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On the Strict Arnold Chord Property and Coisotropic Submanifolds of Complex Projective Space

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Let α be a contact form on a manifold M, and $L \subseteq M$ a closed Legendrian submanifold. I prove that L intersects some characteristic for α at least twice if all characteristics are closed and of the same period, and α embeds nicely into the product of $\mathbb{R}^{2n}$ and an exact symplectic manifold. As an application of the method of proof, the minimal action of a regular closed coisotropic submanifold of complex projective space is at most $\pi/2$. This yields an obstruction to presymplectic embeddings, and in particular to Lagrangian embeddings.

1 Main Results

1.1 The strict chord property

Let M be a manifold (possibly noncompact or with boundary) and α a contact form on M. We say that (M, α) has the *strict chord property* iff for every nonempty closed Legendrian submanifold $L \subseteq M$ there exists a characteristic for α that intersects L at least twice. (Here "closed" means "compact and without boundary", and "characteristic" means a leaf of the foliation determined by the integrable distribution $\ker(d\alpha : TM \to T^*M)$ on M, i.e., an *unparametrized* Reeb trajectory.) To explain this terminology, note that parametrizing part of such a characteristic, we obtain a *strict Reeb chord*, that is, an integral curve of the Reeb vector field that starts and ends at different points of L.

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Such chords arise in classical mechanics as libration motions, that is, oscillations of a mechanical system between two rest points, see [11, p. 118]. The present article is concerned with the following problem.

Problem (strict chord problem)**.** Find conditions on (M, α) under which it has the strict chord property. □

In [4, p. 11], Arnol'd conjectured that for $n \geq 2$ any contact form on S^{2n-1} inducing the standard structure has the strict chord property. The main result of this article roughly is that this property holds for every contact form on a manifold if all its characteristics are closed and of the same period and the contact form nicely embeds into the product of $\mathbb{R}^{2n}$ and an exact symplectic manifold. In particular, this confirms Arnol'd's conjecture for the standard form on S^{2n-1}.

To state the result, let M be a manifold and α a contact form on M. The period of a closed characteristic C for α is the number

$$\left| \int_C \iota^* \alpha \right|,$$

where $\iota : C \to M$ denotes the inclusion and we equip C with either orientation. We denote by $q^1, p_1, \ldots, q^n, p_n$ the standard coordinates of $\mathbb{R}^{2n}$, and define

$$\lambda_0 := \frac{1}{2} \sum_{i=1}^{n} (q^i dp_i - p_i dq^i).$$

Theorem 1 (strict chord property)**.** The pair (M, α) has the strict chord property if all characteristics for α are closed and of equal period T, and there exist a manifold W together with a one-form λ, an integer $n \geq \frac{1}{2} \dim W + 2$, and an embedding $\varphi : M \to \mathbb{R}^{2n} \times W$, such that $d\lambda$ is a geometrically bounded symplectic form, and

$$\dim M = 2n + \dim W - 1, \tag{1}$$

$$\varphi(M) \subseteq \bar{B}^{2n}(T) \times W, \tag{2}$$

$$\varphi^*(\lambda_0 \oplus \lambda) = \alpha. \tag{3}$$

Here $\bar{B}^{2n}(a)$ denotes the closed ball in $\mathbb{R}^{2n}$ of radius $\sqrt{a/\pi}$. □

We call a symplectic form ω on a manifold W geometrically bounded iff there exists an ω-compatible almost complex structure J on W such that the metric $\omega(\cdot, J\cdot)$ is complete with bounded sectional curvature and injectivity radius bounded away from 0. Examples are closed symplectic manifolds, cotangent bundles of closed manifolds, and symplectic vector spaces.

The proof of Theorem 1 is based on a result by Chekanov, which implies that the displacement energy of a closed Lagrangian submanifold in a geometrically bounded symplectic manifold is at least the minimal symplectic action of the Lagrangian.

Assuming by contradiction that there is no strict Reeb chord, such a Lagrangian is constructed from the given Legendrian submanifold by moving it with the Reeb flow. This technique is a variation on the approach used by Mohnke in [23]. (In that article the Lagrangian was obtained by moving the Legendrian both with the Reeb flow and with the Liouville flow.)

A crucial ingredient of the proof is the fact that the displacement energy of a compact subset X of the closed unit ball $\bar{B}^{2n}$ is strictly less than π, provided that X does not contain the unit sphere (see Lemma 12).

Theorem 1 has the following immediate application. We denote by $\iota : S^{2n-1} \to \mathbb{R}^{2n}$ the inclusion.

