PARTIAL PARAMETRIZED PRESENTABILITY AND THE UNIVERSAL PROPERTY OF EQUIVARIANT SPECTRA

BASTIAAN CNOSSEN, TOBIAS LENZ, AND SIL LINSKENS

ABSTRACT. We introduce a notion of *partial presentability* in parametrized higher category theory and investigate its interaction with the concepts of parametrized semiadditivity and stability from [CLL23]. In particular, we construct the free partially presentable *T*-categories in the unstable, semiadditive, and stable contexts and explain how to exhibit them as full subcategories of their fully presentable analogues.

Specializing our results to the setting of (global) equivariant homotopy theory, we obtain a notion of *equivariant presentability* for the global categories of [CLL23], and we show that the global category of genuine equivariant spectra is the free global category that is both equivariantly presentable and equivariantly stable. As a consequence, we deduce the analogous result about the *G*-category of genuine *G*-spectra for any finite group *G*, previously formulated by [Nar17].

Contents

1.	Introduction	1
2.	Preliminaries on parametrized higher categories	5
3.	Cleft categories	9
4.	Partial presentability	22
5.	The universal property of equivariant spaces	29
6.	The semiadditive story	33
7.	The universal property of equivariant special $\Gamma\text{-spaces}$	41
8.	The stable story	48
9.	The universal property of equivariant spectra	51
References		55

1. INTRODUCTION

The term *equivariant mathematics* was coined by Balmer and dell'Ambrogio [BD20] to refer in a unified way to the study of objects with group actions across a wide range of mathematical disciplines, for example in representation theory or equivariant homotopy theory. Given a group homomorphism $\alpha: H \to G$, any *G*-action on an object *X* can naturally be restricted to an *H*-action, and accordingly most notions of 'equivariant objects' give rise to global categories: collections

of $(\infty$ -)categories¹ $\mathcal{C}(G)$ for every finite group G equipped with suitably coherent restriction functors $\alpha^* \colon \mathcal{C}(G) \to \mathcal{C}(H)$, or more precisely categories parametrized over the 2-category Glo of finite connected groupoids.

Many fundamental concepts of (higher) category theory have analogues in the world of global categories, leading for instance to notions of *presentability*, *equivariant semiadditivity*, and *equivariant* stability. These properties were introduced and studied by the present authors in the previous article [CLL23], where we in particular showed that the universal presentable, presentable equivariantly semiadditive, and presentable equivariantly stable global categories all admit explicit models in terms of global homotopy theory in the sense of [Sch18, Hau19, Len20].

The presentability condition on a global category \mathcal{C} used in these results is quite strong: in particular, it demands the existence of left adjoints to *all* restriction functors $\alpha^* \colon \mathcal{C}(G) \to \mathcal{C}(H)$. This is in fact too strong for certain applications: several interesting examples, like the global category sending G to the category of *genuine G-spectra*, only admit such adjoints for *injective* homomorphisms.

In this article, we will therefore introduce and study a weaker notion of presentability for global categories called *equivariant presentability*, which emphasizes the role of the subgroup inclusions among all group homomorphisms and allows one to capture these additional examples. As our main results, we will show that the universal examples of equivariantly presentable global categories in the unstable, semiadditive, and stable contexts are given by equivariant homotopy theory:

Theorem A (Universal property of equivariant spaces, Theorem 5.3). The global category $\underline{\mathscr{S}}$ which associates to a finite group G the category \mathscr{S}_G of G-spaces is the free equivariantly presentable global category on one generator: for every equivariantly presentable global category \mathcal{D} , evaluation at the 1-point space $* \in \underline{\mathscr{S}}(1)$ induces an equivalence of global categories

$$\underline{\operatorname{Fun}}_{\operatorname{Glo}}^{\operatorname{eq-cc}}(\underline{\mathscr{G}}, \mathcal{D}) \xrightarrow{\sim} \mathcal{D}$$

where the left hand side denotes a certain global category of 'equivariantly cocontinuous' functors.

Theorem B (Universal property of equivariant special Γ -spaces, Theorem 7.17). The global category $\underline{\Gamma \mathscr{P}}_{*}^{\text{spc}}$ which associates to each finite group G the category of special Γ -G-spaces in the sense of Shimakawa [Shi89] is the free equivariantly presentable equivariantly semiadditive global category on one generator: for every equivariantly presentable equivariantly semiadditive global category \mathcal{D} evaluation at the free commutative monoid $\mathbb{P}(*)$ provides an equivalence of global categories

$$\underline{\operatorname{Fun}}_{\operatorname{Glo}}^{\operatorname{eq-cc}}(\underline{\Gamma}_{\ast}^{\operatorname{spc}},\mathcal{D}) \xrightarrow{\sim} \mathcal{D}$$

Theorem C (Universal property of genuine equivariant spectra, Theorem 9.4). The global category $\underline{\mathscr{S}p}$ which associates to a finite group G the category $\mathscr{S}p_G$ of genuine G-spectra is the free equivariantly presentable equivariantly stable global category on one generator: for any other such \mathcal{D} evaluation at the sphere spectrum \mathbb{S} defines an equivalence

$$\operatorname{Fun}_{\operatorname{Glo}}^{\operatorname{eq-cc}}(\underline{\mathscr{G}p}, \mathcal{D}) \xrightarrow{\sim} \mathcal{D}.$$

¹We work in the context of higher category theory throughout, and so we will refer to ∞ -categories simply as 'categories.'

In this sense, the original, stronger notion of presentability from [CLL23] can be viewed as a characteristic feature of global homotopy theory, distinguishing it from classical equivariant homotopy theory, and we will accordingly use the term *global presentability* for it below.

Partial presentability in parametrized higher category theory. The above notions of equivariant presentability, semiadditivity, and stability are in fact instances of more general notions defined in the setting of *parametrized higher category theory* as introduced in [BDG⁺16]. Such parametrized notions usually come in various degrees of 'parametrized refinement': in particular, [CLL23] studied various levels of semiadditivity and stability that can exist in a parametrized category, encoded in the choice of a so-called *atomic orbital* subcategory of the parametrizing category T. Equivariant stability and semiadditivity of global categories correspond to the case of the wide subcategory Orb \subset Glo of faithful functors.

To study the analogous situation for presentability of parametrized categories, we introduce *clefts* $S \subset T$ in the present article and associate to each of them a notion of presentability, interpolating between naïve, or 'fiberwise,' presentability and the full parametrized presentability considered e.g. in [MW22, Hil22, CLL23]. The aforementioned atomic orbital subcategories are examples of clefts, and equivariant presentability of global categories is again recovered from the case $Orb \subset Glo$.

Definition (Definition 4.3, Lemma 4.9). A *T*-category $C: T^{\text{op}} \to \text{Cat}$ is said to be *S*-presentable if the following conditions are satisfied:

- (1) C is fiberwise presentable, i.e. it factors through the non-full subcategory $Pr^{L} \subset Cat$ of presentable categories and colimit-preserving functors.
- (2) For every morphism $f: A \to B$ in S, the restriction $f^*: \mathcal{C}(B) \to \mathcal{C}(A)$ admits a left adjoint $f_!: \mathcal{C}(A) \to \mathcal{C}(B)$, and these left adjoints satisfy base change for pullbacks along arbitrary maps in T (see Lemma 4.9 for a precise definition).

As one of our key technical results (Theorem 3.9), we moreover show how clefts give rise to *fractured* ∞ -*topoi* in the sense of [Lur18, Definition 20.1.2.1], which allows us to investigate the behavior of partial presentability under changing the parametrizing category along a cleft. Using this 'change of parameter' yoga, we then establish analogues of the results from [MW21, CLL23] in the partial parametrized world by constructing the free unstable, semiadditive, and stable examples of *S*-presentable *T*-categories, and relating them both to the corresponding universal *S*-presentable *S*-categories as well as *T*-presentable *T*-categories:²

Theorem D (Theorem 8.11). Let $S \subset T$ be a cleft, and let $P \subset T$ be an atomic orbital subcategory such that $P \subset S$. Then there exists an S-presentable P-stable T-category $\underline{Sp}_{S \triangleright T}^P$ with the following universal property: for any S-presentable Pstable T-category \mathcal{D} , evaluation at a certain object S induces an equivalence

$$\underline{\operatorname{Fun}}_{T}^{S\operatorname{-cc}}(\underline{\operatorname{Sp}}_{S\triangleright T}^{P},\mathcal{D}) \xrightarrow{\sim} \mathcal{D}$$

Moreover, the underlying S-category of $\underline{\mathrm{Sp}}_{S \triangleright T}^{P}$ agrees with the free S-presentable P-stable S-category $\underline{\mathrm{Sp}}_{S}^{P}$.

 $^{^2\}mathrm{For}$ brevity we only state the stable cases of these theorems here, and we refer the reader to Lemma 3.17 and Corollary 4.27 resp. Theorems 6.18 and 6.19 for the unstable and semiadditive versions.

Theorem E (Theorem 8.12). Let $P \subset S \subset T$ be as above and consider the unique S-cocontinuous T-functor

$$\iota_! \colon \underline{\operatorname{Sp}}^P_{S \triangleright T} \to \underline{\operatorname{Sp}}^P_T$$

sending S to S. Then $\iota_!$ is fully faithful, and its underlying S-functor sits in a sequence of S-adjoints $\iota_! \dashv \iota^* \dashv \iota_*$.

This then allows us to deduce Theorems A, B, and C from their global analogues established in [CLL23]: building on the model categorical results of [Len20], we show that the global categories $\underline{\mathscr{S}}$ of equivariant spaces, $\underline{\Gamma}\underline{\mathscr{S}}_*^{\mathrm{spc}}$ of equivariant special Γ -spaces, and $\underline{\mathscr{S}p}$ of equivariant spectra likewise embed into their global counterparts, and furthermore that the images of these embeddings match up with those on the parametrized side.

Outlook. While Theorem E above (together with its unstable and semiadditive versions) explains how to obtain the S-presentable universal examples as full subcategories of their T-presentable analogues, it is sometimes also possible to go the other way round, and to actually reconstruct the universal fully presentable categories from the partially presentable ones: namely, as the third author will show in [Lin23], under somewhat more restrictive conditions on the pair $S \subset T$ the forgetful functor from T-presentable to S-presentable T-categories admits a left adjoint, which can be explicitly computed in terms of certain partially lax limits. Furthermore this left adjoint preserves the subcategories of P-stable T-categories for $P \subset S$. Specializing to the inclusion $Orb \subset Glo$ again, the main results of the present paper as well as its prequel [CLL23] then yield a description of G-global spectra as a partially lax limit of H-equivariant spectra over all homomorphisms $H \to G$, generalizing the result for G = 1 proven in [LNP22].

Organization. We begin by recalling the necessary background on parametrized higher category theory in Section 2. We then introduce the notion of a cleft in Section 3 and explain its connection to fractured ∞ -topoi. We moreover show that any atomic orbital subcategory and any right class of a factorization system give rise to a cleft, in particular establishing our key example Orb \subset Glo.

Section 4 explains how a cleft S of T yields a well-behaved theory of partial presentability for T-categories, and how general (co)limits behave under changing the parametrizing category along a cleft. This allows us to reinterpret and extend work of Martini and Wolf [MW21] on freely adding S-colimits, in particular identifying the free S-presentable T-category with a full subcategory of the free T-presentable T-category. In Section 5 we use this to describe the free equivariantly presentable global category as the underlying global category of a diagram of model categories of equivariant spaces, proving Theorem A.

In Section 6 we recall the notion of P-semiadditivity from [CLL23] for atomic orbital subcategories $P \subset T$. Given a cleft S with $P \subset S$, we then construct the free S-presentable P-semiadditive T-category as an extension of the corresponding S-category, and we once again exhibit it as a full subcategory of the free T-presentable P-semiadditive T-category. Combining this with results from [CLL23], we then prove Theorem B describing the free equivariantly presentable equivariantly semi-additive global category in terms of equivariant Γ -spaces in Section 7.

The final two sections are then devoted to the stable case: In Section 8 we construct the free S-presentable P-stable T-category, and relate it to the corresponding presentable S- and T-categories, proving Theorems D and E. From this we then deduce Theorem C in Section 9, giving an explicit model of the free equivariantly presentable equivariantly stable global category via equivariant stable homotopy theory.

Conventions. We work in the context of higher category theory throughout, and refer to ∞ -categories as 'categories.' We fix a chain of Grothendieck universes $\mathfrak{U} \in \mathfrak{V} \in \mathfrak{W}$, and we will use the terms 'small category,' (large) category,' and 'very large category' to refer to \mathfrak{U} -small, \mathfrak{V} -small, and \mathfrak{W} -small categories, respectively. A 'locally small category' will mean a \mathfrak{V} -small category such that all its mapping spaces have \mathfrak{U} -small homotopy groups.

Acknowledgements. B.C. and S.L. are associate members of the Hausdorff Center for Mathematics at the University of Bonn. B.C. is supported by the Max Planck Institute for Mathematics in Bonn. S.L. is supported by the DFG Schwerpunktprogramm 1786 "Homotopy Theory and Algebraic Geometry" (project ID SCHW 860/1-1).

2. Preliminaries on parametrized higher categories

We begin by recalling the necessary background on parametrized higher category theory, as developed in [BDG⁺16, Nar16, Sha21] and, from the perspective of categories internal to ∞ -topoi, in [Mar21, MW21, MW22]. Throughout this section, let us fix a small category T.

Definition 2.1. A *T*-category is a functor $C: T^{\text{op}} \to \text{Cat}$ into the (very large) category of categories. If C and D are *T*-categories, then a *T*-functor $F: C \to D$ is a natural transformation from C to D. The category Cat_T of *T*-categories is defined as the functor category $\text{Cat}_T \coloneqq \text{Fun}(T^{\text{op}}, \text{Cat})$.

Example 2.2. Define Glo as the (2, 1)-category of finite groups, group homomorphisms, and conjugations, i.e. a 2-morphism $h: f \Rightarrow f'$ in Glo between group homomorphisms $f, f': G \to H$ is an element $h \in H$ such that $f'(g) = hf(g)h^{-1}$ for all $g \in G$. In particular, Glo comes with a fully faithful functor $B: \text{Glo} \hookrightarrow \text{Grpd}$ into the (2, 1)-category of groupoids which sends a finite group G to the corresponding 1-object groupoid BG. We will use the term global category for a Glo-category, global functor for a Glo-functor, etc.

Example 2.3. For a finite group G, let $T = \operatorname{Orb}_G$ be the orbit category of G, the full subcategory of the 1-category of G-sets spanned by the transitive G-sets. Following [BDG⁺16], we will refer to Orb_G -categories as G-categories.

Let us mention some common examples of *T*-categories:

Example 2.4. Every presheaf X on T gives rise to a T-category $\underline{X}: T^{\text{op}} \to \text{Cat}$ by postcomposing with the inclusion Spc \hookrightarrow Cat of spaces into categories. In particular, every object $A \in T$ yields a T-category A via the Yoneda embedding.

Example 2.5. Every category \mathcal{E} gives rise to a *T*-category of *T*-objects $\underline{\mathcal{E}}_T$, given by $\underline{\mathcal{E}}_T(B) = \operatorname{Fun}((T_{/B})^{\operatorname{op}}, \mathcal{E})$ where the functoriality of $T_{/B}$ is given by straightening the cocartesian target fibration $T^{[1]} \to T$.

Example 2.6. Any category \mathcal{E} gives rise a constant T-category const $_{\mathcal{E}} : A \mapsto \mathcal{E}$. The construction $\mathcal{E} \mapsto \text{const}_{\mathcal{E}}$ is left adjoint to the underlying category functor $\Gamma : \text{Cat}_T \to \text{Cat}$ which sends \mathcal{C} to $\Gamma(\mathcal{C}) := \lim_{B \in T^{\text{op}}} \mathcal{C}(B)$.

Convention 2.7 (cf. [CLL23, Convention 2.1.15]). Any *T*-category $\mathcal{C}: T^{\text{op}} \to \text{Cat}$ admits a unique extension to a limit-preserving functor $\text{PSh}(T)^{\text{op}} \to \text{Cat}$, which we will abusively denote by \mathcal{C} again. By convention, all limits and colimits of objects in *T* are taken in the presheaf category PSh(T).

Example 2.8. Viewing \mathcal{C} as a functor $PSh(T)^{op} \to Cat$ as above, its value at the terminal presheaf 1 is given by the underlying category $\Gamma(\mathcal{C})$ of \mathcal{C} , in the sense of Example 2.6.

Example 2.9. The category Cat_T is cartesian closed, i.e. given *T*-categories \mathcal{C} and \mathcal{D} , there is a *T*-category $\operatorname{Fun}_T(\mathcal{C}, \mathcal{D})$ of *T*-functors, characterized by the property that there is a natural equivalence

$$\operatorname{Hom}(\mathcal{E} \times \mathcal{C}, \mathcal{D}) \simeq \operatorname{Hom}(\mathcal{E}, \underline{\operatorname{Fun}}_T(\mathcal{C}, \mathcal{D}))$$

for every third T-category \mathcal{E} . We let

$$\operatorname{Fun}_T(\mathcal{C}, \mathcal{D}) \coloneqq \Gamma(\underline{\operatorname{Fun}}_T(\mathcal{C}, \mathcal{D}))$$

denote the underlying category of $\underline{\operatorname{Fun}}_T(\mathcal{C}, \mathcal{D})$. By adjunction, its objects can be identified with *T*-functors $\mathcal{C} \simeq \mathcal{C} \times \operatorname{const}_{[0]} \to \mathcal{D}$, while its morphisms are *natural* transformations of *T*-functors, i.e. functors $\mathcal{C} \times \operatorname{const}_{[1]} \to \mathcal{D}$.

To describe these functor categories more explicitly, we will use:

Lemma 2.10 (Categorical Yoneda lemma, [CLL23, Corollary 2.2.8]). For every presheaf $B \in PSh(T)$ and every T-category C, there is an equivalence of categories

$$\operatorname{Fun}_T(B, \mathcal{C}) \xrightarrow{\sim} \mathcal{C}(B),$$

natural in both variables, determined by the fact that for $B \in T$ it is given by evaluation at the identity $id_B \in Hom_T(B, B) = \underline{B}(B)$.

Combining this with the (internal) adjunction equivalence for $\underline{\operatorname{Fun}}_T$ we immediately get:

Corollary 2.11 (cf. [CLL23, Corollary 2.2.9]). Let $C, D \in \operatorname{Cat}_T$ and $X \in PSh(T)$. There are natural equivalences

$$\underline{\operatorname{Fun}}_{T}(\mathcal{C},\mathcal{D})(X)\simeq\operatorname{Fun}_{T}(\mathcal{C}\times\underline{X},\mathcal{D})\simeq\operatorname{Fun}_{T}(\mathcal{C},\underline{\operatorname{Fun}}_{T}(\underline{X},\mathcal{D})).$$

In particular, we can (and will) identify objects of $\underline{\operatorname{Fun}}_T(\mathcal{C}, \mathcal{D})(X)$ with *T*-functors $\mathcal{C} \times \underline{X} \to \mathcal{D}$ or equivalently $\mathcal{C} \to \underline{\operatorname{Fun}}_T(\underline{X}, \mathcal{D})$.

Example 2.12. As PSh(T) has pullbacks, the target map $PSh(T)^{[1]} \to PSh(T)$ is a *cartesian* fibration, so we can straighten it to a functor

$$\operatorname{Spc}_T := \operatorname{PSh}(T)_{/\bullet} \colon \operatorname{PSh}(T)^{\operatorname{op}} \to \operatorname{Cat}.$$

Explicitly, this sends $X \in PSh(T)$ to the slice $PSh(T)_{/X}$ and a map $f: Y \to X$ to the pullback functor $f^* \colon PSh(T)_{/Y} \to PSh(T)_{/X}$. By [Lur09, Theorem 6.1.3.9 and Proposition 6.1.3.10], this functor preserves limits, so it defines a *T*-category via our convention.

As the notation suggests, this can be identified with the *T*-category of *T*-objects (Example 2.5) in Spc: [CLL23, Remark 2.1.16] constructs an equivalence between the two which is given in degree $A \in T$ by the colimit extension $PSh(T_{A}) \rightarrow PSh(T)_{A}$ of the slice of the Yoneda embedding $T \rightarrow PSh(T)$ over A.

2.1. Adjunctions. In Cat_T there is a natural notion of (internal) *adjunctions*: a T-functor $F: \mathcal{C} \to \mathcal{D}$ is left adjoint to $G: \mathcal{D} \to \mathcal{C}$ if there are natural transformations $\eta: \text{ id} \Rightarrow GF$ and $\varepsilon: FG \Rightarrow \text{ id satisfying the triangle identities up to homotopy. We will frequently rely on the following 'pointwise criterion' for adjoints:$

Proposition 2.13 (see [MW21, Proposition 3.2.8 and Corollary 3.2.10]). A functor $F: C \to D$ of T-categories admits a right adjoint if and only if the following hold:

- (1) For every $A \in T$ the functor $F_A \colon C(A) \to D(A)$ admits a right adjoint G_A .
- (2) For every $f: A \to B$ in T the Beck-Chevalley transformation $f^* \circ G_B \Rightarrow G_A \circ f^*$ given by the composite

$$f^*G_B \xrightarrow{\eta} G_A F_A f^*G_B \xrightarrow{\sim} G_A f^*F_B G_B \xrightarrow{\varepsilon} G_A f^*$$

is an equivalence.

Moreover, in this case the following hold:

- (1') For every $X \in PSh(T)$ the functor $F_X : C(X) \to D(X)$ (cf. Convention 2.7) admits a right adjoint G_X .
- (2') For every $f: X \to Y$ in PSh(T) the Beck-Chevalley map $G_X f^* \Rightarrow f^*G_Y$ is an equivalence.

Finally, the right adjoint G is given in degree $X \in PSh(T)$ by G_X as above and the unit and counit are given pointwise by the unit and counit of $F_X \dashv G_X$. \Box

2.2. Limits and colimits. Next, we come to parametrized notions of limits and colimits. While this can be developed 'internally' using the notions of parametrized adjunctions and parametrized functor categories, we will instead take a purely 'pointwise' perspective in the spirit of the previous proposition in this paper.

Remark 2.14. Below we will for simplicity restrict ourselves to the case of colimits; the theory of limits is then formally dual.

Definition 2.15. A *T*-category C is called *fiberwise cocomplete* if C(A) is cocomplete for every $A \in T$ and the restriction $f^* : C(B) \to C(A)$ is cocontinuous for every $f : A \to B$. A *T*-functor $F : C \to D$ is called *fiberwise cocontinuous* if $F_A : C(A) \to D(A)$ is cocontinuous for every $A \in T$.

Note that in the above situation $\mathcal{C}(X)$ is more generally cocomplete for any $X \in PSh(T)$, and for any $f: X \to Y$ in PSh(T) the restriction $f^*: \mathcal{C}(Y) \to \mathcal{C}(X)$ is cocontinuous, see [Lur09, Corollary 5.1.2.3 and Lemma 5.4.5.5].

Definition 2.16. Let $\mathbf{U} \subset \underline{\operatorname{Spc}}_T$ be any *T*-subcategory. We say that a *T*-category \mathcal{C} admits \mathbf{U} -colimits if the following conditions are satisfied:

(1) For every $D \in PSh(T)$ and every $(f: C \to D) \in \mathbf{U}(D)$ the restriction $f^*: \mathcal{C}(D) \to \mathcal{C}(C)$ admits a left adjoint $f_!$.

(2) For any pullback

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ \downarrow^{u} & \downarrow^{t} & \downarrow^{t} \\ C & \xrightarrow{f} & D \end{array}$$

in PSh(T) such that $f \in U(D)$ (and hence $g \in U(B)$ as U is a T-subcategory), the Beck–Chevalley transformation $g_!u^* \Rightarrow t^*f_!$ is an equivalence.

If \mathcal{D} is another U-cocomplete T-category, then a T-functor $F: \mathcal{C} \to \mathcal{D}$ is called U-cocontinuous if for every $(f: C \to D) \in \mathbf{U}(D)$ the Beck–Chevalley map $f_!F_C \Rightarrow F_D f_!$ is an equivalence.

Remark 2.17. In the definition of a U-cocomplete T-category, it suffices that the above conditions are satisfied whenever B and D are representable, see [CLL23, Remark 2.3.15], and likewise for U-cocontinuity.

Definition 2.18. A *T*-category C is called *T*-cocomplete if it is fiberwise cocomplete (Definition 2.15) and <u>Spc_T</u>-cocomplete (Definition 2.16).

Similarly, a *T*-functor $F: \mathcal{C} \to \mathcal{D}$ between *T*-cocomplete *T*-categories is called *T*-cocontinuous if it is fiberwise cocontinuous and <u>Spc_T</u>-cocontinuous.

Example 2.19. The *T*-category Spc_T is *T*-cocomplete, see [MW21, Example 5.2.11].

Example 2.20. If \mathcal{D} is U-cocomplete for some $U \subset \underline{\operatorname{Spc}}_T$, and \mathcal{C} is any T-category, then $\underline{\operatorname{Fun}}_T(\mathcal{C}, \mathcal{D})$ is again U-cocomplete, see [CLL23, Corollary 2.3.25].

Example 2.21. Any left adjoint $F: \mathcal{C} \to \mathcal{D}$ of *T*-cocomplete *T*-categories is *T*-cocontinuous: indeed, it is clearly fiberwise cocontinuous, and the Beck–Chevalley map from Definition 2.16 is simply the total mate of the Beck–Chevalley map from Proposition 2.13. Conversely, a functor of *T*-cocomplete categories is a *T*-left adjoint if and only if it is *T*-cocontinuous and admits a pointwise right adjoint.

Definition 2.22. Let $\mathbf{U} \subset \underline{\operatorname{Spc}}_T$ be any T-subcategory. For any \mathbf{U} -cocomplete Tcategories \mathcal{C}, \mathcal{D} and any $X \in \operatorname{PSh}(T)$ we write $\underline{\operatorname{Fun}}_T^{\mathbf{U}-\operatorname{cc}}(\mathcal{C}, \mathcal{D})(X) \subset \underline{\operatorname{Fun}}_T(\mathcal{C}, \mathcal{D})(X)$ for the full subcategory spanned by those functors $F \colon \mathcal{C} \to \underline{\operatorname{Fun}}_T(X, \mathcal{D})$ that are
U-cocontinuous.

Similarly, we define $\operatorname{Fun}_T^{T-\operatorname{cc}}(\mathcal{C},\mathcal{D})(X)$ whenever \mathcal{C} and \mathcal{D} are T-cocomplete.

By [MW21, Remark 4.2.1 and Proposition 4.3.1] the above define *T*-subcategories of $\underline{\operatorname{Fun}}_T(\mathcal{C}, \mathcal{D})$.

Remark 2.23. The articles [MW21] and [CLL23] use an a priori different definition of $\underline{\operatorname{Fun}}_T^{\mathbf{U}\text{-cc}}$ and $\underline{\operatorname{Fun}}_T^{T\text{-cc}}$, see [CLL23, Proposition 2.3.26 and Remark 2.3.27] for the equivalence to the above.

With this terminology at hand, we can now formulate the universal property of T-spaces:

Theorem 2.24 ([MW21, Theorem 7.1.1]). For any *T*-cocomplete \mathcal{D} , evaluation at the terminal object defines an equivalence of *T*-categories

$$\operatorname{Fun}_{T}^{T\operatorname{-cc}}(\operatorname{Spc}_{T}, \mathcal{D}) \xrightarrow{\sim} \mathcal{D}.$$

2.3. **Presentability.** Finally, we come to the notion of *presentability* for *T*-categories from [MW22]:

Definition 2.25. A *T*-category $C: T^{\text{op}} \to \text{Cat}$ is called *fiberwise presentable* if it factors through the non-full subcategory $\text{Pr}^{\text{L}} \subset \text{Cat}$ of presentable categories and left adjoint functors.

In this case the limit extension again factors through Pr^{L} , i.e. for any $X \in PSh(T)$ the category $\mathcal{C}(X)$ is presentable, and for any map $f: X \to Y$ of presheaves the restriction $f^*: \mathcal{C}(Y) \to \mathcal{C}(X)$ is a left adjoint, see [Lur09, Proposition 5.5.3.13].

Definition 2.26. A *T*-category is called *T*-presentable if it is fiberwise presentable and *T*-cocomplete.

Remark 2.27. Any *T*-presentable category is also *T*-complete, see [MW22, Corollary 6.2.5].

Example 2.28. The *T*-category $\underline{\operatorname{Spc}}_T$ of *T*-spaces is *T*-presentable: clearly each $\operatorname{PSh}(T)_{/X}$ is presentable, each $f^* \colon \operatorname{PSh}(T)_{/Y} \to \operatorname{PSh}(T)_{/X}$ is a left adjoint by local cartesian closedness, and finally Spc_T is *T*-cocomplete by Example 2.19.

