certain total variation of R that takes into account the manifold structure, and we refer to [28] for its the precise definition. On the other hand, an appropriate truncation of $\tilde{\varphi}_\varepsilon$ near $t = a$ and $t = b$, which we do not relabel, leads to a function $\tilde{\varphi}_\varepsilon$ mapping $\bar{I}$ into I and satisfying (3.16). In particular, $R_{\varepsilon,\delta} := S_\delta \circ \tilde{\varphi}_\varepsilon \in W^{1,\infty}(I; \mathrm{SO}(N))$, and a similar diagonalization argument as in Lemma 3.3 yields a sequence $(R_\varepsilon)_\varepsilon \subset W^{1,\infty}(I; \mathrm{SO}(N))$ that converges to R weakly* in $\mathrm{BV}(I; \mathbb{R}^{N\times N})$ and satisfies $R'_\varepsilon = 0$ on $\varepsilon I_{\mathrm{rig}} \cap I$.

We conclude this section by characterizing two special subsets of $\mathcal{B}$ (see (3.3)), which will be useful in the following. Using (2.6), it can be checked that

$$\begin{aligned}\mathcal{B} \cap W^{1,1}(\Omega; \mathbb{R}^2) = \{u \in W^{1,1}(\Omega; \mathbb{R}^2) : u(x) = R(x_2)x + \psi(x_2) \text{ for a.e. } x \in \Omega,\\ \text{with } R \in W^{1,1}(a_\Omega, b_\Omega; \mathrm{SO}(2)) \text{ and } \psi \in W^{1,1}(a_\Omega, b_\Omega; \mathbb{R}^2)\}\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}\mathcal{B} \cap \mathrm{SBV}(\Omega; \mathbb{R}^2) = \{u \in \mathrm{SBV}(\Omega; \mathbb{R}^2) : u(x) = R(x_2)x + \psi(x_2) \text{ for a.e. } x \in \Omega,\\ \text{with } R \in \mathrm{SBV}(a_\Omega, b_\Omega; \mathrm{SO}(2)) \text{ and } \psi \in \mathrm{SBV}(a_\Omega, b_\Omega; \mathbb{R}^2)\}.\end{aligned} \tag{3.20}$$

By definition, and accounting for the fact that R takes values in SO(2), the jump set of $u \in \mathcal{B} \cap \mathrm{SBV}(\Omega; \mathbb{R}^2)$ is related to the jump sets of R and ψ via

$$J_u = [(c_\Omega, d_\Omega) \times (J_R \cup J_\psi)] \cap \Omega,$$

cf. (1.2).

4 Asymptotic behavior of admissible layered deformations

In this section, we prove Theorem 1.3, which characterizes the asymptotic behavior of deformations of bilayered materials that coincide with rigid body rotations on the stiffer layers, and are subject to a single slip constraint on the softer layers. The latter is described with the help of the set

$$\begin{aligned}\mathcal{M}_{e_1} &= \{F \in \mathbb{R}^{2\times 2} : \det F = 1 \text{ and } |Fe_1| = 1\}\\ &= \{F \in \mathbb{R}^{2\times 2} : F = R(\mathbb{I} + \gamma e_1 \otimes e_2) \text{ with } R \in \mathrm{SO}(2) \text{ and } \gamma \in \mathbb{R}\}.\end{aligned} \tag{4.1}$$

As in the previous section, we consider $\Omega = (0, 1) \times (-1, 1)$ for simplicity. The results for general x_1-connected domains follow as in the proof of Theorem 1.1.

Using the representations of $\mathcal{M}_{e_1}$ in (4.1) and recalling the sets $\mathcal{B}_\varepsilon$ introduced in (3.1), the sets of admissible layered deformations defined in (1.6) admit the equivalent representations

$$\begin{aligned}\mathcal{A}_\varepsilon &= \mathcal{B}_\varepsilon \cap \{u \in W^{1,1}(\Omega; \mathbb{R}^2) : \nabla u \in \mathcal{M}_{e_1} \text{ a.e. in } \Omega\}\\ &= \{u \in W^{1,1}(\Omega; \mathbb{R}^2) : \nabla u = R(\mathbb{I} + \gamma e_1 \otimes e_2) \text{ with } R \in L^\infty(\Omega; \mathrm{SO}(2)) \text{ and}\\ &\qquad \gamma \in L^1(\Omega) \text{ such that } \gamma = 0 \text{ in } \varepsilon Y_{\mathrm{rig}} \cap \Omega\}.\end{aligned} \tag{4.2}$$

In the sequel, according to the context, we will always adopt the most convenient representation.

In analogy with $\mathcal{B}_0$ defined in (3.2), we introduce the set

$$\begin{aligned}\mathcal{A}_0 := \{u \in \mathrm{BV}(\Omega; \mathbb{R}^2) : \text{there exists } (u_\varepsilon)_\varepsilon \subset W^{1,1}(\Omega; \mathbb{R}^2) \text{ with } u_\varepsilon \in \mathcal{A}_\varepsilon \text{ for all } \varepsilon\\ \text{such that } u_\varepsilon \overset{*}{\rightharpoonup} u \text{ in } \mathrm{BV}(\Omega; \mathbb{R}^2)\}\end{aligned} \tag{4.3}$$

of *asymptotically admissible deformations*. We aim at characterizing $\mathcal{A}_0$, or suitable subclasses thereof, in terms of the set $\mathcal{A}$ introduced in (1.11). Note that

$$\mathcal{A} = \mathcal{B} \cap \{u \in \mathrm{BV}(\Omega; \mathbb{R}^2) : \det \nabla u = 1 \text{ a.e. in } \Omega\}, \tag{4.4}$$

where $\mathcal{B}$ is given by (3.3). Moreover, recalling the notation for the distributional derivative of one-dimensional BV-functions discussed in Section 2.2, we can equivalently express $\mathcal{A}$ as follows.

Proposition 4.1. *Let $\Omega = (0,1) \times (-1,1)$. Then, $\mathcal{A}$ from* (1.11) *admits the following two alternative representations:*

$$\begin{aligned}\mathcal{A} = \{u \in \mathrm{BV}(\Omega;\mathbb{R}^2) : \nabla u(x) = R(x_2)(\mathbb{I} + \gamma(x_2)e_1 \otimes e_2) \text{ for a.e. } x \in \Omega, \text{ with } R \in \mathrm{BV}(-1,1;\mathrm{SO}(2))\\ \text{such that } R' = 0 \text{ a.e. in } (-1,1), \gamma \in L^1(-1,1), \text{ and } (D^s u)e_1 = 0\}\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}\mathcal{A} = \{u \in \mathrm{BV}(\Omega;\mathbb{R}^2) : u(x) = R(x_2)x + \psi(x_2) \text{ for a.e. } x \in \Omega, \text{ with } R \in \mathrm{BV}(-1,1;\mathrm{SO}(2))\\ \text{and } \psi \in \mathrm{BV}(-1,1;\mathbb{R}^2) \text{ such that } \psi' \cdot Re_2 = 0 \text{ and } R' = 0 \text{ a.e. in } (-1,1)\}.\end{aligned} \tag{4.6}$$

Proof. Let $\tilde{\mathcal{A}}$ and $\hat{\mathcal{A}}$ denote the sets on the right-hand side of (4.5) and (4.6), respectively. We will show that $\mathcal{A} \subset \tilde{\mathcal{A}} \cap \hat{\mathcal{A}}$, $\hat{\mathcal{A}} \subset \mathcal{A}$, and $\tilde{\mathcal{A}} \subset \hat{\mathcal{A}}$, from which (4.5) and (4.6) follow.

We start by proving that $\mathcal{A} \subset \tilde{\mathcal{A}} \cap \hat{\mathcal{A}}$. Fix $u \in \mathcal{A}$. Due to (2.6), we have $(D^s u)e_1 = 0$ and

$$\nabla u = R + (R'x + \psi') \otimes e_2 = R(\mathbb{I} + R^T(R'x + \psi') \otimes e_2). \tag{4.7}$$

We first observe that the condition $\det \nabla u = 1$ becomes

$$1 + R^T(R'x + \psi') \cdot e_2 = 1$$

or, equivalently,

$$(R'x + \psi') \cdot Re_2 = 0.$$

This condition, together with the independence of R, R', and ψ' on x_1, yields

$$R'e_1 \cdot Re_2 = 0 \quad \text{and} \quad (x_2 R'e_2 + \psi') \cdot Re_2 = 0. \tag{4.8}$$

Let $\theta \in \mathrm{BV}(-1,1;[-\pi,\pi])$ be as in (2.7). Then, the first condition in (4.8) gives $\theta' = 0$; consequently, also $R' = 0$. Thus, the second equation in (4.8) becomes $\psi' \cdot Re_2 = 0$, which shows that $u \in \hat{\mathcal{A}}$. Moreover, $\psi' \cdot Re_2 = 0$ is equivalent to $R^T\psi' \cdot e_2 = 0$; hence, $u \in \tilde{\mathcal{A}}$ with $\gamma := Re_1 \cdot \psi'$. Thus, $\mathcal{A} \subset \tilde{\mathcal{A}} \cap \hat{\mathcal{A}}$.

Next, we observe that if $u \in \hat{\mathcal{A}}$, then, using (4.7), we have

$$\det \nabla u = 1 + R^T(R'x + \psi') \cdot e_2 = 1 + R^T\psi' \cdot e_2 = 1 + \psi' \cdot Re_2 = 1.$$

Hence, $u \in \mathcal{A}$, which shows that $\hat{\mathcal{A}} \subset \mathcal{A}$.

Finally, we prove that $\tilde{\mathcal{A}} \subset \hat{\mathcal{A}}$. Let $u \in \tilde{\mathcal{A}}$. Then, $(Du)e_1 = (\nabla u)e_1 \mathcal{L}^2 \lfloor \Omega + (D^s u)e_1 = Re_1 \mathcal{L}^2 \lfloor \Omega$. By this identity and the Du Bois–Reymond lemma (see [32], for instance), we can find $\phi \in \mathrm{BV}(-1,1;\mathbb{R}^2)$ such that

$$u(x) = R(x_2)x_1 e_1 + \phi(x_2).$$

In particular, $\nabla u(x) = R(x_2)e_1 \otimes e_1 + (R'(x_2)x_1 e_1 + \phi'(x_2)) \otimes e_2 = R(x_2)e_1 \otimes e_1 + \phi'(x_2) \otimes e_2$, considering that $R' = 0$. Consequently, using the expression for ∇u given by the definition of $\tilde{\mathcal{A}}$, together with the independence of R, γ, and ϕ' on x_1, we conclude that

$$\phi' = Re_2 + \gamma Re_1.$$

Finally, set $\psi(x_2) := \phi(x_2) - R(x_2)x_2 e_2$ for $x_2 \in (-1,1)$. Then, we have $\psi \in \mathrm{BV}(-1,1;\mathbb{R}^2)$, which satisfies $\psi' \cdot Re_2 = \gamma Re_1 \cdot Re_2 = 0$, because $R \in \mathrm{SO}(2)$ in $(-1,1)$, and also $u(x) = R(x_2)x + \psi(x_2)$. Thus, $u \in \hat{\mathcal{A}}$, which implies $\tilde{\mathcal{A}} \subset \hat{\mathcal{A}}$. □

The following lemma on weak continuity of Jacobian determinants for gradients in $W^{1,1}(\Omega;\mathbb{R}^2)$ with suitable additional properties will be instrumental in the proof of the inclusion $\mathcal{A}_0 \subset \mathcal{A}$.

Lemma 4.2 (Weak continuity properties of Jacobian determinants). *Suppose that $\Omega \subset \mathbb{R}^2$ is a bounded Lipschitz domain, and let $(u_\varepsilon)_\varepsilon \subset W^{1,1}(\Omega;\mathbb{R}^2)$ be a uniformly bounded sequence satisfying* $\det \nabla u_\varepsilon = 1$ *a.e. in Ω for all ε and*

$$\|\partial_1 u_\varepsilon\|_{L^\infty(\Omega;\mathbb{R}^2)} \le C, \tag{4.9}$$

where C is a positive constant independent of ε. If $u_\varepsilon \to u$ in $L^1(\Omega;\mathbb{R}^2)$ for some $u \in \mathrm{BV}(\Omega;\mathbb{R}^2)$, then $\det \nabla u = 1$ *a.e. in Ω.*

Proof. The claim in Lemma 4.2 would be an immediate consequence of [26, Theorem 2] if in place of (4.9), we required

$$(\operatorname{adj} \nabla u_\varepsilon)_\varepsilon \subset L^2(\Omega; \mathbb{R}^{2\times 2}), \tag{4.10}$$

which, because of the structure of the adjoint matrix in this two-dimensional setting, is equivalent to $\nabla u_\varepsilon \in L^2(\Omega; \mathbb{R}^{2\times 2})$ for all ε. Even though we are not assuming this here, it is still possible to validate the arguments of [26, Proof of Theorem 2] in our context, as we detail next.

Since $|\operatorname{adj} \nabla u_\varepsilon| = |\nabla u_\varepsilon|$, it can be checked that in order to mimic the proof of [26, Theorem 2] with $N = 2$, we are only left to prove the following: If $(\varphi_j)_{j\in\mathbb{N}}$ is a sequence of standard mollifiers and Ω' is an arbitrary open set compactly contained in Ω, then $(\det \nabla u_{\varepsilon,j})_{j\in\mathbb{N}}$ converges to $\det \nabla u_\varepsilon$ in $L^1(\Omega')$ as $j \to \infty$ for all ε, where $u_{\varepsilon,j} := \varphi_j * u_\varepsilon$.