Corollary 2 (sphere)**.** For $n \geq 2$ the standard contact form $\alpha_0 := \iota^* \lambda_0$ on S^{2n-1} has the strict chord property. □

More examples are obtained by the following construction. By an *exact Hamiltonian S^1-manifold,* we mean a triple consisting of a smooth manifold W, a smooth S^1-action ρ on W, and an ρ-invariant one-form λ on W, such that $d\lambda$ is non-degenerate. We fix such a triple and numbers $c \in (0, \infty)$ and $n \in \mathbb{N} \cup \{0\}$. We denote by X the vector field generated by ρ. (This is the infinitesimal action of the element $1 \in \mathbb{R} = \mathrm{Lie} S^1$, where we identify S^1 with $\mathbb{R}/2\pi\mathbb{Z}$.) We define

$$H_0 : \mathbb{R}^{2n} \to \mathbb{R}, \quad H_0(x_0) := \frac{1}{2}|x_0|^2, \tag{4}$$

$$H := \lambda(X) : W \to \mathbb{R}, \tag{5}$$

$$M := \{(x_0, x) \in (\bar{B}^{2n}(2\pi c) \setminus \{0\}) \times W \mid H_0(x_0) + H(x) = c\}. \tag{6}$$

We denote by

$$\iota : M \to \tilde{W} := \mathbb{R}^{2n} \times W$$

the inclusion.

Proposition 3 (contact form)**.** The set M is a (smoothly embedded) hypersurface in $\tilde{W}$, and

$$\alpha := \iota^*(\lambda_0 \oplus \lambda) \tag{7}$$

is a contact form on M all of whose characteristics are closed and of period $2\pi c$. (M may have a boundary.) □

Corollary 4 (strict chord property)**.** If $(W, d\lambda)$ is geometrically bounded and $n \geq \frac{1}{2}\dim W + 2$ then (M, α) has the strict chord property. □

Proof of Corollary 4. This follows from Theorem 1, using Proposition 3 and the facts that $(\mathbb{R}^{2n}, \omega_0)$ is geometrically bounded, geometric boundedness is invariant under products, and that conditions (1–3) are satisfied with $\varphi := \iota$. ■

Example 5. Let X be a manifold. We define $W := T^*X$ and λ to be the canonical one-form on W. We fix a smooth S^1-action σ on X and define

$$\rho : S^1 \to \mathrm{Diff}(W), \quad \rho(z)(q, p) := (\sigma_z(q), p\, d\sigma_z(q)^{-1}),$$

where $\sigma_z := \sigma(z)$. The triple (W, ρ, λ) is an exact Hamiltonian S^1-manifold, and $(W, d\lambda)$ is geometrically bounded. Hence by Corollary 4 the pair (M, α), defined as in (6, 7), is a contact manifold that has the strict chord property, provided that $n \geq \frac{1}{2}\dim W + 2$. □

Remark. There exist contact forms on closed manifolds that do not have the strict chord property. The simplest example is the standard contact form on S^1.

Another example, which is taken from [23], goes as follows. We denote by γ the standard angular form on S^1 that integrates to 2π. Consider the contact one-form on $M := S^1 \times S^2$ given by

$$\alpha := x_1\gamma + \tfrac{1}{2}(x_2 dx_3 - x_3 dx_2),$$

where $x \in S^2 \subseteq \mathbb{R}^3$. Each Legendrian loop $S^1 \times \{(0, x_2, x_3)\}$ intersects each Reeb orbit $\{z\} \times \{x_1 = 0\}$ (with $z \in S^1$) only once. □

1.2 Minimal action of a regular coisotropic submanifold of complex projective space

The bound on the displacement energy of a compact subset of $\bar{B}^{2n}$, which is used in the proof of Theorem 1, and a coisotropic version of Chekanov's theorem have the following application. Let (M, ω) be a symplectic manifold and $N \subseteq M$ a coisotropic submanifold. (This means that for every $x \in N$ the symplectic complement $T_xN^\omega := \{v \in T_xM \mid \omega(v, w) = 0, \forall w \in T_xN$ is contained in T_xN.)

The set

$$TN^\omega = \{(x, v) \mid x \in N,\ v \in T_xN^\omega\} \subseteq TN$$

is an involutive distribution on N. Hence by Frobenius' theorem it gives rise to a foliation on N. Its leaves are called the isotropic (or characteristic) leaves of N. We denote by N_ω the set of all these leaves, and by $\mathbb{D} \subseteq \mathbb{R}^2$ the closed unit disk. We define the *action* *spectrum* and the *minimal action (or area)* of N to be

$$S(N) := S(M,N) := S(M,\omega,N) := \left\{ \int_{\mathbb{D}} u^*\omega \;\middle|\; u \in C^\infty(\mathbb{D}, M) : \exists F \in N_\omega : u(S^1) \subseteq F \right\}, \tag{8}$$

$$A(N) := A(M,N) := A(M,\omega,N) := \inf(S(N) \cap (0,\infty)) \in [0,\infty]. \tag{9}$$

Remarks.