Example 2.29. If C is small and D is T-presentable, then $\underline{\operatorname{Fun}}_{T}(C, D)$ is again T-presentable, see [MW22, Corollary 6.2.6].

Remark 2.30. Let C be T-presentable and D be locally small and T-cocomplete. Combining Example 2.21 with the usual non-parametrized Special Adjoint Functor Theorem [Lur09, Corollary 5.5.2.9(1)], we see that a T-functor $C \to D$ is a left adjoint if and only if it is T-cocontinuous.

3. Cleft categories

Let T be a small category and let $S \subset T$ be a (wide) subcategory. Associated to the inclusion $\iota: S \hookrightarrow T$ we have a natural restriction functor $\iota^*: \operatorname{Cat}_T \to \operatorname{Cat}_S$, which admits both a left adjoint ι_1 as well as a right adjoint ι_* , given by left and right Kan extension, respectively. One of the central questions of the present paper is under which conditions the adjunction $\iota^* \dashv \iota_*$ interacts nicely with parametrized concepts, and in particular with the notions of parametrized colimits for T-categories and S-categories discussed above.

To address this question, we make use of a more 'geometric' description of the adjunction $\iota^* \dashv \iota_*$. By identifying *T*-categories with limit-preserving functors on $PSh(T)^{op}$ as in Convention 2.7, we see that precomposition with any colimit-preserving functor $f \colon PSh(S) \to PSh(T)$ determines a functor $f^* \colon Cat_T \to Cat_S$. Applying this to the left Kan extension functor $f = \iota_1 \colon PSh(S) \to PSh(T)$ recovers $\iota^* \colon Cat_T \to Cat_S$, and consequently the right adjoint $\iota_* \colon Cat_S \to Cat_T$ of ι^* is obtained by precomposition with $\iota^* \colon PSh(T) \to PSh(S)$, with the unit and counit of the adjunction $\iota^* \colon Cat_T \rightleftharpoons Cat_S : \iota_*$ given by plugging in the unit and counit of the adjunction $\iota_1 \colon PSh(S) \rightleftharpoons PSh(T) : \iota^*$.

The above description suggests that we can understand the category theoretic behavior of the adjunction ι^* : $\operatorname{Cat}_T \rightleftharpoons \operatorname{Cat}_S :\iota_*$ in terms of the geometric, or topostheoretic, behavior of the adjunction $\iota_1: \operatorname{PSh}(S) \rightleftharpoons \operatorname{PSh}(T) :\iota^*$. As a concrete example, consider the question of whether $\iota^*: \operatorname{Cat}_T \to \operatorname{Cat}_S$ preserves cocompleteness. If $\mathcal{C} \in \operatorname{Cat}_T$ is *T*-cocomplete, it is easy to see that the *S*-category $\iota^* \mathcal{C}$ is fiberwise cocomplete and that its restriction functors admit pointwise left adjoints, without any restrictions on ι . However, the Beck–Chevalley condition for these adjoints does not always hold: it translates to the requirement that $\iota_1 \colon PSh(S) \to PSh(T)$ preserves pullbacks. Similarly, one can translate cocontinuity of the unit $\mathcal{C} \to \iota_* \iota^* \mathcal{C}$ into a pullback condition on the *counit* of $\iota_1 \dashv \iota^*$. Upon closer inspection, it turns out that all the required conditions we will need for a well-behaved theory can be nicely summed up in Lurie's notion of a *fractured* ∞ -topos [Lur18, Definition 20.1.2.1]:

Definition 3.1. Let \mathcal{X} be an ∞ -topos. A functor $j_!: \mathcal{Y} \to \mathcal{X}$ is called a *fracture subcategory* if the following conditions are satisfied:

- (F0) The functor $j_{!}$ is a monomorphism of categories, i.e. it is faithful and the induced functor on groupoid cores is even fully faithful.
- (F1) The functor $j_! \colon \mathcal{Y} \to \mathcal{X}$ preserves pullbacks.
- (F2) The functor $j_!: \mathcal{Y} \to \mathcal{X}$ admits a right adjoint $j^*: \mathcal{X} \to \mathcal{Y}$ which is conservative and preserves colimits.
- (F3) For every morphism $f: X \to Y$ in \mathcal{Y} , the naturality square

of the counit transformation $\varepsilon: j_! j^* \to \mathrm{id}$ is a pullback square in \mathcal{X} .

An ∞ -topos \mathcal{X} equipped with a fracture subcategory \mathcal{Y} is called a *fractured* ∞ -topos.

However, these axioms are quite strong, making them somewhat hard to check directly. Accordingly, before coming to the parametrized applications of fractured ∞ -topoi sketched above, we devote the present section to their construction from simpler, less geometric data. Namely, as in the introductory example we will be interested in the special case of functors $PSh(S) \rightarrow PSh(T)$ arising as left Kan extension along the inclusion $S \hookrightarrow T$ of a wide subcategory. It turns out that in this can case we can give a more explicit characterization in terms of the indexing categories S and T:

Definition 3.2. Let T be a small category. A wide subcategory $S \subset T$ is called a *cleft* of T if the following conditions are satisfied:

- (C1) The subcategory S contains all equivalences of T and is left-cancellable, i.e. whenever f and g are composable maps in T with $g \in S$ and $gf \in S$, then $f \in S$.
- (C2) For any map $f: A \to B$ in S and any map $g: B' \to B$ in T there exists a map $f': X' \to B'$ in PSh(S) and a pullback square



in PSh(T), where $\iota_1 \colon PSh(S) \to PSh(T)$ denotes left Kan extension along the inclusion $\iota \colon S \hookrightarrow T$.

(C3) If $\alpha: A \to B$, $\beta: B \to A$ are maps in T such that $\beta \alpha = id_A$ and $\alpha\beta$ is a map in S, then also α belongs to S (whence so does β by left cancellability).

We call a small category T equipped with a cleft $S \subset T$ a *cleft category*.

Remark 3.3. As Axiom (C3) might look somewhat exotic, we record several more familiar properties that imply this axiom:

- (C3') Any idempotent $e: B \to B$ in S is the identity.
- (C3'') The morphisms of S are closed under retracts in the arrow category of T.
- (C3''') The morphisms of S satisfy the restricted 2-out-of-6 property: given composable f, g, h in T such that hg and gf belong to S, so does f.

Indeed, to see that (C3') implies (C3), note that the map $\alpha\beta: B \to B$ is an idempotent in S and thus the identity. It follows that α and β are (mutually inverse) equivalences, hence belong to S by (C1). In case of (C3"), it suffices to observe that the diagram

$$\begin{array}{ccc} A & \stackrel{\alpha}{\longrightarrow} B & \stackrel{\beta}{\longrightarrow} A \\ \alpha \downarrow & & \stackrel{i}{\alpha\beta} & & \downarrow \alpha \\ B & \stackrel{s}{\longrightarrow} B & \stackrel{s}{\longrightarrow} B \end{array}$$

expresses α as a retract of $\alpha\beta$. Finally, applying (C3''') to the chain α, β, α also implies (C3).

Remark 3.4. Axiom (C2) is a relaxation of the following more familiar condition:

(C2') Pullbacks of maps in S along maps in T exist in T and belong to S.

Wide subcategories $S \subset T$ satisfying axioms (C1), (C2') and (C3'') are called *admissibility structures* in [Lur18, Definition 20.2.1.1]. In particular, every admissibility structure on T is also a cleft in the above sense.

Let us mention some examples of cleft categories:

Example 3.5 (Trivial clefts). Every category T admits two extremal clefts: letting S consist of all maps in T constitutes the maximal cleft on T, while letting S consist of only the equivalences of T constitutes the minimal cleft on T.

Example 3.6 (Factorization systems). Let (E, M) be a *factorization system* on T. We will prove in Proposition 3.33 below that the right class M is a cleft.

Example 3.7 (Atomic orbital subcategories). Let $P \subset T$ be an *atomic orbital sub*category in the sense of [CLL23, Definition 4.3.1]. We will prove in Proposition 3.36 below that $P \subset T$ is a cleft category.

Example 3.8. Recall the global indexing category Glo from Example 2.2. We define a wide subcategory $\text{Orb} \subset \text{Glo}$ spanned by the *injective* homomorphisms. Then $\text{Orb} \subset \text{Glo}$ is a cleft category: this follows either from Example 3.7 together with [CLL23, Example 4.3.3] or from Example 3.6 with [LNP22, Proposition 6.14].

3.1. Clefts vs. fractures. As promised, we will prove as the main result of this section:

Theorem 3.9. For a wide subcategory $S \subset T$, the following are equivalent:

- (1) The subcategory S is a cleft (Definition 3.2).
- (2) The left Kan extension functor $\iota_1 \colon PSh(S) \to PSh(T)$ along the inclusion $\iota \colon S \hookrightarrow T$ is a fracture subcategory (Definition 3.1).

Remark 3.10. In the special case where $S \subset T$ defines an admissibility structure on T, cf. Remark 3.4, the implication $(1) \Rightarrow (2)$ was already proved by Lurie in [Lur18, Theorem 20.2.4.1]. In the examples we care about, and in particular for the inclusion $Orb \subset Glo$, the stronger Axiom (C2') of an admissibility structure is not satisfied: the required pullbacks do *not* exist before passing to presheaves. The above strengthening of Lurie's result will therefore be crucial for our purposes.

The proof of Theorem 3.9 will occupy this whole subsection; it is somewhat involved and may be skipped on a first reading.

For the remainder of this subsection, we fix a cleft category $\iota: S \hookrightarrow T$. We start with some elementary consequences of the axioms.

Lemma 3.11. The functor $(\iota_!)_{/A}$: $PSh(S)_{/A} \to PSh(T)_{/A}$ is fully faithful for any object $A \in S$.

Proof. As recalled in Example 2.12, $(\iota_!)_{/A}$ may be identified with the functor $(\iota_{/A})_!$: $PSh(S_{/A}) \to PSh(T_{/A})$ given by left Kan extension along $\iota_{/A}: S_{/A} \to T_{/A}$. By Axiom (C1), S is left cancellable, so that $\iota_{/A}$ is fully faithful. Thus, also the Kan extension $(\iota_{/A})_!$ is fully faithful, whence so is $(\iota_!)_{/A}$.

Lemma 3.12. For every $g: A \to B$ in T, the pullback functor $g^*: PSh(T)_{/B} \to PSh(T)_{/A}$ sends the essential image of $(\iota_1)_{/B}$ to the essential image of $(\iota_1)_{/A}$.

Proof. The functor g^* : PSh(T)_{/B} → PSh(T)_{/A} preserves colimits as PSh(T) is an ∞-topos. Since the essential image of the fully faithful left adjoint $(\iota_1)_{/A}$ is closed under colimits, it will be enough to show that g^* maps any element of the form $\iota_1 f: \iota_1 X \to \iota_1 B$ for $f: X \to B$ a map in S into the essential image of $(\iota_1)_{/A}$. This is precisely Axiom (C2), finishing the proof.

Construction 3.13. The functor $\iota^* \colon PSh(T) \to PSh(S)$ preserves pullbacks, so it induces a map

$$\operatorname{PSh}(T)^{[1]} \to \operatorname{PSh}(S)^{[1]} \times_{\operatorname{PSh}(S)} \operatorname{PSh}(T)$$
 (1)

of cartesian fibrations over PSh(T).

We now define the S-functor $\iota^* \colon \iota^* \underline{\operatorname{Spc}}_T \to \underline{\operatorname{Spc}}_S$ as the composite

$$\iota^* \underline{\operatorname{Spc}}_T = \operatorname{PSh}(T)_{/\iota_!(\bullet)} \longrightarrow \operatorname{PSh}(S)_{/\iota^*\iota_!(\bullet)} \xrightarrow{\eta^*} \operatorname{PSh}(S)_{/\bullet} = \underline{\operatorname{Spc}}_S,$$

where the first map is obtained from the straightening of (1) by restricting along $\iota_1 \colon PSh(S) \to PSh(T)$, while the second map is obtained by pullback along the unit transformation $\eta \colon id \Rightarrow \iota^* \iota_1$.

Lemma 3.14. The left Kan extension functor $\iota_1 \colon PSh(S) \to PSh(T)$ preserves pullbacks.

Proof. For any $X \in PSh(S)$, the above functor $\iota^* \colon PSh(T)_{\iota_1X} \to PSh(S)_{/X}$ admits a left adjoint given by $(\iota_1)_{/X}$. The lemma then precisely amounts to saying that the Beck–Chevalley map $(\iota_1)_{/X}g^* \Rightarrow g^*(\iota_1)_{/Y}$ is an equivalence for any $g \colon X \to Y$ in PSh(S). By Propositions 2.13 it suffices to check this in the case that X and Y are representable, i.e. g is a map in S. Since the functors $(\iota_1)_{/X}$ and $(\iota_1)_{/Y}$ are fully faithful in this case by Lemma 3.11, the Beck–Chevalley condition is equivalent to the condition that g^* preserves their essential images, which is an instance of Lemma 3.12. □

Construction 3.15. As a consequence of the previous lemma, ι_1 induces a map $PSh(S)^{[1]} \to PSh(T)^{[1]} \times_{PSh(T)} PSh(S)$ of cartesian fibrations, which we straighten to an S-functor $\iota_1 \colon \underline{Spc}_S \to \iota^* \underline{Spc}_T$. For any presheaf X in PSh(S), this is given by $(\iota_1)_{X} \colon PSh(S)_{X} \to PSh(T)_{\iota_1X}$.

Lemma 3.16. The S-functor ι_1 is left adjoint to the S-functor ι^* from Construction 3.13.

Proof. We have already seen in the proof of Lemma 3.14 that ι^* admits a left adjoint L which agrees *pointwise* with $\iota_!$. In the same way, one shows that $\iota_!$ is indeed a left adjoint (with adjoint agreeing pointwise with ι^*). But then $L \simeq \iota_!$ because left adjoint functors out of $\underline{\text{Spc}}_S$ are characterized by their value on the terminal presheaf by Theorem 2.24, so $\iota_!$ is left adjoint to ι^* as claimed.

Lemma 3.17. The fully faithful S-functor $\iota_1 \colon \underline{\operatorname{Spc}}_S \hookrightarrow \iota^* \underline{\operatorname{Spc}}_T$ extends uniquely to a T-functor $\iota_1 \colon \underline{\operatorname{Spc}}_{S \triangleright T} \hookrightarrow \underline{\operatorname{Spc}}_T$ (which is again fully faithful).

Proof. The statement is equivalent to the claim that the essential image of the inclusion $\iota_1 \colon \underline{\operatorname{Spc}}_S \hookrightarrow \iota^* \underline{\operatorname{Spc}}_T$ is in fact a *T*-subcategory of $\underline{\operatorname{Spc}}_T$, which is precisely the content of Lemma 3.12.

As an upshot, Axiom (C2) holds without any representability assumptions on A, B, or B'.

Lemma 3.18. For any presheaf $X \in PSh(S)$, the unit map $\eta_X : X \to \iota^* \iota_! X$ is a monomorphism. Put differently, the functor $\iota_! : PSh(S) \to PSh(T)$ is faithful.

Proof. This works in exactly the same way as for admissibility structures [Lur18, Proposition 20.2.4.5-(a)]: By Kan's pointwise formula, the presheaf $\iota^*\iota_!X$ is given in degree $A \in S$ by $\operatorname{colim}_{B \in (A/\iota)^{\operatorname{op}}} X(B)$ with $A/\iota := A/T \times_T S$, and the unit map $\eta: X(A) \to (\iota^*\iota_!X)(A)$ corresponds under this identification with the structure map of the term $\operatorname{id}_A \in A/\iota$. Since this term is contained in the full subcategory $A/S \subset A/\iota$ of maps in S, we may factor η as

 $X(A) \to \operatorname{colim}_{B \in (A/S)^{\operatorname{op}}} X(B) \to \operatorname{colim}_{B \in (A/\iota)^{\operatorname{op}}} X(B).$

The first map is an equivalence (the object $\operatorname{id}_A \in A/S$ being terminal), and thus it remains to show that the second map is a monomorphism. For this, we claim that the category A/ι is a disjoint union of the full subcategory A/S and its complement (consisting of maps not in S), i.e. any object $t: A \to B$ in A/ι mapping to or from an object in A/S must itself be in A/S. Indeed, let $s: A \to B'$ be any map in S: if there is a map $s \to t$ in A/ι , then t belongs to S as the latter is a subcategory; on the other hand, if there is a map $t \to s$, then t belongs to S by left cancellability. It follows that $\operatorname{colim}_{B \in (A/\iota)^{\operatorname{op}}} X(B)$ splits as a disjoint union $\operatorname{colim}_{B \in (A/S)^{\operatorname{op}}} X(B) \amalg Y$, finishing the proof.

Lemma 3.19. Let $f: X \to Y$ be a map in PSh(S). Then the naturality square



of the unit transformation η : id $\Rightarrow \iota^* \iota_1$ is a pullback square.

Proof. Again, this is analogous to the proof for admissibility structures [Lur18, Proposition 20.2.4.5-(b)]. The proof of the previous lemma shows that after evaluating at $A \in T$ the naturality square is equivalent to a square of the form

$$\begin{array}{c} X(A) & \xrightarrow{f(A)} & Y(A) \\ & & \downarrow \\ X(A) \amalg X' & \xrightarrow{f(A)\amalg f'} & Y(A) \amalg Y', \end{array}$$

which is evidently a pullback.

Our next goal is to prove the following sharpening of Lemma 3.18:

Proposition 3.20. The functor $\iota_1 \colon PSh(S) \to PSh(T)$ is fully faithful on groupoid cores, and thus a monomorphism of categories.

The proof of this proposition is surprisingly subtle and will require some further preparations.

Definition 3.21. Let $X, Y \in PSh(S)$. We call $f: \iota_! X \to \iota_! Y$ in PSh(T) admissible if it lies in the image of the inclusion $\iota_!: Hom_{PSh(S)}(X, Y) \hookrightarrow Hom_{PSh(T)}(\iota_! X, \iota_! Y)$.

Beware that a priori this depends on the equivalence classes of X and Y in PSh(S), not only on the equivalence classes of their left Kan extensions in PSh(T), and only once we have proven Proposition 3.20 will we know that this independent of the choices of preimages.

Lemma 3.22. Let $X, Y \in PSh(S)$ and let $f: \iota_! X \to \iota_! Y$ be a map in PSh(T).

- (1) The map f is admissible if and only if its adjunct $\tilde{f}: X \to \iota^* \iota_! Y$ factors through the monomorphism $\eta: Y \to \iota^* \iota_! Y$.
- (2) Let $(g_i)_{i \in I}$: $\coprod_{i \in I} X_i \to X$ be an effective epimorphism in PSh(S). Then f is admissible if and only if the composite $f \circ \iota_!(g_i) : \iota_! X_i \to \iota_! Y$ is admissible for every $i \in I$.
- (3) Let $(h_i)_{i \in I}$: $\coprod_{i \in I} Y_i \twoheadrightarrow Y$ be an effective epimorphism in PSh(S). Then f is admissible if and only if for every $i \in I$ there exists a pullback diagram in PSh(T) of the form

$$\begin{array}{ccc} \iota_! X_i & \stackrel{\iota_! h'_i}{\longrightarrow} & \iota_! X \\ f_i & {} & {} & {} & {} \\ f_i & {} & {} & {} \\ \iota_! Y_i & \stackrel{\iota_! h_i}{\longrightarrow} & \iota_! Y \end{array}$$

such that f_i is admissible.

Proof. Part (1) is immediately clear from the definitions. Using (1), we see that part (2) is equivalent to the statement that the adjunct map $\tilde{f}: X \to \iota^* \iota_! Y$ factors through the unit $\eta: X \to \iota^* \iota_! Y$ if and only if each of the composites $\tilde{f} \circ g_i: X_i \to \iota^* \iota_! Y$ do, which is immediate. For part (3), the 'only if'-direction follows directly from Lemma 3.14. For the 'if'-direction, observe that the map $(h'_i)_{i \in I}: \coprod_{i \in I} X_i \to X$ from part (3) is an effective epimorphism in PSh(S): by Lemma 3.19 it is a pullback of the effective epimorphism $\iota^* \iota_! (h_j)_{j \in J}: \iota^* \iota_! \coprod_{j \in J} Y_j \to \iota^* \iota_! Y$. The claim thus follows from part (2), as for every $i \in I$ the composite $f \circ \iota_! (h'_i) = \iota_! h_i \circ f_i$ is admissible by assumption.

Lemma 3.23. Let $X, Y, Z \in PSh(S)$ and let $f: \iota_! X \to \iota_! Y$ and $g: \iota_! Y \to \iota_! Z$ be maps in PSh(T) such that g and gf are admissible. Then also f is admissible.

Proof. By the previous lemma, we have to show that the composite $\iota^*(f)\eta: X \to \iota^* \iota_! Y$ factors through $\eta: Y \to \iota^* \iota_! Y$. However, by Lemma 3.19 and admissibility of g the latter is pulled back from the unit $\eta: Z \to \iota^* \iota_! Z$ along $\iota^*(g)$. It therefore suffices to show that $\iota^*(g)\iota^*(f)\eta$ factors accordingly. However, this is immediate from admissibility of gf.

We are now ready for the proof of Proposition 3.20:

Proof of Proposition 3.20. In light of the faithfulness of $\iota_1 \colon PSh(S) \to PSh(T)$ from Lemma 3.18, it remains to show that ι_1 is full on cores. Note that it suffices to prove that for presheaves $X, Y \in PSh(S)$ any equivalence $f \colon \iota_1 X \longrightarrow \iota_1 Y$ is admissible. We will prove this in two steps:

Step 1: We will first treat the special case where $X = A \in S$ is a representable presheaf. Consider the image $f_A(\operatorname{id}_A) \in (\iota_! Y)(A)$ of the identity $\operatorname{id}_A \in \iota_!(A)(A)$ under f. Because of the equivalence $(\iota_! Y)(A) \simeq \operatorname{colim}_{B \in (A/\iota)^{\operatorname{op}}} Y(B)$, we may represent $f(\operatorname{id}_A)$ by a class $[\alpha, y]$ for some morphism $\alpha \colon A \to B$ in T and some object $y \in Y(B)$. By Lemma 3.22, we have to prove that this class $[\alpha, y]$ lies in the image of the monomorphism

 $\eta_A \colon Y(A) \simeq \operatorname{colim}_{B \in (A/S)^{\operatorname{op}}} Y(B) \hookrightarrow \operatorname{colim}_{B \in (A/\iota)^{\operatorname{op}}} Y(B) = (\iota^* \iota! Y)(A)$

induced by the disjoint summand inclusion $A/S \hookrightarrow A/\iota$ (see the proof of Lemma 3.18). In other words, we have to show that α is a morphism in S.

Since the map $f_B: \operatorname{Hom}_T(B, A) = (\iota_! A)(B) \xrightarrow{\sim} (\iota_! Y)(B)$ is an equivalence, there exists a map $\beta: B \to A$ in T satisfying $f_B(\beta) = [\operatorname{id}_B, y]$. We thus have $f_A(\beta \alpha) = \alpha^* f_B(\beta) = [\alpha, y] = f_A(\operatorname{id})$, and since also f_A is an equivalence we deduce that $\beta \alpha = \operatorname{id}$. On the other hand, we have $[\operatorname{id}_B, y] = f_B(\beta) = \beta^* f_A(\operatorname{id}_A) = \beta^* [\alpha, y] = [\alpha\beta, y]$, and since $A/S \hookrightarrow A/\iota$ is a disjoint summand inclusion we see that $\alpha\beta$ belongs to S. It follows from Axiom (C3) that also α belongs to S, finishing Step 1.

Step 2: We will now deduce the statement for an arbitrary presheaf $X \in PSh(S)$. Pick an effective epimorphism $(h_j)_{j \in J} : \coprod_{j \in J} Y_j \twoheadrightarrow Y$ in PSh(S) for representable Y_i , and choose for each $j \in J$ a pullback

$$\begin{array}{ccc} \iota_! P_j & \xrightarrow{\iota_! h'_j} & \iota_! X \\ f_j & {}^{\smile} & {}^{\smile} & {}^{\downarrow} f \\ \iota_! Y_j & \xrightarrow{\iota_! h_j} & \iota_! Y \end{array}$$

in PSh(T) using Axiom (C2). As f is an equivalence, so is each f_j . As Y_j is representable, it follows from Step 1 that $f_j^{-1} : \iota_! Y_j \xrightarrow{\sim} \iota_! P_j$ is admissible, and thus by Lemma 3.23 also f_j is admissible. It thus follows from Lemma 3.22 that also f is admissible, completing the proof of the proposition.

Remark 3.24. Axiom (C3) is necessary for the previous proposition: every wide subcategory $\iota: S \hookrightarrow T$ for which the left Kan extension functor $\iota_1: PSh(S) \to PSh(T)$ is a monomorphism of categories automatically satisfies (C3). To see this, consider morphisms α, β as in Axiom (C3), and define $X \in PSh(S)$ to be the colimit of the diagram

$$A \xrightarrow{\alpha\beta} A \xrightarrow{\alpha\beta} \cdots$$

Since ι_1 preserves colimits, it follows that $\iota_1 X$ is the colimit of the analogous diagram in PSh(T). But since α and β are maps in T, the maps $\alpha: A \to B$ exhibit B as another colimit of this diagram, and thus we get an equivalence $\iota_1 X \simeq \iota_1 B$ in PSh(T) compatible with the colimit structure maps. Assuming that ι_1 is a monomorphism, it follows that $X \simeq B$ is a representable presheaf on S, and thus the map $\alpha: A \to B$ in T agrees up to equivalence in T with the structure maps $A \to X$, which belong to S by construction. As S contains all equivalences, this shows that also α belongs to S, finishing the argument.

Note moreover that (C3) is not implied by the remaining two axioms as the following example shows:

Example 3.25. Let R be a commutative ring. We let T = Perf(R) be the stable category of perfect R-chain complexes, and we let S consist of those $f: X \to Y$ such that $[X] = [Y] \in K_0(R)$, or equivalently (by the defining relations of K_0) such that the fiber of f vanishes in K_0 .

The first description makes it clear that S is a subcategory, contains all equivalences, and even satisfies 2-out-of-3, proving (C1). On the other hand, the second description shows that S is closed under pullbacks, proving (C2'). However, (C3) does not hold: $0 \to R \to 0$ is the identity and $R \to 0 \to R$ belongs to S as [R] = [R], but neither $0 \to R$ nor $R \to 0$ are contained in S as $[R] \neq 0$ in $K_0(R)$.

Definition 3.26. Following Lurie's notation and terminology for fractured ∞ -topoi, we let $PSh(T)^{corp} \subset PSh(T)$ denote the (non-full) essential image of the left Kan extension functor $\iota_1 \colon PSh(S) \to PSh(T)$. A presheaf on T is called *corporeal* if it is an object of $PSh(T)^{corp}$, and a morphism between two corporeal presheaves on T is called *admissible* if it is a morphism in $PSh(T)^{corp}$.

Note that for two $X, Y \in PSh(S)$ a map $f: \iota_! X \to \iota_! Y$ is admissible in the sense of Definition 3.26 if and only if it is admissible in the sense of Definition 3.21 above.

Lemma 3.27. Let $X, Y, Z \in PSh(T)$ be corporeal presheaves.

- (1) Let $f: X \to Z$ be an admissible morphism, and let $g: Y \to Z$ be arbitrary. Then the base change $g^*(f): g^*(X) \to Y$ of f along g is again an admissible morphism of corporeal presheaves.
- (2) Let $f: X \to Y$ be an effective epimorphism, and let $g: Y \to Z$ be arbitrary. Assume that f and gf are admissible. Then also g is admissible.
- (3) Let $f: X \to Y$, $g: Y \to Z$ be maps such that g and gf are admissible. Then f is admissible.

Proof. The first statement is a consequence of Lemma 3.17, while the second statement follows from Lemma 3.22. Finally, the third statement follows from Lemma 3.23. $\hfill \square$

We now come to the final missing ingredient of the proof of Theorem 3.9:

Proposition 3.28. Let $f: X \to Y$ be a map in $PSh(T)^{corp}$. Then the naturality square



is a pullback in PSh(T).

For the proof we will use:

Lemma 3.29. Let $Y \in PSh(T)$ be an arbitrary presheaf. Then the composite

$$\operatorname{PSh}(S)_{/\iota^*Y} \xrightarrow{\iota_!} \operatorname{PSh}(T)_{/\iota_!\iota^*Y} \xrightarrow{\operatorname{PSh}(T)_{/\varepsilon}} \operatorname{PSh}(T)_{/Y}$$
(2)

induces an equivalence onto the non-full subcategory $(PSh(T)_{/Y})^{corp}$ whose objects are those $X \to Y$ where X is corporeal (but there is no condition on the map to Y) and whose morphisms are the admissible maps in PSh(T).

