STABLE HOMOLOGY ISOMORPHISMS FOR THE PARTITION AND JONES ANNULAR ALGEBRAS

GUY BOYDE

ABSTRACT. We show that the homology of the Jones annular algebras is isomorphic to that of the cyclic groups below a line of gradient $\frac{1}{2}$. We also show that the homology of the partition algebras is isomorphic to that of the symmetric groups below a line of gradient 1, strengthening a result of Boyd-Hepworth-Patzt. Both isomorphisms hold in a range exceeding the stability range of the algebras in question. Along the way, we prove the usual odd-strand and invertible parameter results for the Jones annular algebras.

1. INTRODUCTION

To say that a family of augmented *R*-algebras

 $A_1 \xrightarrow{\iota_1} A_2 \xrightarrow{\iota_2} A_3 \xrightarrow{\iota_3} A_4 \xrightarrow{\iota_4} \cdots$

exhibits *homological stability* is to say that the maps

 $\operatorname{Tor}_{*}^{A_{n}}(\mathbb{1},\mathbb{1}) \to \operatorname{Tor}_{*}^{A_{n+1}}(\mathbb{1},\mathbb{1})$

induced on homology by ι_n are isomorphisms in a range of degrees which increases with n. Here $\mathbb{1}$ is the trivial module obtained as the quotient of A by the augmentation ideal.

Various authors have proven stability results for Temperley-Lieb algebras [BH20; BH21; Sro22], Brauer algebras [BHP21], Iwahori-Hecke algebras of types A and B [Hep22; Mos22], and, most recently, partition algebras [BHP23]. In many of these cases, the homology is stably identified with that of an algebra whose homology is known to be stable, and the authors show that these identifications actually hold in all degrees under the assumption that some defining parameter is invertible. For the Temperley-Lieb algebras [Sro22] and Brauer algebras [Boy22], these global identifications have been shown to hold independently of the parameter when the number of strands is odd. In parallel, Patzt [Pat20] has studied representation stability for these algebras.

²⁰²⁰ Mathematics Subject Classification. Primary 16E40, 20J06; Secondary 20B30.

Key words and phrases. Homological stability, Partition algebras, Jones annular algebras.

Often, in the existing literature, one is not proving homological stability for the algebras A_n directly, but rather proving a stable isomorphism to some family B_n whose homology is known to be stable, and then getting stability for the A_n as a corollary. This paper is intended to advertise the virtues of approaching such questions by *fixing* n and attempting to resolve B_n over A_n , forgetting altogether that A_n and B_n each belong to family.

1.1. Results: Partition algebras. The partition algebras $P_n(\delta)$ (Definition 3.2) originate independently in work of Jones [Jon94b] and Martin [Mar94]. Boyd, Hepworth, and Patzt [BHP23] have proven an optimal homology stability range for the partition algebras: the map

$$\operatorname{Tor}_{q}^{P_{n-1}(\delta)}(\mathbb{1},\mathbb{1}) \to \operatorname{Tor}_{q}^{P_{n}(\delta)}(\mathbb{1},\mathbb{1})$$

is an isomorphism for $q \leq \frac{n-1}{2}$. They accomplish this by showing that the homology of the partition algebras is naturally isomorphic to that of the symmetric groups in this range. Our result is as follows.

Theorem 1.1. The natural map

$$\operatorname{Tor}_{q}^{P_{n}(\delta)}(\mathbb{1},\mathbb{1}) \to \operatorname{Tor}_{q}^{R\Sigma_{n}}(\mathbb{1},\mathbb{1}) =: H_{q}(\Sigma_{n};R)$$

is an isomorphism for $q \leq n-3$, and a surjection for q = n-2.

The point is that although the partition algebras and the symmetric groups are both known to have best-possible stability ranges given by a line of gradient $\frac{1}{2}$, the isomorphism between their homologies actually holds in a larger range, below a line of gradient 1.

1.2. **Results: Jones annular algebras.** The second family that we will study is the Jones annular algebras [Jon94a] (Definition 4.1). We will write \underline{n} for the set $\{1, 2, ..., n\}$

Let R be a commutative ring with unit, and let $\delta \in R$. Informally, the Jones annular algebra $J_n(\delta)$ has a basis consisting of partitions of the set $\underline{n} \cup \underline{n'} = \{1, 2, \dots, n, 1', 2' \dots, n'\}$ into parts of cardinality 2, such that these partitions can be represented by non-intersecting edges when the vertices are embedded in the ends of the cylinder. In other words, it is the 'cylindrical' or 'annular' version of the Temperley-Lieb algebra.

Recalling the definition of the Brauer algebra $Br_n(\delta)$ ([Bra37], but see [BHP21] for a definition that suits us better), and the Temperley-Lieb algebra $TL_n(\delta)$ ([TL71], see [BH20]), there are inclusions:

$$\operatorname{TL}_n(\delta) \subset \operatorname{J}_n(\delta) \subset \operatorname{Br}_n(\delta) \subset \operatorname{P}_n(\delta).$$

 $\mathbf{2}$

All of these algebras are *cellular* in the sense of Graham and Lehrer [GL96; Xi99], but we will not make much explicit use of this.

There is one important subtlety in the definition of $J_n(\delta)$: pictorial representatives which differ 'by a Dehn twist' correspond to the same basis element. That is, unlike the Temperley-Lieb algebra, a single basis element (i.e. a single pairing on the vertices) can be represented by multiple non-isotopic pictures. If this were not so then $J_n(\delta)$ would be infinite dimensional, and would not be a subalgebra of the partition algebra. This makes it a slight abuse of terminology to call the basis elements *annular diagrams*, but we will do so nonetheless.

Example 1.2. Here are two non-isotopic pictorial representatives of a single basis element $J_{11}(\delta)$ - they differ only by a Dehn twist. The connections are coloured only to more clearly indicate which is which - this has no meaning in the algebra. The cylinder is drawn as a pasting diagram (the dotted lines to be identified).



Example 1.3. Here is a sample multiplication in $J_{11}(\delta)$. Let α be the diagram from the previous example, and let



Then $\alpha\beta$ is computed by forming the 'composed diagram'



and (just like in the Temperley-Lieb algebra) replacing each loop appearing in the middle with a factor of $\delta \in R$. This gives:



where we simplify the result as in the Temperley-Lieb algebra, but with an additional Dehn twist.

An annular diagram having n left-to-right connections may be identified with an element of the cyclic group C_n . This gives a retraction

$$RC_n \to J_n(\delta) \to RC_n,$$

where the second map is the quotient by the ideal $I_{\leq n-1}$ spanned by diagrams with fewer than the maximal number n of left-to-right connections.

Our main result for the Jones annular algebras is as follows.