In Step 4 of the proof of [26, Theorem 2], this convergence is a consequence of the Vitali–Lebesgue lemma using (4.10), the bound $|\det A| \le |\operatorname{adj} A|^2$ for all $A \in \mathbb{R}^{2\times 2}$ (see [26, (7)]), and well-known properties of mollifiers. Here, similar arguments can be invoked, but instead of the estimate $|\det A| \le |\operatorname{adj} A|^2$ for $A \in \mathbb{R}^{2\times 2}$, we use the fact that (4.9) yields

$$|\det \nabla u_{\varepsilon,j}| = |(\partial_1 u_{\varepsilon,j})^\perp \cdot \partial_2 u_{\varepsilon,j}| \le C|\partial_2 u_{\varepsilon,j}| \le C|\nabla u_{\varepsilon,j}|$$

a.e. in Ω. Hence, since $u_{\varepsilon,j} \to u_\varepsilon$ in $W^{1,1}(\Omega'; \mathbb{R}^2)$ and pointwise a.e. in Ω as $j \to \infty$, we conclude that $(\det \nabla u_{\varepsilon,j})_{j\in\mathbb{N}}$ converges to $\det \nabla u_\varepsilon$ in $L^1(\Omega')$ as $j \to \infty$ for all ε by the Vitali–Lebesgue lemma. □

We obtain from the following proposition that weak* limits of sequences in $\mathcal{A}_\varepsilon$ belong to $\mathcal{A}$.

Proposition 4.3 (Asymptotic behavior of sequences in $\mathcal{A}_\varepsilon$). *Let* $\Omega = (0, 1) \times (-1, 1)$. *Then,*

$$\mathcal{A}_0 \subset \mathcal{A}, \tag{4.11}$$

where $\mathcal{A}_0$ *and* $\mathcal{A}$ *are the sets introduced in* (4.3) *and* (1.11), *respectively.*

Proof. The statement follows from the inclusion $\mathcal{A}_\varepsilon \subset \mathcal{B}_\varepsilon$ (see (4.2)) and the identity (4.4) in conjunction with Proposition 3.1 and Lemma 4.2, observing that the condition $\nabla u_\varepsilon \in \mathcal{M}_{e_1}$ a.e. in Ω guarantees $|\partial_1 u_\varepsilon| = |\nabla u_\varepsilon e_1| = 1$ a.e. in Ω, and hence $\|\partial_1 u_\varepsilon\|_{L^\infty(\Omega;\mathbb{R}^2)} = 1$ for any ε. □

The question whether the set $\mathcal{A}$ can be further identified as limiting set for sequences in $\mathcal{A}_\varepsilon$, namely, whether the equality $\mathcal{A}_0 = \mathcal{A}$ is true, cannot be answered at this point. However, as stated in Theorem 1.3, the inclusions $\mathcal{A}_0 \supset \mathcal{A} \cap SBV_<(\Omega; \mathbb{R}^2)$ and $\mathcal{A}_0 \supset \mathcal{A}^\parallel$ hold. Before proving these inclusions, we discuss a further characterization of some special subsets of $\mathcal{A}$.

Remark 4.4 (Structure of subsets of $\mathcal{A}$). Similarly to (3.19) and (3.20), using fine properties of one-dimensional BV-functions, the sets $\mathcal{A} \cap W^{1,1}(\Omega; \mathbb{R}^2)$, $\mathcal{A} \cap SBV(\Omega; \mathbb{R}^2)$, $\mathcal{A} \cap SBV_<(\Omega; \mathbb{R}^2)$, and $\mathcal{A} \cap SBV_\infty(\Omega; \mathbb{R}^2)$ can be characterized as follows.

(a) In view of (2.6) and (4.6), one observes that

$$\begin{aligned}\mathcal{A} \cap W^{1,1}(\Omega; \mathbb{R}^2) &= \{u \in W^{1,1}(\Omega; \mathbb{R}^2) : u(x) = Rx + \theta(x_2)Re_1 + c \text{ for a.e. } x \in \Omega,\\ &\qquad \text{with } R \in SO(2),\ \theta \in W^{1,1}(-1, 1),\ c \in \mathbb{R}^2\}\\ &= \{u \in W^{1,1}(\Omega; \mathbb{R}^2) : \nabla u(x) = R(\mathbb{I} + \gamma(x_2)e_1 \otimes e_2) \text{ for a.e. } x \in \Omega,\\ &\qquad \text{with } R \in SO(2),\ \gamma \in L^1(-1, 1)\}.\end{aligned}$$

Additionally, as a consequence of the construction of the recovery sequence in the Γ-convergence homogenization result of [12], we also know that

$$\begin{aligned}\mathcal{A} \cap W^{1,1}(\Omega; \mathbb{R}^2) = \{u \in W^{1,1}(\Omega; \mathbb{R}^2) : &\text{ there exists } (u_\varepsilon)_\varepsilon \subset W^{1,1}(\Omega; \mathbb{R}^2) \text{ with } u_\varepsilon \in \mathcal{A}_\varepsilon \text{ for all } \varepsilon\\ &\text{ such that } u_\varepsilon \rightharpoonup u \text{ in } W^{1,1}(\Omega; \mathbb{R}^2)\}.\end{aligned}$$

To see this, it suffices to follow Step 2 in the proof of [12, Theorem 4.1], replacing the spaces involving square integrable functions by their counterparts with integrable functions.

(b) Using (2.6) and (4.6) once more, we have

$$\begin{aligned}\mathcal{A}\cap \mathrm{SBV}(\Omega;\mathbb{R}^2) = \{u \in \mathrm{SBV}(\Omega;\mathbb{R}^2) : u(x) = R(x_2)x + \psi(x_2) \text{ for a.e. } x \in \Omega,\\ \text{with } R \in \mathrm{SBV}(-1,1;\mathrm{SO}(2)) \text{ and } \psi \in \mathrm{SBV}(-1,1;\mathbb{R}^2)\\ \text{such that } R' = 0 \text{ and } \psi' \cdot Re_2 = 0 \text{ a.e. in } (-1,1)\}.\end{aligned}$$

Note that both J_R and J_ψ are given by an at most countable union of points in $(-1, 1)$, which implies that J_u consists of at most countably many segments parallel to e_1. It is not possible to conclude that the functions R are piecewise constant according to [2, Definition 4.21], as we have, a priori, no control on $\mathcal{H}^0(J_R)$ (cf. [2, Example 4.24]).

(c) With (b) and recalling (2.9), it follows that

$$\begin{aligned}\mathcal{A}\cap \mathrm{SBV}_<(\Omega;\mathbb{R}^2) = \{u \in \mathrm{SBV}_<(\Omega;\mathbb{R}^2) : u(x) = R(x_2)x + \psi(x_2) \text{ for a.e. } x \in \Omega,\\ \text{with } R \in \mathrm{PC}(-1,1;\mathrm{SO}(2)) \text{ and } \psi \in \mathrm{SBV}_<(-1,1;\mathbb{R}^2)\\ \text{such that } \psi' \cdot Re_2 = 0 \text{ a.e. in } (-1,1)\}.\end{aligned} \tag{4.12}$$

Here, both J_R and J_ψ are finite sets of points in $(-1, 1)$, and J_u is given by a finite union of segments parallel to e_1. Alternatively, one can express $\mathcal{A}\cap \mathrm{SBV}_<(\Omega;\mathbb{R}^2)$ with the help of a finite Caccioppoli partition of Ω into finitely many horizontal strips; precisely,

$$\begin{aligned}\mathcal{A}\cap \mathrm{SBV}_<(\Omega;\mathbb{R}^2) = \{u \in \mathrm{SBV}_<(\Omega;\mathbb{R}^2) : \nabla u|_{E_i} = R_i(\mathbb{I} + \gamma_i e_1 \otimes e_2), \text{ with } \{E_i\}_{i=1}^n \text{ a partition of } \Omega\\ \text{such that } E_i = (\mathbb{R}\times I_i)\cap\Omega \text{ with } I_i \subset (-1,1) \text{ for } i = 1,\dots,n,\\ R_i \in \mathrm{SO}(2) \text{ and } \gamma_i \in L^1(E_i) \text{ with } \partial_1\gamma_i = 0 \text{ for } i = 1,\dots,n,\\ \text{and } J_u = \textstyle\bigcup_{i=1}^n(\partial E_i)\cap\Omega\}\end{aligned} \tag{4.13}$$

or

$$\begin{aligned}\mathcal{A}\cap \mathrm{SBV}_<(\Omega;\mathbb{R}^2) = \{u \in \mathrm{SBV}_<(\Omega;\mathbb{R}^2) : u = \textstyle\sum_{i=1}^n g_i \mathbb{1}_{E_i}, \text{ where } \{E_i\}_{i=1}^n \text{ is a partition of } \Omega \text{ such that,}\\ \text{for each } i = 1,\dots,n,\ E_i = (\mathbb{R}\times I_i)\cap\Omega \text{ with } I_i \subset (-1,1),\\ \text{and } g_i \in W^{1,1}(E_i;\mathbb{R}^2) \text{ is such that } \nabla g_i = R_i(\mathbb{I} + \gamma_i e_1 \otimes e_2)\\ \text{for some } R_i \in \mathrm{SO}(2) \text{ and } \gamma_i \in L^1(E_i) \text{ with } \partial_1\gamma_i = 0\}.\end{aligned} \tag{4.14}$$

The same equality as (4.12) holds replacing $\mathrm{SBV}_<$ by SBV_∞. Analogously, (4.13) holds if we replace $\mathrm{SBV}_<$ by SBV_∞ and take $\gamma_i \in L^\infty(E_i)$. Also, (4.14) holds if we replace $\mathrm{SBV}_<$ by SBV_∞ and take $g_i \in W^{1,\infty}(E_i;\mathbb{R}^2)$ and $\gamma_i \in L^\infty(E_i)$.

In the following lemma, we construct an admissible piecewise affine approximation for basic limit deformations in $\mathcal{A}\cap \mathrm{SBV}_\infty(\Omega;\mathbb{R}^2)$ with a non-trivial jump along the horizontal line at $x_2 = 0$. Based on this construction, we will then establish the inclusion $\mathcal{A}_0 \supset \mathcal{A}\cap \mathrm{SBV}_<(\Omega;\mathbb{R}^2)$ in Proposition 4.7 below.

Lemma 4.5 (Approximation of maps in $\mathcal{A}\cap \mathrm{SBV}_\infty$ with a single jump). *Let $\Omega = (0, 1)\times(-1, 1)$, and suppose that $u \in \mathcal{A}\cap \mathrm{SBV}_\infty(\Omega;\mathbb{R}^2)$ is such that $u(x) = R(x_2)x + \psi(x_2)$ for a.e. $x \in \Omega$, where*

$$R(t) := \begin{cases} R^+ & \text{if } t \in [0,1),\\ R^- & \text{if } t \in (-1,0),\end{cases} \quad \text{and} \quad \psi(t) := \begin{cases} \psi^+ & \text{if } t \in [0,1),\\ \psi^- & \text{if } t \in (-1,0),\end{cases} \quad \text{for } t \in (-1,1),$$

with some $R^\pm \in \mathrm{SO}(2)$ and $\psi^\pm \in \mathbb{R}^2$. Then, there exists a sequence $(u_\varepsilon)_\varepsilon \subset W^{1,1}(\Omega;\mathbb{R}^2)$ with $\int_\Omega u_\varepsilon\,dx = \int_\Omega u\,dx$ and $u_\varepsilon \in \mathcal{A}_\varepsilon$ for all ε, and such that $u_\varepsilon \overset{*}{\rightharpoonup} u$ in $\mathrm{BV}(\Omega;\mathbb{R}^2)$.

Proof. We start by observing that for u as in the statement of the lemma, there holds

$$Du = R\mathcal{L}^2\lfloor\Omega + [(R^+ - R^-)e_1x_1 + (\psi^+ - \psi^-)] \otimes e_2\mathcal{H}^1\lfloor((0,1)\times\{0\}). \tag{4.15}$$

Let $S \in \mathrm{SO}(2)$ be such that

(i) $S \neq R^\pm$,

(ii) Se_1 and R^+e_1 are linearly independent,

(iii) $\theta^\pm \in (-\pi, \pi)\setminus\{0\}$ is the rotation angle of $S^TR^\pm$, cf. (2.7).

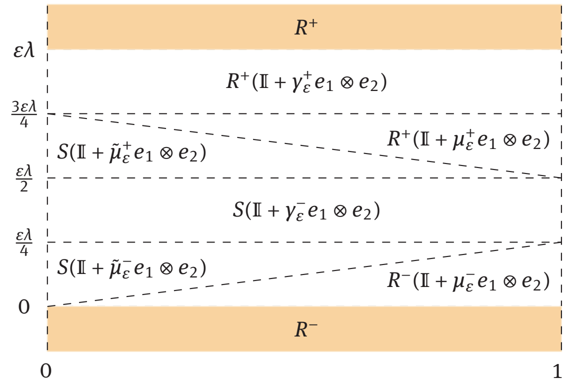


Figure 3: Construction of V_ε.

Due to (ii), there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\psi^+ - \psi^- = \alpha R^+ e_1 + \beta S e_1. \tag{4.16}$$

For each $\varepsilon > 0$, set

$$\gamma_\varepsilon^+ := \frac{4\alpha}{\varepsilon\lambda}, \quad \gamma_\varepsilon^- := \frac{4\beta}{\varepsilon\lambda}, \quad \mu_\varepsilon^\pm := \pm\frac{4}{\varepsilon\lambda} + \tan\Big(\frac{\theta^\pm}{2}\Big), \quad \tilde{\mu}_\varepsilon^\pm := \pm\frac{4}{\varepsilon\lambda} - \tan\Big(\frac{\theta^\pm}{2}\Big), \tag{4.17}$$

and let $V_\varepsilon \in L^1(\Omega;\mathbb{R}^{2\times 2})$ be the function defined by

$$V_\varepsilon(x) = \begin{cases} R^+ & \text{if } x \in (0,1)\times(\varepsilon\lambda, 1),\\ R^+(\mathbb{I}+\gamma_\varepsilon^+ e_1\otimes e_2) & \text{if } x \in (0,1)\times(\frac{3\varepsilon\lambda}{4}, \varepsilon\lambda),\\ R^+(\mathbb{I}+\mu_\varepsilon^+ e_1\otimes e_2) & \text{if } x_1 \in (0,1) \text{ and } x_2 \in (-\frac{\varepsilon\lambda}{4}x_1 + \frac{3\varepsilon\lambda}{4}, \frac{3\varepsilon\lambda}{4}),\\ S(\mathbb{I}+\tilde{\mu}_\varepsilon^+ e_1\otimes e_2) & \text{if } x_1 \in (0,1) \text{ and } x_2 \in (\frac{\varepsilon\lambda}{2}, -\frac{\varepsilon\lambda}{4}x_1 + \frac{3\varepsilon\lambda}{4}),\\ S(\mathbb{I}+\gamma_\varepsilon^- e_1\otimes e_2) & \text{if } x \in (0,1)\times(\frac{\varepsilon\lambda}{4}, \frac{\varepsilon\lambda}{2}),\\ S(\mathbb{I}+\tilde{\mu}_\varepsilon^- e_1\otimes e_2) & \text{if } x_1 \in (0,1) \text{ and } x_2 \in (\frac{\varepsilon\lambda}{4}x_1, \frac{\varepsilon\lambda}{4}),\\ R^-(\mathbb{I}+\mu_\varepsilon^- e_1\otimes e_2) & \text{if } x \in (0,1) \text{ and } x_2 \in (0, \frac{\varepsilon\lambda}{4}x_1),\\ R^- & \text{if } x \in (0,1)\times(-1,0), \end{cases} \tag{4.18}$$

see Figure 3.