Proof. It is clear that (2) factors through $(PSh(T)_Y)^{corp}$, so it only remains to show that the induced functor is essentially surjective and fully faithful. For this we observe that since ι_1 and ι^* are adjoint, the map $Hom(X, \iota^*Y) \to Hom(\iota_1X, Y), g \mapsto \varepsilon \circ$ $\iota_1(g)$ is an equivalence for any $X \in PSh(S)$. This immediately implies essential surjectivity, while for full faithfulness we observe that for objects $X, X' \in PSh(S)_{/\iota^*Y}$ the induced map on mapping spaces fits in the following diagram of fiber sequences:

$$\operatorname{Hom}_{\operatorname{PSh}(S)_{/\iota^*Y}}(X, X') \longrightarrow \operatorname{Hom}_{\operatorname{PSh}(S)}(X, X') \longrightarrow \operatorname{Hom}_{\operatorname{PSh}(S)}(X, \iota^*Y)$$

$$\downarrow^{\iota_!} \qquad \qquad \downarrow^{\simeq}$$

$$\operatorname{Hom}_{\operatorname{PSh}(T)_{/Y}}(\iota_!X, \iota_!X') \longrightarrow \operatorname{Hom}_{\operatorname{PSh}(T)}(\iota_!X, \iota_!X') \longrightarrow \operatorname{Hom}_{\operatorname{PSh}(T)}(\iota_!X, Y).$$

We now simply note that the middle map is a monomorphism by Lemma 3.18, with image the admissible maps. $\hfill \Box$

Proposition 3.30. The *T*-functor $\iota^* \colon PSh(T)_{\bullet} \to PSh(S)_{\iota^*(\bullet)}$ admits an *S*-left adjoint (that is, the underlying *S*-functor admits a parametrized left adjoint) given pointwise by the composites (2).

Proof. It is clear that the composites (2) yield a pointwise left adjoint, so it only remains to check the Beck–Chevalley condition. By the previous lemma, this amounts to saying that the adjunction $PSh(T)_{f}$: $PSh(T)_{\iota_{1}X} \rightleftharpoons PSh(T)_{\iota_{1}Y} : f^{*}$ restricts to an adjunction $(PSh(T)_{\iota_{1}X})^{corp} \rightleftharpoons (PSh(T)_{\iota_{1}Y})^{corp}$ for any admissible $f: \iota_{1}X \to \iota_{1}Y$, i.e.

- (1) The right adjoint f^* restricts to $(PSh(T)_{/\iota_! Y})^{corp} \to (PSh(T)_{/\iota_! X})^{corp}$.
- (2) For each $Z \in (PSh(T)_{\iota_1 Y})^{corp}$ the counit $PSh(T)_{/f} f^* Z \to Z$ is admissible.
- (3) For each $W \in (PSh(T)_{/\iota X})^{corp}$ the unit $W \to f^* PSh(T)_{/f} W$ is admissible.

For this, let $g: Z \to Z'$ be a map in $(PSh(T)_{\ell \mid Y})^{corp}$ and consider the coherent cube



Lemma 3.27-(1) then shows that the objects f^*Z and f^*Z' are corporeal and that the maps $\varepsilon \colon f^*Z \to Z$ and $\varepsilon \colon f^*Z' \to Z'$ are admissible, proving the second claim and one half of the first claim. Together with Lemma 3.27-(3) we then conclude that f^*g is again admissible, proving the remaining half of the first claim.

Finally, if $W \in (PSh(T)_{\iota \wr X})^{corp}$, then as a morphism in PSh(T) the unit $\eta \colon W \to f^* PSh(T)_{f}Y$ is right inverse to the counit ε . Thus, η is admissible by another application of Lemma 3.27-(3).

Proof of Proposition 3.28. We may assume without loss of generality that f is of the form $\iota_! f'$ for some $f': X' \to Y'$ in PSh(S). In this case, the previous proposition shows that the Beck–Chevalley transformation

$$\begin{array}{ccc} \operatorname{PSh}(S)_{/\iota^*Y} & \xrightarrow{(\iota^*f)^*} \operatorname{PSh}(S)_{/\iota^*X} \\ & & & & \downarrow \\ \operatorname{PSh}(T)_{/\varepsilon} \circ \iota_! & & & \downarrow \\ & & & \downarrow \\ \operatorname{PSh}(T)_{/Y} & \xrightarrow{f^*} \operatorname{PSh}(T)_{/X} \end{array}$$

is an equivalence. Chasing through the identity of $\iota^* Y$ precisely yields the claim. \Box

We are now ready to prove Theorem 3.9.

Proof of Theorem 3.9. If $\iota: S \hookrightarrow T$ is a cleft category, then $\iota_!: PSh(S) \to PSh(T)$ is a fracture subcategory:

- (F0) The functor ι_1 is a monomorphism by Proposition 3.20.
- (F1) The functor $\iota_!$ preserves pullbacks by Lemma 3.14.
- (F2) The right adjoint ι^* of $\iota_!$ is clearly cocontinuous, and it is conservative as S contains all objects of T.
- (F3) The pullback condition for the counit was verified in Proposition 3.28.

Conversely, assume that $\iota_1 \colon PSh(S) \to PSh(T)$ is a fracture subcategory. Then $(\iota_1)_{/X} \colon PSh(S)_{/X} \to PSh(T)_{/\iota_1X}$ is fully faithful for any X by [Lur18, Proposition 20.1.3.1]; specializing to $X = A \in S$, we see that left Kan extension along $\iota_{/A} \colon S_{/A} \hookrightarrow T_{/A}$ is fully faithful, whence so is $\iota_{/A}$ itself by the Yoneda Lemma. Letting A vary, this precisely amounts to saying that S is left cancellable, proving (C1).

For (C2), consider a map $f: X \to Y$ in PSh(S) and a map $g: \iota_! Y' \to \iota_! Y$ in PSh(T). Write $\tilde{g}: Y' \to \iota^* \iota_! Y$ for the adjunct of g, and define X' via the following pullback square in PSh(S):

$$\begin{array}{ccc} X' & \longrightarrow \iota^* \iota_! X \\ f' & & & \downarrow \iota^* \iota_! f \\ Y' & & & \downarrow \iota^* \iota_! Y. \end{array}$$

In the diagram

$$\begin{array}{ccc} \iota_!(X') & \longrightarrow \iota_! \iota^* \iota_!(X) & \stackrel{\varepsilon}{\longrightarrow} \iota_!(X) \\ \iota_!(f') & & \iota_! \iota^* \iota_! f & & \downarrow \iota_!(f) \\ \iota_!(Y') & \stackrel{\iota_!(\tilde{a})}{\longrightarrow} \iota_! \iota^* \iota_!(Y) & \stackrel{\varepsilon}{\longrightarrow} \iota_!(Y), \end{array}$$

the left-hand square is a pullback as ι_1 preserves pullbacks by (F1), while the righthand square is a pullback square by (F3). Thus, the total square expresses $\iota_1(f')$ as a pullback of $\iota_1(f)$ along g, showing (C2).

Finally, Axiom (C3) holds because ι_1 is a monomorphism, see Remark 3.24.

3.2. **Examples.** We close this section by establishing our two key examples of cleft categories. We begin with Example 3.6, for which we recall:

Definition 3.31. A factorization system on an category T consists of two wide subcategories $E, M \subset T$ satisfying the following conditions:

- (1) Both E and M contain all equivalences.
- (2) Every morphism in E is *left orthogonal* to every morphism in M in the following sense: for every pair of morphisms $e: A \to B$ in E and $m: X \to Y$ in M and every solid square

$$\begin{array}{c} A \longrightarrow X \\ e \downarrow & \swarrow^{\rtimes} \downarrow^{m} \\ B \longrightarrow Y \end{array}$$

there is a contractible space of dotted lifts making both triangles commute, i.e. the square

$$\begin{array}{ccc} \operatorname{Hom}_{T}(A, X) & \xrightarrow{m \circ -} & \operatorname{Hom}_{T}(A, Y) \\ & & & & \downarrow \\ & & & \downarrow - \circ e \\ & & & \downarrow - \circ e \\ & & & & \downarrow - \circ e \\ & & & & & Hom_{T}(B, X) \end{array}$$

is a pullback square in the category of spaces;

(3) Every morphism $f \in T$ admits a factorization f = me, with e in E and m in M.

Remark 3.32. The above definition follows [ABFJ22, Definition 3.1.6]. By Lemma 3.1.9 of *op. cit.*, the class E in a factorization system is *exactly* the class of morphisms in T which are left orthogonal to all morphisms in M, and vice-versa. In particular, this implies that both E and M are closed under retracts, so that the above is equivalent to [Lur09, Definition 5.2.8.8] (where this condition is assumed a priori).

Proposition 3.33. Let (E, M) be a factorization system on T. Then the right class M is a cleft of T.

If T has pullbacks, this proposition appears (with a rather different proof) as [Lur18, Proposition 20.2.2.1].

Proof. By assumption $M \subset T$ is a wide subcategory containing all equivalences, and it is left cancellable by [Lur09, Proposition 5.2.8.6-(3)], proving (C1). Moreover, Axiom (C3") was noted in the previous remark.

It remains to verify (C2), i.e. that for every $f: A \to B$ in T the pullback functor $f^*: \operatorname{PSh}(T)_{/B} \to \operatorname{PSh}(T)_{/A}$ maps the image of $M_{/B}$ into $\operatorname{PSh}(M)_{/A}$. We will prove this more generally for $\operatorname{PSh}(M)_{/B}$. For this we observe that the diagram

$$\begin{array}{ccc} \operatorname{PSh}(M_{/A}) & \xrightarrow{\sim} & \operatorname{PSh}(M)_{/A} \\ & & & & \\ (\iota_{/A})_! & & & & \downarrow (\iota_!)_{/A} \\ & & & & & \\ \operatorname{PSh}(T_{/A}) & \xrightarrow{\sim} & \operatorname{PSh}(T)_{/A} \end{array}$$

$$(3)$$

with the horizontal equivalences as in Example 2.12 commutes up to equivalence since both paths are cocontinuous and agree on the Yoneda image. Arguing in the same way for B, it then suffices to show: if $X \in PSh(T_{/B})$ is left Kan extended from $PSh(M_{/B})$, then its restriction to $PSh(T_{/A})$ is left Kan extended from $PSh(M_{/A})$. To this end, let $f: X \to A$ be any map in T, and fix a factorization

$$X \xrightarrow{e} Y$$

$$f \xrightarrow{} A$$

$$f \xrightarrow{} A$$

with e in E and m in M. Viewing this as a map in $T_{/A}$, [Lur09, Remark 5.2.8.3] shows that for every other other $t \in T_{/A}$ the map $e^* \colon \operatorname{Hom}(m, t) \to \operatorname{Hom}(f, t)$ is an equivalence. It follows that $\iota_{/A} \colon S_{/A} \hookrightarrow T_{/A}$ admits a left adjoint $\lambda_A \colon T_{/A} \to S_{/A}$ sending f to m with unit $f \to m$ given by the above triangle. In particular, all units live in the subcategory $T_{/A}^E \coloneqq E \times_T T_A$; conversely, an easy 2-out-of-3 argument shows that λ_A inverts all maps in $T_{/A}^E$. By abstract nonsense about Bousfield localizations, it follows that λ_A is a localization at $T_{/A}^E$, so that $X \in \operatorname{PSh}(T_{/A})$ is left Kan extended if and only if it inverts $T_{/A}^E$. Arguing in the same way for B, the proposition follows as $T_{/f}$ obviously restricts to $T_{/A}^E \to T_{/B}^E$.

Next, we recall atomic orbital subcategories from [CLL23, Definition 4.3.1]:

Definition 3.34. A wide subcategory $P \subset T$ containing all equivalences is called *atomic orbital* if the following conditions are satisfied:

(1) For every $p: C \to D$ in P and $t: B \to D$ in T there exists a pullback



in PSh(T) such that each $p_i: A_i \to B$ belongs to P.

(2) For every $p: A \to B$ in P the diagonal $A \to A \times_B A$ is a disjoint summand inclusion in PSh(T), i.e. it is equivalent to an inclusion of the form $A \hookrightarrow A \sqcup C$ for some $C \in PSh(T)$.

Remark 3.35. By [CLL23, Lemma 4.3.2] we can equivalently replace (2) by the following axiom:

(2') Every map in P that admits a section in T is an equivalence.

Atomic orbital subcategories were introduced in [CLL23] to encode different degrees of 'parametrized semiadditivity,' and we will revisit them from this perspective in Section 6. For now we are interested in them as examples of clefts:

Proposition 3.36. Any atomic orbital subcategory $P \subset T$ is a cleft.

For the proof we will use:

Lemma 3.37. Let P be atomic orbital (say, as a subcategory of itself) and consider an object $A \in P$. Then any endomorphism in $P_{/A}$ is invertible.

Proof. Let $B \in P_{A}$ and fix a decomposition $B \times_A B = \coprod_{i=1}^n X_i$ into representables. We introduce the following terminology:

- (A) Given any map $g: C \to B \times_A B$ from a representable, it factors through a unique X_i , and we call i =: idx(g) the *index* of g.
- (B) An index $i \in \{1, ..., n\}$ is called *good* if the projection $\operatorname{pr}_2: B \times_A B \to B$ to the second factor restricts to an equivalence $X_i \to B$.

Now let f be an endomorphism of B, inducing a map $(1, f): B \to B \times_A B$. We claim that idx(1, f) is good, which will then imply the lemma as (1, f) induces an equivalence onto $X_{idx(1,f)}$, being a section to the map $pr_1: X_{idx(1,f)} \to B$ in P.

To prove the claim, we make the following basic observations:

- (1) Given any endomorphism g of B, the index idx(g, 1) is good (arguing as above using that $pr_2(g, 1) = 1$).
- (2) Given any map $\alpha \colon X \to Y$ of representables and a map $\beta \colon Y \to B \times_A B$, we have $idx(\beta \alpha) = idx(\beta)$.
- (3) If $\alpha, \beta \colon X \rightrightarrows B \times_A B$ are maps from a representable with $\operatorname{idx}(\alpha) = \operatorname{idx}(\beta)$ and γ is any endomorphism of $B \times_A B$, then $\operatorname{idx}(\gamma \alpha) = \operatorname{idx}(\gamma \beta)$.

By (2), we have

$$\operatorname{idx}(1,f) = \operatorname{idx}(f^k, f^{k+1})$$

for any $k \ge 0$. Now by the pigeonhole principle we find $\ell > k \ge 0$ with $idx(f^k, 1) = idx(f^{\ell}, 1)$ and hence also

$$\operatorname{idx}(f^k, f^{k+1}) = \operatorname{idx}(f^\ell, f^{k+1})$$

by (3) applied to $1 \times_A f^{k+1}$. However, by construction $\ell \geq k+1$, whence

$$\operatorname{idx}(f^{\ell}, f^{k+1}) = \operatorname{idx}(f^{\ell-k-1}, 1)$$

by another application of (2). Altogether we therefore get

$$dx(1, f) = idx(f^k, f^{k+1}) = idx(f^{\ell}, f^{k+1}) = idx(f^{\ell-k-1}, 1)$$

and the right hand side is good by (1), finishing the proof.

Proof of Proposition 3.36. Axiom (C1) follows from [CLL23, Lemma 4.3.5], while (C2) is immediate from Definition 3.34-(1). To prove (C3'), we note that any idempotent $e: A \to A$ defines an endomorphism of itself considered as an object of $P_{/A}$. By the previous lemma, we conclude that e is invertible, hence homotopic to the identity.

4. PARTIAL PRESENTABILITY

Given a small category T, there is a natural notion of T-presentability for a T-category, recalled in Definition 2.26. This is quite a strong condition on C: it in particular requires that the restriction functors $f^*: C(B) \to C(A)$ admit left adjoints for all morphisms $f: A \to B$, which is unfortunately not satisfied in several naturally occurring examples, see for example Warning 9.8 about the global category of equivariant spectra.

The goal of this section is to introduce and study relaxations of the notion of presentability for a *T*-category C. While we still demand that C be fiberwise presentable, we will weaken the cocompleteness assumption: more precisely, for any cleft $S \subset T$, we will introduce notions of *S*-cocompleteness and *S*-presentability, see Subsection 4.1. In Subsection 4.2 we discuss the relation between *S*-presentable *T*-categories and *S*-presentable *S*-categories. We end this section in Subsection 4.3 with a discussion of the *S*-cocompletion of a small *T*-category and the relation to the *S*-cocompletion of its underlying *S*-category.

4.1. S-(co)limits and S-presentability. We fix a cleft category $S \subset T$ and we write $\iota: S \hookrightarrow T$ for the inclusion. In this subsection we study what it means for a T-category C to be S-(co)complete or S-presentable.

Definition 4.1. We define the *T*-subcategory $\mathbf{U}_S \subset \underline{\operatorname{Spc}}_T$ as the essential image of the fully faithful *T*-functor $\iota_1 \colon \underline{\operatorname{Spc}}_{S \triangleright T} \hookrightarrow \underline{\operatorname{Spc}}_T$ from Lemma 3.17: for an object $A \in T$, the subcategory $\mathbf{U}_S(A) \subset \underline{\operatorname{Spc}}_T(A) = \operatorname{PSh}(T)_{/A}$ is the full subcategory spanned by the admissible maps.

Definition 4.2 (S-(co)completeness). A T-category C is called S-cocomplete if it is fiberwise cocomplete and admits all U_S -colimits in the sense of Definition 2.16. Dually, C is called S-complete if it is fiberwise complete and admits all U_S -limits.

Definition 4.3 (S-presentability). A T-category C is called S-presentable if it is S-cocomplete and fiberwise presentable (Definition 2.25).

Warning 4.4. As recalled in Remark 2.27 any T-presentable T-category is also T-complete. In contrast, there are interesting examples of T-categories that are S-presentable in the above sense, but not S-complete, see Warnings 9.6 and 9.8.

i

We will now provide a description of S-(co)completeness in terms of pointwise conditions. For this we first introduce:

Definition 4.5. A morphism $f: X \to Y$ in PSh(T) is called *admissible* if it defines an object in $\mathbf{U}(Y) \subset \underline{Spc}_T$, i.e. for every $A \in T$ and $t: A \to Y$ in PSh(T) the pulled back map $t^*(f): t^*(X) \to A$ is an admissible map of corporeal objects in the sense of Definition 3.26.

Remark 4.6. Note that for a corporeal object Y this recovers the previous definition by Lemma 3.27.

By the pasting law, the admissible maps are closed under composition, and it is clear that every equivalence is admissible; in particular, the admissible maps define a wide subcategory $PSh(T)^{ad} \subset PSh(T)$. By another application of the pasting law, this is closed under pulling back along arbitrary maps in PSh(T).

Lemma 4.7. Let C be a fiberwise cocomplete T-category. Then the following are equivalent:

- (1) For every $m: A \to B$ in S, the map $m^*: \mathcal{C}(B) \to \mathcal{C}(A)$ admits a left adjoint m_1 .
- (2) For every $B' \in T$ and any admissible $n: A' \to B'$ the functor n^* admits a left adjoint n_1 .

Proof. It is immediate that (2) implies (1). Conversely, let $B' \in T$ and consider an object $n: A' \to B'$ in $\mathbf{U}_S(B')$. Decomposing a preimage in $\mathrm{PSh}(S)_{/B}$ into representables, we get an equivalence $(k_i)_{i\in I}: \operatorname{colim}_{i\in I} A'_i \simeq A'$ for a functor $A'_{\bullet}: I \to T$ such that for every $i \in I$ the composite $n_i = nk_i: A'_i \to A' \to B'$ lies in S. Then n^* agrees up to equivalence with the functor $\mathcal{C}(B') \to \lim_{i\in I} \mathcal{C}(A'_i)$ induced by the n^*_i . Now each of these n^*_i admits a left adjoint by assumption and moreover $\mathcal{C}(B)$ is cocomplete; thus, also n^* admits a left adjoint by [HY17, Theorem B^{\mathrm{op}}]. \square

Remark 4.8. For later use, we make the construction of the left adjoint given in *loc. cit.* semi-explicit, keeping the notation from the previous proof:

- (1) For $X \in \mathcal{C}(A')$, $n_! X$ is the colimit of a suitable I^{op} -diagram with $i \mapsto n_{i!} k_i^*(X)$.
- (2) The counit $n_! n^* X = \operatorname{colim}_{i \in I^{\operatorname{op}}} n_! k_i^* n^* X = \operatorname{colim}_{i \in I^{\operatorname{op}}} n_i! n_i^* X \to X$ is induced by a cocone given at $i \in I^{\operatorname{op}}$ by the counit of $n_{i!} \dashv n_i^*$.
- (3) The unit $Y \to n^* n_! Y$ is given after restricting along k_i by the composite $k_i^* Y \to n_i^* n_{i!} k_i^* Y \to n_i^* \operatorname{colim}_{j \in J} n_{j!} k_j^* Y$ of the unit and the structure map of the colimit.

Using this we can now prove:

Lemma 4.9. Let C be a T-category. Then C is S-cocomplete if and only if the following conditions are satisfied:

- (1) The T-category C is fiberwise cocomplete,
- (2) For every morphism $m: A \to B$ in S, the restriction $m^*: \mathcal{C}(B) \to \mathcal{C}(A)$ admits a left adjoint m_1 ,

(3) For every pullback square

$$\begin{array}{ccc} A & \stackrel{m}{\longrightarrow} & B \\ {}^{t} \downarrow & \stackrel{}{\longrightarrow} & \downarrow^{u} \\ A' & \stackrel{}{\longrightarrow} & B' \end{array}$$

in PSh(T) where n belongs to S and u is a map in T, the Beck-Chevalley map $m_1t^* \rightarrow u^*n_1$ is an equivalence (note that m_1 exists by Lemma 4.7).

The dual characterization for S-completeness also holds.

Proof. By definition, S-cocompleteness implies all of the above conditions. Conversely, if these three conditions are satisfied, it only remains by the previous lemma together with Remark 2.17 to show that the Beck–Chevalley condition (3) actually holds without representability assumption on A'.

For this we fix a decomposition $(k_i)_{i \in I}$: $\operatorname{colim}_{i \in I} A'_i \simeq A'$ in $\operatorname{PSh}(T)$ into representables as before. We now pull back each individual $n_i = nk_i$ along u to an m_i , and then appeal to universality of colimits to obtain a pullback

$$\operatorname{colim}_{i \in I}(A'_{i} \times_{B'} B) \xrightarrow{m=(m_{i})} B$$
$$\underset{t=\operatorname{colim} t_{i}}{\overset{ \ }{\longrightarrow}} \begin{array}{c} \downarrow \\ u \\ \operatorname{colim}_{i \in I} A'_{i} \xrightarrow{(n_{i})} B'. \end{array}$$

It then follows from cocontinuity of u^* and the above description of unit and counit, that the Beck–Chevalley map $m_!t^*X \to u^*n_!X$ is given for any $X \in \mathcal{C}(\operatorname{colim}_{i \in I} A_i)$ as a colimit (over I^{op}) of the Beck–Chevalley maps

$$m_{i!}t_i^*k_i^*X \to u^*n_{i!}k_i^*X,$$

each of which is an equivalence by assumption.

Warning 4.10. Even for a fiberwise cocomplete *T*-category, being *S*-cocomplete is not just a property of the underlying *S*-category: the former includes more Beck–Chevalley conditions.

Lemma 4.11. Let $F: \mathcal{C} \to \mathcal{D}$ be a *T*-functor of *S*-cocomplete *T*-categories. Then the following are equivalent:

- (1) The T-functor F preserves fiberwise colimits and U_S -colimits.
- (2) The T-functor F preserves fiberwise colimits and for every map m in S the Beck-Chevalley map $m_1F \to Fm_1$ is an equivalence.
- (3) The S-functor ι^*F is S-cocontinuous.

The dual statement for S-complete categories also holds.

Proof. The equivalence between (1) and (2) follows just as in Lemma 4.9. Since the conditions in (2) only depend on the underlying S-functor ι^*F , the equivalence between (2) and (3) is clear.

Definition 4.12. An S-functor F satisfying the above equivalent conditions is called S-cocontinuous. We write $\operatorname{Cat}_T^{S-\operatorname{cc}} \subset \operatorname{Cat}_T$ for the very large category of

S-cocomplete T-categories and S-cocontinuous functors, and $\operatorname{Pr}_T^S \subset \operatorname{Cat}_T^{S-\operatorname{cc}}$ for the full subcategory spanned by the S-presentable T-categories.

Lemma 4.13. Let $\mathcal{C}, \mathcal{D} \in \operatorname{Cat}_T$ such that \mathcal{D} is *S*-cocomplete. Then $\underline{\operatorname{Fun}}_T(\mathcal{C}, \mathcal{D})$ is again *S*-cocomplete. Moreover, for any $F \colon \mathcal{C} \to \mathcal{C}'$ the restriction $\underline{\operatorname{Fun}}_T(\mathcal{C}', \mathcal{D}) \to \underline{\operatorname{Fun}}_T(\mathcal{C}, \mathcal{D})$ is *S*-cocontinuous.

Proof. This is a special case of [CLL23, Corollary 2.3.25].

Definition 4.14. We write $\underline{\operatorname{Fun}}_T^{S-\operatorname{cc}}(\mathcal{C}, \mathcal{D}) \subset \underline{\operatorname{Fun}}_T(\mathcal{C}, \mathcal{D})$ for the full subfunctor spanned in degree $X \in \operatorname{PSh}(T)$ by the S-cocontinuous functors $\mathcal{C} \to \underline{\operatorname{Fun}}_T(X, \mathcal{D})$.

Lemma 4.15. <u>Fun</u>^{S-cc}_T(C, D) defines a T-subcategory of <u>Fun</u>_T(C, D).

Proof. If $X \to Y$ is any map in PSh(T), then Lemma 4.13 shows that composing with the restriction $\underline{Fun}_T(\underline{Y}, \mathcal{D}) \to \underline{Fun}_T(\underline{X}, \mathcal{D})$ preserves S-cocontinuous functors. To see that this subfunctor is limit preserving, it suffices to observe that the functors $\underline{Fun}_T(\underline{Y}, \mathcal{D}) \to \underline{Fun}_T(\underline{A}, \mathcal{D})$ for all $A \to Y$ with A representable are jointly conservative and hence detect S-cocontinuity, cf. the proof of [CLL23, Proposition 2.3.28].

4.2. Colimits in Kan extensions. Recall that for any functor $\alpha : S \to T$ the restriction $\alpha^* : \operatorname{Cat}_T \to \operatorname{Cat}_S$ admits a right adjoint α_* , which can be computed via restriction along $\alpha^* : \operatorname{PSh}(T) \to \operatorname{PSh}(S)$. We will now study the interplay of these adjoints with parametrized colimits and limits in the case that $\alpha = \iota$ is a cleft category.

Convention 4.16. For the rest of this subsection let us fix a cleft category $\iota: S \hookrightarrow T$ and a *T*-subcategory $\mathbf{V}^{(T)} \subset \mathbf{U}_S \subset \underline{\operatorname{Spc}}_T$. We will write $\mathbf{V}^{(S)}$ for the *S*-subcategory defined as the preimage of $\iota^* \mathbf{V}^{(T)}$ along the inclusion $\operatorname{Spc}_S \hookrightarrow \iota^* \operatorname{Spc}_T$.

Lemma 4.17. Let $A \in T$. Then $\iota^* \colon PSh(T)_{/A} \to PSh(S)_{/\iota^*A}$ restricts to a map $\mathbf{V}^{(T)}(A) \to \mathbf{V}^{(S)}(\iota^*A)$.

Proof. Let $(u: X \to A) \in \mathbf{V}^{(T)}(A)$ arbitrary. By assumption on $\mathbf{V}^{(T)}$, u is admissible, so we have a pullback

in PSh(T) by Proposition 3.28; in particular $\iota_{!}\iota^{*}u \in \mathbf{V}^{(T)}(\iota_{!}\iota^{*}A)$ as $\mathbf{V}^{(T)}$ is a T-subcategory of $\underline{\operatorname{Spc}}_{T}$. But then $\iota^{*}u \in \mathbf{V}^{(S)}(\iota^{*}A)$ as desired. \Box

From now on we will confuse $\mathbf{V}^{(S)}$ and $\mathbf{V}^{(T)}$ and simply write \mathbf{V} for both of them.

Theorem 4.18. The adjunction ι^* : $\operatorname{Cat}_T \rightleftharpoons \operatorname{Cat}_S : \iota_*$ restricts to an adjunction $\operatorname{Cat}_T^{\mathbf{V}\text{-}\operatorname{cc}} \rightleftharpoons \operatorname{Cat}_S^{\mathbf{V}\text{-}\operatorname{cc}}$ between the categories of \mathbf{V} -cocomplete T- and S-categories, respectively, and \mathbf{V} -cocontinuous functors.