Theorem 1.4. The natural map

$$\operatorname{Tor}_{q}^{\operatorname{J}_{n}(\delta)}(\mathbb{1},\mathbb{1}) \to \operatorname{Tor}_{q}^{RC_{n}}(\mathbb{1},\mathbb{1}) =: H_{q}(C_{n};R)$$

is an isomorphism for $q \leq \frac{n}{2} - 3$, and a surjection for $q = \frac{n}{2} - 2$.

The cyclic groups (hence also the Jones annular algebras) cannot exhibit any kind of homological stability (for example $H_1(C_n; \mathbb{Z}) \cong \mathbb{Z}/n$), but the two families are nonetheless stably isomorphic.

We also prove the following.

Theorem 1.5. If n is odd or δ is invertible, then the map

$$\operatorname{Tor}_{a}^{J_{n}(\delta)}(\mathbb{1},\mathbb{1}) \to H_{q}(C_{n};R)$$

of Theorem 1.4 is an isomorphism for all q.

1.3. Methods. Each of our two main theorems asserts that a certain map of algebras $A \rightarrow B$ is a homology isomorphism in a range. Morally, what we are doing in each case is

- constructing a partial resolution of B as an A-module, and
- using this as input to a change of rings spectral sequence.

In practice, we will make (the usual) more elementary argument (see the proof of Theorem 2.8) instead of using the change of rings spectral sequence. The range of degrees in which we succeed in constructing the resolution will become our stability range.

For a positive integer w we will write \underline{w} for the set $\{1, \ldots, w\}$.

Definition 1.6. Let A be an R-algebra, let I be a twosided ideal of A, and let $w \ge h \ge 1$. An *idempotent (left) cover* of I of *height* h and width w is a finite collection of left ideals J_1, \ldots, J_w of A, which cover I in the sense that $J_1 + \cdots + J_w = I$, and such that for each $S \subset \underline{w}$ with $|S| \le h$, the intersection

$$\bigcap_{i\in S} J_i$$

is either zero or is a principal left ideal generated by an idempotent. If I is free as an R-module, then an idempotent cover is said to be R-free if there is a choice of R-basis for I such that each J_i is free on a subset of this basis.

Our main technical theorem is as follows.

Theorem 1.7. Let A be an augmented R-algebra with trivial module $\mathbb{1}$. Let I be a twosided ideal of A which is free as an R-module and acts trivially on $\mathbb{1}$. Suppose that there exists an R-free idempotent left cover of I of height h. Then the natural map

$$\operatorname{Tor}_q^A(\mathbb{1},\mathbb{1}) \to \operatorname{Tor}_q^{A/I}(\mathbb{1},\mathbb{1})$$

is an isomorphism for $q \leq h-2$, and a surjection for q = h-1. If h is equal to the width w of the cover, then this map is an isomorphism for all q.

This theorem will be proven in Section 2.3; it is essentially the combination of Theorem 2.8 and Proposition 2.5 in the case $M = N = \mathbb{1}$. We should emphasise that this theorem is fairly elementary, and almost certainly already known in some guise. It nonetheless seems to be quite an effective tool.

The 'absolute' version of our approach originates with Sroka's *cellular* Davis complex [Sro22]. Sroka describes this complex as a resolution of the trivial module $\mathbb{1}$ over TL_n , and Sroka uses this to attack the homology

$$\operatorname{Tor}^{\operatorname{TL}_n}_*(1,1)$$

directly, showing that it vanishes below a line of gradient $\frac{1}{2}$, and providing an alternative to the original vanishing proof of Boyd-Hepworth [BH20], which uses their technique of *inductive resolutions* to obtain an optimal vanishing range of slope 1.

This paper began with the observation that Sroka's complex may be thought of as a partial resolution of the trivial R-algebra R as an TL_nmodule, and the conclusion of his theorem may then be interpreted as saying that the map

$$\operatorname{Tor}^{\operatorname{TL}_n}_*(\mathbb{1},\mathbb{1}) \to \operatorname{Tor}^R_*(\mathbb{1},\mathbb{1}) \cong \begin{cases} R & *=0\\ 0 & \text{otherwise,} \end{cases}$$

induced on homology by the map of algebras $\operatorname{TL}_n \to R$, is an isomorphism below a line of gradient $\frac{1}{2}$. One is then drawn to ask whether such an argument can be made to work when the second algebra is not isomorphic to the trivial module of the first. Theorem 1.7 says that the answer is yes: the additional input is essentially a change of rings spectral sequence (though we will use the usual elementary argument instead). This is what we mean when we say that this paper takes a 'relative' point of view.

It is interesting that inductive resolutions give better results for the Temperley-Lieb algebras, while a relative version of Sroka's approach seems to works better on the partition algebras.

Acknowledgements. I would like to thank Richard Hepworth for his early encouragement. The author's postdoc is funded by Gijs Heuts' ERC Starting Grant 'Chromatic homotopy theory of spaces'.

2. Algebra

In this section we will prove the main technical result, Theorem 1.7.

2.1. The Mayer-Vietoris Complex. Let A be an R-algebra, and let $N \subset M$ be A-modules. Let N_1, \ldots, N_w be submodules of N such that

$$N_1 + \dots + N_w = N.$$

In this situation, we have a 'Mayer-Vietoris' chain complex of left A-modules C_* , which we will now describe. It is essentially the 'concentrated in a single degree' version of the double complex on the (e.g. singular) chains of a topological covering, but approached from a purely algebraic point of view.

We will be interested in this complex in the case that $N_i = J_i$ is an idempotent cover of an ideal N = I of M = A. It is essentially an abstract version of Sroka's *cellular Davis complex* [Sro22, Definition 8].

We set

$$C_p := \bigoplus_{\substack{S \subset \underline{w} \\ |S| = p}} \bigcap_{i \in S} N_i$$

for p = 1, ..., w, with $C_0 := M$, and (the augmentation) $C_{-1} := M / N$. We adopt the convention that $C_p = 0$ for p > w. The differential $d_p: C_p \to C_{p-1}$ is defined as follows. For p = 0 it is just the projection $M \to M_N$. For p = 1 it is the direct sum of the inclusions. For $p \ge 2$, we define d_p on the summand $\bigcap_{i \in S} N_i$ to be the map

$$\bigcap_{i \in S} N_i \to \bigoplus_{j \in S} \bigcap_{i \in S \setminus \{j\}} N_i,$$
$$x \mapsto \sum_{j \in S} (-1)^{\#(S,j)} \iota_{(S,j)}(x),$$

where #(S, j) is the number of elements of S which are less than j, and $\iota_{(S,j)}$ is the inclusion $\bigcap_{i \in S} N_i \to \bigcap_{i \in S \setminus \{j\}} N_i$.

