HOOKE AND COULOMB ENERGY OF TRIPOD SPIDERS

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ABSTRACT. Tripod spiders are the simplest examples of arachnoid mechanisms. Their workspaces and configuration spaces are well known. For Hooke potential, we give a complete description of the Morse theory and treat the robust control of the spider. For the Coulomb energy, we use stationary charges and the trapping domain to study the robust control of spiders. We show that, for a regular triangle and positive charges, the domain of robust control is non-void. This relates to questions about the Maxwell conjecture about point charges. We end with several natural problems and research perspectives suggested by our results.

1. INTRODUCTION

The geometry and topology of mechanical linkages play an important and increasing role in applied problems. Most of the previous studies were concerned with the workspace and the topology of the configuration space, which is only known in a few cases summarized in [KM], [Oh], [Mo]. The aim of this paper is to extend the known results to new classes of linkages and enrich them by considering potential functions on the workspace, with aplications to control of the linkages considered. Our approach is based on Morse theory which yields an explicit connection between the topology of configuration space and the critical points of potential.

We will be basically concerned with the so-called arachnoid mechanisms, the topology of which is a largely unexplored topic. Nowadays the same type of objects is often called "spidery linkage". Detailed information on the topology of arachnoid linkages is important for the design and control of certain types of spider robots. More concretely, we study the simplest arachnoid mechanism, the 3-leg spider also known as tripod spider [Oh], [Mo].

In Section 2 we use Hooke energy as a potential function and describe in detail the workspace and the critical point theory. This implies, in particular, that a weighted version of Hooke energy can be used to control such a linkage.

In section 3 we study the Coulomb potential of point charges placed at the foots of a 3-leg spider linkage. We deal with the so-called stationary charges of the spider's center (the common point of legs) and the so-called trapping domain of stationary charges determined in [GK1], [GK2]. This enables us to determine the domain in the workspace of spider, where the position of spider's center can be robustly controlled by the values of stationary charges using the Coulomb control scenario developed in [KPS], [GK1]. For a symmetric spider based on a regular triangle and having a contractible workspace, we show that the

domain of robust Coulomb control is a non-void open subset of the workspace containing the center of the reference triangle.

In the sequel we use Morse theory for manifolds with boundaries and corners. Most of it is "folklore" hidden in the literature. We mention here [JJT] and [GM] as the most general reference. For critical points of functions on manifolds with boundary and corners we refer to [Si]. For brevity, we refer to criteria of such critical points and corresponding topological changes in level surfaces as "standard rules".

In conclusion we mention several related problems and perspectives suggested by our results. In general, this paper may be considered as a first step in applying our approach to spider linkages and creating a paradigm for further research in this direction.

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2. Hooke Energy as a potential function of tripod spider

We will consider the Hooke potential for three points in several situations. First without any constraint, next with constraints on maximal an minimal distances, and finally for a 3-leg spider.

2.1. **No constraints.** We start with the following simplified situation. Given 3 points A, B, C (in vector notation a, b, c), the foot points of the spider, and its center (joint) the point X. The legs AX, BX, CX are completely flexible around the foot, their length is allowed to change.
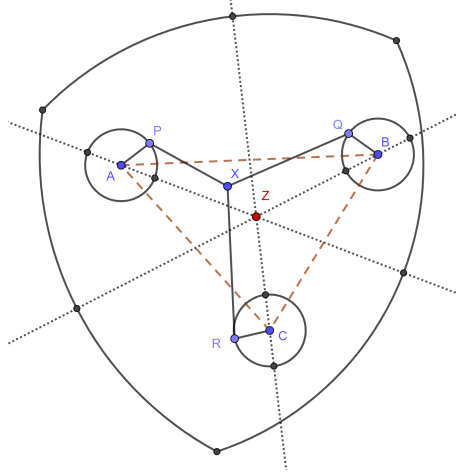
The Hooke Energy is defined by

$$H(x) = ||x - a||^2 + ||x - b||^2 + ||x - c||^2$$

The stationary points are determined by: $\nabla H = 6x - 2(a+b+c) = 0$, so $x = \frac{1}{3}(a+b+c)$, the center of gravity. There are no other stationary points. Note that $H(X) = ||x - z||^2 + K$, where $z = \frac{1}{3}(a + b + c)$, the center of gravity Z and $K = ||a||^2 + ||b||^2 + ||c||^2 - ||z||^2$. All level curves are circles.