By construction, each function V_ε takes values only in $\mathcal{M}_{e_1}$, and its piecewise definition is chosen such that neighboring matrices in Figure 3 are rank-one-connected along their separating lines according to [12, Lemma 3.1]. Hence, there exists a Lipschitz function $u_\varepsilon \in W^{1,\infty}(\Omega;\mathbb{R}^2)$ such that $\nabla u_\varepsilon = V_\varepsilon$. By adding a suitable constant, we may assume that

$$\int_\Omega u_\varepsilon \,dx = \int_\Omega u \,dx.$$

In view of the Poincaré–Wirtinger inequality and (4.18), $(u_\varepsilon)_\varepsilon$ is a uniformly bounded sequence in $W^{1,1}(\Omega;\mathbb{R}^2)$ satisfying $u_\varepsilon \in \mathcal{A}_\varepsilon$ for all ε (cf. (4.2)).

To prove that $u_\varepsilon \overset{*}{\rightharpoonup} u$ in $\mathrm{BV}(\Omega;\mathbb{R}^2)$, it suffices to show that

$$Du_\varepsilon \overset{*}{\rightharpoonup} Du \text{ in } \mathcal{M}(\Omega;\mathbb{R}^{2\times 2}), \tag{4.19}$$

or, equivalently, in view of (4.15), that for every $\varphi \in C_0(\Omega;\mathbb{R}^2)$,

$$\lim_{\varepsilon\to 0}\int_\Omega \nabla u_\varepsilon(x)\varphi(x)\,dx = \int_\Omega R(x_2)\varphi(x)\,dx + \int_0^1 [(R^+ - R^-)e_1 x_1 + (\psi^+ - \psi^-)]\otimes e_2\varphi(x_1,0)\,dx_1. \tag{4.20}$$

Clearly,

$$\lim_{\varepsilon\to 0}\int_{(0,1)\times[(-1,0)\cup(\varepsilon\lambda,1)]} \nabla u_\varepsilon(x)\varphi(x)\,dx = \lim_{\varepsilon\to 0}\int_{(0,1)\times[(-1,0)\cup(\varepsilon\lambda,1)]} R(x_2)\varphi(x)\,dx = \int_\Omega R(x_2)\varphi(x)\,dx. \tag{4.21}$$

Moreover, using (4.17), a change of variables, and Lebesgue's dominated convergence theorem together with the continuity and boundedness of φ, we have

$$\lim_{\varepsilon\to 0}\int_{(0,1)\times(0,\frac{\varepsilon\lambda}{4}x_1)}\nabla u_\varepsilon(x)\varphi(x)\,\mathrm{d}x=\lim_{\varepsilon\to 0}\int_0^1\int_0^{\frac{\varepsilon\lambda}{4}x_1}R^-(\mathbb{I}+\tan(\tfrac{\theta^-}{2})e_1\otimes e_2-\tfrac{4}{\varepsilon\lambda}e_1\otimes e_2)\varphi(x)\,\mathrm{d}x_2\,\mathrm{d}x_1$$
$$=\lim_{\varepsilon\to 0}\int_0^1\int_0^{x_1}R^-(\tfrac{\varepsilon\lambda}{4}\mathbb{I}+\tfrac{\varepsilon\lambda}{4}\tan(\tfrac{\theta^-}{2})e_1\otimes e_2-e_1\otimes e_2)\varphi(x_1,\tfrac{\varepsilon\lambda}{4}z)\,\mathrm{d}z\,\mathrm{d}x_1$$
$$=-\int_0^1\int_0^{x_1}R^-e_1\otimes e_2\varphi(x_1,0)\,\mathrm{d}z\,\mathrm{d}x_1=-\int_0^1 x_1R^-e_1\otimes e_2\varphi(x_1,0)\,\mathrm{d}x_1. \tag{4.22}$$

Similarly,

$$\lim_{\varepsilon\to 0}\int_{(0,1)\times(\frac{\varepsilon\lambda}{4}x_1,\frac{\varepsilon\lambda}{4})}\nabla u_\varepsilon(x)\varphi(x)\,\mathrm{d}x=\lim_{\varepsilon\to 0}\int_0^1\int_{x_1}^1 S(\tfrac{\varepsilon\lambda}{4}\mathbb{I}-\tfrac{\varepsilon\lambda}{4}\tan(\tfrac{\theta^-}{2})e_1\otimes e_2-e_1\otimes e_2)\varphi(x_1,\tfrac{\varepsilon\lambda}{4}z)\,\mathrm{d}z\,\mathrm{d}x_1$$
$$=\int_0^1(x_1-1)Se_1\otimes e_2\varphi(x_1,0)\,\mathrm{d}x_1, \tag{4.23}$$

$$\lim_{\varepsilon\to 0}\int_{(0,1)\times(\frac{\varepsilon\lambda}{4},\frac{\varepsilon\lambda}{2})}\nabla u_\varepsilon(x)\varphi(x)\,\mathrm{d}x=\lim_{\varepsilon\to 0}\int_0^1\int_1^2 S(\tfrac{\varepsilon\lambda}{4}\mathbb{I}+\beta e_1\otimes e_2)\varphi(x_1,\tfrac{\varepsilon\lambda}{4}z)\,\mathrm{d}z\,\mathrm{d}x_1$$
$$=\int_0^1\beta Se_1\otimes e_2\varphi(x_1,0)\,\mathrm{d}x_1, \tag{4.24}$$

$$\lim_{\varepsilon\to 0}\int_{(0,1)\times(\frac{\varepsilon\lambda}{2},-\frac{\varepsilon\lambda}{4}x_1+\frac{3\varepsilon\lambda}{4})}\nabla u_\varepsilon(x)\varphi(x)\,\mathrm{d}x=\lim_{\varepsilon\to 0}\int_0^1\int_2^{3-x_1} S(\tfrac{\varepsilon\lambda}{4}\mathbb{I}-\tfrac{\varepsilon\lambda}{4}\tan(\tfrac{\theta^+}{2})e_1\otimes e_2+e_1\otimes e_2)\varphi(x_1,\tfrac{\varepsilon\lambda}{4}z)\,\mathrm{d}z\,\mathrm{d}x_1$$
$$=\int_0^1(1-x_1)Se_1\otimes e_2\varphi(x_1,0)\,\mathrm{d}x_1, \tag{4.25}$$

$$\lim_{\varepsilon\to 0}\int_{(0,1)\times(-\frac{\varepsilon\lambda}{4}x_1+\frac{3\varepsilon\lambda}{4},\frac{3\varepsilon\lambda}{4})}\nabla u_\varepsilon(x)\varphi(x)\,\mathrm{d}x=\lim_{\varepsilon\to 0}\int_0^1\int_{3-x_1}^3 R^+(\tfrac{\varepsilon\lambda}{4}\mathbb{I}+\tfrac{\varepsilon\lambda}{4}\tan(\tfrac{\theta^+}{2})e_1\otimes e_2+e_1\otimes e_2)\varphi(x_1,\tfrac{\varepsilon\lambda}{4}z)\,\mathrm{d}z\,\mathrm{d}x_1$$
$$=\int_0^1 x_1R^+e_1\otimes e_2\varphi(x_1,0)\,\mathrm{d}x_1, \tag{4.26}$$

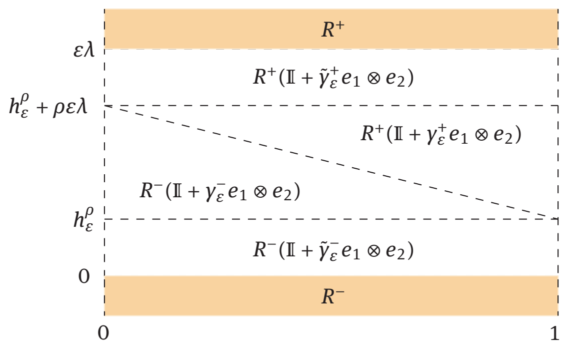
$$\lim_{\varepsilon\to 0}\int_{(0,1)\times(\frac{3\varepsilon\lambda}{4},\varepsilon\lambda)}\nabla u_\varepsilon(x)\varphi(x)\,\mathrm{d}x=\lim_{\varepsilon\to 0}\int_0^1\int_3^4 R^+(\tfrac{\varepsilon\lambda}{4}\mathbb{I}+\alpha e_1\otimes e_2)\varphi(x_1,\tfrac{\varepsilon\lambda}{4}z)\,\mathrm{d}z\,\mathrm{d}x_1$$
$$=\int_0^1\alpha R^+e_1\otimes e_2\varphi(x_1,0)\,\mathrm{d}x_1. \tag{4.27}$$

Combining (4.21)–(4.27) and (4.16), we finally obtain (4.20). □

Remark 4.6 (On the construction in Lemma 4.5). Notice that the main idea of the construction in the proof of Lemma 4.5 for dealing with jumps is to use piecewise affine functions that are as simple as possible to accommodate them. Since triple junctions where two of the three angles add up to π are not compatible (compare with [12, Lemma 3.1]), we work with inclined interfaces that stretch over the full width of Ω.

Let $u \in \mathcal{A} \cap \mathrm{SBV}_\infty(\Omega; \mathbb{R}^2)$ be as in Lemma 4.5, and assume that either $R^+ \neq \pm R^-$ or $R^+ = R^-$. In these cases, we can simplify the construction of $(u_\varepsilon)_\varepsilon$ in the previous proof. We focus here on stating the counterparts of Figure 3 and (4.17), and omit the detailed calculations, which are very similar to (4.21)–(4.27). Note further that these constructions are not just simpler, but also energetically more favorable, see Remark 5.2 below for more details.

(i) If $R^+ \neq \pm R^-$, we may replace the construction depicted in Figure 3 by:



$$\psi^+ - \psi^- = \alpha R^+ e_1 + \beta R^- e_1$$

$$\theta \in (-\pi, \pi) \setminus \{0\} \text{ rotation angle of } (R^-)^T R^+$$

$$\rho \in (0,1), \quad h_\varepsilon^\rho := \tfrac{\varepsilon\lambda - \rho\varepsilon\lambda}{2}$$

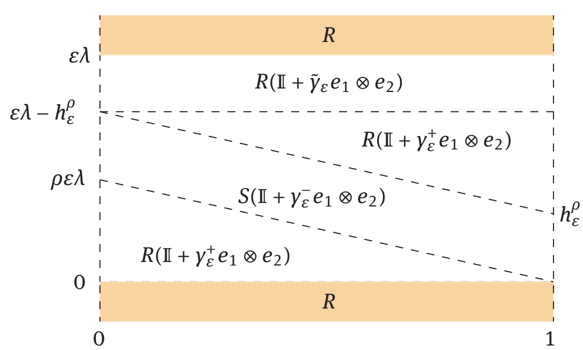
$$\gamma_\varepsilon^+ := \tfrac{1}{\rho\varepsilon\lambda} + \tan(\tfrac{\theta}{2}), \quad \gamma_\varepsilon^- := \tfrac{1}{\rho\varepsilon\lambda} - \tan(\tfrac{\theta}{2})$$

$$\tilde{\gamma}_\varepsilon^+ \text{ satisfies } \alpha = \lim_{\varepsilon\to 0} \tilde{\gamma}_\varepsilon^+ (\varepsilon\lambda - h_\varepsilon^\rho - \rho\varepsilon\lambda)$$

$$\tilde{\gamma}_\varepsilon^- \text{ satisfies } \beta - 1 = \lim_{\varepsilon\to 0} \tilde{\gamma}_\varepsilon^- h_\varepsilon^\rho$$

Figure 4: Alternative construction of V_ε if $R^+ \neq \pm R^-$.

(ii) If R is constant, i.e., $R^+ = R^-$, and $\psi^+ - \psi^-$ is not parallel to Re_1, the construction in Figure 3 can be replaced by:



$$S \in \mathrm{SO}(2) : Re_1 \text{ and } Se_1 \text{ are linearly independent}$$

$$\psi^+ - \psi^- = \alpha Re_1 + \beta Se_1, \quad \beta \neq 0, \quad \iota := \mathrm{sign}(\beta)$$

$$\theta \in (-\pi, \pi) \setminus \{0\} \text{ rotation angle of } R^T S$$

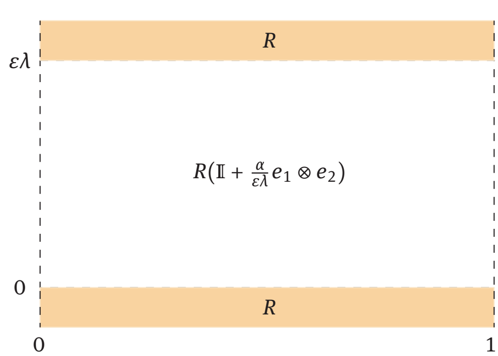
$$\rho := \tfrac{\iota}{2\beta + \iota} \in (0,1), \quad h_\varepsilon^\rho := \tfrac{\varepsilon\lambda - \rho\varepsilon\lambda}{2}$$

$$\gamma_\varepsilon^+ := \iota\tfrac{1}{\rho\varepsilon\lambda} + \tan(\tfrac{\theta}{2}), \quad \gamma_\varepsilon^- := \iota\tfrac{1}{\rho\varepsilon\lambda} - \tan(\tfrac{\theta}{2})$$

$$\tilde{\gamma}_\varepsilon \text{ satisfies } \alpha - \iota = \lim_{\varepsilon\to 0} \tilde{\gamma}_\varepsilon h_\varepsilon^\rho$$

Figure 5: Alternative construction of V_ε if R is constant and $\psi^+ - \psi^-$ is not parallel to Re_1.