Although we presented C_0 and d_1 as special cases, they are not. One may think of C_0 as the intersection of *none* of the J_i , and under this convention the definition of d_1 coincides with the definition of the higher differentials.

Lemma 2.1. The Mayer-Vietoris complex associated to a cover is a chain complex. That is, the differential satisfies $d^2 = 0$.

The proof is standard, and exactly parallels Sroka's proof that his cellular Davis complex is a chain complex [Sro22, Lemma 9].

Proof. Since the N_i are all contained in N, $d_{-1} \circ d_0 = 0$.

For $p \ge 1$, it suffices to argue that the restiction of $d_{p-1} \circ d_p$ to each summand $\bigcap_{i \in S} N_i$ (with |S| = p) of C_p is zero.

Fixing such a set S, let $x \in \bigcap_{i \in S} N_i \subset C_p$. We compute:

$$d_{p-1} \circ d_p(x) = d_{p-1} \left(\sum_{j \in S} (-1)^{\#(S,j)} \iota_{(S,j)}(x) \right)$$

=
$$\sum_{i \in S \setminus \{j\}} \sum_{j \in S} (-1)^{\#(S \setminus \{j\},i) + \#(S,j)} \iota_{(S \setminus \{j\},i)} \iota_{(S,j)}(x)$$

=
$$\sum_{i \neq j \in S} (-1)^{\#(S \setminus \{j\},i) + \#(S,j)} \iota_{(S \setminus \{j\},i)} \iota_{(S,j)}(x)$$

We have

 $\iota_{(S\setminus\{j\},i)}\iota_{(S,j)} = \iota_{(S\setminus\{i\},j)}\iota_{(S,i)},$

so it suffices to show that

$$(-1)^{\#(S\setminus\{j\},i)+\#(S,j)} = -(-1)^{\#(S\setminus\{i\},j)+\#(S,i)}$$

This holds because i < j implies that $\#(S \setminus \{i\}, j) = \#(S, j) - 1$, and $\#(S \setminus \{j\}, i) = \#(S, i)$, and this completes the proof. \Box

Lemma 2.2. If N is free as an R-module on some basis \mathcal{B} , such that the N_i are free R-modules on subsets of \mathcal{B} , then the Mayer-Vietoris complex is acyclic.

Again, the proof runs exactly parallel to Sroka's proof that his cellular Davis complex is acyclic [Sro22, Theorem 10].

Proof. By construction, $d_0: M \to M/N$ is a surjection, so the Mayer-Vietoris complex is exact in degree -1. Exactness in degree 0 follows from the assumption that $N_1 + \cdots + N_w = N$. We may therefore restrict attention to $p \ge 1$.

Write $\mathcal{B}_i \subset \mathcal{B}$ for the basis of N_i . Then each intersection $\bigcap_{i \in S} N_i$ is free on the corresponding intersection $\bigcap_{i \in S} \mathcal{B}_i$.

For each $v \in \mathcal{B}$, and $S \subset \underline{w}$, let $\chi_v(S)$ be defined as follows.

- If $v \in \bigcap_{i \in S} \mathcal{B}_i$, then $\chi_v(S) \cong R$ is the submodule of $\bigcap_{i \in S} N_i$ generated by v.
- If $v \notin \bigcap_{i \in S} \mathcal{B}_i$, then $\chi_v(S) = 0$.

It is then tautological that for each $S \subset \underline{w}$ we have

$$\bigcap_{i\in S} N_i = \bigoplus_{v\in\mathcal{B}} \chi_v(S),$$

so we get a decomposition of C_p as an *R*-module:

$$C_p = \bigoplus_{\substack{S \subseteq \underline{w} \\ |S| = p}} \bigcap_{i \in S} N_i = \bigoplus_{\substack{S \subseteq \underline{w} \\ |S| = p}} \bigoplus_{v \in \mathcal{B}} \chi_v(S) = \bigoplus_{v \in \mathcal{B}} \bigoplus_{\substack{S \subseteq \underline{w} \\ |S| = p}} \chi_v(S) = \bigoplus_{v \in \mathcal{B}} C_p^v,$$

where we define C^v_* to be the chain complex with $C^v_p := \bigoplus_{\substack{S \subset \underline{w} \\ |S|=p}} \chi_v(S)$.

The boundary map d_p is a linear combination of inclusions, so it automatically respects this decomposition. That is, as chain complexes in *R*-modules we have

$$C_* = \bigoplus_{v \in \mathcal{B}} C^v_*,$$

in degrees ≥ 1 , and in degree zero $\bigoplus_{v \in \mathcal{B}} C^v_* = N = \text{Im}(d_1) \subset M$, so it suffices to establish that each C^v_* is acyclic.

Let S(v) be the subset of \underline{w} consisting of those *i* such that $v \in \mathcal{B}_i$. Then, notice that C^v_* is exactly the augmented chain complex (over R, shifted up by one degree) of the simplex $\Delta^{|S(v)|-1}$. This complex is acyclic, so we are done.

Lemma 2.3. Let X be a right A-module. Let J be a left ideal of A which is generated by idempotents and acts trivially on X. Then we have $X \otimes_A J = 0$.

Proof. It suffices to show that $x \otimes \alpha e = 0$ for e an idempotent generator of $J, x \in X$ and $\alpha \in A$. We have $x \otimes \alpha e = x \otimes \alpha e^2 = x \cdot (\alpha e) \otimes e$, which is zero since the action is trivial, as required. \Box

A principal left ideal generated by an idempotent is a projective left A-module (this is standard, but see for example [Boy22, Lemma 3.1]). We therefore have:

Lemma 2.4. Let A be an R-algebra, and let I be a twosided ideal of A, with an idempotent cover of height h. The truncation $C_*^{\leq h}$ in degree h of the Mayer-Vietoris complex associated to the cover is a projective complex over A_{I} , with $C_0 = A$.

Lemmas 2.4, 2.2 and 2.3 combine to give the following proposition.

Proposition 2.5. Let A be an R-algebra, let I be a twosided ideal of A, and let J_1, \ldots, J_w be an R-free idempotent left cover of I of height $h \leq w$. The truncation $C_*^{\leq h}$ of the Mayer-Vietoris complex associated to J_1, \ldots, J_w is an (h-1)-connected projective complex over A/I, with $C_0 = A$, and with the additional property that $X \otimes_A C_p = 0$ for $p \geq 1$ for any right A-module X on which I acts trivially. If h = w then $C_*^{\leq h} = C_*$ is actually acyclic.

In the wild, we will recognise principal idempotent ideals as retracts, in the following manner.