For the proof of the theorem we will use:

Lemma 4.19 (See [CLL23, Lemma 2.3.17]). Let $f: PSh(S) \to PSh(T)$ be a left adjoint functor that preserves pullbacks, let $\mathbf{V}' \subset \underline{Spc}_S$, and let $\mathbf{V} \subset \underline{Spc}_T$ such that for every $A \in S$ and every $v \in \mathbf{V}(A)$ also $f(v) \in \mathbf{V}'(f(A))$. Then $f^*: Cat_T \to Cat_S$ restricts to $Cat_T^{\mathbf{V}\text{-cc}} \to Cat_S^{\mathbf{V}^{\text{-cc}}}$.

Proof of Theorem 4.18. The functor $\iota_1: PSh(S) \to PSh(T)$ preserves pullbacks by Lemma 3.14, so the previous lemma shows that $\iota^*: Cat_T \to Cat_S$ preserves **V**cocomplete categories and **V**-cocontinuous functors. In the same way, we deduce from Lemma 4.17 that ι_* restricts accordingly.

Now let \mathcal{C} be a **V**-cocomplete *T*-category. Then the unit $\mathcal{C} \to \iota_* \iota^* \mathcal{C}$ is given by restriction along the counit $\varepsilon \colon \iota_! \iota^* \Rightarrow$ id of the adjunction $\iota_! \colon PSh(S) \rightleftharpoons PSh(T) : \iota^*$. Thus, if $A \in T$ and $(u \colon X \to A) \in \mathbf{V}(A)$ are arbitrary, then the Beck–Chevalley map $u_! \eta \to \eta u_!$ is given by the Beck–Chevalley map $(\iota_! \iota^* u)_! \varepsilon^* \to \eta^* u_!$ associated to the pullback (4) and hence is an equivalence by **V**-cocompleteness of \mathcal{C} .

Similarly, if \mathcal{D} is a **V**-cocomplete *S*-category, then the counit $\varepsilon \colon \iota^*\iota_*\mathcal{D} \to \mathcal{D}$ is given by restricting along the unit of $PSh(S) \rightleftharpoons PSh(T)$, and the Beck–Chevalley transformation $u_!\varepsilon \to \varepsilon u_!$ for $(u: X \to Y) \in \mathbf{V}(Y)$ is simply the Beck–Chevalley transformation for the pullback

$$\begin{array}{ccc} X & \stackrel{\eta}{\longrightarrow} \iota^* \iota_! X \\ \downarrow^{u} & \stackrel{\downarrow}{\longrightarrow} & \downarrow^{\iota^* \iota_! u} \\ Y & \stackrel{\eta}{\longrightarrow} \iota^* \iota_! Y \end{array}$$

in PSh(S) from Lemma 3.19, hence an equivalence as claimed.

Corollary 4.20. The adjunction $\iota^* \colon \operatorname{Cat}_T \rightleftharpoons \operatorname{Cat}_S : \iota_*$ restricts to adjunctions $\operatorname{Cat}_T^{S\operatorname{-cc}} \rightleftharpoons \operatorname{Cat}_S^{S\operatorname{-cc}}$ and $\operatorname{Pr}_T^S \rightleftharpoons \operatorname{Pr}_S^S$.

Proof. Clearly, ι^* and ι_* preserve fiberwise cocompleteness and cocontinuity; moreover, the unit $\mathcal{C} \to \iota_* \iota^* \mathcal{C}$ and counit $\iota^* \iota_* \mathcal{D} \to \mathcal{D}$ are simply given by restricting along suitable maps in PSh(T) or PSh(S) respectively, hence fiberwise cocontinuous.

The first claim now follows from the special case $\mathbf{V} = \mathbf{U}_S$ of the previous theorem. For the second one it then only remains to observe that ι^* and ι_* preserve fiberwise presentability by the same reasoning as for fiberwise cocompleteness.

We close this discussion by giving an 'internal' version of the above adjunction, for which we introduce:

Construction 4.21. For any $\mathcal{C}, \mathcal{D} \in \operatorname{Cat}_T$ we get a natural map $\iota^* \colon \iota^* \underline{\operatorname{Fun}}_T(\mathcal{C}, \mathcal{D}) \to \underline{\operatorname{Fun}}_S(\iota^*\mathcal{C}, \iota^*\mathcal{D})$ as the composite

$$\iota^* \underline{\operatorname{Fun}}_T(\mathcal{C}, \mathcal{D}) \xrightarrow{\eta} \underline{\operatorname{Fun}}_S(\iota^* \mathcal{C}, \iota^* \mathcal{C} \times \iota^* \underline{\operatorname{Fun}}_T(\mathcal{C}, \mathcal{D}))$$
$$\xrightarrow{\sim} \underline{\operatorname{Fun}}_S(\iota^* \mathcal{C}, \iota^* (\mathcal{C} \times \underline{\operatorname{Fun}}_T(\mathcal{C}, \mathcal{D})))$$
$$\xrightarrow{\iota^* \varepsilon} \underline{\operatorname{Fun}}_T(\iota^* \mathcal{C}, \iota^* \mathcal{D}),$$

where the unlabelled equivalence is the canonical one. Put differently, for any fixed C, this is the mate of the canonical natural equivalence filling the square

$$\begin{array}{ccc} \operatorname{Cat}_{T} & \xrightarrow{\mathcal{C} \times -} & \operatorname{Cat}_{T} \\ & & \iota^{*} \downarrow & & \downarrow \iota^{*} \\ \operatorname{Cat}_{S} & \xrightarrow{\iota^{*} \mathcal{C} \times -} & \operatorname{Cat}_{S} . \end{array}$$

$$(5)$$

Explicitly, this sends an object in degree $A \in T$ corresponding to $F: \underline{\iota(A)} \times \mathcal{C} \to \mathcal{D}$ to the composite

$$\underline{A} \times \iota^* \mathcal{C} \xrightarrow{\eta} \iota^* \underline{\iota(A)} \times \iota^* \mathcal{C} \simeq \iota^* (\underline{\iota(A)} \times \mathcal{C}) \xrightarrow{\iota^* F} \iota^* D,$$

where η now refers to the adjunction $\iota_1 \dashv \iota^*$.

Passing to mates once more, we also obtain an equivalence $\Phi: \underline{\operatorname{Fun}}_T(\mathcal{C}, \iota_*\mathcal{D}) \simeq \iota_*\underline{\operatorname{Fun}}_S(\iota^*\mathcal{C}, \mathcal{D})$ natural in $\mathcal{C} \in \operatorname{Cat}_T$ and $\mathcal{D} \in \operatorname{Cat}_S$, given for any $A \in T$ by sending a functor $F: \mathcal{C} \to \underline{\operatorname{Fun}}_T(\underline{A}, \iota_*\mathcal{D})$ to the composite

$$\iota^* \mathcal{C} \xrightarrow{\iota^* F} \iota^* \underline{\operatorname{Fun}}_T(\underline{A}, \iota_* \mathcal{D}) \xrightarrow{\iota^*} \underline{\operatorname{Fun}}_T(\iota^* \underline{A}, \iota^* \iota_* \mathcal{D}) \xrightarrow{\varepsilon \circ -} \underline{\operatorname{Fun}}(\iota^* \underline{A}, \mathcal{D}).$$

The composition of the two rightmost arrows agrees with $\varepsilon \circ \iota^* \Phi$ by the triangle identity, i.e. $\Phi(F)$ is the adjunct of the composite

$$\mathcal{C} \xrightarrow{F} \underline{\operatorname{Fun}}_{T}(\underline{A}, \iota_{*}\mathcal{D}) \xrightarrow{\Phi} \iota_{*}\underline{\operatorname{Fun}}_{S}(\iota^{*}\underline{A}, \mathcal{D}).$$

The equivalence Φ can accordingly be viewed as an 'internal' version of the adjunction equivalence for ι^* : $\operatorname{Cat}_T \rightleftharpoons \operatorname{Cat}_S : \iota_*$.

Corollary 4.22. Let C be an S-cocomplete T-category and D an S-cocomplete S-category. Then the previous construction restricts to an equivalence

$$\underline{\operatorname{Fun}}_{T}^{S\operatorname{-cc}}(\mathcal{C},\iota_{*}\mathcal{D}) \xrightarrow{\sim} \iota_{*}\underline{\operatorname{Fun}}_{S}^{S\operatorname{-cc}}(\iota^{*}\mathcal{C},\mathcal{D}).$$

Proof. It only remains to show that $F: \mathcal{C} \to \underline{\operatorname{Fun}}_T(X, \iota_* \mathcal{D})$ is S-cocontinuous if and only if $\Phi(F)$ is so. However, by the above explicit description of $\Phi(F)$, this is precisely the statement of Corollary 4.20.

Remark 4.23. One can deduce from the previous corollary that the functor ι^* : $\operatorname{Cat}_T^{S\operatorname{-cc}} \to \operatorname{Cat}_S^{S\operatorname{-cc}}$ from Corollary 4.20 is symmetric monoidal with respect to the symmetric monoidal structures defined in [MW22, Section 8.2], applied to \mathbf{U}_S . It follows in particular that the subcategory $\operatorname{Pr}_T^S \subset \operatorname{Cat}_T^{S\operatorname{-cc}}$ is closed under tensor products, being the preimage along ι^* of the symmetric monoidal subcategory $\operatorname{Pr}_S^{\mathrm{L}} \subset \operatorname{Cat}_S^{S\operatorname{-cc}}$. Since we will not make use of these symmetric monoidal structures in this paper, we will leave the details to the interested reader.

4.3. **S-cocompletion.** As an application of the above theory we can now reinterpret and extend work of Martini and Wolf on parametrized cocompletions:

Theorem 4.24. Let I be any small T-category. Then the unique S-cocontinuous S-functor $\iota_1: \underline{PSh}_S(\iota^*I) = \underline{Fun}_S(\iota^*I^{\mathrm{op}}, \underline{\mathrm{Spc}}_S) \to \iota^*\underline{Fun}_T(I, \underline{\mathrm{Spc}}_T) = \iota^*\underline{PSh}_T(I)$ compatible with the Yoneda embeddings is fully faithful. Moreover, its essential image is actually a T-subcategory, and this is the T-subcategory generated under S-colimits by the Yoneda image.

Proof. Write C for the full *T*-subcategory of $\underline{PSh}_T(I)$ generated under *S*-colimits by the Yoneda image. Then [MW21, Theorem 7.1.11] (for U the union of U_S and the constant *T*-categories) shows that restriction along the Yoneda embedding *y* defines an equivalence

$$\operatorname{Hom}_{\operatorname{Cat}_{\mathcal{F}}^{S-\operatorname{cc}}}(\mathcal{C},\mathcal{D}) \xrightarrow{\sim} \operatorname{Hom}_{\operatorname{Cat}_{\mathcal{T}}}(I,\mathcal{D})$$

for any S-cocomplete \mathcal{D} . Specializing to $\mathcal{D} = \iota_* \mathcal{E}$ for $\mathcal{E} \in \operatorname{Cat}_S^{S-cc}$ and appealing to Corollary 4.20 we see that restricting along $\iota^*(y)$ defines an equivalence

$$\operatorname{Hom}_{\operatorname{Cat}^{S-\operatorname{cc}}}(\iota^*\mathcal{C},\mathcal{E}) \xrightarrow{\sim} \operatorname{Hom}_{\operatorname{Cat}_S}(\iota^*I,\mathcal{E}).$$

However, the Yoneda embedding $\iota^* I \to \underline{PSh}_S(\iota^* I)$ has the same property by [MW21, Theorem 7.1.1], so comparing corepresented functors shows that $\iota_!$ defines an equivalence $\underline{PSh}_S(\iota^* I) \simeq \mathcal{C}$.

Construction 4.25. We let $\iota_!: \underline{PSh}_{S \triangleright T}(I) \to \underline{PSh}_T(I)$ denote the unique extension of $\iota_!: \underline{PSh}_S(\iota^*I) \to \iota^*\underline{PSh}_T(I)$ to a *T*-functor obtained from Theorem 4.24. Note that for I = 1 the terminal presheaf, this recovers the functor $\iota_!: \underline{Spc}_{S \triangleright T} \to \underline{Spc}_T$ from Lemma 3.17.

By full faithfulness of ι_1 , there is then a unique lift of the *S*-parametrized Yoneda embedding $\iota^*I \to \underline{PSh}_S(I)$ to a *T*-functor $y: I \to \underline{PSh}_{S \triangleright T}(I)$ together with an equivalence between $\iota_1 y$ and the *T*-parametrized Yoneda embedding $I \to \underline{PSh}_T(I)$.

Corollary 4.26. In the above situation, $\underline{PSh}_{S \triangleright T}(I)$ is S-presentable. For any S-cocomplete T-category \mathcal{D} , restriction along y defines an equivalence

$$\underline{\operatorname{Fun}}_{T}^{S\operatorname{-cc}}(\underline{\operatorname{PSh}}_{S\triangleright T}(I),\mathcal{D}) \xrightarrow{\sim} \underline{\operatorname{Fun}}_{T}(I,\mathcal{D}).$$

Proof. For S-presentability, we observe that $\underline{PSh}_{S \triangleright T}(I)$ is S-cocomplete as it is equivalent to a subcategory of \underline{Spc}_T closed under S-colimits, and that for any $A \in T$, $\underline{PSh}_{S \triangleright T}(I)(A) = PSh(S)_{/A}$ is clearly presentable.

The universal property is an instance of [MW21, Theorem 7.1.11] as before.

Corollary 4.27. The T-category $\underline{\operatorname{Spc}}_{S \triangleright T}$ is S-presentable. For any S-cocomplete T-category \mathcal{D} , evaluation at the terminal object defines an equivalence

$$\underline{\operatorname{Fun}}_{T}^{S\operatorname{-cc}}(\underline{\operatorname{Spc}}_{S\triangleright T},\mathcal{D}) \xrightarrow{\sim} \mathcal{D}.$$

In the situation of the previous corollary we actually have further right adjoints:

- **Proposition 4.28.** (1) The S-functor $\iota^* \colon \iota^* \underline{\operatorname{Spc}}_T \to \underline{\operatorname{Spc}}_S$ right adjoint to $\iota_!$ admits an S-right adjoint ι_* .
 - (2) The adjunct $\tilde{\iota}^* \colon \underline{\operatorname{Spc}}_T \to \iota_* \underline{\operatorname{Spc}}_S$ of ι^* admits a T-right adjoint $\tilde{\iota}_*$.

Proof. We will prove the second statement. Corollary 4.20 then implies that ι^* is S-cocontinuous, so that the first statement is an instance of the Special Adjoint Functor Theorem.

Recalling the definition (Construction 3.13), ι^* is given by the composite

$$\operatorname{PSh}(T)_{\iota_{1}(\bullet)} \xrightarrow{\iota^{*}} \operatorname{PSh}(S)_{\iota^{*}\iota_{1}(\bullet)} \xrightarrow{\eta^{*}} \operatorname{PSh}(S)_{\prime \bullet},$$

i.e. it is adjunct to $(\iota^*)_{\bullet}$: $PSh(T)_{\bullet} \to PSh(S)_{\iota^*(\bullet)}$. The latter obviously preserves T-colimits and has a pointwise right adjoint given by the composites

$$\operatorname{PSh}(S)_{/\iota^*(\bullet)} \xrightarrow{\iota_*} \operatorname{PSh}(T)_{/\iota_*\iota^*(\bullet)} \xrightarrow{\eta^*} \operatorname{PSh}(T)_{/\bullet}.$$

Remark 4.29. We close this discussion by giving a different interpretation of the functor $\iota^* \colon \iota^* \underline{\operatorname{Spc}}_T \to \underline{\operatorname{Spc}}_S$. For this we recall once more the equivalences $\underline{\operatorname{Spc}}_T \simeq \operatorname{PSh}(T_{\bullet}), \underline{\operatorname{Spc}}_S \simeq \operatorname{PSh}(S_{\bullet})$ from Example 2.12; we claim that under these equivalences our functor ι^* is given by the restriction $f \colon \iota^* \operatorname{PSh}(T_{\bullet}) \to \operatorname{PSh}(S_{\bullet})$ along the S-natural transformation $\iota^* T_{\bullet} \to S_{\bullet}$.

While this can be carefully proven by hand, we will instead resort to a sequence of cheap tricks that avoids ever talking about coherences. Namely, by the universal property of S-spaces it suffices to show that f admits a left adjoint and that this preserves terminal objects. Indeed, f admits a pointwise left adjoint given by the left Kan extension functors $(\iota_A)_!$: $PSh(S_A) \to PSh(T_A)$, and each of these preserves terminal objects (as they are simply represented by the respective identity maps). It remains to show that for every $f: A \to B$ in S the Beck–Chevalley transformation $(\iota_A)_! f^* \Rightarrow f^*(\iota_B)_!$ is an equivalence. By full faithfulness of $(\iota_A)_!$ and $(\iota_B)_!$, this is equivalent to demanding that f^* preserves the essential images, for which it is turn enough to show that there is some equivalence $(\iota_A)_! f^* \simeq f^*(\iota_B)_!$. This however follows simply from the equivalences (3) and the fact that ι^* has a left adjoint.

Note that this argument more generally shows that ι^* corresponds under any pair of equivalences to the above restriction functor $\iota^* \operatorname{PSh}(T_{\ell}) \to \operatorname{PSh}(S_{\ell})$.

5. The universal property of equivariant spaces

Building on the above, we will establish a universal property of equivariant unstable homotopy theory in this section. We begin by introducing the object of study:

Construction 5.1. Write **SSet** for the 1-category of simplicial sets. We define a strict 2-functor \bullet -**SSet**: Glo^{op} \rightarrow Cat₁ into the (2, 1)-category of 1-categories as the composite

$$\operatorname{Glo}^{\operatorname{op}} \xrightarrow{B} \operatorname{Grpd}^{\operatorname{op}} \xrightarrow{\operatorname{Fun}(-, \operatorname{\mathbf{SSet}})} \operatorname{Cat}_1$$

This lifts to a functor into the (2, 1)-category RelCat of relative categories, homotopical functors, and natural isomorphisms by equipping G-SSet := Fun(BG, SSet)with the *G*-equivariant weak equivalences, i.e. the class of those maps f such that f^H is a weak equivalence for every subgroup $H \subset G$, or equivalently such that the geometric realization |f| is a *G*-equivariant homotopy equivalence.

Postcomposing with the localization functor RelCat \rightarrow Cat, we obtain a global category \mathcal{G} : Glo^{op} \rightarrow Cat. We call \mathcal{G} the global category of equivariant spaces.

Note that $\underline{\mathscr{G}}(G) \coloneqq \mathscr{G}_G$ is the usual category of *G*-spaces, and for any $\alpha \colon G \to G'$ the structure map $\alpha^* \colon \mathscr{G}_{G'} \to \mathscr{G}_G$ is the usual restriction functor.

Notation 5.2. Recall from Example 3.8 that $\text{Orb} \subset \text{Glo}$ is an example of a cleft category, giving rise to notions of Orb-cocompleteness and Orb-presentability. To emphasize the connections to equivariant homotopy theory obtained in this article we will refer to these as *equivariant cocompleteness* and *equivariant presentability*.

Similarly an Orb-cocontinuous functor $F: \mathcal{C} \to \mathcal{D}$ between equivariantly cocomplete global categories will be called *equivariantly cocontinuous*, and we will write $\underline{\operatorname{Fun}}_{\operatorname{Glo}}^{\operatorname{eq-cc}}(\mathcal{C}, \mathcal{D})$ for the global category $\underline{\operatorname{Fun}}_{\operatorname{Glo}}^{\operatorname{Orb-cc}}(\mathcal{C}, \mathcal{D})$ of equivariantly cocontinuous functor.

We can now state the main result of this section:

Theorem 5.3. The global category $\underline{\mathcal{S}}$ is equivariantly presentable. Moreover, it is the free equivariantly cocomplete global category in the following sense: for any equivariantly cocomplete global category \mathcal{D} , evaluating at the 1-point space provides an equivalence

 $\underline{\operatorname{Fun}}_{\operatorname{Glo}}^{\operatorname{eq-cc}}(\underline{\mathscr{G}}, \mathcal{D}) \xrightarrow{\sim} \mathcal{D}.$

Remark 5.4. While we will not prove this here, we remark that $\underline{\mathscr{P}}$ is in fact even globally presentable: this is a rather straightforward model categorical computation using that the left adjoints $\alpha_1: \mathbf{H}$ -SSet $\rightarrow \mathbf{G}$ -SSet of the restrictions are again homotopical and that the Beck–Chevalley conditions hold on the pointset level by smooth and proper base change.

However, this 'extra presentability' should be considered as an anomaly for two reasons: firstly, it is something rather specific to $\text{Orb} \subset \text{Glo}$, and does not hold for general cleft categories $S \subset T$; secondly, it breaks down as soon as we pass to the semiadditive and stable world, cf. Warning 9.8.

In view of Corollary 4.27, the second half of the theorem can be reformulated as follows:

Theorem 5.5. The essentially unique equivariantly cocontinuous global functor $\underline{\operatorname{Spc}}_{\operatorname{Orb} \triangleright \operatorname{Glo}} \to \underline{\mathscr{G}}$ preserving the terminal object is an equivalence.

In fact, our proof of these two theorems below will proceed the other way round by first establishing an equivalence $\underline{\text{Spc}}_{\text{Orb} \triangleright \text{Glo}} \simeq \underline{\mathscr{S}}$ and then deducing all the remaining statements from this.

5.1. **G-global spaces.** To do so, we begin by recalling the *global category of global spaces*:

Construction 5.6. We write *I* for the category of finite sets and injections and \mathcal{I} for the simplicial category obtained by applying the right adjoint $E: \mathbf{Set} \to \mathbf{SSet}$ of the evaluation functor $ev_0: \mathbf{SSet} \to \mathbf{Set}$ to all hom sets. We write \mathcal{I} -**SSet** for the category of enriched functors $\mathcal{I} \to \mathbf{SSet}$, and for any *G* we denote the category of *G*-objects in \mathcal{I} -**SSet** by $G-\mathcal{I}$ -**SSet**. Analogously to Construction 5.1 these assemble into a functor $\bullet -\mathcal{I}$ -**SSet**: Glo^{op} \to Cat₁.

We can evaluate a G- \mathcal{I} -simplicial set X at any (not necessarily finite) set A via

$$X(A) \coloneqq \operatorname{colim}_{B \subset A \text{ finite}} X(B),$$

and this acquires an action of the symmetric group Σ_A via permuting the factors. In particular, if A is a G-set, then we can equip X(A) with the diagonal G-action, yielding a functor $ev_A: \mathbf{G-I-SSet} \to \mathbf{G-SSet}$.

We now call a map $f: X \to Y$ of $G \cdot \mathcal{I}$ -simplicial sets a G-equivariant weak equivalence if $f(\mathcal{U})$ is a G-equivariant weak equivalence in G-SSet for some, hence any complete G-set universe \mathcal{U} (i.e. a countable G-set into which any other countable G-set embeds equivariantly). Finally, we call f a G-global weak equivalence if $\varphi^* f$ is an H-equivariant weak equivalence of H- \mathcal{I} -simplicial sets for any homomorphism $\varphi: H \to G$ from a finite group to G.

Clearly, for any $\alpha: G \to G'$ the restriction functor $\alpha^*: \mathbf{G'} - \mathcal{I} - \mathbf{SSet} \to \mathbf{G} - \mathcal{I} - \mathbf{SSet}$, sends G'-global weak equivalences to G-global weak equivalences, lifting $\bullet - \mathcal{I} - \mathbf{SSet}$ to $\mathrm{Glo}^{\mathrm{op}} \to \mathrm{RelCat}$. Localizing, we then again get a global category, which we denote by $\mathcal{P}^{\mathrm{gl}}$.

We write $\mathscr{P}_{G}^{\mathrm{gl}} \coloneqq \underline{\mathscr{P}}^{\mathrm{gl}}(G)$ and call it the *category of G-global spaces*. Note that [CLL23] uses the notation $\underline{\mathscr{P}}_{\mathcal{I}}^{\mathrm{gl}}$, for the above global category and reserves $\underline{\mathscr{P}}^{\mathrm{gl}}$ for a different, but equivalent, model based on actions of a certain 'universal finite group.' In the present paper, however, we will only be interested in the above approach.

Remark 5.7. The *G*-global weak equivalences are part of several model structures on G- \mathcal{I} -SSet, see [Len20, Section 1.4]. We will not need them explicitly in this section, but they will make an indirect appearance in Section 7.

Our interest in \mathscr{S}^{gl} comes from the following 'global' version of Theorem 5.5:

Theorem 5.8 (See [CLL23, Theorem 3.3.1 and Corollary 3.2.5]). The global category $\underline{\mathscr{P}}^{gl}$ is globally presentable. The unique globally cocontinuous functor $\underline{\operatorname{Spc}}_{\operatorname{Glo}} \rightarrow \underline{\mathscr{P}}^{gl}$ preserving the terminal object is an equivalence.

On the other hand we can relate the global categories of global and equivariant spaces as follows:

Lemma 5.9. There exists a global functor const: $\underline{\mathscr{G}} \to \underline{\mathscr{G}}^{gl}$ with the following properties:

- (1) const is fully faithful and sends the terminal object of \mathscr{G} to the terminal object of $\mathscr{G}^{\mathrm{gl}}$,
- (2) it admits an Orb-right adjoint $\mathbf{R} \operatorname{ev}: \underline{\mathscr{S}}^{\mathrm{gl}}|_{\mathrm{Orb}} \to \underline{\mathscr{S}}|_{\mathrm{Orb}}.$

Once the above two theorems have been established, we will see that this adjunction is actually uniquely described by the requirement that the left adjoint preserve the terminal object.

Proof. The functor const: G-SSet $\rightarrow G$ - \mathcal{I} -SSet is homotopical and strictly natural in G, so it induces a global functor const: $\mathcal{I} \rightarrow \mathcal{I}^{\text{gl}}$. By [Len20, Corollary 1.4.56] this functor is fully faithful, and it admits a pointwise right adjoint (given by the right derived functor of ev_{\emptyset} : G- \mathcal{I} -SSet $\rightarrow G$ -SSet).

To complete the proof, it only remains to establish the Beck–Chevalley condition for the pointwise right adjoint, or equivalently that for any injective $\alpha: G \to G'$ the mate transform $\alpha_1 \circ \text{const} \Rightarrow \text{const} \circ \alpha_1$ is an equivalence of functors $\mathscr{P}_G \to \mathscr{P}_{G'}^{\text{gl}}$. But indeed, this holds on the pointset level by direct inspection, so the claim follows as $\alpha_1: \mathbf{G}$ - \mathcal{I} -SSet $\to \mathbf{G'}$ - \mathcal{I} -SSet is homotopical by [Len20, Lemma 1.4.42] while $\alpha_1: \mathbf{G}$ -SSet $\to \mathbf{G'}$ -SSet is so by (a well-known special case of) Proposition 1.1.18 of *op. cit.*

Proof of Theorems 5.3 and 5.5. Lemma 5.9 provides a fully faithful global functor const: $\mathcal{P} \to \mathcal{P}^{\text{gl}}$. Now the right hand side is globally cocomplete, hence in particular equivariantly cocomplete. Moreover, the essential image of the functor const is closed under all equivariant colimits as it is an Orb-left adjoint. Thus, also \mathcal{P} is equivariantly cocomplete.

Appealing to Corollary 4.27, we therefore see that there is an essentially unique equivariantly cocontinuous functor $F: \underline{\text{Spc}}_{\text{Orb} \triangleright \text{Glo}} \rightarrow \underline{\mathscr{I}}$ preserving the terminal object, and we claim that this is an equivalence. For this, we consider the diagram



of global functors where the lower equivalence is as in Theorem 5.8; both paths through this diagram are equivariantly cocontinuous and preserve the terminal object, so this commutes by the universal property of $\underline{\text{Spc}}_{\text{Orb} \triangleright \text{Glo}}$. Moreover, the vertical arrows are fully faithful by Theorem 4.24 and Lemma 5.9, respectively. It follows that also F is fully faithful.

To see that each $F_G: \underline{\operatorname{Spc}}_{\operatorname{Orb}}(G) \to \mathscr{S}_G$ is essentially surjective, we observe that F_G is a fully faithful left adjoint, so that its essential image is closed under all colimits. On the other hand, by Elmendorf's Theorem [Elm83] (or simply looking at the standard generating cofibrations), \mathscr{S}_G is generated under colimits by the G/H's for subgroups $H \subset G$, so it is enough that each G/H is contained in the essential image. However, $G/H = i_!(*)$, where $i: H \hookrightarrow G$ denotes the inclusion, so $F_G(i_!(*)) \simeq i_!F_H(*) \simeq i_!(*) \simeq G/H$ by the defining properties of F.