(iii) If R is constant, i.e., $R^+ = R^-$, and $\psi^+ - \psi^-$ is parallel to Re_1, then we can use the following construction in place of Figure 3:



$$\psi^+ - \psi^- = \alpha Re_1$$

$$\alpha = \iota|\psi^+ - \psi^-|, \quad \iota := \mathrm{sign}((\psi^+ - \psi^-) \cdot Re_1)$$

Figure 6: Alternative construction of V_ε if R is constant and $\psi^+ - \psi^-$ is parallel to Re_1.

Note that in case (i), the slope ρ of the interfaces can attain any value between 0 and 1, while in (ii), ρ is determined by the value of β. In terms of the energies, the construction in case (iii) provides an optimal approximation, which will be detailed in Section 6.

We proceed by extending Lemma 4.5 to arbitrary functions $u \in \mathcal{A} \cap \mathrm{SBV}_<(\Omega;\mathbb{R}^2)$.

Proposition 4.7. *Let $\Omega = (0,1)\times(-1,1)$. Then, for every function $u \in \mathcal{A}\cap\mathrm{SBV}_<(\Omega;\mathbb{R}^2)$, there exists a sequence $(u_\varepsilon)_\varepsilon \subset W^{1,1}(\Omega;\mathbb{R}^2)$ with $\int_\Omega u_\varepsilon\,dx = \int_\Omega u\,dx$ and $u_\varepsilon \in \mathcal{A}_\varepsilon$ for all ε, and such that $u_\varepsilon \overset{*}{\rightharpoonup} u$ in $\mathrm{BV}(\Omega;\mathbb{R}^2)$ or, in other words,*

$$\mathcal{A}\cap\mathrm{SBV}_<(\Omega;\mathbb{R}^2) \subset \mathcal{A}_0,$$

cf. (4.3).

Proof. In view of Remark 4.4 (c), it holds that $J_u = \bigcup_{i=1}^{\ell}(0,1)\times\{a_i\}$ for some $\ell \in \mathbb{N}$ and $a_i \in (-1,1)$ with $a_1 < a_2 < \dots < a_\ell$, and setting $a_0 := -1$ and $a_{\ell+1} := 1$, gives

$$\begin{aligned} Du = \sum_{i=0}^{\ell} R_i(\mathbb{I}+\gamma e_1\otimes e_2)\mathcal{L}^2\lfloor((0,1)\times(a_i,a_{i+1})) \\ + \sum_{i=1}^{\ell}[(R_i - R_{i-1})x_1e_1 + (R_ia_ie_2 + \psi_i^+ - R_{i-1}a_ie_2 - \psi_i^-)]\otimes e_2\mathcal{H}^1\lfloor((0,1)\times\{a_i\}), \end{aligned} \tag{4.28}$$

where $\gamma \in L^1(-1,1)$, and $R_i \in \mathrm{SO}(2)$ and $\psi_i^\pm \in \mathbb{R}^2$ for $i = 0,\dots,\ell$.

We now perform a similar construction as in Lemma 4.5 in a convenient softer layer *near* each a_i, accounting for the possibility that one or more of the jump lines may not intersect $\varepsilon Y_{\mathrm{soft}} \cap \Omega$, and replacing R^+ by R_i, R^- by R_{i-1}, ψ^+ by $R_ia_ie_2 + \psi_i^+$, and ψ^- by $R_{i-1}a_ie_2 + \psi_i^-$.

To be precise, fix $\varepsilon > 0$ and $i \in \{1,\dots,\ell\}$. Let $S_i \in \mathrm{SO}(2)$ be such that
(i) $S_i \notin \{R_{i-1}, R_i\}$,
(ii) S_ie_1 and R_ie_1 are linearly independent,
(iii) $\theta_i^-, \theta_i^+ \in (-\pi,\pi)\setminus\{0\}$ are the rotation angles of $S_i^TR_{i-1}$ and $S_i^TR_i$, respectively.
By (ii), there exist $\alpha_i, \beta_i \in \mathbb{R}$ such that

$$R_ia_ie_2 + \psi_i^+ - R_{i-1}a_ie_2 - \psi_i^- = \alpha_iR_ie_1 + \beta_iS_ie_1. \tag{4.29}$$

Moreover, we set

$$\gamma_{\varepsilon,i}^+ := \frac{4\alpha_i}{\varepsilon\lambda}, \quad \gamma_{\varepsilon,i} := \frac{4\beta_i}{\varepsilon\lambda}, \quad \mu_{\varepsilon,i}^\pm := \pm\frac{4}{\varepsilon\lambda} + \tan\Big(\frac{\theta_i^\pm}{2}\Big), \quad \tilde\mu_{\varepsilon,i}^\pm := \pm\frac{4}{\varepsilon\lambda} - \tan\Big(\frac{\theta_i^\pm}{2}\Big),$$

and let $\kappa_\varepsilon^i \in \mathbb{Z}$ be the unique integer such that $a_i \in \varepsilon[\kappa_\varepsilon^i, \kappa_\varepsilon^i + 1)$. Observing that $a_i \neq a_j$ for $i,j \in \{1,\dots,\ell\}$ with $i \neq j$ and $a_i \in (-1,1)$ for all $i \in \{1,\dots,\ell\}$, we may assume that the sets $\{\varepsilon[\kappa_\varepsilon^i, \kappa_\varepsilon^i+1)\}_{i=1,\dots,\ell}$ are pairwise disjoint, and that $\bigcup_{i=1}^{\ell}\varepsilon[\kappa_\varepsilon^i,\kappa_\varepsilon^i+1] \subset (-1,1)$ (this is true for sufficiently small $\varepsilon > 0$). Finally, with $\kappa_\varepsilon^0 := -\lambda - \frac{1}{\varepsilon}$ and $\kappa_\varepsilon^{\ell+1} := \frac{1}{\varepsilon}$, let $V_\varepsilon \in L^1(\Omega;\mathbb{R}^{2\times2})$ be the function defined by

$$V_\varepsilon(x) := \begin{cases} R_i(\mathbb{I} + \frac{\gamma}{\lambda}\mathbb{1}_{\varepsilon Y_{\mathrm{soft}}}e_1\otimes e_2) & \text{if } x \in (0,1)\times(\varepsilon\lambda+\varepsilon\kappa_\varepsilon^i, \varepsilon\kappa_\varepsilon^{i+1}) \text{ for some } i \in \{0,\dots,\ell\},\\ R_i(\mathbb{I}+\gamma_{\varepsilon,i}^+e_1\otimes e_2) & \text{if } x \in (0,1)\times(\frac{3\varepsilon\lambda}{4}+\varepsilon\kappa_\varepsilon^i, \varepsilon\lambda+\varepsilon\kappa_\varepsilon^i) \text{ for some } i \in \{1,\dots,\ell\},\\ R_i(\mathbb{I}+\mu_{\varepsilon,i}^+e_1\otimes e_2) & \text{if } x_1 \in (0,1) \text{ and } x_2 \in (-\frac{\varepsilon\lambda}{4}x_1 + \frac{3\varepsilon\lambda}{4}+\varepsilon\kappa_\varepsilon^i, \frac{3\varepsilon\lambda}{4}+\varepsilon\kappa_\varepsilon^i) \text{ for some } i \in \{1,\dots,\ell\},\\ S_i(\mathbb{I}+\tilde\mu_{\varepsilon,i}^+e_1\otimes e_2) & \text{if } x_1 \in (0,1) \text{ and } x_2 \in (\frac{\varepsilon\lambda}{2}+\varepsilon\kappa_\varepsilon^i, -\frac{\varepsilon\lambda}{4}x_1 + \frac{3\varepsilon\lambda}{4}+\varepsilon\kappa_\varepsilon^i) \text{ for some } i \in \{1,\dots,\ell\},\\ S_i(\mathbb{I}+\gamma_{\varepsilon,i}e_1\otimes e_2) & \text{if } x \in (0,1)\times(\frac{\varepsilon\lambda}{4}+\varepsilon\kappa_\varepsilon^i, \frac{\varepsilon\lambda}{2}+\varepsilon\kappa_\varepsilon^i) \text{ for some } i \in \{1,\dots,\ell\},\\ S_i(\mathbb{I}+\tilde\mu_{\varepsilon,i}^-e_1\otimes e_2) & \text{if } x_1 \in (0,1) \text{ and } x_2 \in (\frac{\varepsilon\lambda}{4}x_1+\varepsilon\kappa_\varepsilon^i, \frac{\varepsilon\lambda}{4}+\varepsilon\kappa_\varepsilon^i) \text{ for some } i \in \{1,\dots,\ell\},\\ R_{i-1}(\mathbb{I}+\mu_{\varepsilon,i}^-e_1\otimes e_2) & \text{if } x_1 \in (0,1) \text{ and } x_2 \in (\varepsilon\kappa_\varepsilon^i, \frac{\varepsilon\lambda}{4}x_1+\varepsilon\kappa_\varepsilon^i) \text{ for some } i \in \{1,\dots,\ell\}. \end{cases}$$

As in the proof of Lemma 4.5, invoking [12, Lemma 3.1] on rank-one connections in $\mathcal{M}_{e_1}$, we find that V_ε is a gradient field, meaning that there is $u_\varepsilon \in W^{1,\infty}(\Omega;\mathbb{R}^2)$ such that $\nabla u_\varepsilon = V_\varepsilon$. Adding a suitable constant allows us to assume that $\int_\Omega u_\varepsilon\,dx = \int_\Omega u\,dx$. By construction, $(u_\varepsilon)_\varepsilon$ is a uniformly bounded sequence in $W^{1,1}(\Omega;\mathbb{R}^2)$ such that $u_\varepsilon \in \mathcal{A}_\varepsilon$ for all ε (see (4.2)). To prove that $u_\varepsilon \overset{*}{\rightharpoonup} u$ in $\mathrm{BV}(\Omega;\mathbb{R}^2)$, it suffices to show that

$$Du_\varepsilon \overset{*}{\rightharpoonup} Du \quad \text{in } \mathcal{M}(\Omega;\mathbb{R}^{2\times2}). \tag{4.30}$$

The proof of (4.30) follows along the lines of (4.19). For this reason, we only highlight the main differences. First, note that the conditions $\varepsilon\kappa_\varepsilon^0 = -\varepsilon\lambda - 1 = -\varepsilon\lambda + a_0$, $\varepsilon\kappa_\varepsilon^{\ell+1} = 1 = a_{\ell+1}$, and $\varepsilon\kappa_\varepsilon^i \le a_i \le \varepsilon(\kappa_\varepsilon^i + 1)$ yield

$$\lim_{\varepsilon\to 0} \varepsilon\kappa_\varepsilon^i = a_i \quad \text{for all } i \in \{0, \dots, \ell+1\}.$$

Hence, $\mathbb{1}_{(0,1)\times(\varepsilon\lambda+\varepsilon\kappa_\varepsilon^i, \varepsilon\kappa_\varepsilon^{i+1})} \to \mathbb{1}_{(0,1)\times(a_i,a_{i+1})}$ and $\gamma\mathbb{1}_{(0,1)\times(\varepsilon\lambda+\varepsilon\kappa_\varepsilon^i, \varepsilon\kappa_\varepsilon^{i+1})} \to \gamma\mathbb{1}_{(0,1)\times(a_i,a_{i+1})}$ in $L^1(\Omega)$ for $i \in \{0, \dots, \ell\}$. On the other hand, by the Riemann–Lebesgue lemma, we have $\mathbb{1}_{\varepsilon Y_{\text{soft}}} \overset{*}{\rightharpoonup} \lambda$ in $L^\infty(\mathbb{R}^2)$; thus,

$$\begin{aligned}\lim_{\varepsilon\to 0} \int\limits_{(0,1)\times(\varepsilon\lambda+\varepsilon\kappa_\varepsilon^i, \varepsilon\kappa_\varepsilon^{i+1})} \nabla u_\varepsilon(x)\varphi(x)\,\mathrm{d}x &= \lim_{\varepsilon\to 0} \int\limits_\Omega R_i(\mathbb{I} + \tfrac{\gamma}{\lambda}\mathbb{1}_{\varepsilon Y_{\text{soft}}} e_1 \otimes e_2)\mathbb{1}_{(0,1)\times(\varepsilon\lambda+\varepsilon\kappa_\varepsilon^i, \varepsilon\kappa_\varepsilon^{i+1})}\varphi(x)\,\mathrm{d}x \\ &= \int\limits_{(0,1)\times(a_i,a_{i+1})} R_i(\mathbb{I} + \gamma(x_2)e_1 \otimes e_2)\varphi(x)\,\mathrm{d}x\end{aligned}$$

for all $i \in \{0, \dots, \ell\}$ and $\varphi \in C_0(\Omega)$. Arguing as in (4.22) with the change of variables $z = \frac{4}{\varepsilon\lambda}(x_2 - \varepsilon\kappa_\varepsilon^i)$ leads to

$$\begin{aligned}&\lim_{\varepsilon\to 0} \int\limits_{(0,1)\times(\varepsilon\kappa_\varepsilon^i, \frac{\varepsilon\lambda}{4}x_1+\varepsilon\kappa_\varepsilon^i)} \nabla u_\varepsilon(x)\varphi(x)\,\mathrm{d}x \\ &= \lim_{\varepsilon\to 0} \int_0^1 \int_{\varepsilon\kappa_\varepsilon^i}^{\frac{\varepsilon\lambda}{4}x_1+\varepsilon\kappa_\varepsilon^i} R_{i-1}(\mathbb{I} + \tan(\tfrac{\theta_i^-}{2})e_1 \otimes e_2 - \tfrac{4}{\varepsilon\lambda}e_1 \otimes e_2)\varphi(x)\,\mathrm{d}x_2\,\mathrm{d}x_1 \\ &= \lim_{\varepsilon\to 0} \int_0^1 \int_0^{x_1} R_{i-1}(\tfrac{\varepsilon\lambda}{4}\mathbb{I} + \tfrac{\varepsilon\lambda}{4}\tan(\tfrac{\theta_i^-}{2})e_1 \otimes e_2 - e_1 \otimes e_2)\varphi(x_1, \tfrac{\varepsilon\lambda}{4}z + \varepsilon\kappa_\varepsilon^i)\,\mathrm{d}z\,\mathrm{d}x_1 \\ &= -\int_0^1 \int_0^{x_1} R_{i-1}e_1 \otimes e_2\varphi(x_1, a_i)\,\mathrm{d}z\,\mathrm{d}x_1 = -\int_0^1 R_{i-1}x_1 e_1 \otimes e_2\varphi(x_1, a_i)\,\mathrm{d}x_1\end{aligned}$$

for all $i \in \{1, \dots, \ell\}$ and $\varphi \in C_0(\Omega)$. Similarly, one can calculate the counterparts to (4.23)–(4.27) in the present setting. In view of (4.28) and (4.29), we deduce (4.30), which ends the proof. □