2.2. **Maximal length constraints.** We require

$$|XA| \leq R_A \ , \ |XB| \leq R_B \ , \ |XC| \leq R_C.$$

The configuration space is the intersection of 3 discs D_A, D_B, D_C with centers resp A, B, C and radii R_A, R_B, R_C. We study the boundary extrema of H.

Lemma 1. *The map H restricted to the boundary of D_A has an extrema in Y iff AY has the direction of $AB + AC$, i.e the two intersection points of the line trough A and Z with the boundary of D_A, a minimum and a maximum.*

Proof. Suppose A is the origin. Let $R_A e_\phi$ a point on δD_A. Use Lagrange multiplyers:

$$\nabla H = 6R_A e_\phi - 2(b+c) = \lambda e_\phi$$

Therefore

$$(-\lambda + 6r_A)e_\phi = 2(b+c)$$

$\square$

Proposition 1. *Let the workspace $W = D_A \cap D_B \cap D_C$ contain an open neighborhood of Z then Z is the only stationary point of H, an absolute minimum. There are no boundary singularities.*

Proof. The center of gravity Z is clearly a minimum. Potential other critical points are the critical points of the restiction of H to the boundary circles (intersections of ZA with the circle around A, etc) and also the 3 'corner points ', where 2 boundary circles intersect. The statement follows now from the standaard rules . For boundary points, they are here as follows:

type of restriction	direction normal	type	cell attaching
minimum	inward	no change	no
minimum	outward	minimum	0-cell
maximum	inward	no change	no
maximum	outward	saddle	1-cell

With no change we mean, that the topological type of the lower level sets don't change. For corner points there are similar rules. They give in our case: no change. Due to the special (circular) form of level curves and boundaries one can obtain the same result by 'inspection of pictures'. $\square$

2.3. **Two sided length constraints.** We require

$$0 < R_A^- \leq |XA| \leq R_A \ , \ 0 < R_B^- \leq |XB| \leq R_B \ , \ 0 < R_C^- \leq |XC| < R_C.$$

The configuration space is the intersection of 3 discs with centers A, B, C and radii R_A, R_B, R_C, where some smaller (open) discs have been taken out. From the many different posibilties, we consider here the case that these 3 discs have a relatively small radii and the workspace W is a disc with 3 holes, containing an open neighborhood of the gravity center Z.

Proposition 2. *In this situation the point Z is an absolute minumum of H on W , moreover there are 3 saddle points on the (outer) intersection points of the small discs with ZA, ZB, ZC.*

Proof. The center of gravity Z is an absolute minimum. Other potential critical points are the critical points of the restiction of H to the boundary circle, the 3 small circles and also the 3 'corner points ', where 2 boundary circles intersect. The statement follows now from standard rules for boundary and corner singularities as explained above.

$\square$

Remark 1. The phase portrait of the gradient of H consists of straight lines to Z, except for those lines that intersect one of the small discs. Their trajectories follow from the moment that they intersect these discs a part of the boundary circle until they become 'visible' from Z and then continue as a straight line. There are 3 intervals between the outer and the inner circles, which are conflict strata for the gradient flow.

Remark 2. The Morse theory in the 3 cases above is as follows:
In 2.1: $b_0 = 1, b_i = 0$ $(i \geq 1)$ and $\mu_1 = 1, \mu_i = 0$ $(i \geq 1)$,
In 2.2: $b_0 = 1, b_i = 0$ $(i \geq 1)$ and $\mu_1 = 1, \mu_i = 0$ $(i \geq 1)$,
In 2.3: $b_0 = 1, b_2 = 3, b_i = 0$ $(i \geq 2)$ and $\mu_1 = 1, \mu_1 = 3, \mu_i = 0$ $(i \geq 2)$.
In these three cases, H is a perfect Morse function.

2.4. **Robust control.** A similar study can be made for the weighted Hooke Energy:
$$H_{\alpha,\beta,\gamma} = \alpha||x - a||^2 + \beta||x - b||^2 + \gamma||x - c||^2$$
Assume $\alpha > 0, \beta > 0, \gamma > 0$. This potential function has the point $Z = (\alpha : \beta : \gamma)$ (barycentric coordinates) as an absolute minimum. The level curves are circles with the center at this point Z. The critical point theory is similar to the case of H, which corresponds to $(\alpha : \beta : \gamma) = (1 : 1 : 1)$.