Finally, the universal property of $\underline{\mathscr{S}}$ follows from combining the above with Corollary 4.27.

5.2. The universal property of G-spaces. Fix a finite group G. Using the previous theorem, we can now give a model categorical description of the universal G-presentable G-category (Example 2.3).

Lemma 5.10. The assignment $\operatorname{Orb}_{/G} \to \operatorname{Orb}_{G}$ sending an object $\varphi \colon H \to G$ to $G/\operatorname{im}(\varphi)$ and a morphism

$$\begin{array}{c} H \longrightarrow K \\ \swarrow & \swarrow \\ \varphi \searrow & \overset{\mathfrak{F}}{g} \swarrow \\ G & \checkmark \\ \psi \end{array} \tag{6}$$

to the map $G/\operatorname{im}(\varphi) \to G/\operatorname{im}(\psi)$ given by right multiplication with g, is well-defined and an equivalence of categories.

Proof. One easily checks that this is well-defined and an essentially surjective functor. To see that it is fully faithful, we may for ease of notation restrict to the essentially wide subcategory of Orb_G spanned by the honest inclusions $H \hookrightarrow G$. We now observe that the map $H \to K$ in a morphism (6) is necessarily given by $h \mapsto ghg^{-1}$; conversely, $g \in G$ defines a map $(H \hookrightarrow G) \to (K \hookrightarrow G)$ if and only if $ghg^{-1} \in K$ for every $h \in H$, i.e. $[g] \in (G/K)^H$. On the other hand, 2-cells $g \Rightarrow g'$ are in bijection with elements $k \in K$ such that gk=g'. Altogether, we see that $\operatorname{Hom}(H \hookrightarrow G, K \hookrightarrow G)$ is discrete and equivalent to $(G/K)^H$ by sending (6) to the class of g.

The claim then follows by observing that also $\text{Hom}(G/H, G/K) \cong (G/K)^H$ via evaluation at the coset of the identity. \Box

Construction 5.11. We write v_G for the composite $\operatorname{Orb}_G \simeq \operatorname{Orb}_{/G} \xrightarrow{\pi_G} \operatorname{Orb} \hookrightarrow$ Glo. Restricting along v_G then yields a functor v_G^* : $\operatorname{Cat}_{\operatorname{Glo}} \to \operatorname{Cat}_{\operatorname{Orb}_G}$ sending a global category to its 'underlying *G*-category.'

Theorem 5.12. (1) The G-category $v_G^* \mathcal{G}$ is G-presentable, and the unique left

adjoint $\underline{\operatorname{Spc}}_{\operatorname{Orb}_G} \to v_G^* \underline{\mathscr{G}}$ preserving the terminal object is an equivalence. (2) For any G-cocomplete \mathcal{D} , evaluation at the terminal object defines an equivalence $\underline{\operatorname{Fun}}_{\operatorname{Orb}_G}^{G-cc}(v_G^* \underline{\mathscr{G}}, \mathcal{D}) \simeq \mathcal{D}$.

More informally, $v_G^* \underline{\mathscr{S}}$ is given by sending G/H to the category \mathscr{S}_H of H-spaces, $G/H \twoheadrightarrow G/K$ to the restriction $\mathscr{S}_K \to \mathscr{S}_H$ for any $K \supset H$, and $-\cdot g \colon G/H \to G/H$ for an element g of the normalizer $N_G H$ to the conjugation $c_q^* \colon \mathscr{S}_H \to \mathscr{S}_H$.

Proof. It suffices to construct an equivalence $v_G^* \underline{\mathscr{G}} \simeq \underline{\operatorname{Spc}}_{\operatorname{Orb}_G}$; the theorem will then follow from the universal property of the right hand side (Theorem 2.24).

But by Theorem 5.5 we have an equivalence of global categories $\underline{\mathscr{S}} \simeq \underline{\operatorname{Spc}}_{\operatorname{Orb} \triangleright \operatorname{Glo}}$, and hence in particular an equivalence $\underline{\mathscr{S}}|_{\operatorname{Orb}} \simeq \underline{\operatorname{Spc}}_{\operatorname{Orb}}$ of Orb-categories. To finish the proof it suffices now to observe that for any small T and any $A \in T$, there is an equivalence $\pi_A^* \underline{\operatorname{Spc}}_T \simeq \underline{\operatorname{Spc}}_{T/A}$ by [MW21, Lemma 7.1.9].

Remark 5.13. Evaluating at $H \subset G$, the above in particular shows $\mathscr{S}_H \simeq PSh(Orb_G)_{(G/H)} \simeq PSh(Orb_H)$. In this sense, the theorems above can be viewed as a 'coherent' version of the classical Elmendorf Theorem [Elm83], additionally taking into account the restriction functors as well as all higher structure between them.

6. The semiadditive story

We continue to fix a cleft category $\iota: S \hookrightarrow T$. In this section we will give a description of the universal S-presentable T-category that is in addition *semiadditive* in a suitable sense.

6.1. *P*-semiadditivity and *P*-commutative monoids. We begin with a recollection of the relevant material from [CLL23]. Throughout, we fix an *atomic orbital subcategory* $P \subset T$ in the sense of Definition 3.34.

Construction 6.1. We write $\mathbb{F}_T \subset PSh(T)$ for the finite coproduct completion of T and \mathbb{F}_T^P for the finite coproduct completion of P, viewed as a subcategory of \mathbb{F}_T . We define a T-subcategory $\underline{\mathbb{F}}_T^P \subset \underline{Spc}_T$ by letting $\underline{\mathbb{F}}_T^P(B)$ be the full subcategory of $PSh(T)_{/B}$ spanned by objects of the form $(p_i): \prod_{i=1}^n A_i \to B$ such that each morphism $p_i: A_i \to B$ is in P; put differently, this is the slice $(\mathbb{F}_T^P)_{/B}$. Note that by atomic orbitality of $P, \underline{\mathbb{F}}_T^P$ indeed forms a T-subcategory of \underline{Spc}_T .

Definition 6.2. We say a *T*-category has *finite P*-products or finite *P*-coproducts if it has $\underline{\mathbb{F}}_{T}^{P}$ -limits or $\underline{\mathbb{F}}_{T}^{P}$ -colimits, respectively, in the sense of Definition 2.16.

Definition 6.3. A *T*-category C is called *pointed* if it factors through the non-full subcategory Cat_{*} \subset Cat of categories with zero objects and functors preserving the zero object.

Construction 6.4. Let \mathcal{C} be a pointed *T*-category which has finite *P*-coproducts, and let \mathcal{D} be a *T*-category with finite *P*-products. For any functor $F: \mathcal{C} \to \mathcal{D}$ and any $p: A \to B$ in \mathbb{F}_T^P , [CLL23, Construction 4.6.1] defines a *relative norm map*

$$\operatorname{Nm}_p^F : F_B \circ p_! \implies p_* \circ F_A$$

If C also has finite *P*-products, we write $\operatorname{Nm}_p: p_! \Rightarrow p_*$ for the relative norm map of $\operatorname{id}_{\mathcal{C}}$, and simply call it the *norm map*.

Definition 6.5. A *T*-category C is called *P*-semiadditive if it is pointed, has finite *P*-products and *P*-coproducts, and they agree in the sense that for every p in \mathbb{F}_T^P the norm map $\operatorname{Nm}_p: p_! \Rightarrow p_*$ is an equivalence.

Example 6.6. When $P \subset T$ equals $Orb \subset Glo$, the previous definition specializes to the notion of *equivariant semiadditivity* from [CLL23].

Example 6.7. When $P \subset T$ equals $\operatorname{Orb}_G \subset \operatorname{Orb}_G$, the notion of semiadditivity obtained agrees with *G*-semiadditivity as defined in [Nar16], see [CLL23, Proposition 4.6.4].

Definition 6.8. Let $F: \mathcal{C} \to \mathcal{D}$ be a functor of *T*-categories, such that \mathcal{C} is pointed and has finite *P*-coproducts, while \mathcal{D} has finite *P*-products. Then *F* is called *P*semiadditive if it sends *P*-coproducts to *P*-products in the sense that the relative norm map $\operatorname{Nm}_p^F: Fp_! \Rightarrow p_*F$ is an equivalence for every p in \mathbb{F}_T^P .

Definition 6.9. We write $\operatorname{Cat}_T^{P \oplus} \subset \operatorname{Cat}_T$ for the non-full subcategory of *P*-semiadditive categories and *P*-semiadditive *T*-functors.

By [CLL23, Proposition 4.6.14], the morphisms of $\operatorname{Cat}_T^{P-\oplus}$ are equivalently the $\underline{\mathbb{F}}_T^P$ -cocontinuous or $\underline{\mathbb{F}}_T^P$ -continuous *T*-functors.

Remark 6.10. Similarly to Warning 4.10, having finite P-(co)products is not just a property of the underlying P-category. On the other hand, if a T-category either has finite P-coproducts or finite P-products, then it is P-semiadditive if and only if its underlying P-category is so [CLL23, Lemma 4.5.2 and Lemma 4.6.4].

Definition 6.11. In the above situation, we write $\underline{\operatorname{Fun}}_T^{P-\oplus}(\mathcal{C},\mathcal{D})$ for the parametrized subcategory spanned in degree $X \in \operatorname{PSh}(T)$ by the *P*-semiadditive functors $\mathcal{C} \to \underline{\operatorname{Fun}}_T(X, \mathcal{D}).$

Note that the above is indeed a T-subcategory by [CLL23, Corollary 4.6.10].

Definition 6.12. We define $\underline{\mathbb{F}}_{T,*}^P$, the *T*-category of *finite pointed P-sets*, to be the essential image of $\underline{\mathbb{F}}_T^P$ under the functor $(-)_+: \underline{\operatorname{Spc}}_T \to \underline{\operatorname{Spc}}_{T,*}$ which adds a disjoint basepoint.

Definition 6.13. Given a *T*-category \mathcal{C} with *P*-products we define $\underline{\mathrm{CMon}}^{P}(\mathcal{C})$, the *T*-category of *P*-commutative monoids in \mathcal{C} , as $\underline{\mathrm{Fun}}_{T}^{P-\oplus}(\underline{\mathbb{F}}_{T,*}^{P}, \mathcal{C})$. If $\mathcal{C} = \underline{\mathrm{Spc}}_{T}$, we write $\mathrm{CMon}_{T}^{P} \coloneqq \mathrm{CMon}_{P}(\mathrm{Spc}_{T})$.

This construction enjoys several universal properties. To express them we introduce:

Construction 6.14. Let \mathcal{C} have finite P-products. Evaluation at the global section $S^0 := (\mathrm{id})_+ \in \mathbb{E}_{T^*}^P(1) \subset (\mathrm{PSh}(T)_{/1})_*$ gives a forgetful functor

$$\mathbb{U}: \mathrm{CMon}_P(\mathcal{C}) \to \mathcal{C}$$
.

Construction 6.15. Assume \mathcal{C} is presentable. Then $\underline{\mathrm{CMon}}^{P}(\mathcal{C}) \hookrightarrow \underline{\mathrm{Fun}}_{T}(\underline{\mathbb{F}}_{T,*}^{P}, \mathcal{C})$ admits a left adjoint $L^{P-\oplus}$ by [CLL23, Proposition 4.6.15]. In particular, the functor \mathbb{U} from the previous construction has a left adjoint given by composing the left Kan extension functor $(S^{0})_{!}: \mathcal{C} \to \mathrm{Fun}_{T}(\mathbb{F}_{T,*}^{P}, \mathcal{C})$ [MW21, Corollary 6.3.7] with $L^{P-\oplus}$.

Theorem 6.16 (See [CLL23, Theorem 4.8.10]). Let C be a T-category with finite P-products. The functor $\mathbb{U}: \underline{\mathrm{CMon}}^{P}(\mathcal{C}) \to \mathcal{C}$ exhibits $\underline{\mathrm{CMon}}^{P}(\mathcal{C})$ as the P-semiadditive envelope of C in the following sense: for every P-semiadditive T-category \mathcal{T} post-composition with \mathbb{U} induces an equivalence

$$\underline{\operatorname{Fun}}^{P-\times}(\mathcal{T},\mathbb{U})\colon\underline{\operatorname{Fun}}^{P-\oplus}(\mathcal{T},\underline{\operatorname{CMon}}^{P}(\mathcal{C}))\xrightarrow{\sim}\underline{\operatorname{Fun}}^{P-\times}(\mathcal{T},\mathcal{C}).$$

Suppose now that C is moreover presentable. Then the left adjoint \mathbb{P} of \mathbb{U} exhibits $\underline{\mathrm{CMon}}^{P}(\mathcal{C})$ as the presentable P-semiadditive completion of C in the following sense: for any presentable P-semiadditive T-category \mathcal{T} precomposition with \mathbb{P} yields an equivalence

$$\underline{\operatorname{Fun}}^{T\operatorname{-cc}}(\mathbb{P},\mathcal{T})\colon\underline{\operatorname{Fun}}^{T\operatorname{-cc}}(\underline{\operatorname{CMon}}^{P}(\mathcal{C}),\mathcal{T})\xrightarrow{\sim}\underline{\operatorname{Fun}}^{T\operatorname{-cc}}(\mathcal{C},\mathcal{T}).$$

Theorem 6.17 (See [CLL23, Theorem 4.8.11]). The *T*-category \underline{CMon}_T^P is *P*-semiadditive and *T*-presentable. Moreover, it has the following universal property: for any locally small *T*-cocomplete *P*-semiadditive \mathcal{D} , evaluation at $\mathbb{P}(*) \simeq L^{P-\oplus}y(S^0)$ induces an equivalence

$$\underline{\operatorname{Fun}}^{T\operatorname{-cc}}(\underline{\operatorname{CMon}}_T^P, \mathcal{D}) \xrightarrow{\sim} \mathcal{D}.$$

6.2. The free *P*-semiadditive *S*-presentable *T*-category. Let now $P \subset S$ be atomic orbital *as a subcategory of T*. As the main results of this section, we will prove the following 'partially presentable' versions of the previous theorem:

Theorem 6.18. There is a unique S-cocontinuous functor $\iota_! : \underline{\mathrm{CMon}}_S^P \to \iota^* \underline{\mathrm{CMon}}_T^P$ sending $\mathbb{P}(*)$ to $\mathbb{P}(*)$. Moreover, $\iota_!$ is fully faithful, and it sits in a sequence of Sadjoints $\iota_! \dashv \iota^* \dashv \iota_*$.

Theorem 6.19. The S-functor $\iota_!$ uniquely extends to a T-functor $\underline{\mathrm{CMon}}_{S \triangleright T}^P \to \underline{\mathrm{CMon}}_{T}^P$. Moreover, $\underline{\mathrm{CMon}}_{S \triangleright T}^P$ is S-presentable, P-semiadditive, and it has the following universal property: for any S-cocomplete P-semiadditive T-category \mathcal{D} evaluation at a certain global section $\mathbb{P}(*)$ defines an equivalence

$$\underline{\operatorname{Fun}}_{T}^{S\operatorname{-cc}}(\underline{\operatorname{CMon}}_{S\triangleright T}^{P},\mathcal{D}) \xrightarrow{\sim} \mathcal{D}.$$

Remark 6.20. Note that in contrast to Theorem 6.17 there is no local smallness condition on \mathcal{D} here anymore; in particular, for S = T this improves upon our result in [CLL23].

The proof of these two theorems will occupy the rest of this section.

6.2.1. Construction of the universal example. As our first step, we will construct some *P*-semiadditive *S*-cocomplete *T*-category C with the correct universal property; more precisely, we want to show:

Proposition 6.21. Write $\mathcal{C} \subset \underline{CMon}_T^P$ for the full subcategory generated under S-colimits by $\mathbb{P}(*)$. Then \mathcal{C} is a \overline{P} -semiadditive S-cocomplete T-category, and for any other such \mathcal{D} evaluating at $\mathbb{P}(*)$ yields an equivalence $\operatorname{Fun}_{T^{-\operatorname{cc}}}^{T^{-\operatorname{cc}}}(\mathcal{C}, \mathcal{D}) \simeq \mathcal{D}$.

The basic idea will be to deduce the universal property of C from the one for \underline{CMon}_T^P . However, we only understand maps from the latter to T-cocomplete categories, so we will have to embed a general P-semiadditive S-cocomplete T-category into a T-cocomplete one first. However, in this process some size issues pop up; to avoid any ambiguities, we will therefore for once refer back to our chosen Grothendieck universes explcitly:

Lemma 6.22. Let \mathcal{D} be an S-cocomplete P-semiadditive \mathfrak{V} -small T-category. Then there exists a \mathfrak{W} -small, locally \mathfrak{V} -small P-semiadditive T-category \mathcal{D}' having all \mathfrak{V} small T-colimits together with a fully faithful functor $j: \mathcal{D} \to \mathcal{D}'$ preserving all \mathfrak{U} -small S-colimits.

Proof. Write \underline{SPC}_T for the \mathfrak{W} -small and locally \mathfrak{V} -small *T*-category of \mathfrak{V} -small spaces. Then the Yoneda embedding $\mathcal{D}^{\mathrm{op}} \to \underline{\mathrm{Fun}}_T(\mathcal{D}, \underline{SPC}_T)$ actually lands in the full subcategory $\mathcal{E} := \underline{\mathrm{Fun}}_T^{P-\times}(\mathcal{D}, \underline{SPC}_T)$ by [MW21, Corollary 4.4.8]. Now \mathcal{E} is closed under all \mathfrak{V} -small *T*-limits and the Yoneda embedding preserves all \mathfrak{U} -small *S*-limits by Proposition 4.4.7 of *op. cit.* Thus, the Yoneda embedding $\mathcal{D}^{\mathrm{op}} \to \mathcal{E}$ is a fully faithful functor into a category with all \mathfrak{V} -small *T*-limits preserving \mathfrak{U} -small *S*-limits. Moreover, as \mathcal{D} is *P*-semiadditive, so is \mathcal{E} by [CLL23, Proposition 4.6.13], and hence so is $\mathcal{E}^{\mathrm{op}}$ by Lemma 4.5.4 of *op. cit.* The dual $\mathcal{D} \to \mathcal{E}^{\mathrm{op}}$ of the Yoneda embedding therefore has the required properties. \Box

As the lemma requires us to pass a larger universe, it is not clear a priori whether $\underline{\mathrm{CMon}}_T^P$ still has the correct universal property (we will see a posteriori that, as a matter of fact, it does). For locally small \mathcal{D} , one might try to avoid this issue by considering $\underline{\mathrm{Fun}}_T^{P-\times}(\mathcal{D},\underline{\mathrm{Spc}}_T)$ instead or even by just closing up the Yoneda image under \mathfrak{U} -small T-limits in there, but even in this case it is not clear whether the result is still locally small—and said local smallness was crucial in the proof of Theorem 6.16 given in [CLL23], which relied on the Special Adjoint Functor Theorem. Accordingly, we will have to consider a \mathfrak{W} -version $\underline{\mathrm{CMON}}_T^P$ of $\underline{\mathrm{CMon}}_T^P$. The crucial technical lemma to relate these two to each other will be the following:

Lemma 6.23. The functor $\underline{\operatorname{Fun}}_T(\underline{\mathbb{F}}_{T,*}^P, \underline{\operatorname{SPC}}_T) \to \underline{\operatorname{CMON}}_T^P \coloneqq \underline{\operatorname{Fun}}_T^{P-\oplus}(\underline{\mathbb{F}}_{T,*}^P, \underline{\operatorname{SPC}}_T)$ left adjoint to the inclusion preserves $\underline{\operatorname{Fun}}_T(\underline{\mathbb{F}}_{T,*}^P, \underline{\operatorname{Spc}}_T)$.

Accordingly, it restricts to a left adjoint $\underline{\operatorname{Fun}}_T(\underline{\mathbb{F}}_{T,*}^P, \underline{\operatorname{Spc}}_T) \to \underline{\operatorname{CMon}}_T^P$ of the inclusion, and there is no harm in denoting both the localization functor in ordinary T-categories and in large T-categories by the same symbol $L^{P-\oplus}$.

Proof. Let $A \in T$ be arbitrary. [CLL23, Remark 2.2.14] provides for any \mathfrak{W} -small category \mathcal{E} a natural equivalence

$$\underline{\operatorname{Fun}}_{T}(\underline{\mathbb{F}}_{T,*}^{P}, \underline{\mathcal{E}}_{T})(A) \simeq \operatorname{Fun}(\int \underline{\mathbb{F}}_{T,*}^{P} \times \underline{A}, \mathcal{E})$$
(7)

where \int denotes the usual Grothendieck construction over T^{op} and $(-)_T$ denotes the T-category of T-objects (Example 2.5). On the other hand, [CLL23, Remark 4.9.9] characterizes the essential image of $\underline{\text{CMon}}^P(\underline{\mathcal{E}}_T)(A)$ under this—it consists precisely of the functors $F^{\dagger}: \int \underline{\mathbb{F}}_{T,*}^P \times \underline{A} \to \mathcal{E}$ satisfying the following:

(1) For every $f: B \to A$ in T the restriction of F^{\dagger} to the non-full subcategory $\underline{\mathbb{F}}_{T,*}^{P}(B) \simeq \underline{\mathbb{F}}_{T,*}^{P}(B) \times \{f\} \subset \int \underline{\mathbb{F}}_{T,*}^{P} \times \underline{A}$ is semiadditive in the usual sense.

(2) For every $p: B \to B'$ in P and $f: B' \to A$ in T a certain natural Segal map $F^{\dagger}(B', B_+, f) \to F^{\dagger}(B, B_+, pf)$ is an equivalence; here we as usual denote objects in the Grothendieck construction by triples $(C \in T^{\text{op}}, X \in$ $\underline{\mathbb{F}}_{T,*}^{P}(C), g \in \operatorname{Hom}(C,A)).$

Specializing to $\mathcal{E} = SPC$ and writing y for the (non-parametrized) Yoneda embedding of $\int \underline{\mathbb{F}}_{T,*}^P \times \underline{A}$, we see that F is P-semiadditive if and only if F^{\dagger} is local with respect to the set U^{\dagger} made up of suitable maps

- (1) $\emptyset \to y(B, *, f)$ for every map $f \colon B \to A$ in T (so that the restriction to each $\underline{\mathbb{F}}_{T,*}^{P}(B) \times \{f\}$ is pointed) (2) $y(B, X_+, f) \coprod y(B, Y_+, f) \to y(B, X_+ \lor Y_+, f)$ for all $f: B \to A$ in T and
- $X_+, Y_+ \in \underline{\mathbb{F}}_{T,*}^P(B)$ (so that each restriction sends coproducts to products)
- (3) $y(B, B_+, pf) \rightarrow y(B', B_+, f)$ for every $p: A \rightarrow B$ in P and $f: B' \rightarrow B$ in T (ensuring that the Segal maps are equivalences).

Transporting U^{\dagger} along the equivalence (7), we then get a set U such that F is P-semiadditive if and only if it is U-local. By direct inspection, each map in U^{\dagger} actually lives in Fun $(\int \underline{\mathbb{F}}_{T,*}^{P} \times \underline{A}, \operatorname{Spc})$ (as opposed to functors into SPC). By naturality of (7) we can therefore also take the set U to consist of maps in $\underline{\operatorname{Fun}}_T(\underline{\mathbb{F}}_{T,*}^P,\underline{\operatorname{Spc}}_T)(A)$. We now write U_1 for the strongly saturated class generated by U in $\underline{\operatorname{Fun}}_{T}(\underline{\mathbb{F}}_{T,*}^{P}, \underline{\operatorname{Spc}}_{T})(A)$ (with respect to \mathfrak{U} -small colimits) and U_2 for the strongly saturated class generated in $\underline{\operatorname{Fun}}_T(\underline{\mathbb{F}}_{T,*}^P,\underline{\operatorname{SPC}}_T)(A)$ (with respect to \mathfrak{V} -small colimits). Clearly, $U_1 \subset U_2$.

By [Lur09, Proposition 5.4.5.15], there exists for any $F \in \underline{\operatorname{Fun}}_T(\underline{\mathbb{F}}_{T,*}^P, \underline{\operatorname{Spc}}_T)(A)$ a map $\eta: F \to F'$ into a U-local F' such that $\eta \in U_1$. But then also $\eta \in U_2$, so it qualifies as the adjunction unit in the larger category by the same result, and in particular the image of F under the localization functor to $\underline{\mathrm{CMON}}_{T}^{P}(A)$ is equivalent to $F' \in \underline{\operatorname{Fun}}_T(\underline{\mathbb{F}}_{T*}^P, \underline{\operatorname{Spc}}_T)(A)$ as desired.

Proof of Proposition 6.21. By the previous lemma, $\mathrm{CMon}_T^P \subset \mathrm{CMON}_T^P$ contains $\mathbb{P}(*)$. We claim that it is closed under \mathfrak{U} -small T-colimits: indeed, fiberwise colimits in $\underline{\mathrm{CMON}}_T^P$ are formed by first computing them in $\underline{\mathrm{Fun}}_T(\underline{\mathbb{F}}_{T,*}^P, \underline{\mathrm{SPC}}_T)$ and then reflecting via $L^{P-\oplus}$, so $\underline{\mathrm{CMon}}_T^P$ is closed under \mathfrak{U} -small fiberwise colimits by the lemma, and similarly the functor $f_!: \operatorname{CMON}^P_T(A) \to \operatorname{CMON}^P_T(B)$ factors for any map $f: A \to B$ in T as

$$\underline{\mathrm{CMON}}_{T}^{P}(A) \hookrightarrow \underline{\mathrm{Fun}}_{T}(\underline{\mathbb{F}}_{T,*}^{P}, \underline{\mathrm{SPC}}_{T})(A) \xrightarrow{f_{!}} \underline{\mathrm{Fun}}_{T}(\underline{\mathbb{F}}_{T,*}^{P}, \underline{\mathrm{SPC}}_{T})(B) \xrightarrow{L^{P-\oplus}} \underline{\mathrm{CMON}}_{T}^{P}(B)$$

In particular, $\mathcal{C} \subset \underline{\mathrm{CMon}}_T^P$ is also closed under all \mathfrak{U} -small S-colimits in $\underline{\mathrm{CMON}}_T^P$, and thus under finite P-coproducts. As CMON_T^P is P-semiadditive, \mathcal{C} is then also closed under finite P-products and moreover P-semiadditive itself. By [CLL23, Corollary 4.7.8° there is then a unique functor $j: (\underline{\mathbb{F}}_{T,*}^{P})^{\mathrm{op}} \to \mathcal{C}$ that preserves finite P-products and sends S^0 to $\mathbb{P}(*)$. We moreover write k for the inclusion $\mathcal{C} \hookrightarrow \underline{\mathrm{CMON}}_{T}^{P}$; then k preserves \mathfrak{U} -small S-colimits, and hence in particular finite P-(co)products.

If now \mathcal{D}' is a \mathfrak{W} -small and locally \mathfrak{V} -small P-semiadditive T-category which admits all \mathfrak{V} -small *T*-colimits, then [MW21, Theorem 6.3.5 and Corollary 6.3.7] show that the left Kan extension functors $j_!: \underline{\operatorname{Fun}}_T((\underline{\mathbb{F}}_{T,*}^P)^{\operatorname{op}}, \mathcal{D}') \to \underline{\operatorname{Fun}}_T(\mathcal{C}, \mathcal{D}')$ and $k_!: \underline{\operatorname{Fun}}_T(\mathcal{C}, \mathcal{D}') \to \underline{\operatorname{Fun}}_T(\underline{\operatorname{CMON}}_T^P, \mathcal{D}')$ exist and that the latter is fully faithful, while [CLL23, Proposition 4.8.12] shows that $k_! j_!$ restricts to an equivalence $\underline{\operatorname{Fun}}_T^{P-\times}(\underline{\mathbb{F}}_{T,*}^P, \mathcal{D}') \simeq \underline{\operatorname{Fun}}_T^{T-\operatorname{CC}}(\underline{\operatorname{CMON}}_T^P, \mathcal{D}')$, where the right hand side denotes functors preserving all \mathfrak{V} -small T-colimits.

We claim that $j_!$ restricts to an equivalence $\underline{\operatorname{Fun}}_T^{P-\times}((\underline{\mathbb{F}}_{T,*}^P)^{\operatorname{op}}, \mathcal{D}') \simeq \underline{\operatorname{Fun}}_T^{S-\operatorname{cc}}(\mathcal{C}, \mathcal{D}')$, for which it is enough by 2-out-of-3 that for any $A \in T$ and any finite P-product preserving $f: (\underline{\mathbb{F}}_{T,*}^P)^{\operatorname{op}} \to \underline{\operatorname{Fun}}_T(\underline{A}, \mathcal{D}')$ the Kan extension $j_! f$ preserves \mathfrak{U} -small S-colimits, and that conversely any S-cocontinuous functor arises this way.