Lemma 2.6. If J is a left ideal of an R-algebra A which is a retract of A via a right multiplication map $A \xrightarrow{\cdot e} J$ for some element $e \in J$, then J is the principal left ideal generated by e, and e is idempotent.

Proof. Since $\cdot e$ is a retraction of the inclusion $J \to A$, we have $x \cdot e = x$ for all $x \in J$. This gives that J is principal and generated by e. In particular, since e is itself in J, we have $e \cdot e = e$, that is, e is idempotent.

2.2. The spectral sequence argument.

Proposition 2.7. Let M be a left A-module, and let N be a right A-module. Suppose that there exists an (h-1)-connected projective complex C_* of left A-modules over M, with $N \otimes_A C_p = 0$ for $p \ge 1$. Then

 $\operatorname{Tor}_q^A(N,M) = 0$ for $0 < q \le h - 2$, and $N \otimes_A M \cong N \otimes_A C_0$.

Proof. Let C_* be such an (h-1)-connected projective complex of left A-modules over M, regarded as augmented by setting $C_{-1} = M$. This

means that C_p is projective for $p \ge 0$. Let P_* be an (unaugmented) right projective resolution of N over A. The double complex $P_* \otimes_A C_*$ gives two spectral sequences converging to the same target. Write ${}^{I}E_{p,q}^{r}$ for the spectral sequence obtained by first taking homology in the C_* direction, and write ${}^{II}E_{p,q}^{r}$ for the spectral sequence obtained by first taking homology in the P_* -direction.

The P_q are projective, so taking homology in the C_* -direction gives

$${}^{l}E^{1}_{p,q} \cong P_{q} \otimes H_{p}(C_{*}),$$

which by assumption vanishes for $p \leq h - 1$, hence vanishes in total degree $p+q \leq h-1$. The E^{∞} page of the other spectral sequence, ${}^{II}E^{r}_{p,q}$ must vanish in the same range. We now examine the consequences of this vanishing for the ${}^{II}E^{1}$ page. This page takes the form

$${}^{II}E^1_{p,q} \cong \operatorname{Tor}^A_q(N, C_p),$$

since P_* is a projective resolution of N.

In the q = 0 row, we have ${}^{II}E^1_{p,0} \cong N \otimes_A C_p$, which by assumption is 0 for $p \ge 1$. Since $N \otimes_A$ is right exact, the tensor product of N with any quotient of C_p is still trivial. Thus, the exact sequence

$$0 \to C_1 /_{\operatorname{Im}(C_2)} \to C_0 \to M \to 0$$

obtained by truncating C_* implies that the map $C_0 \to M$ becomes an isomorphism $N \otimes_A C_0 \to N \otimes_A M$, which is to say that the d^1 -differential $E_{0,0}^1 \to E_{-1,0}^1$ is an isomorphism. This establishes the second claim, and shows that both groups vanish on the ${}^{II}E^2$ -page.

For q > 0, since the C_p are projective for $p \ge 0$, the E^1 -page ${}^{II}E^1_{p,q} \cong \operatorname{Tor}_q^A(N, C_p)$ vanishes outside of the p = -1 column. In this column, we have

$${}^{II}E^1_{-1,q} \cong \operatorname{Tor}_q^A(N, C_{-1}) = \operatorname{Tor}_q^A(N, M).$$

For degree reasons, then, ${}^{II}E$ collapses at the r = 2 page, which has $\operatorname{Tor}_q^A(N, M)$ in the (-1, q) position for q > 0. Since the E^{∞} -page vanishes in total degree $\leq h - 1$, the result follows.

2.3. The stable isomorphism.

Theorem 2.8. Let N be a left A-module, and let M be a right Amodule. Let I be a twosided ideal of A which acts trivially on M and N. Suppose that there exists an (h-1)-connected projective complex C_* of left A-modules over A/I, with $C_0 = A$ and $N \otimes_A C_p = 0$ for $p \ge 1$. Then the natural map

$$\operatorname{Tor}_q^A(N, M) \to \operatorname{Tor}_q^{A_{/I}}(N, M)$$

is an isomorphism for $q \leq h-2$ and a surjection for q = h-1.

Proof. We will argue that $\operatorname{Tor}_q^A(N, M)$ and $\operatorname{Tor}_q^{A_{/I}}(N, M)$ are the homology of the same chain complex.

Let P_* be a free A-resolution of N. Then $\operatorname{Tor}^A_*(N, M)$ is the homology of $P_* \otimes_A M$. Since I acts trivially on M and N, we may write

$$P_* \otimes_A M \cong P_* \otimes_A A / I \otimes_{A / I} M.$$

Then, since P_* is free over A, the tensor product $P_* \otimes_A A/I$ is free over A/I. By Proposition 2.7, we have $\operatorname{Tor}_q^A(N, A/I) = 0$ for $0 < q \le h - 2$, which is to say that the homology of $P_* \otimes_A A/I$ vanishes in degrees $0 < q \le h - 2$.

By the same proposition,

$$\operatorname{Tor}_0^A(N, A/I) := N \otimes_A A/I \cong N \otimes_A C_0 \cong N,$$

since $C_0 = A$ by assumption.

Thus, $P_* \otimes_A A'_I$ is a free (h-2)-connected complex over N, so $P_* \otimes_A A'_I \otimes_{A'_I} M$ computes $\operatorname{Tor}_q^{A'_I}(N, M)$ for $q \leq h-2$, and in dimension h-1 we get a surjection $H_{h-1}(P_* \otimes_A A'_I \otimes_{A'_I} M) \to \operatorname{Tor}_{h-1}^{A'_I}(N, M)$. This completes the proof. \Box

We are now ready to prove our main algebraic result.

Proof of Theorem 1.7. Combine Theorem 2.8 and Proposition 2.5. \Box

3. PARTITION ALGEBRAS

We will think of the partition algebras $P_n(\delta)$ in more or less the same way as Boyd-Hepworth-Patzt [BHP23, Definition 2.1]. We will use slightly different notation for our vertex set, inspired by Xi's paper [Xi99]. Xi proves that the partition algebras have a *cellular* structure, and our approach is sometimes motivated by this, but we will not need to make it explicit.

We write \mathbb{N}_0 for the set of non-negative integers, and \mathbb{N} for the set of positive integers.

Definition 3.1. Consider the set $\mathbb{N} \times \mathbb{N}_0$. We will write $(x, y) \in \mathbb{N} \times \mathbb{N}_0$ as x, together with a superscript consisting of y instances of the symbol

For example, (3, 2) is written 3'', (1, 0) is written 1, and (5, 4) is written 5''''. For a subset S of $\mathbb{N} \times \mathbb{N}_0$, let S' be the set $\{(x, y + 1) | (x, y) \in S\}$; in our notation, this is the set $\{x' | x \in S\}$.