Remark 4.8 (On the construction in Proposition 4.7). We observe that the sequence of Lipschitz functions $(u_\varepsilon)_\varepsilon$ constructed in Proposition 4.7 to approximate a given $u \in \mathcal{A} \cap \mathrm{SBV}_<(\Omega; \mathbb{R}^2)$ is such that

$$\lim_{\varepsilon\to 0} \int\limits_\Omega |\nabla u_\varepsilon|\,\mathrm{d}x \sim |Du|(\Omega) + 2\ell.$$

In other words, the asymptotic behavior of the total variation of $(u_\varepsilon)_\varepsilon$ incorporates a positive term that is proportional to the number of jumps of the limit function. This fact prevents us from bootstrapping the argument in Proposition 4.7 to generalize it to an arbitrary function in $\mathcal{A} \cap \mathrm{SBV}(\Omega; \mathbb{R}^2)$.

An analogous statement to Proposition 4.7 holds in $\mathcal{A}^\parallel$.

Proposition 4.9. *Let $\Omega = (0,1) \times (-1,1)$. If $u \in \mathcal{A}^\parallel$, then there exists a sequence $(u_\varepsilon)_\varepsilon \subset W^{1,1}(\Omega; \mathbb{R}^2)$ such that $u_\varepsilon \in \mathcal{A}_\varepsilon$ for all ε and $u_\varepsilon \overset{*}{\rightharpoonup} u$ in $\mathrm{BV}(\Omega; \mathbb{R}^2)$; that is,*

$$\mathcal{A}^\parallel \subset \mathcal{A}_0.$$

Proof. Let $u \in \mathcal{A}^\parallel$. Based on (1.13) and (2.4), we can split u into $u = v + w$, where

$$v(x) := Rx + \vartheta_a(x_2)Re_1 + c \quad \text{and} \quad w(x) := \vartheta_s(x_2)Re_1 \quad \text{for } x \in \Omega, \tag{4.31}$$

with $R \in \mathrm{SO}(2)$, $c \in \mathbb{R}^2$, $\vartheta_a \in W^{1,1}(-1,1)$, and $\vartheta_s \in \mathrm{BV}(-1,1)$ such that $\vartheta_s' = 0$. By construction, we have that $v \in W^{1,1}(\Omega; \mathbb{R}^2)$ with $\nabla v(x) = R(\mathbb{I} + \vartheta_a'(x_2)e_1 \otimes e_2)$.

For every $\varepsilon > 0$, let $v_\varepsilon \in W^{1,1}(\Omega; \mathbb{R}^2)$ be the function satisfying $\int_\Omega v_\varepsilon\,\mathrm{d}x = \int_\Omega v\,\mathrm{d}x$ and

$$\nabla v_\varepsilon(x) = R\Big(\mathbb{I} + \frac{\vartheta_a'(x_2)}{\lambda}\mathbb{1}_{\varepsilon Y_{\text{soft}}}(x)e_1 \otimes e_2\Big). \tag{4.32}$$

By the Riemann–Lebesgue lemma,

$$v_\varepsilon \rightharpoonup v \quad \text{in } W^{1,1}(\Omega;\mathbb{R}^{2\times 2}). \tag{4.33}$$

On the other hand, applying Lemma 3.3 to ϑ_s, we can find a sequence $(\vartheta_\varepsilon)_\varepsilon \subset W^{1,\infty}(-1,1)$ such that $\vartheta_\varepsilon \overset{*}{\rightharpoonup} \vartheta_s$ in $\mathrm{BV}(-1,1)$ and $\vartheta_\varepsilon' = 0$ on $\varepsilon I_{\mathrm{rig}} \cap (-1,1)$. Then, setting $w_\varepsilon(x) := \vartheta_\varepsilon(x_2)Re_1 + \int_\Omega (w - \vartheta_\varepsilon(x_2)Re_1)\,\mathrm{d}x$ yields

$$\nabla w_\varepsilon(x) = \vartheta_\varepsilon'(x_2)Re_1 \otimes e_2 = \vartheta_\varepsilon'(x_2)\mathbb{1}_{\varepsilon Y_{\mathrm{soft}}} Re_1 \otimes e_2 \tag{4.34}$$

and

$$w_\varepsilon \overset{*}{\rightharpoonup} w \quad \text{in } \mathrm{BV}(\Omega;\mathbb{R}^2). \tag{4.35}$$

We define the maps $u_\varepsilon := v_\varepsilon + w_\varepsilon$ in $W^{1,1}(\Omega;\mathbb{R}^2)$ for every ε, and infer from (4.32) and (4.34) that

$$\nabla u_\varepsilon = R(\mathbb{I} + \gamma_\varepsilon e_1 \otimes e_2),$$

where $\gamma_\varepsilon(x) := (\frac{\vartheta_a'(x_2)}{\lambda} + \vartheta_\varepsilon'(x_2))\mathbb{1}_{\varepsilon Y_{\mathrm{soft}}}(x)$ is a function in $L^1(\Omega)$ satisfying $\gamma_\varepsilon = 0$ in $\varepsilon Y_{\mathrm{rig}} \cap \Omega$. In particular, $u_\varepsilon \in \mathcal{A}_\varepsilon$ for all ε.

Combining (4.33) and (4.35) shows that $u_\varepsilon \overset{*}{\rightharpoonup} v + w = u$ in $\mathrm{BV}(\Omega;\mathbb{R}^2)$, which finishes the proof. □

Finally, we prove Theorem 1.3.

Proof of Theorem 1.3. In view of the discussion in Section 2.4, it suffices to prove the statement on a rectangle of the form $(c_\Omega, d_\Omega) \times (a_\Omega, b_\Omega)$, where we recall (1.1) and (1.2). A simple modification of the proofs of Propositions 4.3, 4.7, and 4.9 shows that these results hold for any such rectangles, from which Theorem 1.3 follows. □

5 A lower bound on the homogenized energy

In this section, we present partial results for the homogenization problem for layered composites with rigid components discussed in the Introduction. More precisely, we establish a lower bound estimate on the asymptotic behavior of the sequence of energies $(E_\varepsilon)_\varepsilon$ (see (1.7)), and highlight the main difficulties in the construction of matching upper bounds. Note that the following analysis is restricted to the case $s = e_1$.

As a start, we first give alternative representations for the involved energies, which will be useful in the sequel.

Remark 5.1 (Equivalent formulations for E_ε and E). In view of the definition of $\mathcal{A}_\varepsilon$ (see (1.6)), it is straightforward to check that the functional E_ε in (1.7) satisfies

$$E_\varepsilon(u) = \begin{cases} \int_\Omega \sqrt{|\partial_2 u|^2 - 1}\,\mathrm{d}x & \text{if } u \in \mathcal{A}_\varepsilon, \\ \infty & \text{otherwise}, \end{cases} = \begin{cases} \int_\Omega \sqrt{|\nabla u|^2 - 2\det \nabla u}\,\mathrm{d}x & \text{if } u \in \mathcal{A}_\varepsilon, \\ \infty & \text{otherwise}, \end{cases}$$

for $u \in L^1_0(\Omega;\mathbb{R}^2)$. Similarly, according to Proposition 4.1, the functional E from (1.10) can be expressed as

$$E(u) = \begin{cases} \int_\Omega |\gamma|\,\mathrm{d}x + |D^s u|(\Omega) & \text{if } u \in \mathcal{A}, \\ \infty & \text{otherwise}, \end{cases}$$

for $u \in L^1_0(\Omega;\mathbb{R}^2)$.

We can now provide a bound from below on $\Gamma\text{-}\liminf_{\varepsilon\to 0} E_\varepsilon$ and prove Theorem 1.2.

Proof of Theorem 1.2. For clarity, we subdivide the proof into two steps. In the first one, we establish the compactness property. In the second step, we prove (1.12) using the weak* lower semicontinuity of the total variation of a measure. We refer to the Appendix for an alternative proof of (1.12), which is based on a Reshetnyak's lower semicontinuity result, and highlights a different feature of the representation of $\mathcal{A}$.

Step 1: Compactness. Assume that $(u_\varepsilon)_\varepsilon \subset L^1_0(\Omega;\mathbb{R}^2)$ is such that $\sup_\varepsilon E_\varepsilon(u_\varepsilon) < \infty$. Then, we have $u_\varepsilon \in \mathcal{A}_\varepsilon$ and $\sup_\varepsilon \|\nabla u_\varepsilon\|_{L^1(\Omega;\mathbb{R}^{2\times 2})} < \infty$. Hence, using the Poincaré–Wirtinger inequality, there exist a subsequence $(u_{\varepsilon_j})_{j\in\mathbb{N}}$ and $u \in L^1_0(\Omega;\mathbb{R}^2) \cap BV(\Omega;\mathbb{R}^2)$ such that $u_{\varepsilon_j} \overset{*}{\rightharpoonup} u$ in $BV(\Omega;\mathbb{R}^2)$. By Proposition 4.3, we conclude that $u \in L^1_0(\Omega;\mathbb{R}^2) \cap \mathcal{A}$.

Step 2: Lower bound. Let $(u_\varepsilon)_\varepsilon \subset L^1_0(\Omega;\mathbb{R}^2)$ and $u \in L^1_0(\Omega;\mathbb{R}^2)$ be such that $u_\varepsilon \to u$ in $L^1(\Omega;\mathbb{R}^2)$. We want to show that

$$\liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) \geq E(u). \tag{5.1}$$

To prove (5.1), one may assume without loss of generality that the limit inferior on the right-hand side of (5.1) is actually a limit and that this limit is finite. Then, $u_\varepsilon \in \mathcal{A}_\varepsilon$ and $E_\varepsilon(u_\varepsilon) < C$ for all ε, where $C > 0$ is a constant independent of ε. Hence, by Step 1, $u_\varepsilon \overset{*}{\rightharpoonup} u$ in $BV(\Omega;\mathbb{R}^2)$ and $u \in \mathcal{A}$.

By the definition of $\mathcal{A}_\varepsilon$ and (4.1),

$$\nabla u_\varepsilon = R_\varepsilon + \gamma_\varepsilon R_\varepsilon e_1 \otimes e_2$$

with $R_\varepsilon \in L^\infty(\Omega;SO(2))$ and $\gamma_\varepsilon \in L^1(\Omega)$. Since $|\gamma_\varepsilon R_\varepsilon e_1 \otimes e_2| = |\gamma_\varepsilon|$ due to $|R_\varepsilon e_1| = 1$, the estimate

$$E_\varepsilon(u_\varepsilon) = \int_\Omega |\gamma_\varepsilon|\,dx < C$$

implies that $(\gamma_\varepsilon R_\varepsilon e_1 \otimes e_2)_\varepsilon$ is uniformly bounded in $L^1(\Omega;\mathbb{R}^{2\times 2})$. Hence, after extracting a subsequence if necessary (not relabeled),

$$(\gamma_\varepsilon R_\varepsilon e_1 \otimes e_2)\mathcal{L}^2\lfloor\Omega \overset{*}{\rightharpoonup} \nu \quad \text{in } \mathcal{M}(\Omega;\mathbb{R}^{2\times 2})$$

for some $\nu \in \mathcal{M}(\Omega;\mathbb{R}^{2\times 2})$. Note further that the convergence $\nabla u_\varepsilon \mathcal{L}^2\lfloor\Omega \overset{*}{\rightharpoonup} Du$ in $\mathcal{M}(\Omega;\mathbb{R}^{2\times 2})$ along with (4.5) yields also $R_\varepsilon \overset{*}{\rightharpoonup} R$ in $L^\infty(\Omega;\mathbb{R}^{2\times 2})$, where $R \in L^\infty(\Omega;SO(2))$ satisfies in particular that $(\nabla u)e_1 = Re_1$. Hence, we have

$$\nu = Du - R\mathcal{L}^2\lfloor\Omega = (\nabla u - R)\mathcal{L}^2\lfloor\Omega + D^s u = (\gamma R e_1 \otimes e_2)\mathcal{L}^2\lfloor\Omega + D^s u,$$

where the last equality follows again from (4.5), and by the lower semicontinuity of the total variation,

$$\begin{aligned}\liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) &= \liminf_{\varepsilon\to 0} \int_\Omega |\gamma_\varepsilon|\,dx = \liminf_{\varepsilon\to 0} \int_\Omega |\gamma_\varepsilon R_\varepsilon e_1 \otimes e_2|\,dx \\ &\geq |\nu|(\Omega) = \int_\Omega |\gamma R e_1 \otimes e_2|\,dx + |D^s u|(\Omega) = \int_\Omega |\gamma|\,dx + |D^s u|(\Omega) = E(u).\end{aligned}$$ □

Remark 5.2 (Discussion regarding optimality of the lower bound). (a) The lower bound (1.12) is optimal in $\mathcal{A} \cap W^{1,1}(\Omega;\mathbb{R}^2) \cap L^1_0(\Omega;\mathbb{R}^2)$ and, more generally (cf. also Remark 4.4), in the set $\mathcal{A}^\parallel \cap L^1_0(\Omega;\mathbb{R}^2)$ introduced in (1.13). Precisely, we have

$$\Gamma(L^1)\text{-}\lim_{\varepsilon\to 0} E_\varepsilon(u) = E(u) \tag{5.2}$$

for all $u \in \mathcal{A}^\parallel \cap L^1_0(\Omega;\mathbb{R}^2)$. In view of (1.12), the proof of (5.2) is directly related to the ability to construct a recovery sequence. We detail two alternative constructions for $u \in \mathcal{A}^\parallel$ in Section 6 below. For illustration, we treat here the simpler special case where $u \in \mathcal{A} \cap W^{1,1}(\Omega;\mathbb{R}^2) \cap L^1_0(\Omega;\mathbb{R}^2)$.