For the first statement, it is enough to observe that $k_! j_! f : \underline{\text{CMON}}_T^P \to \underline{\text{Fun}}_T(A, \mathcal{D}')$ is in particular S-cocontinuous, whence so is $k^* k_! j_! f \simeq j_! f$ as k is S-cocontinuous. Conversely, if $F : \mathcal{C} \to \underline{\text{Fun}}_T(A, \mathcal{D}')$ is S-cocontinuous, then its restriction to $(\underline{\mathbb{F}}_{T,*}^P)^{\text{op}}$ preserves finite P-products. Consider now the subcategory of \mathcal{C} of all objects for which the counit $\varepsilon : j_! j^* F \to F$ is an equivalence. Then this is closed under \mathfrak{U} -small S-colimits as both sides are S-cocontinuous, and it moreover contains $\mathbb{P}(*)$ as the unit $j^* F \to j^* j_! j^* F$ is an equivalence by full faithfulness of $j_!$. The claim then follows as \mathcal{C} is generated by $\mathbb{P}(*)$ under \mathfrak{U} -small S-colimits by construction.

Let now \mathcal{D} be a *P*-semiadditive *S*-cocomplete *T*-category, and use Lemma 6.22 to obtain an *S*-cocontinuous embedding into a large \mathcal{D}' as above. Then the Kan extension $j_!: \underline{\operatorname{Fun}}_T^{P^-\times}((\underline{\mathbb{F}}_{T,*}^P)^{\operatorname{op}}, \mathcal{D}') \to \underline{\operatorname{Fun}}_T^{T^-\operatorname{CC}}(\mathcal{C}, \mathcal{D}')$ restricts to $\underline{\operatorname{Fun}}_T^{P^-\times}((\underline{\mathbb{F}}_{T,*}^P)^{\operatorname{op}}, \mathcal{D}) \to$ $\underline{\operatorname{Fun}}_T(\mathcal{C}, \mathcal{D})$ as $\mathcal{D} \subset \mathcal{D}'$ is closed under \mathfrak{U} -small *S*-colimits and \mathcal{C} is generated under them by $\mathbb{P}(*)$. In particular, $j^*: \underline{\operatorname{Fun}}_T^{S\text{-cc}}(\mathcal{C}, \mathcal{D}) \to \underline{\operatorname{Fun}}_T^{P^-\times}((\underline{\mathbb{F}}_{T,*}^P)^{\operatorname{op}}, \mathcal{D})$ is an equivalence. The proposition follows as the right hand side is further equivalent to \mathcal{D} via evaluation at S^0 by [CLL23, Corollary 4.7.8^{\operatorname{op}}].

Remark 6.24. Running the same argument in an even larger universe \mathfrak{X} , the above proof (without the penultimate paragraph) shows that $\underline{\operatorname{Fun}}_T^{T-\operatorname{cc}}(\underline{\operatorname{CMon}}_T^P, \mathcal{D}) \simeq \mathcal{D}$ via evaluation at $\mathbb{P}(*)$ for any \mathfrak{W} -small *P*-semiadditive *T*-category \mathcal{D} with \mathfrak{U} -small *S*-colimits.

As an upshot, we can now stop thinking about universes.

6.2.2. Relation to $\underline{\mathrm{CMon}}_{S}^{P}$. Next, we want to understand the underlying S-category of the universal P-semiadditive S-cocomplete T-category \mathcal{C} constructed above. As in the unstable situation this will be some formal Yoneda yoga.

Proposition 6.25. The adjunction ι^* : $\operatorname{Cat}_T \rightleftharpoons \operatorname{Cat}_S : \iota_*$ restricts to give adjunctions $\operatorname{Cat}_T^{P-\oplus} \rightleftharpoons \operatorname{Cat}_S^{P-\oplus}$, $\operatorname{Cat}_T^{P-\oplus,S-\operatorname{cc}} \rightleftharpoons \operatorname{Cat}_S^{P-\oplus,S-\operatorname{cc}}$, and $\operatorname{Pr}_T^{S,P-\oplus} \rightleftharpoons \operatorname{Pr}_S^{S,P-\oplus}$.

Proof. If will suffice to prove the first statement; the second one will then follow from Corollary 4.20.

By Lemma 3.14, $\iota_1: \operatorname{PSh}(S) \to \operatorname{PSh}(T)$ preserves pullbacks, while its right adjoint restricts to $\mathbb{F}_T^P \to \iota_* \mathbb{F}_S^P$ by Lemma 4.17 for $\mathbf{V} = \mathbb{F}_T^P$. [CLL23, Lemma 4.6.5] therefore shows that both ι^* and ι_* restrict accordingly. Moreover, Theorem 4.18 shows that the unit and counit are *P*-cocontinuous and in particular *P*-semiadditive. \Box

In fact, the above argument also shows slightly more generally:

Proposition 6.26. Let C be a pointed T-category with finite P-coproducts and let D be an S-category with finite P-products. Then a T-functor $C \to \iota_* D$ is P-semiadditive if and only if its adjunct $\iota^* C \to D$ is so.

Arguing as in Corollary 4.22 we immediately deduce:

Corollary 6.27. Let C be a pointed T-category with finite P-coproducts and let \mathcal{D} be an S-category with finite P-products. Then the equivalence $\Phi: \underline{\operatorname{Fun}}_T(\mathcal{C}, \iota_* \mathcal{D}) \xrightarrow{\sim} \iota_* \operatorname{Fun}_S(\iota^* \mathcal{C}, \mathcal{D})$ from Construction 4.21 restricts to an equivalence

$$\underline{\operatorname{Fun}}_{T}^{P-\oplus}(\mathcal{C},\iota_{*}\mathcal{D}) \xrightarrow{\sim} \iota_{*}\underline{\operatorname{Fun}}_{S}^{P-\oplus}(\iota^{*}\mathcal{C},\mathcal{D}).$$

On the other hand, we now easily get the following result subsuming Theorem 6.19 and one half of 6.18:

- **Theorem 6.28.** (1) There is a unique S-cocontinuous functor $\iota_1 \colon \underline{\mathrm{CMon}}_S^P \to \iota^*\mathrm{CMon}_T^P$ sending $\mathbb{P}(*)$ to $\mathbb{P}(*)$.
 - (2) $\iota_!$ is fully faithful and extends uniquely to a T-functor $\underline{\mathrm{CMon}}_{S \triangleright T}^P \to \underline{\mathrm{CMon}}_T^P$.
 - (3) $\underline{\mathrm{CMon}}_{S \triangleright T}^{P}$ is S-presentable and P-semiadditive.
 - (4) Let $\mathbb{P}(*) \in \Gamma(\underline{\mathrm{CMon}}_{S \triangleright T}^{P})$ denote the preimage of the object of $\Gamma(\underline{\mathrm{CMon}}_{T}^{P})$ of the same name. Then $\underline{\mathrm{CMon}}_{S \triangleright T}^{P}$ has the following universal property: for any P-semiadditive S-cocomplete T-category \mathcal{D} evaluation at $\mathbb{P}(*)$ defines an equivalence $\underline{\mathrm{Fun}}_{T}(\underline{\mathrm{CMon}}_{S \triangleright T}^{P}, \mathcal{D}) \simeq \mathcal{D}$.

Proof. Let $C \subset \underline{\mathrm{CMon}}_T^P$ again be generated under S-colimits by $\mathbb{P}(*)$. Arguing as in the proof of Theorem 4.24, Proposition 6.25 together with Proposition 6.21 shows that there is a unique S-cocontinuous functor $\underline{\mathrm{CMon}}_S^P \to \iota^* C$ preserving $\mathbb{P}(*)$, and that this is an equivalence. Thus, $\iota_1: \underline{\mathrm{CMon}}_S^P \to \iota^* \underline{\mathrm{CMon}}_T^P$ extends uniquely to a fully faithful T-functor $\underline{\mathrm{CMon}}_{S \triangleright T}^P \to \underline{\mathrm{CMon}}_T^P$, and this induces an equivalence onto \mathcal{C} . The universal property then follows by another application of Proposition 6.21.

It only remains to show that the category \mathcal{C} (and hence also $\underline{\mathrm{CMon}}_{S \triangleright T}^{P}$) is S-presentable. But indeed, \mathcal{C} is S-cocomplete as $\underline{\mathrm{CMon}}_{T}^{P}$ is so, and $\mathcal{C}(A) \simeq \underline{\mathrm{CMon}}_{S}^{P}(A)$ is presentable for any $A \in T$.

6.3. An additional adjoint. Our goal in this subsection will be to understand the right adjoint ι^* of the above S-functor $\iota_1: \underline{\mathrm{CMon}}_S^P \to \iota^* \underline{\mathrm{CMon}}_T^P$ better, and to use this to show that it in turn admits another right adjoint ι_* , finishing the proof of Theorem 6.18. We begin with the following observation:

Lemma 6.29. The diagram



commutes up to natural equivalence.

Note that ι^* is *P*-semiadditive as it is right adjoint; by the universal property of $\underline{\text{CMon}}_S^P$ from Theorem 6.16 the above then actually characterizes ι^* completely.

Proof. All functors in the diagram

$$\frac{\operatorname{Spc}_{S}}{\operatorname{P}} \xrightarrow{\iota_{1}} \iota^{*} \underbrace{\operatorname{Spc}_{T}}{\downarrow_{\iota^{*} \operatorname{P}}} \tag{8}$$

$$\underline{\operatorname{CMon}}_{S}^{P} \xrightarrow{\iota_{1}} \iota^{*} \underbrace{\operatorname{CMon}}_{T}^{P}$$

are left adjoints, and both paths through this diagram send the terminal object to the same object by the defining property of the horizontal maps. Thus, the universal property of $\underline{\text{Spc}}_S$ shows that (8) commutes up to equivalence. The claim follows by passing to total mates.

This suggests a natural strategy to get a more explicit description of ι^* and to prove that it has a right adjoint: construct *some* left adjoint $\iota^*\underline{\mathrm{CMon}}_T^P \to \underline{\mathrm{CMon}}_S^P$ and then show that it is compatible with the forgetful functors. Indeed, this is precisely what we will do now, using the restriction functor from Construction 4.21:

Proposition 6.30. The composite

$$\iota^* \underline{\operatorname{Fun}}_T(\underline{\mathbb{F}}_{T,*}^P, \underline{\operatorname{Spc}}_T) \xrightarrow{\iota^*} \underline{\operatorname{Fun}}_S(\underline{\mathbb{F}}_{S,*}^P, \iota^* \underline{\operatorname{Spc}}_T) \xrightarrow{\underline{\operatorname{Fun}}_S(\underline{\mathbb{F}}_{S,*}^P, \iota^*)} \underline{\operatorname{Fun}}_S(\underline{\mathbb{F}}_{S,*}^P, \underline{\operatorname{Spc}}_S)$$
(9)

restricts to the functor $\iota^*\underline{\mathrm{CMon}}_T^P \to \underline{\mathrm{CMon}}_S^P$ right adjoint to $\iota_!$. Moreover, (9) admits a right adjoint $\iota_* : \underline{\mathrm{Fun}}_S(\underline{\mathbb{F}}_{S,*}^P, \underline{\mathrm{Spc}}_S) \to \iota^*\underline{\mathrm{Fun}}_T(\underline{\mathbb{F}}_{T,*}^P, \underline{\mathrm{Spc}}_T)$, which again restricts to $\underline{\mathrm{CMon}}_S^P \to \iota^*\underline{\mathrm{CMon}}_T^P$.

Proof. For (9) to restrict as claimed, it will be enough to show that its adjunct $\underline{\operatorname{Fun}}_T(\underline{\mathbb{F}}_{T,*}^P, \underline{\operatorname{Spc}}_T) \to \iota_* \underline{\operatorname{Fun}}_S(\underline{\mathbb{F}}_{S,*}^P, \underline{\operatorname{Spc}}_S)$ restricts to $\underline{\operatorname{CMon}}_T^P \to \iota_* \underline{\operatorname{CMon}}_S^P$. However, unravelling the definitions, the adjunct is precisely given by

$$\operatorname{Fun}_{T}(\underline{\mathbb{F}}_{T,*}^{P}, \underline{\operatorname{Spc}}_{T}) \xrightarrow{\operatorname{Fun}_{T}(\underline{\mathbb{F}}_{T,*}^{P}, \iota^{*})} \underline{\operatorname{Fun}}_{T}(\underline{\mathbb{F}}_{T,*}^{P}, \iota_{*}\underline{\operatorname{Spc}}_{S}) \xrightarrow{\Phi} \iota_{*}\underline{\operatorname{Fun}}_{S}(\underline{\mathbb{F}}_{S,*}^{P}, \underline{\operatorname{Spc}}_{S}), (10)$$

where $\tilde{\iota}^*$ is the adjunct of ι^* , as in Proposition 4.28. The first functor restricts to semiadditive functors as $\tilde{\iota}^*$ is *S*-continuous by Corollary 4.20^{op}, and so does the second functor by Corollary 6.27.

Proposition 4.28 then shows that (10) has a right adjoint $\tilde{\iota}_*$ given by the composite

$$\iota_* \underline{\operatorname{Fun}}_S(\underline{\mathbb{F}}^P_{S,*}, \underline{\operatorname{Spc}}_S) \xrightarrow{\Phi^{-1}} \underline{\operatorname{Fun}}_T(\underline{\mathbb{F}}^P_{T,*}, \iota_* \underline{\operatorname{Spc}}_S) \xrightarrow{\underline{\operatorname{Fun}}_T(\underline{\mathbb{F}}^P_{T,*}, \tilde{\iota}_*)} \operatorname{Fun}_T(\underline{\mathbb{F}}^P_{T,*}, \underline{\operatorname{Spc}}_T)$$

which restricts to $\iota_* \underline{\mathrm{CMon}}_S^P \to \underline{\mathrm{CMon}}_T^P$ by the same argument as before. We can now show that (9) has a right adjoint ι_* : namely, as it is adjunct to (10), it factors as

$$\iota^*\underline{\operatorname{Fun}}_T(\underline{\mathbb{F}}^P_{T,*},\underline{\operatorname{Spc}}_T) \xrightarrow{\iota^*(10)} \iota^*\iota_*\underline{\operatorname{Fun}}_S(\underline{\mathbb{F}}^P_{S,*},\underline{\operatorname{Spc}}_S) \xrightarrow{\varepsilon} \underline{\operatorname{Fun}}_S(\underline{\mathbb{F}}^P_{S,*},\underline{\operatorname{Spc}}_S),$$

and the first map has a right adjoint given by $\iota^*(\tilde{\iota}_*)$ as ι^* obviously preserves adjunctions, while the second one has a right adjoint as it S-cocontinuous by (the proof of) Corollary 4.20.

Next, let us show that ι_* restricts to $\underline{\mathrm{CMon}}_S^P \to \iota^*\underline{\mathrm{CMon}}_T^P$. By the above, $\iota^*(\tilde{\iota}_*)$ restricts to $\iota^*\iota_*\underline{\mathrm{CMon}}_S^P \to \iota^*\underline{\mathrm{CMon}}_T^P$, so it only remains to show that the right adjoint of the counit $\varepsilon \colon \iota^*\iota_* \to \operatorname{id} \operatorname{at} \underline{\mathrm{Fun}}_S(\underline{\mathbb{F}}_{S,*}^P, \underline{\mathrm{Spc}}_S)$ restricts to $\underline{\mathrm{CMon}}_S^P \to \iota^*\iota_*\underline{\mathrm{CMon}}_S^P$. But ε is simply given by restricting along the unit of $\iota_! \colon \mathrm{PSh}(S) \rightleftharpoons \mathrm{PSh}(T) : \iota^*$, so the claim follows as $\underline{\mathrm{CMon}}_S^P \subset \underline{\mathrm{Fun}}_S(\underline{\mathbb{F}}_{S,*}^P, \underline{\mathrm{Spc}}_S)$ is closed under all S-limits. It now only remains to show that the restriction of (9) to $\iota^* \underline{\mathrm{CMon}}_T^P \to \underline{\mathrm{CMon}}_S^P$ indeed agrees with the functor ι^* considered before. For this we consider the diagram

$$\iota^* \underbrace{\operatorname{Fun}_T(\mathbb{F}^P_{T,*}, \underline{\operatorname{Spc}}_T) \xrightarrow{\iota^*} \underline{\operatorname{Fun}}_S(\mathbb{F}^P_{S,*}, \iota^* \underline{\operatorname{Spc}}_T) \xrightarrow{\underline{\operatorname{Fun}}_S(\mathbb{F}^P_{S,*}, \iota^*)} \underline{\operatorname{Fun}}_S(\mathbb{F}^P_{S,*}, \underline{\operatorname{Spc}}_S) \xrightarrow{\iota^* \operatorname{ev}_{S^0}} \underbrace{\downarrow^{\operatorname{ev}}_{S^0}} \underbrace{\downarrow^{\operatorname{ev}}_{S^0}} \underbrace{\downarrow^{\operatorname{ev}}_{S^0}} \underbrace{\downarrow^{\operatorname{ev}}_{S^0}} \underline{\downarrow^{\operatorname{ev}}_{S^0}} \underbrace{\downarrow^{\operatorname{ev}}_{S^0}} \underline{\operatorname{Fun}}_{\iota^*} \xrightarrow{\underline{\operatorname{Spc}}} \underline{\operatorname{Spc}}_S$$

with top row (9). The right hand square commutes by naturality, as does the left hand square by a straightforward mate argument. Thus, the restricted functor $\iota^* \underline{\mathrm{CMon}}_T^P \to \underline{\mathrm{CMon}}_S^P$ lies over $\iota^* \colon \iota^* \underline{\mathrm{Spc}}_T \to \underline{\mathrm{Spc}}_S$. But it is also *P*-semiadditive (it is even an *S*-left adjoint by the above), so the claim follows from Lemma 6.29 and the universal property of $\underline{\mathrm{CMon}}_S^P$.

Proof of Theorem 6.18. Combine Theorem 6.28 with the previous proposition. \Box

7. The universal property of equivariant special Γ -spaces

In this section we will identify the universal equivariantly semiadditive equivariantly presentable global category in terms of Shimakawa's *special* Γ -*G*-*spaces* [Shi89, Shi91].

7.1. Model categories of equivariant Γ -spaces. We begin by introducing the main players, which will require a bit more model categorical sophistication than the unstable case.

Definition 7.1. We write Γ for the category of finite pointed sets and based maps. For any $n \ge 0$ we let $n^+ \coloneqq \{0, \ldots, n\}$ with basepoint 0.

Definition 7.2. Let G be a finite group. A Γ -G-space is a functor $\Gamma \rightarrow G$ -SSet that sends the singleton set 0^+ to the 1-point space. We write Γ -G-SSet_{*} for the category of Γ -G-spaces.

By [MMO17, Lemma 1.17] we can equivalently think of a Γ -*G*-space as an **Set**_{*}enriched functor $\Gamma \rightarrow \mathbf{G}$ -**SSet**_{*} into the category of *pointed G*-spaces, with the equivalence given by forgetting the basepoints and the enrichment.

7.1.1. Level model structures. Next, we will equip Γ -G-SSet_{*} with a suitable level model structure. To put this into context, we recall the standard equivariant model structures on G-SSet:

Proposition 7.3. Let G be a finite group and let \mathcal{F} be a family of subgroups of G, *i.e.* a non-empty collection of subgroups that is closed under taking subconjugates. Then **G-SSet** carries a model structure with

- (1) weak equivalences the \mathcal{F} -weak equivalences, *i.e.* those maps f such that f^H is a weak homotopy equivalence for every $H \in \mathcal{F}$;
- (2) fibrations the \mathcal{F} -fibrations, i.e. those maps f such that f^H is a Kan fibration for every $H \in \mathcal{F}$;
- (3) cofibrations the \mathcal{F} -cofibrations: those maps f that are levelwise injective and such that the isotropy of any simplex outside the image of f belongs to \mathcal{F} .

We call this the \mathcal{F} -model structure. It is combinatorial, simplicial, and proper.

Proof. See e.g. [Len20, Proposition 1.1.2] and [Ste16, Proposition 2.16].

Example 7.4. In the special case that $\mathcal{F} = \mathcal{A}\ell\ell$ consists of all subgroups, we call this the *G*-equivariant model structure; its weak equivalences are the *G*-equivariant weak equivalences considered before.

Proposition 7.5. The category Γ -G-SSet_{*} admits a unique model structure with

- (1) weak equivalences those maps $f: X \to Y$ such that $f(S_+): X(S_+) \to Y(S_+)$ is a G-weak equivalence for any finite G-set S; here we equip both sides with the diagonal G-action induced from the actions on X, Y, and S;
- (2) fibrations those f such that $f(S_+)$ is a G-fibration for any finite G-set S.

We call this the G-equivariant level model structure and its weak equivalences the G-equivariant level weak equivalences. It is simplicial, proper, and combinatorial with generating cofibrations the maps

$$\Gamma(S_+, -) \wedge G/H_+ \wedge (\partial \Delta^n \hookrightarrow \Delta^n)_+$$

for all $n \geq 0$, all $H \subset G$, and all finite G-sets S, while its generating acyclic cofibrations are similarly given by the maps $\Gamma(S_+, -) \wedge G/H_+ \wedge (\Lambda_k^n \hookrightarrow \Delta^n)_+$.

Remark 7.6. By [Ost16, Remark 4.11], we could equivalently ask for $f(S_+)$ to be an *H*-weak equivalence or *H*-fibration for any $H \subset G$ and any finite *H*-set *S*. Put differently, if \mathcal{G}_{G,Σ_S} denotes the family of graph subgroups of $G \times \Sigma_S$ (i.e. the subgroups of the form $\Gamma_{H,\varphi} := \{(h,\varphi(h)) : h \in H\}$ for $H \subset G$ and $\varphi : H \to \Sigma_S$, or equivalently those subgroups intersecting $1 \times \Sigma_S$ trivially), then a map *f* is a weak equivalence or fibration in the above model structure if and only if $f(S_+)$ is a \mathcal{G}_{G,Σ_S} -weak equivalence or fibration, respectively, for any finite set *S*.

Proof of Proposition 7.5. The model structure appears without proof as [Ost16, Theorem 4.7]; see [Len20, Proposition 2.2.36] for a complete argument. \Box

Lemma 7.7. Let $\alpha : G \to G'$ be a homomorphism of finite groups. Then the restriction $\alpha^* : \Gamma$ -G'-SSet_{*} $\to \Gamma$ -G-SSet_{*} is left Quillen for the level model structures.

Proof. It suffices that the right adjoint α_* preserves (acyclic) fibrations. As the latter are defined levelwise, this amounts to saying that

$$(\alpha \times \Sigma_S)^* \colon (G' \times \Sigma_S) \operatorname{-SSet}_{\mathcal{G}_{G',\Sigma_S}} \rightleftharpoons (G \times \Sigma_S) \operatorname{-SSet}_{\mathcal{G}_{G,\Sigma_S}} : (\alpha \times \Sigma_S)_*$$

is a Quillen adjunction for every finite set S. But clearly $(\alpha \times \Sigma_S)^*$ preserves cofibrations and sends generating acyclic cofibrations to weak equivalences.

Remark 7.8. If $\alpha: G \to G'$ is injective, then $\alpha^*: \Gamma - G' - SSet_* \to \Gamma - G - SSet$ is easily seen to preserve weak equivalences and fibrations; in particular, it is also right Quillen.

Beware that the previous remark does *not* hold for non-injective α , see e.g. [Len20, Example 2.2.15], and accordingly the composition

$$\Gamma$$
-•-SSet_{*}: Glo^{op} \hookrightarrow Grpd^{op} $\xrightarrow{\text{Fun}(-,\Gamma\text{-SSet}_*)}$ Cat₁

does not lift to RelCat via the above weak equivalences. However, by Ken Brown's Lemma we can fix this by restricting to the subcategories of *cofibrant* objects:

Definition 7.9. We write Γ -•-SSet^{cof}_{*} for the resulting functor $\text{Glo}^{\text{op}} \to \text{RelCat}$ and $\Gamma \mathscr{S}_*$ for the global category obtained by pointwise localization.

We can (and will) equivalently think of $\underline{\Gamma}\mathscr{G}_*$ as sending a finite group G to the localization of Γ -G- $SSet_*$ and a homomorphism α to the left derived functor $\mathbf{L}\alpha^*$.

Lemma 7.10. Let G be a finite group. Then we have a Quillen adjunction

 $\Gamma(1^+, -) \land (-)_+ : \mathbf{G}\text{-}\mathbf{SSet} \rightleftharpoons \Gamma\text{-}\mathbf{G}\text{-}\mathbf{SSet}_* : \mathrm{ev}_{1^+}$

in which both adjoints are homotopical.

Proof. It is clear that both adjoints are homotopical, and that the right adjoint moreover preserves fibrations, so that it is in particular right Quillen. \Box

Thus, $\Gamma(1^+, -) \wedge (-)_+$ induces a natural transformation \bullet -**SSet** $\rightarrow \Gamma$ - \bullet -**SSet**^{cof}_{*}, and hence a global functor $\underline{\mathscr{I}} \rightarrow \underline{\Gamma}\underline{\mathscr{I}}_*$, which we denote by \mathbb{P} . It is not hard to check that \mathbb{P} admits a global right adjoint (induced by ev_{1^+}); as we will not need this below, we leave the details to the interested reader.

7.1.2. *Specialness*. In order to study equivariant commutative monoids, we have to Bousfield localize the above level model structures. For this we recall:

Definition 7.11. A Γ -*G*-space is called *special* if for every finite *G*-set *S* the *Segal* map $X(S_+) \to X(1^+)^{\times S}$ induced by the characteristic maps $\chi_s \colon S_+ \to 1^+$ for varying $s \in S$, is a *G*-weak equivalence.

Similarly to the different characterizations of the *G*-equivariant level weak equivalences, specialness is equivalent to asking more generally for the Segal maps to be *H*-equivariant weak equivalences for all $H \subset G$ and all finite *H*-sets *S*, or for them to be \mathcal{G}_{G,Σ_S} -weak equivalences for every finite set *S*, see [Len20, Lemma 2.2.10].

Proposition 7.12 (See [Len20, Proposition 2.2.60]). The *G*-equivariant level model structure on Γ -*G*-SSet_{*} admits a Bousfield localization with fibrant objects precisely the level fibrant special Γ -*G*-spaces. We call this the *G*-equivariant model structure and its weak equivalences the G-equivariant weak equivalences. It is combinatorial, simplicial, and left proper.

Remark 7.13. The above model structure is obtained from the level model structure by localizing with respect to the maps $S_+ \wedge \Gamma(1^+, -) \wedge G/H_+ \rightarrow \Gamma(S_+, -) \wedge G/H_+$ induced by the map $S_+ \rightarrow \Gamma(S_+, 1^+)$ sending $s \in S$ to its characteristic map $\chi_S \colon S_+ \rightarrow 1^+$ for all finite *G*-sets *S*. In particular, all of these maps are *G*-equivariant weak equivalences.

Lemma 7.14. Let $\alpha \colon G \to G'$ be a homomorphism. Then

 $\alpha^* \colon \mathbf{\Gamma}\text{-}\mathbf{G'}\text{-}\mathbf{SSet}_* \rightleftarrows \mathbf{\Gamma}\text{-}\mathbf{G}\text{-}\mathbf{SSet}_*$

is left Quillen with respect to the above model structures. If α is injective, then α^* is also right Quillen.

Proof. For the first statement, it will suffice by [Lur09, Corollary A.3.7.2] that α^* preserves cofibrations and α_* preserves fibrant objects. The first statement is clear from Lemma 7.7, while for the second statement it is enough by adjunction to show that $\mathbf{L}\alpha^*$ sends the maps from the previous remark to weak equivalences.

As these are maps between cofibrant objects, it is enough to prove the same for α^* . However, decomposing $\alpha^*(G'/H)$ into *G*-orbits expresses $\alpha^*(S_+ \wedge \Gamma(1^+, -) \wedge G'/H_+ \rightarrow \Gamma(S_+, -) \wedge G'/H_+)$ as a coproduct of weak equivalences between cofibrant objects, so the claim follows.

The second statement follows similarly from Remark 7.8 as α^* clearly preserves specialness for injective α .

In particular, we get a functor Γ -•-SSet^{cof, spc}: Glo^{op} \rightarrow RelCat that sends G to Γ -G-SSet^{cof}_{*} with the above weak equivalences. The identity of underlying categories Γ -•-SSet^{cof}_{*} $\rightarrow \Gamma$ -•-SSet^{cof, spc}_{*} then induces a localization $L: \underline{\Gamma} \mathcal{S}_* \rightarrow \underline{\Gamma} \mathcal{S}_*^{\text{spc}}$. We will write $\mathbb{P}: \mathcal{S} \rightarrow \underline{\Gamma} \mathcal{S}_*^{\text{spc}}$ for $L \circ \mathbb{P}$; note that this is again induced by the homotopical left Quillen functors $\Gamma(1^+, -) \land (-)_+$.