For a fixed n, we write \underline{n} for the subset $\{1, 2, \ldots, n\}$ of $\mathbb{N} \times \mathbb{N}_0$, so that $\underline{n}' = \{1', 2', \ldots, n'\}$ is in bijection with \underline{n} via the map which 'primes each element'.

With this notation in hand, we define the partition algebras as follows.

Definition 3.2. Let R be a commutative ring with unit, let $\delta \in R$, and let $n \geq 1$. The *partition algebra* $P_n(\delta)$ is the free R-module on the set of partitions ρ of

$$\underline{n} \cup \underline{n'} = \{1, 2, \dots, n, 1', 2', \dots, n'\}.$$

The multiplication is defined on partitions, then extended bilinearly, as follows. For partitions μ and ν of $\underline{n} \cup \underline{n'}$, let ν' be the partition of $(\underline{n} \cup \underline{n'})' = \underline{n'} \cup \underline{n''}$ corresponding to ν under the priming-unpriming correspondence of Definition 3.1. Let the *composed partition*

 $\mu * \nu$

be the finest partition of $\underline{n} \cup \underline{n}' \cup \underline{n}''$ whose restriction to $\underline{n} \cup \underline{n}'$ is coarser than μ and whose restriction to $\underline{n}' \cup \underline{n}''$ is coarser than ν' . Let $j \in \mathbb{N}_0$ be the number of parts of $\mu \circ \nu'$ containing only elements of \underline{n}' , and let $\mu * \nu$ be the partition of $\underline{n} \cup \underline{n}'$ obtained by restricting $\mu \circ \nu'$ to $\underline{n} \cup \underline{n}''$ and then 'unpriming' \underline{n}'' . The product $\mu\nu$ in the partition algebra is then defined to be

$$\mu\nu = \delta^j \cdot \mu * \nu \in \mathcal{P}_n(\delta).$$

We will feel free to call $\mu * \nu$ the *underlying partition* of the product $\mu\nu$, since it is well defined even if $\delta^j = 0$.

In this section we will construct an idempotent left cover of height n-1 for the the ideal $I_{\leq n-1}$ of the partition algebra $P_n(\delta)$. Recall that $I_{\leq n-1}$ is the *R*-span of partitions having at most n-1 (i.e. fewer than the maximal number n) parts containing both primed and unprimed elements.

Definition 3.3. For i in \underline{n} let K_i be the left ideal of $P_n(\delta)$ spanned by partitions where the vertex i' on the right is a singleton. For distinct elements i < j in \underline{n} , let $L_{i,j}$ be the left ideal spanned by diagrams where the vertices i' and j' on the right belong to the same part of the partition.

Lemma 3.4. The ideals K_i and $L_{i,j}$ cover $I_{\leq n-1}$.

Proof. It suffices to argue that

(1) each K_i or $L_{i,j}$ is contained in $I_{\leq n-1}$, and

(2) any partition with fewer than n parts containing both primed and unprimed elements belongs to some K_i or to some $L_{i,j}$.

For the first point, simply note that the a partition belonging to K_i or $L_{i,j}$ has at most n-1 parts that contain both primed and unprimed elements, so the partition itself belongs to $I_{\leq n-1}$. For the second point, suppose that a partition ρ has fewer than n parts with both primed and unprimed elements. The restriction q of ρ to the primed vertices \underline{n}' must have fewer than n parts which lie in a part of ρ also containing unprimed vertices. Thus, either two primed vertices must lie in the same part of q, or if all parts of q are singletons, then one of them must also be a singleton in ρ . In the first case, ρ is contained in some $L_{i,j}$, and in the second case ρ is contained in some K_i . This completes the proof. \Box

Unsurprisingly, we will be concerned with iterated intersections of these ideals. The following lemma is immediate. Write $\underline{n}_{<}^2$ for the subset of $\underline{n} \times \underline{n}$ consisting of pairs (i, j) with i < j (i.e. the indexing set for the ideals $L_{i,j}$).

Lemma 3.5. Let $S \subset \underline{n}$, and let $T \subset \underline{n}^2_{\leq}$. The intersection

$$\bigcap_{i \in S} K_i \cap \bigcap_{(i,j) \in T} L_{i,j}$$

is the R-span of those partitions for which

• i' is a singleton whenever $i \in S$, and

• i' and j' lie in the same part of the partition whenever $(i, j) \in T$ In particular, the intersection is zero if and only if there exists $(i, j) \in T$ so that either $i \in S$ or $j \in S$.

Lemma 3.6. Let $S \subset \underline{n}$, and let $T \subset \underline{n}^2_{\leq}$. Let

$$J = \bigcap_{i \in S} K_i \cap \bigcap_{(i,j) \in T} L_{i,j},$$

and let $a \in \underline{n} \setminus S$ be such that $S \cup \{a\}$ is a proper subset of \underline{n} . Choose $b \in \underline{n} \setminus (S \cup \{a\})$, and let μ be the partition of $\underline{n} \cup \underline{n'}$ whose parts are

- the singleton $\{a'\}$,
- the triple $\{a, b, b'\}$
- the pair $\{i, i'\}$ for each $i \in \underline{n} \setminus \{a, b\}$.

Then either $K_a \cap J = 0$ or $J \cdot \mu \subset K_a \cap J$.

Of course, μ depends on a and b, but we take the liberty of omitting this from the notation.

Proof. Assume $K_a \cap J \neq 0$, and let $\rho \in J$ be a partition. We must show that $\rho \mu \in K_a \cap J$.

If $i \in S$ then, since $\rho \in J$, i' is a singleton in ρ . By construction, i is not equal to a or b, so i' is still a singleton in (the underlying partition of) $\rho\mu$. It follows that $\rho\mu$ is in K_i .

If $(i, j) \in T$ then, since $\rho \in J$, i' and j' lie in the same part of ρ . If either i = a or j = a then by Lemma 3.5 we have $K_a \cap J = 0$, which contradicts our initial assumption, so in fact neither i nor j can be equal to a. This means that i and i' lie in the same part of μ , and likewise for j and j'. It follows that i' and j' lie in the same part of $\rho\mu$. It follows that $\rho\mu$ is in $L_{i,j}$.

Lastly, a' is a singleton in μ , hence also in $\rho\mu$, so $\rho\mu \in K_a$. In total, we have shown that $\rho\mu \in \bigcap_{i \in S \cup \{a\}} K_i \cap \bigcap_{(i,j) \in T} L_{i,j} = K_a \cap J$, as required.