A proper choice of the controls (α, β, γ) can be used to move the 3-leg spider to any point in the triangle ABC via minimum points of Hooke energy. This procedure yields a *robust control* of the spider.

2.5. **The 3-leg spider.** In this case the telescopic connections are replaced by 2-arms with fixed arm lengths and flexible turning point. We assume that the two parts of the arm have different lengths. So AX is replaced by the arm $AP \cdot PX$, BX by $BQ \cdot QX$ and CX by $CR \cdot RX$. The configuration space $\mathcal{C}$ is an 8-fold cover of the workspace W with certain identifications at the boundaries. The topology has been studied in full generality by P. Mounod [Mo]. He showed that an n-leg spider with generic arm lengths has a smooth two-dimensional configuration space, and gave a formula for its Euler characteristic. In his paper he also solved some questions of J. O'hara [Oh].

The configuration space of 3-leg spider projects now clearly to the workspace W from 2.3 (with the 3 small holes). R_A^- is equal to the difference of arm-lengths in the $AP \cdot PX$ and R_A is equal to their sum. Similar for the points B and C. We assume here that the arm lengths are such, that they give the workspace mentioned above.

According to the formula of [Mo] the Euler characteristic of $\mathcal{C}$ is -22. So we have a smooth surface with genus 12.

We consider the quadratic distance function H (see above).

Proposition 3. *The Hooke energy H on the configuration space $\mathcal{C}$ of the 3-leg spider has:*
  * *8 minima*

- *36 saddles*
- *6 maxima*

Proof. The potential H is defined on the configuration space via projection to the work space. We will use the results from Propostion 2 and use (branched) covering arguments. The covering is 8-fold above the open part of the workspace. The branching takes place at the boundaries and corners. The potential critical points are just the preimages of the special points, which have been studied in Proposition 2.

We find in this way 8 critical pre-images of the center of gravity Z, these are all absolute minima (with the same value). The 36 saddles come from the 9 special points on boundary circles, where we have covering degree 4. Their types can be deduced by topological study of the gluing of the two boundary pieces. Finally, we have 6 maxima, which corespond to the three corner points, where the covering is 2-fold and at each such point such four pieces are glued together.

There is is a dictionary between the Morse theory on the workspace treated above and the configuration space. Compare the level curves in both cases via gluing and this results in the Morse indices mentioned in the proposition. □

3. Coulomb Energy as a potential function of tripod spider

From the viewpoint of control theory, it is also interesting and practically important to consider the Coulomb potential of point charges which are placed at the fixed foot points of a spider and can be varied in order to change the position of its body endowed with a fixed charge. A similar scenario was studied in big detail in the papers [GK1], [GK2] and our exposition in this section relies on the constructions and results given in those two papers.

We begin with recalling several concepts and constructions in the form needed in the sequel. Recall that the Coulomb potential $E = E(Q@A)$ of n point charges of non-zero magnitudes q_i placed at n points A_i is a function of point X in the complement of points A_i by the formula

$$E = \sum \frac{q_i}{d_i},$$

where $d_i = d(X, A_i)$ is the Euclidean distance between the points P and A_i. As usual a point X is called a stationary point (or equilibrium point) of potential E if its gradient ∇E vanishes at X. There exists a huge number of results and problems concerned with the Coulomb potential and equilibrium points of point charges. For us it is important to mention that, as was proven by M. Morse himself, for a generic configuration and generic values of point charges Coulomb potential is a Morse function [CM]. This suggests that Coulomb potential is a reasonable candidate for developing Morse theory for a spider linkage along the same lines as in the case of Hooke's energy considered above. The aim of this section is to present a number of results in this direction and present an application in the spirit of robust Coulomb control discussed in [KPS], [GK1], [GK2]. Another relevant topic in our context is the so-called Maxwell's conjecture on point charges stating that the number of isolated equilibrium points of n point charges in 2-dimensional Euclidean space does not exceed $(n-1)^2$ (see, e.g., a recent review in [GNS]).

We next focus on the case of 3 point charges. Given a triangle $\triangle$ with vertices A, B, C, referred to as a reference triangle (or base triangle), and a point X in the same plane, the triple of normalized stationary charges $Q(X) = \{q_i(X; \triangle)\}$ is defined as a triple of non-zero real numbers q_i such that $\nabla E(X, Q) = 0$. As was proven in [GK1] these charges are uniquely determined by the ratio's

$$q_1 : q_2 : q_3 = d_1^3 \mathcal{A}_1 : d_2^3 \mathcal{A}_2 : d_3^2 \mathcal{A}_3,$$

where $\mathcal{A}_1$ is the area of $\triangle BCX$, $\mathcal{A}_2$ of $\triangle CAX$, $\mathcal{A}_3$ of $\triangle ABX$. We denote the angles at X of these triangles by $\alpha_1, \alpha_2, \alpha_3$ respectively.