Warning 7.15. The functors $\mathbf{L}\alpha^*$ do not preserve specialness for non-injective α , i.e. the pointwise right adjoints of L do not assemble into a global right adjoint. This is hard to see directly (as we know so few cofibrant objects in the above model structure, making it hard to compute $\mathbf{L}\alpha^*$), so we use a trick and a bit of equivariant infinite loop space theory instead:

Let $\Gamma(1^+, -) \to S$ be an acyclic cofibration to a special Γ -space. In particular, S is cofibrant, so if $\mathbf{L}\alpha^*$ preserved specialness, then S with the trivial G-action would be a special Γ -G-space for any finite G. On the other hand, as restrictions are left Quillen by the above, it would be equivalent to $\Gamma(1^+, -)$ with trivial G-action. We show that already for $G = \mathbb{Z}/2$ this is impossible: no special Γ - $\mathbb{Z}/2$ -space equivalent to $\Gamma(1^+, -)$ can have trivial action.

For this we use that the delooping of $\Gamma(1^+, -)$ (and hence of any $\Gamma \cdot \mathbb{Z}/2$ -space equivalent to it) is the equivariant sphere spectrum. Now the zeroth stable homotopy groups of the latter are given by the Burnside ring, and hence in particular $\pi_0^1(\mathbb{S}) \cong \mathbb{Z} \not\cong \mathbb{Z}^2 \cong \pi_0^{\mathbb{Z}/2}(\mathbb{S})$. However, for a *special* $\Gamma \cdot \mathbb{Z}/2$ -space the homotopy groups of its delooping are simply given as the group completions of the original homotopy monoids. In the case of a trivial *G*-action, the restriction homomorphisms between these homotopy monoids are clearly isomorphisms, and in particular their group completions are isomorphic, yielding the desired contradiction.

Note that the same argument shows that also the underived functors α^* do not preserve specialness, although there are much more concrete counterexamples available in this case.

We can now finally state the main results of this section.

Theorem 7.16. The global category $\underline{\Gamma} \mathcal{Y}_*^{\text{spc}}$ is equivariantly presentable and equivariantly semiadditive. Moreover, the unique equivariantly cocontinuous global functor $\underline{\text{CMon}}_{\text{Orb} \rhd \text{Glo}}^{\text{Orb}} \rightarrow \underline{\Gamma} \mathcal{Y}_*^{\text{spc}}$ sending $\mathbb{P}(*)$ to $\mathbb{P}(*)$ is an equivalence.

Theorem 7.17. The global category $\underline{\Gamma} \mathscr{L}^{\text{spc}}_*$ is the free equivariantly cocomplete equivariantly semiadditive global category on one generator in the following sense: for every other such \mathcal{D} evaluation at $\mathbb{P}(*)$ provides an equivalence

$$\operatorname{Fun}_{\operatorname{Glo}}^{\operatorname{eq-cc}}(\Gamma \mathscr{G}_*^{\operatorname{spc}}, \mathcal{D}) \xrightarrow{\sim} \mathcal{D}_*$$

7.2. **G-global vs. G-equivariant** Γ -spaces. In order to prove the above theorems, we will again reduce to our identification of the universal *globally cocomplete* equivariantly semiadditive category from [CLL23].

7.2.1. Model categories of G-global Γ -spaces. We begin by introducing the relevant model categories.

Definition 7.18. A *G*-global Γ -space is a functor $X: \Gamma \to G$ - \mathcal{I} -SSet such that $X(0^+) = *$.

Proposition 7.19 (See [Len20, Theorem 2.2.31]). The category Γ -G- \mathcal{I} -SSet_{*} of G-global Γ -spaces carries a unique model structure with

- (1) weak equivalences those maps f such that $f(S_+)$ is a $(G \times \Sigma_S)$ -global weak equivalence for every finite set S
- (2) acyclic fibrations those maps f such that $f(S_+)(A)$ is a $\mathcal{G}_{\Sigma_A,G\times\Sigma_S}$ -acyclic fibration for all finite sets S and A.

We call this the G-global level model structure and its weak equivalences the G-global level weak equivalences. It is combinatorial, simplicial, and proper. Moreover, pushouts along injective cofibrations (i.e. levelwise injections) are homotopy pushouts in this model structure; in particular, they preserve weak equivalences. \Box

For any $\alpha: G \to G'$ the functor $\alpha^*: \mathbf{\Gamma} \cdot \mathbf{G'} \cdot \mathbf{\mathcal{I}} \cdot \mathbf{SSet}_* \to \mathbf{\Gamma} \cdot \mathbf{G} \cdot \mathbf{\mathcal{I}} \cdot \mathbf{SSet}_*$ preserves weak equivalences, so the above yields a global category $\underline{\Gamma} \cdot \underline{\mathcal{I}}_*^{\mathrm{gl}}$ by the usual procedure. Note that [CLL23] uses the notation $(\underline{\Gamma} \cdot \underline{\mathcal{I}}_{\mathcal{I},*}^{\mathrm{gl}})$, instead; however, the above is equivalent to the category denoted by the same symbols in *op. cit.* by [Len20, Theorem 2.2.33].

Construction 7.20. For any G, we have a homotopical adjunction

$$\Gamma(1^+, -) \land (-)_+ : G - \mathcal{I} - SSet \rightleftharpoons \Gamma - G - \mathcal{I} - SSet_* : ev_{1^+}.$$

As both adjoints are moreover strictly compatible with restriction, we obtain an induced adjunction $\mathbb{P} \colon \underline{\mathscr{G}}_*^{\mathrm{gl}} \rightleftharpoons \underline{\Gamma} \underline{\mathscr{G}}_*^{\mathrm{gl}} : \mathbb{U}$. We will refer to \mathbb{U} as the *forgetful functor*.

Remark 7.21. We can also consider the category Γ -*G*- \mathcal{I} -**SSet** of all functors $\Gamma \rightarrow G$ - \mathcal{I} -**SSet** and equip this with the analogue of the *G*-global level weak equivalence. For varying *G*, these again assemble into a global category, which we denote by $\underline{\Gamma}\mathcal{I}^{g^{l}}$.

The inclusion Γ -*G*- \mathcal{I} -**SSet**_{*} $\hookrightarrow \Gamma$ -*G*- \mathcal{I} -**SSet** admits a left adjoint Λ given by taking the cofibers of the maps $X(0^+) \to X(S_+)$ induced by the unique maps $i: 0^+ \to S_+$ in Γ . As each X(i) is an injective cofibration (as *i* admits a retraction), this is actually a homotopy cofiber and Λ is homotopical. It follows easily that the map $\underline{\Gamma}\mathcal{F}_*^{\text{gl}} \to \underline{\Gamma}\mathcal{F}^{\text{gl}}$ induced by the inclusions is fully faithful with essential image given in degree *G* by those *X* with $X(0^+) \simeq *$ in $\mathcal{F}_G^{\text{gl}}$.

Definition 7.22. A *G*-global Γ -space *X* is called *special* if the Segal map $X(S_+) \to X(1^+)^{\times S}$ is a $(G \times \Sigma_S)$ -global weak equivalence for every finite set *S*.

Note that unlike their equivariant counterparts, these are stable under arbitrary restrictions, so they form a global subcategory $\underline{\Gamma \mathscr{G}_{*}^{gl, spc}}_{*}$.

Theorem 7.23 (See [CLL23, Corollary 5.3.6]). There exists an equivalence of global categories Ξ : $\underline{\Gamma} \underline{\mathscr{P}}^{\text{gl}} \simeq \underline{\operatorname{Fun}}_{\operatorname{Glo}}(\underline{\mathbb{F}}_{\operatorname{Glo},*}^{\operatorname{Orb}}, \underline{\operatorname{Spc}}_{\operatorname{Glo}})$ compatible with the forgetful functors and restricting to an equivalence $\underline{\Gamma} \underline{\mathscr{P}}^{\text{gl}}_{*} \overset{\operatorname{spc}}{\cong} \simeq \underline{\operatorname{CMon}}_{\operatorname{Glo}}^{\operatorname{Orb}}$.

Corollary 7.24. The inclusions $\underline{\Gamma}\mathscr{L}^{\mathrm{gl}, \mathrm{spc}}_* \hookrightarrow \underline{\Gamma}\mathscr{L}^{\mathrm{gl}}_*$ and $\underline{\Gamma}\mathscr{L}^{\mathrm{gl}, \mathrm{spc}}_* \hookrightarrow \underline{\Gamma}\mathscr{L}^{\mathrm{gl}}$ admit global left adjoints.

Proof. The second statement follows from the previous theorem as $\underline{\text{CMon}_{\text{Glo}}^{\text{Orb}} \hookrightarrow \underline{\text{Fun}_{\text{Glo}}}(\underline{\mathbb{F}}_{\text{Glo},*}^{\text{Orb}}, \underline{\text{Spc}_{\text{Glo}}})$ admits a left adjoint. The first one then follows from this together with Remark 7.21.

Remark 7.25. One can also prove the corollary via purely model categorical arguments: by [Len20, Proposition 2.2.61], the *G*-global level model structure admits a Bousfield localization with fibrant objects the level fibrant *special G*-global Γ -spaces. In particular, we get a pointwise left adjoint, and the Beck–Chevalley condition then translates to demanding that each $\alpha^* \colon \underline{\Gamma} \mathscr{F}^{\mathrm{gl}}_*(G') \to \underline{\Gamma} \mathscr{F}^{\mathrm{gl}}_*(G)$ preserve the weak equivalences of these Bousfield localizations, or equivalently that the restriction functors

$$\alpha^* \colon \mathbf{\Gamma} \textbf{-} \mathbf{\mathcal{G}}' \textbf{-} \mathbf{\mathcal{I}} \textbf{-} \mathbf{SSet}_* \to \mathbf{\Gamma} \textbf{-} \mathbf{\mathcal{G}} \textbf{-} \mathbf{\mathcal{I}} \textbf{-} \mathbf{SSet}_*$$
(11)

be homotopical for the localized model structures. While this is doable by careful inspection, it is actually more work than in the equivariant case (as the maps we localize at are more complicated), and hence deliberately avoided in [Len20], which is why we went via the above route instead.

Note however that conversely the above corollary now shows that the functor $\alpha^* \colon \underline{\Gamma} \mathscr{P}^{\mathrm{gl}}_*(G') \to \underline{\Gamma} \mathscr{P}^{\mathrm{gl}}_*(G)$ and hence also (11) is homotopical for any α , yielding an ∞ -categorical proof of a model categorical statement.

Composing the above with the adjunction from Construction 7.20, we get an adjunction $\mathcal{G}^{\text{gl}} \rightleftharpoons \underbrace{\Gamma \mathcal{G}^{\text{gl}}_*}_*^{\text{gl}, \text{ spc}}$ that we again denote by $\mathbb{P} \dashv \mathbb{U}$. The (inverse) equivalence $\underbrace{\text{CMon}_{\text{Glo}}^{\text{Orb}}}_{\text{Glo}} \simeq \underbrace{\Gamma \mathcal{G}^{\text{gl}}_*}^{\text{gl}}$ from Theorem 7.23 can then be described (by some easy mate yoga) as the unique left adjoint that sends $\mathbb{P}(*)$ to $\mathbb{P}(*)$.

7.2.2. The comparison. Finally, let us relate G-global and G-equivariant Γ -spaces to each other:

Proposition 7.26. There is a global functor $\mathbf{L} \operatorname{const}: \underline{\Gamma} \mathscr{P}^{\operatorname{spc}}_* \to \underline{\Gamma} \mathscr{P}^{\operatorname{gl}, \operatorname{spc}}_*$ with the following properties:

- (1) It is fully faithful and sends $\mathbb{P}(*)$ to $\mathbb{P}(*)$.
- (2) It admits an Orb-right adjoint.

Once again, after the universal property of $\underline{\Gamma}\mathscr{G}_*$ is established, we will see a posteriori that the above adjunction is actually unique.

For the proof of the proposition we will need another model structure:

Lemma 7.27 (See [Len20, Corollary 2.2.40 and proof of Proposition 2.2.42]). The category Γ -G- \mathcal{I} -SSet_{*} admits a model structure with

- (1) weak equivalences the G-global level weak equivalences
- (2) cofibrations the injective cofibrations.

We call this the injective G-global level model structure. It is combinatorial, simplicial, and proper. Moreover, if $\alpha: G \to G'$ is an injective homomorphism, then $\alpha^*: (\Gamma\text{-}G'\text{-}\mathcal{I}\text{-}\mathbf{SSet}_*)_{injective} \to (\Gamma\text{-}G\text{-}\mathcal{I}\text{-}\mathbf{SSet}_*)_{injective}$ is right Quillen. \Box

Proof of Proposition 7.26. For every G, we have a Quillen adjunction

const:
$$\Gamma$$
-G-SSet_{*} \rightleftharpoons (Γ -G- \mathcal{I} -SSet_{*})_{injective} : ev_{\varnothing}, (12)

see [Len20, Proposition 2.2.25]. By Ken Brown's Lemma, we in particular see that const sends *G*-equivariant weak equivalences between cofibrant objects to *G*-global level weak equivalences, so we get an induced global functor $\mathbf{L} \operatorname{const}: \underline{\Gamma} \mathcal{P}_* \to \underline{\Gamma} \mathcal{P}_*^{\mathrm{gl}}$, which we can postcompose with the localization to $\underline{\Gamma} \mathcal{P}_*^{\mathrm{gl}, \mathrm{spc}}$. Note that this sends $\Gamma(1^+, -)$ to $\mathbb{P}(*)$ by direct inspection.

We now claim that this descends to $\underline{I\mathscr{Y}}^{\mathrm{spc}}_{*}$, which amounts to saying that the left adjoint in (12) sends *G*-equivariant weak equivalences of cofibrant objects to *G*global weak equivalences, for which it is in turn enough that the right derived functor $\mathbf{Rev}_{\varnothing}$ preserve specialness. However, by *loc. cit.* this right adjoint is equivalent to $\mathrm{ev}_{\mathcal{U}}$ for our favourite complete *G*-set universe \mathcal{U} , and it is clear that the latter has the required property (also see Lemma 2.2.51 of *op. cit.*). Altogether, we therefore get a functor $\mathbf{L} \operatorname{const}: \underline{I\mathscr{Y}}^{\mathrm{spc}}_{*} \to \underline{I\mathscr{Y}}^{\mathrm{gl}, \mathrm{spc}}_{*}$ sending $\mathbb{P}(*)$ to $\mathbb{P}(*)$; moreover, this is fully faithful as the right adjoint \mathcal{R} of the right adjoint $\operatorname{Rev}_{\varnothing}$ is fully faithful by Theorem 2.2.59 of *op. cit.*

It only remains to show that the pointwise right adjoints $\operatorname{Rev}_{\varnothing}$ assemble into an Orb-right adjoint, i.e. that for any *injective* homomorphism $\alpha \colon G \to G'$ the Beck–Chevalley transformation $\operatorname{L}_{\alpha^*} \circ \operatorname{Rev}_{\varnothing} \Rightarrow \operatorname{Rev}_{\varnothing} \circ \alpha^*$ is an equivalence.

However, the *pointset level* Beck–Chevalley map $\alpha^* \circ ev_{\emptyset} \Rightarrow ev_{\emptyset} \circ \alpha^*$ is clearly an isomorphism, and all functors in question are right Quillen by the above together with Lemmas 7.14 and 7.27, so this already models the derived Beck–Chevalley map when restricted to injectively fibrant objects.

7.3. Proof of Theorems 7.16 and 7.17. Finally, we turn to the universal property of $\underline{\Gamma}\mathcal{S}_*^{\text{spc}}$.

Lemma 7.28. The category $\underline{\Gamma}\mathcal{G}^{\text{spc}}_{*}(G)$ is generated under (non-parametrized) colimits by the G-equivariant Γ -spaces $\Gamma(1^+, -) \wedge G/H_+$ for $H \subset G$.

Proof. Inspecting the generating cofibrations from Proposition 7.5 we see that $\underline{\Gamma}\mathscr{P}_*(G)$ is generated under colimits by the $\Gamma(S_+, -) \wedge G/H_+$ for finite *G*-sets *S* and subgroups $H \subset G$. Thus, these objects also generate $\underline{\Gamma}\mathscr{P}^{\text{spc}}_*(G)$. However, in the latter $\Gamma(S_+, -) \wedge G/H_+ \simeq S_+ \wedge \Gamma(1^+, -) \wedge G/H_+$ by Remark 7.13. The claim follows by decomposing the *G*-set $G/H \times S$ into its orbits.

Note that $\underline{\Gamma}\mathscr{P}^{\mathrm{spc}}_{*}(G) \ni \Gamma(1^+, -) \wedge G/H_+ \simeq i_! p^* \Gamma(1^+, -)$ where $i: H \hookrightarrow G$ denotes the inclusion and $p: H \to 1$ the unique map. Thus, once we know that $\underline{\Gamma}\mathscr{P}^{\mathrm{spc}}_{*}$ is equivariantly cocomplete, the lemma will tell us that it is generated under equivariant colimits by $\mathbb{P}(*) = \Gamma(1^+, -) \in \underline{\Gamma}\mathscr{P}^{\mathrm{spc}}_{*}(1)$.

Proof of Theorems 7.16 and 7.17. The fully faithful functor **L** const from Proposition 7.26 identifies $\underline{\Gamma \mathscr{Y}}_*^{\text{spc}}$ with a full subcategory of $\underline{\Gamma \mathscr{Y}}_*^{\text{gl, spc}}$, and the latter is globally presentable by Theorem 7.23. However, the essential image of **L** const is closed under all equivariant colimits as **L** const has an Orb-right adjoint, so $\underline{\Gamma \mathscr{Y}}_*^{\text{spc}}$ is equivariantly cocomplete.

In particular, there is a unique equivariantly cocontinuous functor $\underline{CMon}_{Orb \succ Glo}^{Orb} \rightarrow \underline{\Gamma} \mathscr{L}_*^{spc}$ sending $\mathbb{P}(*)$ to $\mathbb{P}(*)$. We claim that this is an equivalence, for which it will

be enough to construct *some* equivalence preserving $\mathbb{P}(*)$. To this end, we will show that the composite

$$\underline{\operatorname{CMon}}_{\operatorname{Orb} \rhd \operatorname{Glo}}^{\operatorname{Orb}} \xrightarrow{\iota_{1}} \underline{\operatorname{CMon}}_{\operatorname{Glo}}^{\operatorname{Orb}} \xrightarrow{\Xi} \underline{\Gamma} \underbrace{\mathcal{F}}_{*}^{\operatorname{gl, spc}}$$
(13)

of the fully faithful functor from Theorem 6.18 and the equivalence from Theorem 7.23 (which sends $\mathbb{P}(*)$ to $\mathbb{P}(*)$ by construction) restricts to an equivalence $\underline{CMon}_{Orb > Glo}^{Orb} \simeq \operatorname{ess\,im}(\mathbf{L}\operatorname{const}) =: \mathcal{E}$. On the one hand, the source of (13) is generated under equivariant colimits by $\mathbb{P}(*)$, so that (13) factors through \mathcal{E} as both functors are in particular Orb-left adjoints. On the other hand, Lemma 7.28 shows that $\underline{\Gamma} \mathcal{F}_*^{\operatorname{spc}}$ and hence also \mathcal{E} is generated by $\mathbb{P}(*)$, so this restriction is also essentially surjective, hence an equivalence.

Finally, the universal property of $\underline{\Gamma}\mathscr{P}^{\text{spc}}_{*}$ follows immediately from this equivalence and the universal property of $\underline{CMon}^{\text{Orb}}_{\text{Orb} \triangleright \text{Glo}}$ established in Theorem 6.19.

7.4. The universal property of special Γ -G-spaces. We close this section by similarly establishing a universal property of special Γ -G-spaces for a fixed finite group G:

Theorem 7.29. Recall the functor v_G : $Orb_G \to Glo$ from Construction 5.11.

- (1) The G-category $v_G^*\underline{\Gamma}\mathcal{P}_*^{\mathrm{spc}}$ (sending G/H to the category of special Γ -H-spaces) is G-presentable and G-semiadditive in the sense of Example 6.7. Moreover, the unique left adjoint $\underline{\mathrm{CMon}}_{\mathrm{Orb}_G} \to v_G^*\underline{\Gamma}\mathcal{P}_*^{\mathrm{spc}}$ preserving $\mathbb{P}(*)$ is an equivalence.
- (2) For any G-cocomplete G-semiadditive \mathcal{D} , evaluation at $\mathbb{P}(*)$ induces an equivalence $\underline{\operatorname{Fun}}_{\operatorname{Orb}_{G}}^{\operatorname{G-cc}}(v_{G}^{*}\underline{\Gamma} \underline{\mathscr{P}}_{*}^{\operatorname{spc}}, \mathcal{D}) \simeq \mathcal{D}.$

For the proof we will need:

Proposition 7.30. Let $P \subset T$ be atomic orbital, let $A \in T$, and write $\pi_A : T_{/A} \to T$ for the forgetful functor. Then $\pi_A^* \underline{\mathrm{CMon}}_T^P$ is $T_{/A}$ -cocomplete and $T_{/A}^P$ -semiadditive, and the unique left adjoint $\underline{\mathrm{CMon}}_{T_{/A}}^{T_{/A}^P} \to \pi_A^* \underline{\mathrm{CMon}}_T^P$ preserving $\mathbb{P}(*)$ is an equivalence.

Proof. By [CLL23, Proposition 2.3.26 and Corollary 4.6.9] $\pi_A^* \dashv \pi_{A*}$ restricts to an adjunction

$$\operatorname{Cat}_{T}^{P-\oplus,T-\operatorname{cc}} \rightleftharpoons \operatorname{Cat}_{T_{/A}}^{T_{/A}^{P}-\oplus,T_{/A}-\operatorname{cc}},$$

so the claim follows as before by comparing corepresented functors.

Proof of Theorem 7.29. As in the unstable case (Theorem 5.12), it will be enough to construct an equivalence $\underline{\text{CMon}}_{\text{Orb}_G} \simeq v_G^* \underline{\Gamma} \mathcal{Y}_*^{\text{spc}}$ preserving $\mathbb{P}(*)$, for which it in turn suffices to combine the previous proposition with Theorem 7.16.

8. The stable story

As in the previous sections, we fix a cleft category $\iota: S \hookrightarrow T$. The goal of this section is to establish the stable analogues of the results from Section 6. We begin with the fiberwise (or naïve) version of stability:

Definition 8.1. A *T*-category C is called *fiberwise stable* if factors through the non-full subcategory Catst \subset Cat of stable categories and exact functors.

Construction 8.2. Recall [Lur17, Proposition 1.4.4.4, Example 4.8.1.23] that the inclusion $Pr^{L, st} \hookrightarrow Pr^{L}$ of presentable stable categories and left adjoints into all presentable categories admits a left adjoint, given by tensoring with the category Sp of spectra.

If now \mathcal{C} is a fiberwise presentable *T*-category, then we write $\operatorname{Sp} \otimes \mathcal{C}$ for the composite

$$T^{\mathrm{op}} \xrightarrow{\mathcal{C}} \mathrm{Pr}^{\mathrm{L}} \xrightarrow{\mathrm{Sp}\otimes -} \mathrm{Pr}^{\mathrm{L, st}} \subset \mathrm{Cat}$$

and call it the *left fiberwise stabilization* of \mathcal{C} . It comes with a functor $\Sigma^{\infty} \colon \mathcal{C} \to$ Sp $\otimes \mathcal{C}$ induced by the unit of the adjunction $\operatorname{Pr}^{\mathrm{L}} \rightleftharpoons \operatorname{Pr}^{\mathrm{L, st}}$.

Remark 8.3. There is another way to fiberwise stabilize suitable *T*-categories, which we will refer to as *right fiberwise stabilization* below: if C factors through the non-full subcategory $\operatorname{Cat}^{\operatorname{lex}}$ of pointed categories with finite limits and left exact functors, then we can define $\underline{\operatorname{Sp}}^{\operatorname{fib}}(C)$ by composing with the *right* adjoint to the inclusion $\operatorname{Cat}^{\operatorname{st}} \hookrightarrow \operatorname{Cat}^{\operatorname{lex}}$ of stable categories. This is the perspective taken in [CLL23, Subsection 6.1].

For the *T*-categories which we would like to stabilize, such as $\underline{\text{CMon}}_{S \triangleright T}^{P}$, it is not clear whether the restriction functors preserve finite limits (as a consequence of the example in Warning 9.8 below, they cannot preserve general limits). Therefore $\underline{\text{Sp}}^{\text{fib}}(\mathcal{C})$ is not well-defined, and we cannot sensibly ask for $\text{Sp} \otimes \mathcal{C}$ to agree with $\underline{\text{Sp}}^{\text{fib}}(\mathcal{C})$.

However, on the category $\Pr^{L, lex}$ of pointed presentable categories and left exact left adjoints, the two stabilization constructions agree [Lur17, Example 4.8.1.23]. Thus, whenever we are given some subcategory $T' \subset T$ such that $\mathcal{C}|_{T'}$ is pointed and restrictions in \mathcal{C} along maps in T' are left exact, then $(\operatorname{Sp} \otimes \mathcal{C})|_{T'}$ agrees with $\operatorname{\underline{Sp}^{fb}}(\mathcal{C}|_{T'})$. This will allow us below to still apply the results from [CLL23, Section 6] to the present situation.

Lemma 8.4. Let C be a fiberwise presentable T-category. Then $\operatorname{Sp} \otimes C$ is fiberwise presentable and fiberwise stable. Moreover, for every fiberwise cocomplete and fiberwise stable D, restriction along Σ^{∞} defines an equivalence

$$\operatorname{Fun}_{T}^{\operatorname{fib-cc}}(\operatorname{Sp} \otimes \mathcal{C}, \mathcal{D}) \to \operatorname{Fun}_{T}^{\operatorname{fib-cc}}(\mathcal{C}, \mathcal{D})$$
(14)

of T-categories of fiberwise cocontinuous functors.

Proof. It is clear that $\operatorname{Sp} \otimes \mathcal{C}$ is fiberwise presentable and fiberwise stable. Replacing \mathcal{D} by $\operatorname{Fun}_T(\mathcal{T}, \mathcal{D})$ for small $\mathcal{T} \in \operatorname{Cat}_T$, it will suffice for the universal property to show that the induced map

$$\operatorname{Hom}_{\operatorname{Cat}_{\operatorname{c}}^{\operatorname{fib-cc}}}(\operatorname{Sp}\otimes\mathcal{C},\mathcal{D}) \to \operatorname{Hom}_{\operatorname{Cat}_{\operatorname{c}}^{\operatorname{fib-cc}}}(\mathcal{C},\mathcal{D})$$

of mapping spaces in the category $\operatorname{Cat}_T^{\operatorname{fib-cc}} \coloneqq \operatorname{Fun}(T^{\operatorname{op}}, \operatorname{Cat}^{\operatorname{cc}})$ of fiberwise cocomplete T-categories and fiberwise colimit-preserving functors is an equivalence.

Writing both sides as the ends of the mapping spaces in $\operatorname{Cat}^{\operatorname{cc}}$, it then suffices to consider the case T = 1, i.e. that for any cocomplete stable \mathcal{D} restriction along $\mathcal{C} \to \operatorname{Sp} \otimes \mathcal{C}$ defines an equivalence $\operatorname{Hom}^{\operatorname{cc}}(\operatorname{Sp} \otimes \mathcal{C}, \mathcal{D}) \simeq \operatorname{Hom}^{\operatorname{cc}}(\mathcal{C}, \mathcal{D})$. Using that the tensor product of presentable categories agrees with the tensor product of cocomplete categories [Lur17, Proposition 4.8.1.15], the tensor-hom adjunction reduces to the case $\mathcal{C} = \text{Spc}$, i.e. we want to show that evaluation at the sphere defines an equivalence $\text{Hom}^{cc}(\text{Sp}, \mathcal{D}) \simeq \iota \mathcal{D}$. This however follows at once by exhibiting Sp as the Ind-completion of the Spanier–Whitehead category [Lur18, Remark C.1.1.6] and noting that right exact functors out of the latter classify objects by [Lur18, Proposition C.1.1.7] together with [Lur17, Proposition 1.4.2.21].