If $u \in \mathcal{A} \cap W^{1,1}(\Omega;\mathbb{R}^2) \cap L^1_0(\Omega;\mathbb{R}^2)$, then $\nabla u = R(\mathbb{I} + \gamma e_1 \otimes e_2)$ for some $R \in SO(2)$ and $\gamma \in L^1(\Omega)$ such that $\partial_1 \gamma = 0$ (see Remark 4.4 (a)). As in the proof of Proposition 4.9, we take $(u_\varepsilon)_\varepsilon \subset W^{1,1}(\Omega;\mathbb{R}^2) \cap L^1_0(\Omega;\mathbb{R}^2)$ such that $\nabla u_\varepsilon = R(\mathbb{I} + \frac{\gamma}{\lambda}\mathbb{1}_{\varepsilon Y_{\text{soft}}} e_1 \otimes e_2)$ for all ε. Then, by the Riemann–Lebesgue lemma, $u_\varepsilon \overset{*}{\rightharpoonup} u$ in $BV(\Omega;\mathbb{R}^2)$ and $\lim_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) = E(u)$.

(b) The question whether (5.2) holds for a larger class than $\mathcal{A}^\parallel$ is open at this point. We observe that the gradient-based constructions in Lemma 4.5, Remark 4.6 (i)–(ii), and Proposition 4.7 yield upper bounds on the Γ-lim sup, which, however, do not match the lower bound of Theorem 1.2. This indicates that, in general, a more tailored approach will be necessary.

(c) The upper bounds on the Γ-lim sup of $(E_\varepsilon)_\varepsilon$ resulting from Lemma 4.5, Remark 4.6 (i)–(ii), and Proposition 4.7 can be quantified. As previously mentioned, the constructions in Remark 4.6 (iii) and Proposition 4.9 are even recovery sequences. This is not the case for the general construction in Lemma 4.5 and for those highlighted in Remark 4.6 (i)–(ii). In the following, we suppose that $u \in \mathcal{A} \cap \mathrm{SBV}_\infty(\Omega;\mathbb{R}^2) \cap L^1_0(\Omega;\mathbb{R}^2)$ has a single jump as in the statement of Lemma 4.5; i.e.,

$$u(x) = \mathbb{1}_{(0,1)\times(0,1)}(x)(R^+x + \psi^+) + \mathbb{1}_{(0,1)\times(-1,0)}(x)(R^-x + \psi^-)$$

with $R^\pm \in SO(2)$ and $\psi^\pm \in \mathbb{R}^2$. Then,

$$E(u) = \int_0^1 |(R^+ - R^-)e_1x_1 + (\psi^+ - \psi^-)|\,dx_1,$$

which can be estimated from above by

$$E(u) \le |R^+e_1 - R^-e_1| \int_0^1 x_1\,dx_1 + |\psi^+ - \psi^-| \le 1 + |\psi^+ - \psi^-|. \tag{5.3}$$

For the sequence $(u_\varepsilon)_\varepsilon$ constructed in Lemma 4.5 (and Lemma 4.7), we obtain, recalling (4.16), that

$$\lim_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) = |\alpha| + |\beta| + 2 > |\alpha| + |\beta| + 1 \ge E(u).$$

Regarding the construction of $(u_\varepsilon)_\varepsilon$ in Remark 4.6 (i), it follows that

$$\lim_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) = |\alpha| + |\beta - 1| + 1.$$

This limit is strictly greater than $E(u)$ as we will show next. If $|\beta - 1| > |\beta|$ (i.e., if $\beta < \frac{1}{2}$), this is an immediate consequence of (5.3). For $\frac{1}{2} \le \beta < 1$, we use that $\psi^+ - \psi^- = \alpha R^+e_1 + \beta R^-e_1$ yields

$$E(u) \le \int_0^1 |x_1 + \alpha|\,dx_1 + \int_0^\beta (\beta - x_1)\,dx_1 + \int_\beta^1 (x_1 - \beta)\,dx_1 \le 1 + |\alpha| + \beta(\beta - 1) < 1 + |\alpha| + |\beta - 1|.$$

If $\beta \ge 1$, we note that $\lim_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) = |\alpha| + \beta$, and subdivide the estimate of $E(u)$ into three cases. Recalling the assumption $R^+ \ne \pm R^-$, we set $c := R^+e_1 \cdot R^-e_1 \in (-1, 1)$ to obtain

$$E(u) = \int_0^1 \sqrt{(x_1 + \alpha)^2 + (\beta - x_1)^2 + 2c(x_1 + \alpha)(\beta - x_1)}\,dx_1.$$

Then, we have for $\alpha \ge 0$ that

$$E(u) < \int_0^1 \sqrt{(x_1 + \alpha)^2 + (\beta - x_1)^2 + 2(x_1 + \alpha)(\beta - x_1)}\,dx_1 = |\alpha + \beta| \le |\alpha| + \beta,$$

for $\alpha \le -1$ that

$$\begin{aligned} E(u) &< \int_0^1 \sqrt{(x_1 + \alpha)^2 + (\beta - x_1)^2 - 2(x_1 + \alpha)(\beta - x_1)}\,dx_1 = \int_0^1 (-2x_1 - \alpha + \beta)\,dx_1 \\ &= -1 - \alpha + \beta < -\alpha + \beta = |\alpha| + \beta, \end{aligned}$$

and for $-1 < \alpha < 0$ that

$$E(u) < \int_0^{-\alpha} (-2x_1 - \alpha + \beta)\,dx_1 + \int_{-\alpha}^1 |\alpha + \beta|\,dx_1 = \alpha + \beta + \alpha^2 < -\alpha + \beta = |\alpha| + \beta.$$

Summing up, we have shown that in the context of Remark 4.6 (i),

$$\lim_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) > E(u).$$

Finally, we consider the sequence $(u_\varepsilon)_\varepsilon$ constructed in Remark 4.6 (ii). Then,

$$\lim_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) = |\alpha - \iota| + |\beta| + 1,$$

and since $R^+ = R^-$ in this case,

$$E(u) = \sqrt{\alpha^2 + \beta^2 + 2\alpha\beta Re_1 \cdot Se_1}.$$

Using the fact that $Re_1 \cdot Se_1 \in (-1, 1)$, it can be checked that, also here, we have

$$\lim_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) > E(u).$$

6 Homogenization of the regularized problem

This section is devoted to the proof of our main Γ-convergence result, Theorem 1.4. We first provide an alternative characterization of the class $\mathcal{A}^\parallel$ of *restricted asymptotically admissible deformations* introduced in (1.13).

Lemma 6.1. *Let $\Omega = (0, 1) \times (-1, 1)$. Then, $\mathcal{A}^\parallel$ as in (1.13) admits the representation*

$$\begin{aligned}\mathcal{A}^\parallel = \{u \in \mathrm{BV}(\Omega; \mathbb{R}^2) : \nabla u = R(\mathbb{I} + \gamma e_1 \otimes e_2) \text{ with } R \in \mathrm{SO}(2),\ \gamma \in L^1(\Omega) \text{ such that } \partial_1\gamma = 0,\\ D^s u = (\varrho \otimes e_2)|D^s u| \text{ with } \varrho \in L^1_{|D^s u|}(\Omega; \mathbb{R}^2) \text{ such that}\\ |\varrho| = 1 \text{ and } \varrho \| Re_1 \text{ for } |D^s u|\text{-a.e. in } \Omega\}.\end{aligned} \tag{6.1}$$

Proof. Let $\tilde{\mathcal{A}}^\parallel$ denote the set on the right-hand side of (6.1). Arguing as in the beginning of the proof of Proposition 4.9 (precisely, with the notation of (1.13), we set $\gamma(x) = \vartheta_a'(x_2)$ for $x \in \Omega$, and observe that $(D^s u)e_2 = \mathcal{L}^1\lfloor(0, 1) \otimes D^s\vartheta_s Re_1)$ and exploiting the polar decomposition of measures (cf. (2.2) and (2.3)) gives rise to $\mathcal{A}^\parallel \subset \tilde{\mathcal{A}}^\parallel$. Conversely, the inclusion $\tilde{\mathcal{A}}^\parallel \subset \mathcal{A}$, which follows from (4.5), along with (4.6) yields that $\tilde{\mathcal{A}}^\parallel \subset \mathcal{A}^\parallel$. □

We are now in a position to prove the Γ-convergence of the energies $(E_\delta^\varepsilon)_\varepsilon$ in (1.14) as $\varepsilon \to 0$. We refer to the Appendix for an alternative proof of Theorem 1.4, in which we explore the representation of $\mathcal{A}^\parallel$ in (6.1) to construct a recovery sequence.

Proof of Theorem 1.4. As before in the proofs of Theorems 1.1 and 1.3, one may assume without loss of generality that $\Omega = (0, 1) \times (-1, 1)$. We proceed in three steps.

Step 1: Compactness. Let $(u_\varepsilon)_\varepsilon \subset W^{1,1}(\Omega; \mathbb{R}^2) \cap L^1_0(\Omega; \mathbb{R}^2)$ be a sequence such that $E_\varepsilon^\delta(u_\varepsilon) \le C$ for all $\varepsilon > 0$. Then, because $u_\varepsilon \in \mathcal{A}_\varepsilon$ for all ε,

$$\nabla u_\varepsilon = R_\varepsilon(\mathbb{I} + \gamma_\varepsilon e_1 \otimes e_2) \in L^1(\Omega; \mathbb{R}^{2\times 2}), \tag{6.2}$$

and $\|\gamma_\varepsilon\|_{L^1(\Omega)} \le C$ for every $\varepsilon > 0$. Additionally, since each map R_ε takes value in the set of proper rotations, it holds that $\|R_\varepsilon\|^2_{L^\infty(\Omega;\mathbb{R}^{2\times 2})} = 2$ for all $\varepsilon > 0$. Consequently, along with the Poincaré–Wirtinger inequality,

$$\|u_\varepsilon\|_{W^{1,1}(\Omega;\mathbb{R}^2)} \le C.$$

We further know that

$$\|\partial_1 u_\varepsilon\|^p_{W^{1,p}(\Omega;\mathbb{R}^2)} = \|R_\varepsilon e_1\|^p_{W^{1,p}(\Omega;\mathbb{R}^2)} \le \frac{C}{\delta}$$

for any ε. Thus, after extracting subsequences if necessary, one can find $u \in \mathrm{BV}(\Omega; \mathbb{R}^2)$ and $\hat{R} \in W^{1,p}(\Omega; \mathbb{R}^{2\times 2})$ such that

$$u_\varepsilon \overset{*}{\rightharpoonup} u \quad \text{in } \mathrm{BV}(\Omega; \mathbb{R}^2), \tag{6.3}$$

$$R_\varepsilon \rightharpoonup \hat{R} \quad \text{in } W^{1,p}(\Omega; \mathbb{R}^{2\times 2}). \tag{6.4}$$

Recalling the compact embedding $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow C^{0,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1 - \frac{2}{p}$, it follows even that

$$\hat{R} \in W^{1,p}(\Omega; \mathrm{SO}(2)) \cap C^{0,\alpha}(\overline{\Omega}; \mathbb{R}^{2\times 2})$$

and

$$R_\varepsilon \to \hat{R} \quad \text{in } L^\infty(\Omega;\mathbb{R}^{2\times 2}). \tag{6.5}$$

As a consequence of Proposition 4.3, it holds that $u \in \mathcal{A}$. In view of (4.5) in Proposition 4.1, this implies for the singular part of Du that

$$(D^s u)e_1 = 0, \tag{6.6}$$

while one obtains for the absolutely continuous part of Du that

$$\nabla u(x) = R(x_2)(\mathbb{I} + \gamma(x_2)e_1 \otimes e_2) \tag{6.7}$$

for a.e. $x \in \Omega$ with $R \in BV(-1,1;SO(2))$ such that $R' = 0$ and $\gamma \in L^1(-1,1)$ with $\partial_1\gamma = 0$; note that R and γ can be identified with elements in $BV(\Omega;SO(2))$ and $L^1(\Omega)$, respectively, via constant extension in the x_1-direction.

We observe that on the one hand, $(\nabla u_\varepsilon)e_1 = R_\varepsilon e_1 \rightharpoonup \hat{R}e_1$ in $W^{1,p}(\Omega;\mathbb{R}^2)$ by (6.4), and on the other hand, $(\nabla u_\varepsilon)e_1\mathcal{L}^2\lfloor\Omega \overset{*}{\rightharpoonup} (Du)e_1 = Re_1\mathcal{L}^2\lfloor\Omega$ in $\mathcal{M}(\Omega;\mathbb{R}^2)$ by (6.3), (6.7), and (6.6). Hence, a comparison of the limit objects yields $R = \hat{R} \in W^{1,p}(\Omega;SO(2))$. Since the absolutely continuous part of DR vanishes, R has to be constant, meaning $R \in SO(2)$.

Due to Alberti's rank-one theorem (cf. Section 2.2), it follows from (6.6) that

$$D^s u = (\varrho \otimes e_2)|D^s u|, \tag{6.8}$$

where $\varrho \in L_{|D^s u|}(\Omega;\mathbb{R}^2)$ with $|\varrho| = 1$ for $|D^s u|$-a.e. in Ω.