In [GK1] , [GK2] *the Coulomb trapping domain $T(\triangle)$ is defined as the set of minimum points of the potential induced by the charges $Q(X)$.* The trapping domain is given by the inequality $h(X) > 0$, where h is the Hessian of $E(X, Q(X))$, which we call the *trapping Hessian.* We have in geometric terms the following formula:

$$h(X) = -2\mathcal{A} + 9(\prod_i^3 \sin\alpha_i)(\sum_{i=1}^3 d_i^2 \mathcal{A}_i).$$

It was conjectured in [GK2] that $T(\triangle)$ is a non-empty subset containing the incenter of $\triangle$. This conjecture has been verified in a number of cases, including the regular triangle.

As was mentioned, a complete description of equilibria (stationary points) of E and their Morse indices is not known even in the case of three charges. However rather complete results for special configurations of three point charges have been obtained in [Ts]. In particular, the Maxwell's conjecture was proven for arbitrary three non-zero point charges at the vertices of a regular triangle and the Morse indices are computed.

For this reason, in the rest of the text we always assume that the reference triangle is regular since it is the most important case for applications and, at the same time, in this case all necessary background results are rigorously proven in [GK2].

Further, given a 3-leg spider S with the center at (moving) point X, (fixed) foot-points A, B, C and identical legs with the links of lengths a (thigh length) and b (foot length), we denote by $W(S)$ its workspace considered above. To simplify the discussion we assume that $R > a > b > 0$, in which case the whole workspace of the spider is the intersection of three circular annuli with centers A, B, C.

Proposition 4. *The domain of robust Coulomb control of spider S in the above scenario is equal to the intersection $D(S) = W(S) \cap T(\triangle)$.*

As said before the topology of the workspace depends on the triangle and the arm lengths. This influences the intersection which could even be void. For the regular triangle, we make the choice that the workspace becomes a contractible region bounded by three circular arcs and containing the incenter. Also we assume that all charges are positive. The trapping domain $T(\triangle)$ is explicitly known as the interior of a compact convex region bounded by $h = 0$ and containing the incenter of the triangle. Under these conditions this yields the following conclusion.

Corollary 1. *The domain of robust Coulomb control of spider S based on a regular triangle is non-void.*

For a spider based on the regular triangle one can also carry out a complete Morse analysis in a similar way as for the Hooke potential. One knows that for given positive charges there is a single minimum of E in the trapping area $T(\triangle)$ and 3 saddle points in $\triangle$ ouside $T(\triangle)$. The poles of E are not in the workspace and therefore not in the configuration space, which is an 8-fold cover of the workspace branched (glued) over boundary components. After obtaining information about the boundary singularities one gets a complete Morse analysis.

4. Concluding remarks

There are several natural problems and research perspectives suggested by our results. We mention some of them which seem most interesting and feasible.

First of all, note that all the concepts used in our paper make sense for n-leg spiders with arbitrary base n-gon and arbitrary lengths of the links. So a natural next step is to generalize the above results to Hooke and Coulomb potential of an n-leg spider.

Next, one can also consider other geometrically or physically meaningful potentials of n-leg spider like Riesz energies or the oriented area of the polygon formed by the moving joints of spider.

Further, one can formulate several extremal problems related to the design of spider linkages with desirable properties like the area or shape of the workspace.

Moreover, one can try to extend our results to the case of n-leg spiders with a moving n-gon platform instead of the center point.

Finally, it is natural to extend our discussion to spatial spiders where the legs can move in a three-dimensional Euclidean space.

It is also possible to consider "legs" with more than two links or with flexible non-extendible tether links. Each of the mentioned possibilities deserves a closer look and careful exploration which we intend to undertake in the future research.

Returning to 3-leg spiders we add that an urgent topic is to clarify what happens with the domain of robust Coulomb control for arbitrary configuration of foots. In this case, there are several feasible problems concerned with the optimization of workspace with given restrictions on the sum of the lengths of links and circumradius of the base triangle. For example, there is good evidence that the maximal area of $W(S)$ arises for the most symmetric spider with equal links and regular configuration of foots.

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