Lemma 3.7. Let $S \subset \underline{n}$, and let $T \subset \underline{n}^2_{\leq}$. Let

$$J = \bigcap_{i \in S} K_i \cap \bigcap_{(i,j) \in T} L_{i,j},$$

and let $a \in \underline{n} \setminus S$ be such that $S \cup \{a\}$ is a proper subset of \underline{n} . If $K_a \cap J$ is nonzero, then right multiplication by the element μ constructed in Lemma 3.6 gives a retraction of the inclusion of $K_a \cap J$ into J.

Proof. We must take a partition $\rho \in K_a \cap J$ and show that $\rho \mu = \rho$.

Every unprimed vertex lies in the same part of μ as some primed vertex, so for any partition ρ the product $\rho\mu$ produces no factors of δ , and is again a partition. We must argue that this partition is ρ .

Let A be the part of ρ containing a', and let B be the part of ρ containing b'. It then follows from the definition of μ that $\rho\mu$ has parts:

- the singleton $\{a'\}$
- $(A \setminus \{a'\}) \cup B$
- the parts of ρ other than A and B.

In other words, μ operates on ρ by removing a' from its part and merging the remainder with the part containing b'.

It follows that if a' is already a singleton in ρ , then $\rho\mu = \rho$.

Lemma 3.8. Let $S \subset \underline{n}$, and let $T \subset \underline{n}^2_{\leq}$. Let

$$J = \bigcap_{i \in S} K_i \cap \bigcap_{(i,j) \in T} L_{i,j},$$

and let $(a,b) \in \underline{n}^2_{\leq} \setminus T$. Let ν be the partition of $\underline{n} \cup \underline{n}'$ whose parts are

• the quadruple $\{a, b, a', b'\}$, and

• the pair $\{i, i'\}$ for each $i \in \underline{n} \setminus \{a, b\}$.

Then either $L_{a,b} \cap J = 0$ or $J \cdot \nu \subset L_{a,b} \cap J$.

Proof. Assume $L_{a,b} \cap J \neq 0$, and let $\rho \in J$ be a partition. We must show that $\rho \nu \in L_{a,b} \cap J$.

If $i \in S$ then i' is a singleton in ρ . By Lemma 3.5, the assumption $L_{a,b} \cap J \neq 0$ gives that a and b are not in S, so i is not equal to a or b, so $\{i, i'\}$ is a part of ν , and i' is again a singleton in $\rho\nu$. This gives that $\rho\nu \in K_i$, as required.

If $(i, j) \in T$ then i' and j' lie in the same part of ρ . For each $k \in \underline{n}$, k and k' lie in the same part of ν , so i' and j' lie in the same part of $\rho\nu$, so $\rho\nu \in L_{i,j}$, as required.

Lastly, a', b' lie in the same part of ν , hence also in the same part of $\rho\nu$, so $\rho\nu \in L_{a,b}$. In total, we have shown that $\rho\nu \in L_{a,b} \cap J$, as required.

Lemma 3.9. Let $S \subset \underline{n}$, and let $T \subset \underline{n}^2_{\leq}$. Let

$$J = \bigcap_{i \in S} K_i \cap \bigcap_{(i,j) \in T} L_{i,j},$$

and let $(a,b) \in \underline{n}^2 \setminus T$. If $L_{a,b} \cap J$ is nonzero, then right multiplication by the element ν constructed in Lemma 3.8 gives a retraction of left $P_n(\delta)$ -modules $L_{a,b} \cap J \to J$.

Proof. We must take a partition $\rho \in L_{a,b}$ and show that $\rho \nu = \rho$.

Again, every unprimed vertex lies in the same part of ν as some primed vertex, so the product $\rho\nu$ produces no factors of δ , and is again a partition. We must argue that this partition is ρ .

For any ρ , the product $\rho\nu$ is the partition obtained from ρ by merging the part containing *a* with the part containing *b*. In particular, if *a* and *b* are already in the same part of ρ (i.e. if $\rho \in L_{a,b}$) then $\rho\nu = \rho$. \Box

Lemma 3.10. Unless $S = \underline{n}$, the left ideal

$$J = \bigcap_{i \in S} K_i \cap \bigcap_{(i,j) \in T} L_{i,j}$$

is (zero or) principal and generated by an idempotent.

Proof. By inductive application of Lemmas 3.7 and 3.9, either J is zero, or J is a retract of $P_n(\delta)$. The retraction map is a composite of maps given by right multiplication by certain elements of $P_n(\delta)$, so is itself given by right multiplication by the product of those elements. This product element ρ must be lie in J because $\rho = 1 \cdot \rho$ lies in $\text{Im}(\cdot \rho) \subset J$. The result then follows from Lemma 2.6.

We are now ready to prove the main result on $P_n(\delta)$. Recall from the introduction of [BHP21] that we have

$$\mathbb{P}_n(\delta) / \mathbb{I}_{\leq n-1} \cong R\Sigma_n$$

the group algebra of the symmetric group.

Proof of Theorem 1.1. By Lemma 3.4, these ideals do indeed form an R-free cover of $I_{\leq n-1}$. By Lemma 3.10, an intersection of at most n-1 ideals from among the K_i and $L_{i,j}$ is either zero or principal idempotent, so this is indeed a principal idempotent cover of height n-1. Certainly $I_{\leq n-1}$ acts trivially on $\mathbb{1}$, so the result follows by Theorem 1.7.

4. Jones Annular Algebras

Again, we will use the 'priming' convention of Definition 3.1 for vertex labelling.

Definition 4.1. Consider a cylinder $C = S^1 \times [0, 1]$. Embed the unprimed vertices \underline{n} , equally spaced, around $S^1 \times \{0\} \subset C$, and embed the primed vertices \underline{n}' around $S^1 \times \{1\} \subset C$. Precisely, regarding S^1 as the complex unit circle, we embed j at $(e^{i\frac{2\pi j}{n}}, 0)$, and j' at $(e^{i\frac{2\pi j}{n}}, 1)$. This naturally identifies the unprimed vertices with a copy of the cyclic group C_n , and likewise for the primed vertices.

Let R be a ring and let $\delta \in R$. Recall (from e.g. [BHP21]) that the Brauer algebra $\operatorname{Br}_n(\delta)$ is the subalgebra of the partition algebra $\operatorname{P}_n(\delta)$ which has a basis consisting of partitions of $\underline{n} \cup \underline{n}'$ such that each part has cardinality 2.

Let ρ be such a partition. Say that a graphical representative of ρ on the annulus is a choice, for each part of ρ , of an embedded curve in C connecting the two vertices of the part. Say that ρ admits an annular representative or that ρ is an annular diagram if there exists a graphical representative of ρ on the annulus for which no two of the embedded curves intersect.

The Jones annular algebra $J_n(\delta)$ is then the subalgebra of $Br_n(\delta)$ spanned by partitions which admit annular representatives.