Lemma 8.5. Assume C is S-presentable. Then $\text{Sp} \otimes C$ is again S-presentable, hence in particular S-cocomplete. Moreover, if also D is S-cocomplete, then (14) restricts to an equivalence

$$\underline{\operatorname{Fun}}_{T}^{S\operatorname{-cc}}(\operatorname{Sp}\otimes\mathcal{C},\mathcal{D})\xrightarrow{\sim}\underline{\operatorname{Fun}}_{T}^{S\operatorname{-cc}}(\mathcal{C},\mathcal{D}).$$

Proof. From the previous lemma we see that $\operatorname{Sp} \otimes \mathcal{C}$ is fiberwise presentable. If $s: A \to B$ is in S, then the adjunction $s_1: \mathcal{C}(A) \rightleftharpoons \mathcal{C}(B) : s^*$ is an internal adjunction in $\operatorname{Pr}^{\mathrm{L}}$ (as s^* is itself a left adjoint by fiberwise presentability), so we get an induced adjunction $\operatorname{Sp} \otimes s_1 \dashv \operatorname{Sp} \otimes s^*$ by 2-functoriality of the tensor product. Moreover, the Beck–Chevalley conditions for $\operatorname{Sp} \otimes \mathcal{C}$ follow immediately from the ones for \mathcal{C} .

Finally, for the universal property it suffices by the previous lemma and replacing \mathcal{D} by \mathcal{D}^A to show that Σ^{∞} preserves S-colimits and that for any S-cocontinuous $F: \mathcal{C} \to \mathcal{D}$ also the lift $\tilde{F}: \operatorname{Sp} \otimes \mathcal{C} \to \mathcal{D}$ is S-cocontinuous.

For the first statement, we observe that Σ^{∞} is clearly fiberwise cocontinuous, and that for any admissible $f: X \to Y$ in PSh(T) the Beck–Chevalley maps are equivalences by the explicit description of the adjoints $f_{!}: (Sp \otimes C)(X) \to (Sp \otimes C)(Y)$ given above.

For the second statement, we first observe that \tilde{F} is fiberwise cocontinuous by definition. Given now any admissible $f: X \to Y$, the mate of the total square in



is the Beck–Chevalley map $f_!F \Rightarrow Ff_!$, hence an equivalence by S-cocontinuity of F, and similarly the mate of the top square is an equivalence by the above. By the compatibility of mates with pastings, we conclude that the Beck–Chevalley map $f_!\tilde{F} \Rightarrow \tilde{F}f_!$ becomes an equivalence after precomposition with $\Sigma^{\infty}: \mathcal{C}(X) \to$ $\operatorname{Sp}(\mathcal{C}(X))$. However, both $f_!\tilde{F}$ and $\tilde{F}f_!$ are cocontinuous functors, so the claim follows from the universal property of $\operatorname{Sp} \otimes -$ (cf. the previous lemma). \Box

Let us restate the key step in the above proof separately for easy reference:

Corollary 8.6. Let \mathcal{C}, \mathcal{D} be as above. Then a fiberwise cocontinuous functor $F: \operatorname{Sp} \otimes \mathcal{C} \to \mathcal{D}$ is S-cocontinuous if and only if $F \circ \Sigma^{\infty}: \mathcal{C} \to \mathcal{D}$ is so. \Box

Finally, let us move to the setting of genuine stability:

Definition 8.7. Let $P \subset T$ be atomic orbital. A *T*-category *C* is called *P*-stable if it is *P*-semiadditive (Definition 6.5) and fiberwise stable (Definition 8.1).

Lemma 8.8. Let $P \subset S$ be atomic orbital in T and let C be a P-semiadditive S-cocomplete T-category. Then $Sp \otimes C$ is P-stable.

Proof. We already know that $\operatorname{Sp} \otimes \mathcal{C}$ is *S*-cocomplete and fiberwise stable. Moreover, its underlying *S*-category is *P*-semiadditive by [CLL23, Lemma 6.2.6], so $\operatorname{Sp} \otimes \mathcal{C}$ is also *P*-semiadditive as a *T*-category by Remark 6.10.

Definition 8.9. We define $\underline{\operatorname{Sp}}_{S \triangleright T}^{P} \coloneqq \operatorname{Sp} \otimes \underline{\operatorname{CMon}}_{S \triangleright T}^{P}$, and we write Σ_{+}^{∞} for the composite

$$\underline{\operatorname{Spc}}_{S \triangleright T} \xrightarrow{\mathbb{P}} \underline{\operatorname{CMon}}_{S \triangleright T}^{P} \xrightarrow{\Sigma^{\infty}} \operatorname{Sp} \otimes \underline{\operatorname{CMon}}_{S \triangleright T}^{P} = \underline{\operatorname{Sp}}_{S \triangleright T}^{P}.$$

Remark 8.10. Note that Σ^{∞}_{+} is by construction an extension of the *S*-functor $\Sigma^{\infty}_{+} \coloneqq \Sigma^{\infty} \circ \mathbb{P} \colon \underline{\mathrm{Spc}}_{S} \to \mathrm{Sp} \otimes \underline{\mathrm{CMon}}_{S}^{P} = \underline{\mathrm{Sp}}_{S}^{P}$ from [CLL23, Definition 6.2.12].

Combining the above fiberwise results with the universal property of $\underline{\text{CMon}}_{S \triangleright T}^{P}$ from Theorem 6.19 we get:

Theorem 8.11. The *T*-category $\underline{Sp}_{S \triangleright T}^P$ is *S*-presentable and *P*-stable. For any *S*-cocomplete *P*-stable *T*-category \mathcal{D} evaluation at $\mathbb{S} \coloneqq \Sigma^{\infty}_{+}(*)$ induces an equivalence $\underline{Fun}_{T}^{S-cc}(\underline{Sp}_{S \triangleright T}^P, \mathcal{D}) \simeq \mathcal{D}.$

We can also compare this to Sp_T^P :

Theorem 8.12. The essentially unique S-cocontinuous functor $\iota_! : \underline{Sp}_{S \triangleright T}^P \to \underline{Sp}_T^P$ preserving \mathbb{S} is fully faithful. Moreover, it admits an S-right adjoint ι^* , which in turn admits a further S-right adjoint ι_* (again fully faithful for formal reasons).

Proof. The functor $\operatorname{Sp} \otimes \iota_{!} \colon \operatorname{Sp}_{S \triangleright T}^{P} = \operatorname{Sp} \otimes \operatorname{CMon}_{S \triangleright T}^{P} \to \operatorname{Sp} \otimes \operatorname{CMon}_{T}^{P} = \operatorname{Sp}_{T}^{P}$ admits an *S*-right adjoint given by $\operatorname{Sp} \otimes \iota^{*}$ (as ι^{*} is itself an *S*-left adjoint). For each $A \in T$ the unit id $\to \operatorname{Sp} \otimes (\iota^{*}\iota_{!})$ is then induced by the unit of $\iota_{!} \dashv \iota^{*}$, so it is an equivalence as $\iota_{!}$ is fully faithful (Theorem 6.18). Thus, also $\operatorname{Sp} \otimes \iota_{!}$ is fully faithful. Moreover, it sends $\Sigma_{+}^{\infty}(*)$ to $\Sigma_{+}^{\infty}(*)$ simply by naturality, so this is the functor $\operatorname{Sp}_{S \triangleright T}^{P} \to \operatorname{Sp}_{T}^{P}$ in question.

It only remains to show that also $\text{Sp} \otimes \iota^*$ admits an *S*-right adjoint. However, by construction it admits a pointwise right adjoint, and it is moreover *S*-cocontinuous as a consequence of Corollary 8.6 (for T = S), so the claim follows.

9. The Universal property of equivariant spectra

In this section, we will describe the universal equivariantly presentable equivariantly stable (i.e. Orb-stable) global category in terms of classical equivariant stable homotopy theory.

9.1. **G-equivariant spectra.** We start by introducing the global category of equivariant spectra, and state our main results.

Definition 9.1. We write **Spectra** for the 1-category of symmetric spectra [HSS00] in simplicial sets. For any finite G, we write **G-Spectra** for the category of G-objects; by slight abuse of language, we will refer to its objects simply as G-spectra.

We refer the reader to [Hau17, Definition 2.35] for the definition of the G-stable weak equivalences of G-spectra. Below, we will simply refer to these as G-equivariant weak equivalences.

Proposition 9.2 (See [Hau17, Theorem 4.8 and Proposition 4.9]). The category G-Spectra carries a model structure with

- (1) weak equivalences the G-equivariant weak equivalences
- (2) acyclic fibrations those maps f such that f_n is a \mathcal{G}_{G,Σ_n} -acyclic fibration for every $n \ge 0$.

We call this the G-equivariant projective model structure. It is combinatorial and stable. $\hfill \Box$

All that we will need to know about this model structure below is that the sphere spectrum is cofibrant, which follows from [Hau17, discussion after Corollary 2.26] or by simply observing that the above acyclic fibrations are surjective in degree 0 and hence have the right lifting property against $0 \rightarrow S$.

Lemma 9.3. Let $\alpha: G \to G'$ be any homomorphism. Then $\alpha^*: G'$ -Spectra $\to G$ -Spectra is left Quillen with respect to the above model structures.

Proof. Factoring α , we may assume that it is either injective or surjective. In the first case, the claim is an instance of [Hau17, 5.2], while in the latter case it follows by combining 5.3 and 5.1 of *op. cit.*

As before, we therefore get a global category $\underline{\mathscr{Sp}}$ with $\mathscr{Sp}_G := \underline{\mathscr{Sp}}(G)$ the localization of (projectively cofibrant) G-spectra at the G-weak equivalences, and with structure maps given by the left derived functors $\mathbf{L}\alpha^*$. We will refer to this as the global category of equivariant spectra. It has a natural section \mathbb{S} given by the equivariant sphere spectra (determined by the usual sphere in \mathscr{Sp}_1).

Using this, we can now state our main results:

Theorem 9.4. The global category $\underline{\mathcal{S}p}$ is equivariantly presentable and equivariantly stable. For any other equivariantly cocomplete equivariantly stable \mathcal{D} evaluation at \mathbb{S} defines an equivalence $\underline{\operatorname{Fun}}_{\operatorname{Glo}}^{\operatorname{eq-cc}}(\underline{\mathcal{S}p}, \mathcal{D}) \simeq \mathcal{D}$.

Theorem 9.5. The essentially unique equivariantly cocontinuous global functor $\underline{Sp}_{Orb}^{Orb} \subseteq \underline{Sp}$ sending \mathbb{S} to \mathbb{S} is an equivalence.

The proof will be given at the end of this section. For now let us stop to observe that some pleasant properties one might have hoped for $\underline{\mathrm{Sp}}_{S\triangleright T}^{P}$ to satisfy do not hold even for $\underline{\mathrm{Sp}}_{\mathrm{Orb}}^{\mathrm{Orb}} = \underline{\mathscr{Sp}}$:

Warning 9.6. For any $f: G \to G'$ the functor $\mathbf{L}f^*: \mathscr{P}p_{G'} \to \mathscr{P}p_G$ admits a right adjoint $\mathbf{R}f_*$ by Lemma 9.3. However, these do *not* satisfy the Beck–Chevalley condition in general (i.e. $\mathscr{P}p$ does not have finite global products). To see this, consider the pullback

$$\begin{array}{ccc} \mathbb{Z}/2 \times \mathbb{Z}/2 & \xrightarrow{\mathrm{pr}_2} & \mathbb{Z}/2 \\ & & & \downarrow q \\ & & \mathbb{Z}/2 & \xrightarrow{q} & 1 \end{array}$$

in Glo, giving rise to a map $\mathbf{L}q^*\mathbf{R}q_*X \to \mathbf{R}\mathrm{pr}_{2^*}\mathbf{L}\mathrm{pr}_1^*X$ for any $X \in \mathscr{P}p_{\mathbb{Z}/2}$; we will now show that this cannot be an equivalence for $X = \mathbb{S}$ by computing the result of applying $\mathbf{R}q_*$ to both sides:

The functor $\mathbf{R}q_*$ is given by taking categorical $\mathbb{Z}/2$ -fixed points, so the tom Dieck-splitting [tD75] tells us that

$$\mathbf{R}q_*\mathbb{S}\simeq\bigvee_{G\subset\mathbb{Z}/2}\Sigma^\infty_+B\bigl((\mathbb{Z}/2)/G\bigr).$$

The right hand side is actually cofibrant, so $\mathbf{L}q^*\mathbf{R}q_*\mathbb{S}$ is simply given by equipping this with the trivial $\mathbb{Z}/2$ -action. Accordingly, another application of the tom Dieck splitting shows

$$\mathbf{R}q_*\mathbf{L}q^*\mathbf{R}q_*\mathbb{S} \simeq \bigvee_{G \subset \mathbb{Z}/2} \bigvee_{H \subset \mathbb{Z}/2} \Sigma^{\infty}_+ \big(B\big((\mathbb{Z}/2)/G\big) \times B\big((\mathbb{Z}/2)/H\big)\big).$$

If we take π_0 , then each wedge summand contributes a summand of \mathbb{Z} (being the unreduced suspension of a connected space), so $\pi_0(\mathbf{R}q_*\mathbf{L}q^*\mathbf{R}q_*\mathbb{S})$ is free abelian of rank 4.

On the other hand, by uniqueness of adjoints $\mathbf{R}q_*\mathbf{R}\mathrm{pr}_{2*}$ agrees with $\mathbf{R}r_*$ for $r: \mathbb{Z}/2 \times \mathbb{Z}/2 \to 1$ the unique map, so $\mathbf{R}q_*\mathbf{R}\mathrm{pr}_{2*}\mathbf{L}\mathrm{pr}_1^*\mathbb{S}$ is given by the categorical $(\mathbb{Z}/2 \times \mathbb{Z}/2)$ -fixed points of S. By another application of the tom Dieck splitting (or using the classical computation of the zeroth equivariant homotopy groups of S as the Burnside ring), we therefore see that $\pi_0(\mathbf{R}q_*\mathbf{R}\mathrm{pr}_{2*}\mathbf{L}\mathrm{pr}_1^*\mathbb{S}) \cong \pi_0^{\mathbb{Z}/2 \times \mathbb{Z}/2}(\mathbb{S})$ is free abelian of rank the number of subgroups of $\mathbb{Z}/2 \times \mathbb{Z}/2$, which is 5 instead of 4.

Remark 9.7. The extra \mathbb{Z} -summand in $\pi_0(\mathbf{R}q_*\mathbf{R}\mathrm{pr}_{2*}\mathbf{L}\mathrm{pr}_1^*\mathbb{S})$ can be attributed to the fact that $\mathbb{Z}/2 \times \mathbb{Z}/2$ has a subgroup that is not given as a product of subgroups of its factors, namely the diagonal subgroup. A similar phenomenon appears for general G, and as observed in [Nic22] this is what prevents the tom Dieck map

$$\bigvee_{(H \subset G)/\text{conj.}} \Sigma^{\infty} \left(E(W_G H) \wedge_{W_G H} X^H \right) \to F^G \Sigma^{\infty} X \tag{15}$$

for a pointed G-simplicial set X from being a global weak equivalence instead of just a non-equivariant weak equivalence: after taking categorical K-fixed points on both sides, the left hand side only contains the wedge summands of the tom Dieck splitting of $F^{K\times G}\Sigma^{\infty}X$ corresponding to subgroups of the form $L \times H \subset K \times G$ for $L \subset K, H \subset G$. In fact, this is the only obstruction to (15) being a global weak equivalence, see *op. cit.* for details.

Warning 9.8. \mathcal{P}_{p} is neither globally cocomplete nor fiberwise complete, and hence neither is $\underline{Sp}_{Orb \ Glo}^{Orb}$ by Theorem 9.5. In fact, already the restriction functor $\mathbf{L}q^*: \mathcal{P}_{p_1} \to \mathcal{P}_{\mathbb{Z}/2}$ induced by the unique map $q: \mathbb{Z}/2 \to 1$ does not preserve all products, and in particular it does not admit a left adjoint. The third author learned the following argument for this fact from Denis Nardin: By [BDS16, Theorem 3.3], $\mathbf{L}q^*$ preserves all products if and only if $\mathbf{R}q_*$ preserves compact objects. However, as observed above $\mathbf{R}q_*\mathbb{S}$ contains $\Sigma^{\infty}_+ B(\mathbb{Z}/2)$ as a wedge summand. As the latter is not compact, neither is $\mathbf{R}q_*\mathbb{S}$, yielding the desired contradiction. A similar argument shows that $\mathbf{L}q^*$ does not have a left adjoint whenever q has a non-trivial kernel. 9.2. **G-global spectra.** As before, the proof of Theorems 9.4 and 9.5 will proceed via comparison with a model of the universal *globally* presentable equivariantly stable category.

Definition 9.9. A map $f: X \to Y$ in *G***-Spectra** is called a *G*-global weak equivalence if $\alpha^* f$ is an *H*-equivariant weak equivalence for every finite group *H* and every homomorphism $\alpha: H \to G$.

We emphasize that we are *not* deriving α^* here with respect to the equivariant model structures (as otherwise this would of course simply recover the *G*-weak equivalences again).

Proposition 9.10 (See [Len20, Corollary 3.1.46–Proposition 3.1.48]). The category **G-Spectra** admits a model structure with

- (1) weak equivalences the G-global weak equivalences
- (2) cofibrations the injective cofibrations.

We call this the injective G-global model structure. It is combinatorial, simplicial, proper, and stable. $\hfill \Box$

Basically by definition, the restriction functors $\alpha^* : \mathbf{G'}$ -Spectra $\rightarrow \mathbf{G}$ -Spectra are homotopical and left Quillen. In particular, we again obtain a global category $\mathscr{Sp}^{\mathrm{gl}}$.

Theorem 9.11 (See [CLL23, Corollary 7.3.3]). <u>Sp</u> is globally presentable and equivariantly stable. The essentially unique globally cocontinuous functor $\underline{Sp}_{Glo}^{Orb} \rightarrow \underline{Sp}$ sending $\Sigma_{+}^{\infty}(*)$ to the global sphere spectrum \mathbb{S} is an equivalence.

9.3. **Proof of Theorems 9.4 and 9.5.** Let us begin with a comparison of the above models complementing Theorem 8.12:

Lemma 9.12. There is a global functor $\mathbf{Lid}: \underline{\mathscr{P}p} \to \underline{\mathscr{P}p}^{gl}$ with the following properties:

- (1) It is fully faithful and sends S to S.
- (2) It admits an Orb-right adjoint.

Proof. For any G, [Len20, Proposition 3.3.1] provides a Quillen adjunction

 $\mathrm{id}: \mathbf{G}\operatorname{-}\mathbf{Spectra}_{G\operatorname{-}\mathrm{equiv.\ proj.}} \rightleftharpoons \mathbf{G}\operatorname{-}\mathbf{Spectra}_{G\operatorname{-}\mathrm{gl.\ inj.}} : \mathrm{id}.$ (16)

In particular, *G*-equivariant weak equivalences between projectively cofibrant spectra are *G*-global weak equivalences (also see Lemma 9.3), so the inclusion of projectively cofibrant objects yields a functor \mathbf{L} id: $\underline{\mathscr{P}p} \to \underline{\mathscr{P}p}^{gl}$ sending \mathbb{S} to \mathbb{S} . Moreover, the right adjoint in (16) evidently induces a localization, so that \mathbf{L} id is fully faithful.

It only remains that the right adjoints assemble into an Orb-right adjoint. However, the pointset level Beck–Chevalley maps $\alpha^* \circ id \Rightarrow id \circ \alpha^*$ are isomorphisms for trivial reasons, and for *injective* α , α^* is also homotopical in the equivariant world [Hau17, 5.2], so that this already models the derived Beck–Chevalley map.

Proof of Theorems 9.4 and 9.5. By Theorem 8.11 it is enough to prove that $\underline{\mathscr{P}p}$ is equivariantly stable and equivariantly cocomplete, and that the preferred map $\underline{\operatorname{Sp}}_{\operatorname{Orb}}^{\operatorname{Orb}} \xrightarrow{\operatorname{Gp}} p$ is an equivalence.

For this, let us write \mathcal{E} for the essential image of \mathbf{L} id: $\underline{\mathscr{P}p} \to \underline{\mathscr{P}p}^{\mathrm{gl}}$; this is then closed under equivariant colimits as \mathbf{L} id admits an Orb-right adjoint, and it is closed under desuspension as each $\mathscr{P}p_G$ is stable. It follows that \mathcal{E} and hence also $\underline{\mathscr{P}p}$ is indeed equivariantly cocomplete and equivariantly stable.

Now let $F: \underline{Sp}_{Orb \triangleright Glo}^{Orb} \to \underline{\mathscr{G}p}$ be the unique equivariantly cocontinuous functor preserving S. Then $\mathbf{L} \operatorname{id} \circ F: \underline{Sp}_{Orb \triangleright Glo}^{Orb} \to \underline{\mathscr{G}p}^{gl}$ is an equivariantly cocontinuous functor sending S to S. The same holds for the composite

$$\underline{\operatorname{Sp}}_{\operatorname{Orb} \rhd \operatorname{Glo}}^{\operatorname{Orb}} \xrightarrow{\iota_!} \underline{\operatorname{Sp}}_{\operatorname{Glo}}^{\operatorname{Orb}} \xrightarrow{\sim} \underline{\mathscr{G}p}^{\operatorname{gl}}$$

of the fully faithful functor from Theorem 7.16 with the equivalence from Theorem 9.11, so they have to agree by the universal property of $\underline{\operatorname{Sp}}_{\operatorname{Orb}}^{\operatorname{Orb}}_{\operatorname{Orb}}_{\operatorname{Glo}}$. In particular, F is fully faithful. To see that it is also essentially surjective, it is by [Hau17, Proposition 4.9] enough to see that it hits the suspension spectra $\Sigma^{\infty}_{+}(G/H)$ for all $H \subset G$. However, as before we have $i_! \mathbb{S} \simeq \Sigma^{\infty}_{+}(G/H)$ for $i: H \hookrightarrow G$ the inclusion, so the claim follows from the defining properties of F.

Again this immediately implies a variant for the G-category of G-spectra for any finite group G:

Theorem 9.13. Recall the functor v_G : $Orb_G \to Glo$ from Construction 5.11.

- (1) The G-category $v_G^* \underline{\mathscr{G}p}$ (sending G/H to $\mathscr{G}p_H$) is G-presentable and G-stable. Moreover, the unique left adjoint $\underline{\operatorname{Sp}}_{\operatorname{Orb}_G} \to v_G^* \underline{\mathscr{G}p}$ preserving \mathbb{S} is an equivalence.
- (2) For any G-cocomplete G-stable \mathcal{D} , evaluation at \mathbb{S} defines an equivalence $\underline{\operatorname{Fun}}_{\operatorname{Orb}_{G}}^{\operatorname{G-cc}}(v_{G}^{*}\underline{\mathscr{G}p}, \mathcal{D}) \simeq \mathcal{D}.$

A proof of this has previously been sketched by Nardin as [Nar16, Theorem A.4].

Proof. Arguing as in the unstable (Theorem 5.12) and semiadditive case (Theorem 7.29), it only remains to show that there is for any atomic orbital $P \subset T$ and $A \in T$ an equivalence $\pi_A^* \underline{\operatorname{Sp}}_T^P \simeq \underline{\operatorname{Sp}}_{T/A}^{T/A}$ preserving \mathbb{S} . This however follows at once from Proposition 7.30 by applying $\operatorname{Sp} \otimes -$ to both sides.

References

- [ABFJ22] Mathieu Anel, Georg Biedermann, Eric Finster, and André Joyal, Left-exact localizations of ∞-topoi. I: Higher sheaves, Adv. Math. 400 (2022), 64, Id/No 108268.
- [BD20] Paul Balmer and Ivo Dell'Ambrogio, Mackey 2-functors and Mackey 2-motives, EMS Monogr. Math., European Mathematical Society (EMS), 2020.
- [BDG⁺16] Clark Barwick, Emanuele Dotto, Saul Glasman, Denis Nardin, and Jay Shah, Parametrized higher category theory and higher algebra: Exposé I – Elements of parametrized higher category theory, arXiv:1608.03657 (2016).
- [BDS16] Paul Balmer, Ivo Dell'Ambrogio, and Beren Sanders, Grothendieck-Neeman duality and the Wirthmüller isomorphism, Compos. Math. 152 (2016), no. 8, 1740–1776.
- [CLL23] Bastiaan Cnossen, Tobias Lenz, and Sil Linskens, Parametrized stability and the universal property of global spectra, arXiv:2301.08240 (2023).
- [Elm83] Anthony D. Elmendorf, Systems of fixed point sets, Trans. Am. Math. Soc. 277 (1983), 275–284.
- [Hau17] Markus Hausmann, G-symmetric spectra, semistability and the multiplicative norm, J. Pure Appl. Algebra 221 (2017), no. 10, 2582–2632.
- [Hau19] _____, Symmetric spectra model global homotopy theory of finite groups, Algebr. Geom. Topol. 19 (2019), no. 3, 1413–1452.

- [Hil22] Kaif Hilman, Parametrised presentability over orbital categories, arXiv:2202.02594 (2022).
- [HSS00] Mark Hovey, Brooke Shipley, and Jeff Smith, Symmetric spectra, J. Am. Math. Soc. 13 (2000), no. 1, 149–208.
- [HY17] Asaf Horev and Lior Yanovski, On conjugates and adjoint descent, Topology Appl. 232 (2017), 140–154.
- [Len20] Tobias Lenz, G-Global Homotopy Theory and Algebraic K-Theory, to appear in Mem. Amer. Math. Soc., arXiv:2012.12676 (2020).
- [Lin23] Sil Linskens, Globalizing and stabilizing global ∞ -categories, in preparation, 2023.
- [LNP22] Sil Linskens, Denis Nardin, and Luca Pol, Global homotopy theory via partially lax limits, arXiv:2206.01556 (2022).
- [Lur09] Jacob Lurie, Higher topos theory, Ann. Math. Stud., vol. 170, Princeton University Press, Princeton, NJ, 2009, updated version available at https://people.math.harvard.edu/~lurie/papers/highertopoi.pdf.
- [Lur17] _____, Higher algebra, https://www.math.ias.edu/~lurie/papers/HA.pdf (2017).
- [Lur18] _____, Spectral Algebraic Geometry, under construction (version dated February 2018), www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf (2018).
- [Mar21] Louis Martini, Yoneda's lemma for internal higher categories, arXiv:2103.17141 (2021).
- [MMO17] J. Peter May, Mona Merling, and Angélica M. Osorno, Equivariant Infinite Loop Space Theory, the Space Level Story, to appear in Mem. Amer. Math. Soc., available as arXiv:1704.03413 (2017).
- [MW21] Louis Martini and Sebastian Wolf, Limits and colimits in internal higher category theory, arXiv:2111.14495 (2021).
- [MW22] _____, Presentable categories internal to an ∞ -topos, arXiv:2209.05103 (2022).
- [Nar16] Denis Nardin, Parametrized higher category theory and higher algebra: Exposé IV Stability with respect to an orbital ∞-category, arXiv:1608.07704 (2016).
- [Nar17] _____, Stability and distributivity over orbital ∞-categories, Ph.D. thesis, Massachusetts Institute of Technology, 2017.
- [Nic22] Jana Nickel, The tom Dieck splitting in global equivariant stable homotopy theory, Master's thesis, Rheinische Friedrich-Wilhelms-Universität Bonn, 2022.
- [Ost16] Dominik Ostermayr, Equivariant Γ-spaces, Homology Homotopy Appl. 18 (2016), no. 1, 295–324.
- [Sch18] Stefan Schwede, Global homotopy theory, New Math. Monogr., vol. 34, Cambridge University Press, 2018.
- [Sha21] Jay Shah, Parametrized higher category theory, to appear in Algebr. Geom. Topol., arXiv:1809.05892 (2021).
- [Shi89] Kazuhisa Shimakawa, Infinite Loop G-Spaces Associated to Monoidal G-Graded Categories, Publ. Res. Inst. Math. Sci. 25 (1989), no. 2, 239–262.
- [Shi91] _____, A note on Γ_G -spaces, Osaka J. Math. 28 (1991), no. 2, 223–228.
- [Ste16] Marc Stephan, On equivariant homotopy theory for model categories, Homology Homotopy Appl. 18 (2016), no. 2, 183–208.
- [tD75] Tammo tom Dieck, Orbittypen und äquivariante Homologie. II, Arch. Math. **26** (1975), 650–662.

B.C.: Mathematisches Institut, Rheinische Friedrich-Wilhelms-Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany

T.L.: MATHEMATICAL INSTITUTE, UNIVERSITY OF UTRECHT, BUDAPESTLAAN 6, 3584 CD UTRECHT, THE NETHERLANDS

S.L.: MATHEMATISCHES INSTITUT, RHEINISCHE FRIEDRICH-WILHELMS-UNIVERSITÄT BONN, EN-DENICHER ALLEE 60, 53115 BONN, GERMANY