To conclude that $u \in \mathcal{A}^\parallel$, in view of Lemma 6.1, it remains to show that

$$\varrho || Re_1 \quad |D^s u|\text{-a.e. in } \Omega. \tag{6.9}$$

To prove (6.9), we first observe that for every ε, the identity $(\nabla u_\varepsilon)e_2 = R_\varepsilon e_2 + \gamma_\varepsilon R_\varepsilon e_1$, which follows from $u_\varepsilon \in \mathcal{A}_\varepsilon$, yields

$$\int_\Omega [(\nabla u_\varepsilon)e_2 \cdot R_\varepsilon e_2 - 1]\varphi \, dx = 0 \tag{6.10}$$

for all $\varphi \in C_c^\infty(\Omega)$. Thus, by (6.3) and (6.5) in combination with a weak-strong convergence argument, taking the limit $\varepsilon \to 0$ in (6.10) leads to

$$\int_\Omega \varphi \, dx = \int_\Omega \varphi Re_2 \cdot \, d((Du)e_2) = \int_\Omega \varphi Re_2 \cdot (\nabla u)e_2 \, dx + \int_\Omega \varphi Re_2 \cdot \, d((D^s u)e_2)$$

for every $\varphi \in C_c^\infty(\Omega)$. Next, we plug in the identities $(\nabla u)e_2 = Re_2 + \gamma Re_1$ and $(D^s u)e_2 = \varrho|D^s u|$ (see (6.8)) to derive that

$$0 = \int_\Omega \varphi Re_2 \cdot \, d((D^s u)e_2) = \int_\Omega \varphi Re_2 \cdot \varrho \, d|D^s u|$$

for every $\varphi \in C_c^\infty(\Omega)$, which completes the proof of (6.9).

Step 2: Lower bound. Let $(u_\varepsilon) \subset L_0^1(\Omega;\mathbb{R}^2)$ and $u \in L_0^1(\Omega;\mathbb{R}^2)$ be such that $u_\varepsilon \to u$ in $L^1(\Omega;\mathbb{R}^2)$. We want to show that

$$E^\delta(u) \le \liminf_{\varepsilon\to 0} E_\varepsilon^\delta(u_\varepsilon). \tag{6.11}$$

To prove (6.11), we proceed as in the proof of (5.1), observing in addition that

$$\liminf_{\varepsilon\to 0} \delta\|\partial_1 u_\varepsilon\|^p_{W^{1,p}(\Omega;\mathbb{R}^2)} = \liminf_{\varepsilon\to 0} \delta\|R_\varepsilon e_1\|^p_{W^{1,p}(\Omega;\mathbb{R}^2)} \ge \delta\|Re_1\|^p_{W^{1,p}(\Omega;\mathbb{R}^2)} = \delta|\Omega|$$

due to (6.2) and (6.4) with $R \in SO(2)$.

Step 3: Upper bound. Let $u \in L_0^1(\Omega;\mathbb{R}^2) \cap \mathcal{A}^\parallel$. We want to show that there is a sequence $(u_\varepsilon) \subset L_0^1(\Omega;\mathbb{R}^2)$ such that $u_\varepsilon \to u$ in $L^1(\Omega;\mathbb{R}^2)$, and

$$E^\delta(u) \ge \limsup_{\varepsilon\to 0} E_\varepsilon^\delta(u_\varepsilon). \tag{6.12}$$

Let $(u_\varepsilon)_\varepsilon \subset W^{1,1}(\Omega;\mathbb{R}^2) \cap L^1_0(\Omega;\mathbb{R}^2)$ be the sequence constructed in the proof of Proposition 4.9, that is, $u_\varepsilon \in \mathcal{A}_\varepsilon$ for every ε with

$$\nabla u_\varepsilon(x) = R\Big(\mathbb{I} + \Big(\frac{\vartheta'_a(x_2)}{\lambda} + \vartheta'_\varepsilon(x_2)\Big)\mathbb{1}_{\varepsilon Y_{\text{soft}}}(x) e_1 \otimes e_2\Big),$$

where $(\vartheta_\varepsilon)_\varepsilon \subset W^{1,\infty}(-1,1)$ satisfies

$$\lim_{\varepsilon\to 0} \int_{-1}^{1} |\vartheta'_\varepsilon|\,\mathrm{d}x_2 = |D^s\vartheta_s|(-1,1) = |D^s u|(\Omega),$$

and $u_\varepsilon \overset{*}{\rightharpoonup} u$ in $\mathrm{BV}(\Omega;\mathbb{R}^2)$. Recalling that $\vartheta' = \vartheta'_a + \vartheta'_s = \vartheta'_a$, we have

$$\begin{aligned}\limsup_{\varepsilon\to 0} E^\delta_\varepsilon(u_\varepsilon) &\le \lim_{\varepsilon\to 0}\Big(\int_\Omega \frac{|\vartheta'_a(x_2)|}{\lambda}\mathbb{1}_{\varepsilon Y_{\text{soft}}}(x)\,\mathrm{d}x + \int_{-1}^{1}|\vartheta'_\varepsilon(x_2)|\,\mathrm{d}x_2 + \delta\|Re_1\|^p_{W^{1,p}(\Omega;\mathbb{R}^2)}\Big)\\ &= \int_\Omega |\vartheta'(x_2)|\,\mathrm{d}x + |D^s u|(\Omega) + \delta|\Omega| = E^\delta(u),\end{aligned}$$

which proves (6.12) and completes the proof of the theorem. □

Remark 6.2 (On compensated compactness). We point out that if $u_\varepsilon \in \mathcal{A}_\varepsilon$, with $\nabla u_\varepsilon = R_\varepsilon(\mathbb{I} + \gamma_\varepsilon e_1 \otimes e_2)$ for $R_\varepsilon \in L^\infty(\Omega;\mathrm{SO}(2))$ and $\gamma_\varepsilon \in L^1(\Omega)$ with $\gamma_\varepsilon = 0$ on $\varepsilon Y_{\text{rig}} \cap \Omega$, is such that $u_\varepsilon \overset{*}{\rightharpoonup} u$ in $\mathrm{BV}(\Omega;\mathbb{R}^2)$, and if in addition,

$$R_\varepsilon \to R \quad \text{in } C(\Omega;\mathbb{R}^{2\times 2}),$$

then a weak-strong convergence argument implies that

$$\gamma_\varepsilon \mathcal{L}^2\lfloor\Omega = [(\nabla u_\varepsilon)e_2 \cdot R_\varepsilon e_1]\mathcal{L}^2\lfloor\Omega \overset{*}{\rightharpoonup} (Du)e_2 \cdot Re_1 \quad \text{in } \mathcal{M}(\Omega).$$

However, if continuity and uniform convergence of R_ε fail, the limit representation above may no longer be true in general, even if $R \in C(\Omega;\mathrm{SO}(2))$. To see this, let us consider the basic construction in Remark 4.6 (ii). In this case,

$$\gamma_\varepsilon \mathcal{L}^2\lfloor\Omega \overset{*}{\rightharpoonup} (\alpha+\beta)\mathcal{H}^1\lfloor((0,1)\times\{0\}) \quad \text{in } \mathcal{M}(\Omega), \tag{6.13}$$

whereas

$$(Du)e_2 \cdot Re_1 = [(\psi^+ - \psi^-)\cdot Re_1]\mathcal{H}^1\lfloor((0,1)\times\{0\}). \tag{6.14}$$

Recalling that $\psi^+ - \psi^- = \alpha Re_1 + \beta Se_1$, the quantities in (6.13) and (6.14) can only match if $Re_1 || Se_1$, which contradicts the assumption that Re_1 and Se_1 are linearly independent.

The role of the higher-order regularization in (1.14) is exactly that it helps overcome the issue discussed above. In fact, it guarantees the desired compactness properties for sequences of deformations with equibounded energies.

A Appendix

In this appendix, we start by presenting an alternative proof of Step 2 in the proof of Theorem 1.2. As we mentioned before, this alternative argument is based on a Reshetnyak's lower semicontinuity result, and highlights a different feature of the representation of $\mathcal{A}$.

Alternative proof of Step 2 in Theorem 1.2. As before, let $(u_\varepsilon)_\varepsilon \subset L^1_0(\Omega;\mathbb{R}^2)$ and $u \in L^1_0(\Omega;\mathbb{R}^2)$ be such that $u_\varepsilon \to u$ in $L^1(\Omega;\mathbb{R}^2)$. We want to show that (5.1) holds, for which we may assume without loss of generality that the limit inferior on the right-hand side of (5.1) is actually a limit and that this limit is finite. Then, $u_\varepsilon \in \mathcal{A}_\varepsilon$ and $E_\varepsilon(u_\varepsilon) < C$ for all ε, where $C > 0$ is a constant independent of ε. Hence, by Step 1, $u_\varepsilon \overset{*}{\rightharpoonup} u$ in $\mathrm{BV}(\Omega;\mathbb{R}^2)$ and $u \in \mathcal{A}$.

Next, we observe that the map $\mathbb{R}^{2\times 2} \ni F \mapsto \sqrt{|F|^2 - 2\det F}$ is convex (see [18]) and one-homogeneous. Consequently, it follows from Remark 5.1 and Reshetnyak's lower semicontinuity theorem (see [2, Theorem 2.38]), under consideration of our notation for the polar decomposition $Du = g_u|Du|$ introduced in Section 2.2, that

$$\liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) = \liminf_{\varepsilon\to 0} \int_\Omega \sqrt{|\nabla u_\varepsilon|^2 - 2\det \nabla u_\varepsilon}\, dx \geq \int_\Omega \sqrt{|g_u|^2 - 2\det g_u}\, d|Du|. \tag{A.1}$$

Since $\nabla u = R(\mathbb{I} + \gamma e_1 \otimes e_2)$ with $R \in BV(\Omega; SO(2))$ and $(D^s u)e_1 = 0$ (see (4.5)), we have $|\nabla u|^2 - 2\det\nabla u = |\gamma|^2$ for $\mathcal{L}^2$-a.e. in Ω and $\det g_u = 0$ for $|D^s u|$-a.e. in Ω. Thus,

$$\begin{aligned}\int_\Omega \sqrt{|g_u|^2 - 2\det g_u}\, d|Du| &= \int_\Omega \sqrt{|\nabla u|^2 - 2\det\nabla u}\, dx + \int_\Omega \sqrt{|g_u|^2 - 2\det g_u}\, d|D^s u| \\ &= \int_\Omega |\gamma|\, dx + |D^s u|(\Omega) = E(u),\end{aligned} \tag{A.2}$$

where we also used that the relation $|g_u| = 1$ holds $|D^s u|$-a.e. in Ω.

From (A.1) and (A.2), we deduce (5.1). □

To conclude, we present an alternative construction for the recovery sequence in Step 3 of the proof of Theorem 1.4 based on the representation of $\mathcal{A}^\parallel$ in (6.1).

Alternative proof of Theorem 1.4. As before, we may assume without loss of generality that $\Omega = (0,1)\times(-1,1)$. Moreover, the compactness property and lower bound can be studied exactly as in the proof of Theorem 1.4 above. We are then left to show that given $u \in L^1_0(\Omega;\mathbb{R}^2) \cap \mathcal{A}^\parallel$, there exists a sequence $(u_\varepsilon)_\varepsilon \subset L^1_0(\Omega;\mathbb{R}^2)$ satisfying $u_\varepsilon \to u$ in $L^1(\Omega;\mathbb{R}^2)$ and (6.12). We will proceed in three steps, building up complexity.

Step 1. We assume first that $u \in L^1_0(\Omega;\mathbb{R}^2) \cap \mathcal{A}^\parallel$ is an SBV-function with a single, constant jump line at $x_2 = 0$.

This case can be treated as highlighted in Remark 4.6 (iii). Let $R \in SO(2)$, $\gamma \in L^1(\Omega)$ with $\partial_1\gamma = 0$, and $\psi^+, \psi^- \in \mathbb{R}^2$ with $(\psi^+ - \psi^-)||Re_1$ be such that

$$Du = R(\mathbb{I} + \gamma e_1 \otimes e_2)\mathcal{L}^2\lfloor\Omega + (\psi^+ - \psi^-)\otimes e_2 \mathcal{H}^1\lfloor((0,1)\times\{0\}).$$

Note that setting $\iota := \operatorname{sign}((\psi^+ - \psi^-)\cdot Re_1) \in \{\pm 1\}$, we have $\psi^+ - \psi^- = \iota|\psi^+ - \psi^-|Re_1$ and

$$|Du|(\Omega) = |D^a u|(\Omega)| + |D^s u|(\Omega)| = |D^a u|(\Omega) + |D^j u|(\Omega) = \int_\Omega |R(\mathbb{I} + \gamma e_1 \otimes e_2)|\, dx + |\psi^+ - \psi^-|.$$

For each $\varepsilon > 0$, set $\tau_\varepsilon := \iota \frac{|D^j u|(\Omega)}{\lambda\varepsilon} = \iota \frac{|\psi^+ - \psi^-|}{\lambda\varepsilon}$. Arguing as, for instance, in the proof of Lemma 4.5, we can find $u_\varepsilon \in L^1_0(\Omega;\mathbb{R}^2) \cap \mathcal{A}_\varepsilon$ such that

$$\nabla u_\varepsilon = \begin{cases} R(\mathbb{I} + \tau_\varepsilon e_1 \otimes e_2) & \text{if } x \in (0,1)\times(0,\lambda\varepsilon), \\ R(\mathbb{I} + \frac{\gamma}{\lambda}\mathbb{1}_{\varepsilon Y_{\text{soft}}\cap\Omega} e_1 \otimes e_2) & \text{otherwise,} \end{cases}$$

and $u_\varepsilon \overset{*}{\rightharpoonup} u$ in $BV(\Omega;\mathbb{R}^2)$. Next, we show that this construction yields convergence of energies. Indeed, we have

$$\begin{aligned}\lim_{\varepsilon\to 0} E^\delta_\varepsilon(u_\varepsilon) &= \lim_{\varepsilon\to 0}\Big(\int_{(0,1)\times(0,\lambda\varepsilon)} |\tau_\varepsilon|\, dx + \int_{\Omega\setminus(0,1)\times(0,\lambda\varepsilon)} \Big|\frac{\gamma}{\lambda}\Big|\mathbb{1}_{\varepsilon Y_{\text{soft}}}\, dx + \delta\|Re_1\|^p_{W^{1,p}(\Omega;\mathbb{R}^2)}\Big) \\ &= |\psi^+ - \psi^-| + \int_\Omega |\gamma|\, dx + \delta|\Omega| = |D^s u|(\Omega) + \int_\Omega |\gamma|\, dx + \delta|\Omega| = E^\delta(u).\end{aligned}$$

Step 2. We assume next that $u \in L^1_0(\Omega;\mathbb{R}^2) \cap \mathcal{A}^\parallel$ is an SBV-function with a finite number of horizontal jump lines and with constant upper and lower approximate limits on each jump line.