For the sake of fluidity we will feel at liberty to think of the vertices as being labelled either by the cyclic group C_n or by <u>n</u>.

By the cyclic interval [a, b] in the cyclic group C_n we mean the set $\{a, a + 1, a + 2, \ldots, b\}$. Open cyclic intervals are defined similarly. Graham and Lehrer [GL96, Proposition 6.14] give a useful description of the canonical basis of $J_n(\delta)$, which we will now describe.

Definition 4.2. An annular link state is a partition p of $C_n \cong \underline{n}$ into parts of cardinality 1 and 2, such that the defect parts are precisely those of cardinality 1, and if $\{i, j\}$ is a part of p, then:

- No part of p having cardinality 2 consists of one element from the cyclic interval (i, j) and one from (j, i). In other words, (i, j) and (j, i) are unions of parts of p.
- Either all defect parts of p (equivalently, all singletons) are contained in (i, j), or all defect parts of p are contained in (j, i).

The t defect vertices may be ordered via the correspondence $C_n \cong \underline{n}$, and we may speak of the *i*-th defect vertex for $i \in \underline{t}$. Write M(t) for the set of annular link states with t defects.

The next proposition is essentially [GL96, Proposition 6.14].

Proposition 4.3. For $0 \le t \le n$, $\sigma \in C_t$, and annular link states p and q having t defects, there is a unique annular diagram $C_{p,a}^{\sigma}$ on $\underline{n} \cup \underline{n}' = C_n \cup C'_n$ such that:

- the restriction of C^σ_{p,q} to <u>n</u> ≅ C_n is p,
 the restriction of C^σ_{p,q} to <u>n'</u> ≅ C'_n is identified with q under the priming-unpriming correspondence, and
- the *i*-th defect vertex of q and the $\sigma(i)$ -th vertex of p are connected by an edge in $C_{p,q}^{\sigma}$

Furthermore, the resulting map

$$\prod_{0 \le t \le n} M(t) \times C_t \times M(t) \to \mathcal{J}_n(\delta)$$

$$(p,q,\sigma) \to C_{p,q}^{o}$$

is an injection onto the R-basis of annular diagrams.

Henceforth, we will typically write $C_n \cup C'_n$ for the set of vertices, remembering the ordering given by $C_n \cong \underline{n}$ only when necessary.

Definition 4.4. For $i \in C_n$, let J_i be the left ideal of $J_n(\delta)$ spanned by diagrams where the vertices i' and (i + 1)' on the right are connected by an edge.

Recall that the twosided ideal $I_{\leq n-1}$ is the *R*-span of the diagrams with fewer than the maximal number n of left-to-right connections.

Lemma 4.5. The ideals J_i cover $I_{\leq n-1}$.

Proof. Any diagram x with fewer than n left-to-right connections must have a primed vertex i' not connected to an unprimed vertex. This vertex must therefore be connected to some other vertex j', so by Proposition 4.3, and the definition of annular link states (Definition 4.2), at

least one of the cyclic intervals (i', j') or (j', i') consists entirely of vertices with right-to-right connections. Without loss of generality, suppose that (i', j') consists entirely of vertices with right-to-right connections. Choose such a right-to-right connection, and call its ends i'_1 and j'_1 (choosing which end receives which name so that $(i'_1, j'_1) \subset (i', j')$). If (i'_1, j'_1) is empty, then i'_1 and j'_1 must be adjacent. If not, then (as a subset of (i', j')) it must consist entirely of vertices with right-to-right connections, and we may choose one of *these* and repeat. At each stage, the cyclic interval becomes smaller, so we eventually reach a connection between two adjacent vertices, say k' and (k + 1)', whence $x \in J_k$. \Box

Let $T \subset C_n$. Borrowing terminology from [Sro22], we will say that T is *innermost* if there do not exist distinct elements $i \neq j$ in T such that i = j + 1 or j = i + 1. As in that paper, the point is that the innermost sets are precisely those T for which there exists an annular diagram where i' and (i + 1)' are connected for every $i \in T$.

Lemma 4.6. Let $T \subset C_n$. The intersection

$$\bigcap_{i \in T} J_i$$

is the R-span of those partitions which have an edge between vertices i'and (i + 1)' whenever $i \in T$. This intersection is nonzero if and only if T is innermost.

Proof. The first claim is immediate. For the second claim, there exists an annular link state q with an arc between i' and (i + 1)' for each $i \in T$ provided that no distinct elements $i \neq j$ in T have i = j + 1or j = i + 1 (both of the conditions of Definition 4.2 holding trivially, since the open cyclic interval (i, i + 1) is empty). If such a q exists, then by Proposition 4.3 there exists a diagram whose right link state is q (for example, $C_{q,q}^{1C_t}$).

For T a subset of C_n , say that the moral support of T is the set $MS(T) = T \cup (T+1) \subset C_n$. For S a subset of C_n , and $b \in S$, say that b is locally (cyclically) minimal in S if $b \in S$, but $b - 1 \notin S$.

The following lemma is immediate.

Lemma 4.7. Let $T \subset C_n$. If MS(T) is a proper subset of C_n , then there exists $a \in C_n$ so that a + 2 is locally minimal in the complement $C_n \setminus MS(T)$.

Lemma 4.8. Let $T \subset C_n$ be innermost. Suppose that $a \in C_n$ is such that a + 2 is locally minimal in the complement of the moral support MS(T). Let ω be the annular diagram where:

• a + 2 is connected to a + 1,

- a is connected to (a+2)',
- (a+1)' is connected to a',
- for $i \in (a+2, a) = C_n \setminus [a, a+2]$, *i* is connected to *i'*.

Then $(\bigcap_{i \in T \setminus \{a\}} J_i) \cdot \omega \subset \bigcap_{i \in T} J_i$.

The picture is the same as the key one in Sroka's paper [Sro22]. If a = 3 in J₈, then:



Proof. Let ρ be a diagram in $\bigcap_{i \in T \setminus \{a\}} J_i$. We must show that $\rho \omega \in \bigcap_{i \in T} J_i$.

First, since ω has an edge connecting (a + 1)' and a', $\rho\omega$ also has an edge connecting these vertices, and hence $\rho\omega$ is in J_a .

Suppose now that $j \in T \setminus a$. Neither j = a + 1 nor j + 1 = a can hold, since this would contradict the assumption that T was innermost. Thus, since j is not in the set $\{a - 1, a, a + 1\}$, and a + 2 lies in the complement of T, so in particular $a + 2 \neq j$, we have that j and j + 1are both in the open cyclic interval (a + 2, a). Thus, j is connected to j' in ω , and likewise j + 1 is connected to (j + 1)'. Since j' and (j + 1)'are assumed to be connected in ρ , it now follows that they are still connected in $\rho\omega$. Thus $\rho\omega$ is in J_j .

Since j was chosen arbitrarily from $T \setminus a$, and we have already established that $\rho \omega \in J_a$, it follows that $\rho \omega$ is contained in the intersection $\bigcap_{i \in T} J_i$, as required.

Lemma 4.9. Let $T \subset C_n$ be innermost. Suppose that $a \in C_n$ is such that a + 2 is locally minimal in the complement of the moral support MS(T). The map

$$\cdot \omega : \bigcap_{i \in T \setminus \{a\}} J_i \to \bigcap_{i \in T} J_i$$

constructed in Lemma 4.8 is a retraction of the inclusion of $\bigcap_{i \in T} J_i$ into $\bigcap_{i \in T \setminus \{a\}} J_i$.

Proof. Suppose $\rho \in \bigcap_{i \in T} J_i$. We must show $\rho \omega = \rho$.

In particular, we have that $\rho \in J_a$, so a' and (a+1)' are connected in ρ . First, the product $\rho\omega$ produces no factors of δ : the only left-to-left connection in ω is the one between (a+2) and a+1, so there can only be a loop if (a+2)' and (a+1)' are connected in ρ . They cannot be, because (a+1)' is connected to a' in ρ . It follows that $\rho\omega$ is a partition

with no prefactor, and we will now argue that this partition is equal to ρ .

Since a' and (a + 1)' are connected in ρ , (a + 2)' is connected to (a + 2)'' in the composed partition $\rho \circ \omega$ (c.f. Definition 3.2). More generally, this means that i' is connected to i'' for i not equal to a or a + 1. It follows that for such i, i' is connected to the same vertex in $\rho\omega$ as it was in ρ , and since (a + 1)' and a' are connected in ω , they are also connected in $\rho\omega$. This establishes that $\rho\omega$ has all of the right-to-right and left-to-right connections from ρ , and it is automatic that the product retains all left-to-left connections from ρ . Thus, $\rho\omega = \rho$, as required.

Lemma 4.10. Suppose that $T \subset C_n$ is innermost. Unless n is even and T consists either of all of the odd or all of the even elements of C_n , the ideal

 $\bigcap_{i \in T} J_i$

is a principal ideal generated by an idempotent. If δ is invertible, then $\bigcap_{i \in T} J_i$ is principal idempotent for any innermost T.

Proof. If T is innermost, then the moral support MS(T) is a proper subset of C_n unless n is even and T consists either of all of the odd or all of the even elements of C_n . The result then follows by combining Lemma 4.7, Lemma 4.9 and Lemma 2.6. The case of invertible δ is left to the reader.

We are now ready to prove the main result on $J_n(\delta)$. Recall from the introduction that we have

$$\mathbf{J}_n(\delta) / \mathbf{I}_{\leq n-1} \cong RC_n,$$

the group algebra of the cyclic group C_n .

Proof of Theorems 1.4 and 1.5. By Lemma 4.5, the ideals J_i form an R-free cover of $I_{\leq n-1}$. By Lemma 4.10, an intersection of at most $\frac{n}{2}-1$ ideals from among the J_i is either zero or principal idempotent, so this is indeed a principal idempotent cover of height $\frac{n}{2} - 1$. Theorem 1.4 then follows by Theorem 1.7.

If n is odd or δ is invertible, then Lemma 4.10 gives that any intersection of ideals from among the J_i is either zero or principal idempotent, so Theorem 1.5 follows by applying Theorem 1.7 for arbitrarily large finite h.

REFERENCES

References

- [BH20] Rachael Boyd and Richard Hepworth. The homology of the Temperley-Lieb algebras. To appear in Geometry and Topology. 2020. URL: https://arxiv.org/abs/2006.04256.
- [BH21] Rachael Boyd and Richard Hepworth. "Combinatorics of injective words for Temperley-Lieb algebras". In: J. Combin. Theory Ser. A 181 (2021), Paper No. 105446, 27.
- [BHP21] Rachael Boyd, Richard Hepworth, and Peter Patzt. "The homology of the Brauer algebras". In: *Selecta Math. (N.S.)* 27.5 (2021), Paper No. 85, 31.
- [BHP23] Rachael Boyd, Richard Hepworth, and Peter Patzt. The homology of the partition algebras. 2023. arXiv: 2303.07979 [math.AT].
- [Boy22] Guy Boyde. Idempotents and homology of diagram algebras. 2022. URL: https://arxiv.org/abs/2212.01826.
- [Bra37] Richard Brauer. "On algebras which are connected with the semisimple continuous groups". In: Ann. of Math. (2) 38.4 (1937), pp. 857–872.
- [GL96] J. J. Graham and G. I. Lehrer. "Cellular algebras". In: *Invent. Math.* 123.1 (1996), pp. 1–34.
- [Hep22] Richard Hepworth. "Homological stability for Iwahori–Hecke algebras". In: *Journal of Topology* 15.4 (2022), pp. 2174– 2215.
- [Jon94a] V.F.R. Jones. "Quotient of the affine Hecke algebra in the Brauer algebra". In: *L'Enseignement Math.* 40.3-4 (1994), pp. 313–344.
- [Jon94b] V.F.R. Jones. "The Potts model and the symmetric group". In: Subfactors (Kyuzeso, 1993) (1994). River Edge, NJ, World Sci. Publishing, pp. 259–267.
- [Mar94] Paul Martin. "Temperley-Lieb algebras for nonplanar statistical mechanics - the partition algebra construction". In: *Journal of Knot Theory and Its Ramifications* 03.01 (1994), pp. 51–82.
- [Mos22] Isaac Moselle. Homological stability for Iwahori-Hecke algebras of type B. 2022. URL: https://arxiv.org/abs/2211.06230.
- [Pat20] Peter Patzt. Representation stability for diagram algebras. 2020. URL: https://arxiv.org/abs/2009.06346.
- [Sro22] Robin J. Sroka. The homology of a Temperley-Lieb algebra on an odd number of strands. 2022. URL: https://arxiv.org/abs/2202.08799.
- [TL71] H. N. V. Temperley and E. H. Lieb. "Relations between the "percolation" and "colouring" problem and other graphtheoretical problems associated with regular planar lattices:

REFERENCES

some exact results for the "percolation" problem". In: *Proc.* Roy. Soc. London Ser. A 322.1549 (1971), pp. 251–280.

[Xi99] Changchang Xi. "Partition algebras are cellular". In: Compositio Math. 119.1 (1999), pp. 99–109.

MATHEMATICAL INSTITUTE, UTRECHT UNIVERSITY, HEIDELBERGLAAN 8 3584 CS UTRECHT, THE NETHERLANDS

Email address: g.boyde@uu.nl