In this setting, $\nabla u = R(\mathbb{I} + \gamma e_1 \otimes e_2)$ with $R \in SO(2)$ and $\gamma \in L^1(\Omega)$ with $\partial_1\gamma = 0$, $J_u = \bigcup_{i=1}^{\ell}(0,1)\times\{a_i\}$ with $\ell \in \mathbb{N}$ and $-1 < a_1 < a_2 < \cdots < a_\ell < 1$, $D^j u = \sum_{i=1}^{\ell}(\psi_i^+ - \psi_i^-)\otimes e_2\mathcal{H}^1\lfloor((0,1)\times\{a_i\})$ with $\psi_i^\pm \in \mathbb{R}^2$ satisfying $(\psi_i^+ - \psi_i^-)||Re_1$ for all $i \in \{1, \dots, \ell\}$, and $D^c u = 0$. Hence,

$$Du = R(\mathbb{I} + \gamma e_1 \otimes e_2)\mathcal{L}^2 \lfloor \Omega + \sum_{i=1}^{\ell} (\psi_i^+ - \psi_i^-) \otimes e_2 \mathcal{H}^1 \lfloor ((0,1) \times \{a_i\}) \tag{A.3}$$

and

$$|D^s u|(\Omega) = \sum_{i=1}^{\ell} |\psi_i^+ - \psi_i^-|.$$

As in the proof of Proposition 4.7, the idea is to perform a construction similar to that in Step 1 around each jump line but accounting for the possibility that one or more of the jump lines may not intersect $\varepsilon Y_{\text{soft}} \cap \Omega$.

Fix $i \in \{1, \dots, \ell\}$ and $\varepsilon > 0$, and let $\kappa_\varepsilon^i \in \mathbb{Z}$ be the integer such that $a_i \in \varepsilon[\kappa_\varepsilon^i, \kappa_\varepsilon^i + 1)$. Since $a_i \neq a_j$ if $i \neq j$, we may assume that the sets $\{\varepsilon[\kappa_\varepsilon^i, \kappa_\varepsilon^i + 1)\}_i$ are pairwise disjoint for all $\varepsilon > 0$ (this is true for all $\varepsilon > 0$ sufficiently small). Then, we take $u_\varepsilon \in L_0^1(\Omega; \mathbb{R}^2) \cap \mathcal{A}_\varepsilon$ such that

$$\nabla u_\varepsilon = \begin{cases} R(\mathbb{I} + \tau_\varepsilon^i e_1 \otimes e_2) & \text{in } (0,1) \times \varepsilon(\kappa_\varepsilon^i, \kappa_\varepsilon^i + \lambda), \\ R(\mathbb{I} + \frac{\gamma}{\lambda} \mathbb{1}_{\varepsilon Y_{\text{soft}} \cap \Omega} e_1 \otimes e_2) & \text{otherwise}, \end{cases}$$

where $\tau_\varepsilon^i = \iota_i \frac{|\psi_i^+ - \psi_i^-|}{\lambda \varepsilon}$ with $\iota_i := \operatorname{sign}((\psi_i^+ - \psi_i^-) \cdot Re_1) \in \{\pm 1\}$. As in the proof of Proposition 4.7, we obtain that

$$\lim_{\varepsilon \to 0} \int_\Omega \nabla u_\varepsilon \varphi \, dx = \sum_{i=1}^{\ell} \int_0^1 \iota_i |\psi_i^+ - \psi_i^-| (Re_1 \otimes e_2) \varphi(x_1, a_i) \, dx_1 + \int_\Omega R(\mathbb{I} + \gamma e_1 \otimes e_2) \varphi \, dx \tag{A.4}$$

for all $\varphi \in C_0(\Omega)$. Recalling (A.3) and the equalities $\psi_i^+ - \psi_i^- = \iota_i |\psi_i^+ - \psi_i^-| Re_1$ for $i \in \{1, \dots, \ell\}$, (A.4) shows that $Du_\varepsilon \overset{*}{\rightharpoonup} Du$ in $\mathcal{M}(\Omega; \mathbb{R}^{2\times 2})$. Hence, $u_\varepsilon \overset{*}{\rightharpoonup} u$ in $BV(\Omega; \mathbb{R}^2)$.

Finally, proceeding exactly as in Step 1, we conclude that this construction also yields convergence of the energies. This ends Step 2.

Step 3. We consider now the general case $u \in L_0^1(\Omega; \mathbb{R}^2) \cap \mathcal{A}^{\parallel}$.

Similarly to the beginning of the proof of Proposition 4.9 (see (4.31)), we can write

$$u(x) = x_1 Re_1 + \phi_a(x_2) + \phi_s(x_2), \quad x \in \Omega,$$

where

$$\phi_a(x_2) := x_2 Re_2 + \vartheta_a(x_2) Re_1 + c \quad \text{and} \quad \phi_s(x_2) := \vartheta_s(x_2) Re_1.$$

Note that $\phi_a \in W^{1,1}(-1, 1; \mathbb{R}^2)$ and $\phi_s \in BV(-1, 1; \mathbb{R}^2)$ is the sum of a jump function and a Cantor function; in particular, $\vartheta' = \vartheta_a'$ and $D\phi_s = D^s \phi_s$ (see (2.4)). Moreover,

$$\begin{aligned} \nabla u &= Re_1 \otimes e_1 + \nabla \phi_a \otimes e_2 = R(\mathbb{I} + \vartheta_a' e_1 \otimes e_2) = R(\mathbb{I} + \vartheta' e_1 \otimes e_2), \\ D^s u &= \mathcal{L}^1 \lfloor (0,1) \otimes D\phi_s, \\ |D^s u| \lfloor \Omega &= \mathcal{L}^1 \lfloor (0,1) \otimes |D\phi_s|. \end{aligned} \tag{A.5}$$

By Lemma 6.1, there exists $\varrho \in L^1_{|D^s u|}(-1, 1; \mathbb{R}^2)$ with $|\varrho| = 1$ such that

$$D^s u = (\varrho \otimes e_2)|D^s u| \quad \text{and} \quad \varrho = (\varrho \cdot Re_1) Re_1. \tag{A.6}$$

Let $\varrho_h \in C^\infty([-1, 1])$ be such that

$$\lim_{h \to \infty} \int_\Omega |\varrho_h(x_2) - \varrho(x_2)| \, d|D^s u|(x) = 0. \tag{A.7}$$

Since $|\varrho| = 1$, we can choose such a sequence so that $|\varrho_h| \le 1$.

Due to the properties of good representatives (see [2, (3.24)]) and [19, Lemma 3.2], for each $n \in \mathbb{N}$, there exists a piecewise constant function $\phi_n \in BV(-1, 1; \mathbb{R}^2)$, of the form

$$\phi_n = \sum_{i=0}^{\ell_n} b_i^n \chi_{A_i^n},$$

where $\ell_n \in \mathbb{N}$, $(b_i^n)_{i=0}^{\ell_n} \subset \mathbb{R}^2$, and $(A_i^n)_{i=0}^{\ell_n}$ is a partition of $(-1,1)$ into intervals with $\sup A_i^n = \inf A_{i+1}^n$, satisfying

$$J_{\phi_n} = \bigcup_{i=1}^{\ell_n} \{a_i^n\} \quad \text{with } a_i^n := \sup A_{i-1}^n,$$

$$\lim_{n\to\infty} \|\phi_n - \phi_s\|_{L^1(-1,1;\mathbb{R}^2)} = 0, \tag{A.8}$$

$$\lim_{n\to\infty} |D\phi_n|(-1,1) = \lim_{n\to\infty} |D^j\phi_n|(-1,1) = |D\phi_s|(-1,1) = |D^s u|(\Omega). \tag{A.9}$$

Indeed, (A.8) and (A.9) mean that $(\phi_n)_{n\in\mathbb{N}}$ converges strictly to ϕ_s in $\mathrm{BV}(-1,1;\mathbb{R}^2)$, which implies that

$$|D\phi_n| \overset{*}{\rightharpoonup} |D\phi_s| \quad \text{in } \mathcal{M}(-1,1), \tag{A.10}$$

see [2, Proposition 3.5].

Finally, for $n \in \mathbb{N}$, we define

$$u_n(x) := x_1 R e_1 + \phi_a(x_2) + \phi_n(x_2) + c_n, \quad x \in \Omega,$$

where $c_n \in \mathbb{R}^2$ are constants chosen so that $\int_\Omega u_n \,\mathrm{d}x = 0$. Note that $c_n \to 0$ as $n \to \infty$ by (A.8). Moreover, for each $n \in \mathbb{N}$, the map $u_n \in L_0^1(\Omega;\mathbb{R}^2)$ has the same structure as in Step 2 apart from the condition $(u_n^+ - u_n^-) \| R e_1$ on J_{u_n}, which a priori is not satisfied. Choosing $\iota_i^n := \varrho_h(a_i^n)\cdot Re_1$, we can invoke Step 2 up to, and including, (A.4) to construct a sequence $(u_\varepsilon^{n,h})_\varepsilon \subset L_0^1(\Omega;\mathbb{R}^2) \cap W^{1,1}(\Omega;\mathbb{R}^2)$ that satisfies for all $\varphi \in C_0(\Omega)$,

$$\begin{aligned}\lim_{\varepsilon\to 0} \int_\Omega \nabla u_\varepsilon^{n,h} \varphi \,\mathrm{d}x = \sum_{i=1}^{\ell_n} \int_0^1 (\varrho_h(a_i^n)\cdot Re_1)|b_i^n - b_{i-1}^n|(Re_1 \otimes e_2)\varphi(x_1, a_i^n)\,\mathrm{d}x_1 \\ + \int_\Omega R(\mathbb{I} + \vartheta_a'(x_2) e_1 \otimes e_2)\varphi \,\mathrm{d}x.\end{aligned} \tag{A.11}$$

We conclude from (A.5), (A.6), (A.7), (A.8), (A.10), and the Lebesgue dominated convergence theorem that

$$\begin{aligned}&\lim_{h\to\infty}\lim_{n\to\infty} \sum_{i=1}^{\ell_n} \int_0^1 (\varrho_h(a_i^n)\cdot Re_1)|b_i^n - b_{i-1}^n|(Re_1\otimes e_2)\varphi(x_1,a_i^n)\,\mathrm{d}x_1 \\ &= \lim_{h\to\infty}\lim_{n\to\infty} \int_0^1 \int_{-1}^1 (\varrho_h(x_2)\cdot Re_1)(Re_1\otimes e_2)\varphi(x_1,x_2)\,\mathrm{d}|D\phi_n|(x_2)\,\mathrm{d}x_1 \\ &= \lim_{h\to\infty} \int_0^1 \int_{-1}^1 (\varrho_h(x_2)\cdot Re_1)(Re_1\otimes e_2)\varphi(x_1,x_2)\,\mathrm{d}|D\phi_s|(x_2)\,\mathrm{d}x_1 \\ &= \int_\Omega (\varrho(x_2)\cdot Re_1)(Re_1\otimes e_2)\varphi \,\mathrm{d}|D^s u| \\ &= \int_\Omega (\varrho(x_2)\otimes e_2)\varphi\,\mathrm{d}|D^s u| \\ &= \int_\Omega \varphi \,\mathrm{d}D^s u.\end{aligned} \tag{A.12}$$

Recalling that $|\varrho_h(a_i^n)\cdot Re_1| \le 1$, we can further argue as in Steps 1 and 2 regarding the convergence of the energies to get

$$\begin{aligned}\limsup_{\varepsilon\to 0} E_\varepsilon^\delta(u_\varepsilon^{n,h}) \le E^\delta(u_n) &= \int_\Omega |\vartheta_a'(x_2)|\,\mathrm{d}x + |D^s\phi_n|(-1,1) + \delta|\Omega| \\ &= \int_\Omega |\vartheta'(x_2)|\,\mathrm{d}x + |D^j\phi_n|(-1,1) + \delta|\Omega|.\end{aligned} \tag{A.13}$$

Letting $n \to \infty$ and $h \to \infty$ in (A.11) and (A.13), from (A.12), (A.9), and (A.5), we conclude that for all $\varphi \in C_0(\Omega)$,

$$\lim_{h\to\infty} \lim_{n\to\infty} \lim_{\varepsilon\to 0} \int_\Omega \nabla u_\varepsilon^{n,h} \varphi \,\mathrm{d}x = \int_\Omega \varphi \,\mathrm{d}Du, \tag{A.14}$$

$$\limsup_{h\to\infty} \limsup_{n\to\infty} \lim_{\varepsilon\to 0} E_\varepsilon^\delta(u_\varepsilon^{n,h}) \le \int_\Omega |\vartheta'(x_2)| \,\mathrm{d}x + |D^s u|(\Omega) + \delta|\Omega| = E^\delta(u). \tag{A.15}$$

Owing to the separability of $C_0(\Omega)$ and (A.14)–(A.15), we can use a diagonalization argument as that in [25, proof of Proposition 1.11 (p. 449)] to find sequences $(h_\varepsilon)_\varepsilon$ and $(n_\varepsilon)_\varepsilon$ such that $h_\varepsilon,\ n_\varepsilon \to \infty$ as $\varepsilon \to 0$ and $\tilde{u}_\varepsilon := u_\varepsilon^{n_\varepsilon,h_\varepsilon} \in L^1_0(\Omega;\mathbb{R}^2) \cap W^{1,1}(\Omega;\mathbb{R}^2)$ has all the desired properties. □

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