- In the case $N = M$ we have

$$A(M) = \inf\left(\left\{ \int_{S^2} u^*\omega \;\middle|\; u \in C^\infty(S^2, M) \right\} \cap (0,\infty)\right).$$

(see [28, Lemma 29])

- If N is Lagrangian then

$$A(N) = \inf\left(\left\{ \int_{\mathbb{D}} u^*\omega \;\middle|\; u \in C^\infty(\mathbb{D}, M) : u(S^1) \subseteq N \right\} \cap (0,\infty)\right). \qquad \square$$

We call N *regular* iff there exists a manifold structure on N_ω, such that the canonical projection $\pi : N \to N_\omega$ is a smooth submersion (Assume that this is the case. Then the topology on N_ω induced by its manifold structure is by definition Hausdorff and second countable. It agrees with the quotient topology. The symplectic quotient of N is well-defined in the sense that the manifold structure on N_ω as above is unique and there exists a unique symplectic form on N_ω that pulls back to $\iota^*\omega$ under π. Here $\iota : N \to M$ denotes the inclusion). Examples are Lagrangian submanifolds, $N = M$, and the sphere $N = S^{2n-1} \subseteq M = \mathbb{R}^{2n}$ (for further examples, see [29]). Regularity is invariant under taking products.

Let $n \in \mathbb{N} = \{1, 2, \ldots\}$. We equip the complex projective space $\mathbb{C}\mathrm{P}^n$ with the Fubini-Study form ω_{FS}. (This form is normalized in such a way that the area of a projective line is π.)

Theorem 6 (minimal action)**.** Let (M,ω) be a geometrically bounded symplectic manifold, and $\emptyset \neq N$ a regular closed coisotropic submanifold of $\mathbb{C}\mathrm{P}^n \times M$ of dimension $< 2n$. Then

$$A(N) \leq \frac{\pi}{2}. \qquad \square$$

Remarks.

- The hypothesis that $\dim N < 2n$ cannot be dropped. Otherwise,

$$M := \{\mathrm{pt}\}, \quad N := \mathbb{C}\mathrm{P}^n$$

and

$$M := \mathbb{C}\mathrm{P}^n, \quad \omega := -\omega_{\mathrm{FS}}, \quad N := \{(x, x) \mid x \in \mathbb{C}\mathrm{P}^n\}$$

are counterexamples.

- The hypothesis that N be regular cannot be dropped. To see this, let $n \geq 2$. Then there exists a closed hypersurface $N_0 \subseteq \mathbb{R}^{2n}$ without any closed characteristic, see [17] and references therein. (In the case $n \geq 3$ this hypersurface may be chosen to be smooth, but for $n = 2$ the hypersurface constructed in [17] is only C^2.) By shrinking N_0 with a homothety and using a Darboux chart, we obtain a hypersurface N inside $\mathbb{C}\mathrm{P}^n$ with the same property. It satisfies $A(N) = \pi$. (but is not regular.) □

Corollary 7 (minimal action). Let (M, ω) be a geometrically bounded symplectic manifold, such that $\dim M < 2n$. Then the minimal action of a closed nonempty Lagrangian submanifold of $\mathbb{C}\mathrm{P}^n \times M$ is bounded above by $\pi/2$. □

Proof. This follows from Theorem 6 and the fact that every Lagrangian submanifold is regular. ■

Remark. By Corollary 7 there is no *exact* Lagrangian submanifold of $\mathbb{C}\mathrm{P}^n \times M$. By this we mean a Lagrangian submanifold with minimal action equal to π. □

To explain a further application of Theorem 6, recall that a presymplectic form on a manifold M is a closed two-form ω on M, such that

$$\mathrm{corank}\,\omega_x := \dim(T_x M)^\omega$$

does not depend on $x \in M$. (Here $(T_x M)^\omega = \{v \in T_x M \mid \omega(v, w) = 0,\ \forall w \in T_x M\}$.) Let ω be such a form. The set

$$TM^\omega = \{(x, v) \mid x \in N,\ v \in T_x M^\omega\} \subseteq TM$$

is an involutive distribution on M. Hence by Frobenius' theorem it gives rise to a foliation on M. We denote by M_ω the set of its leaves. We call (M, ω) regular iff there exists a manifold structure on M_ω for which the canonical projection $\pi : M \to M_\omega$ is a smooth submersion.

A presymplectic embedding of a presymplectic manifold into another one is by definition a smooth embedding that intertwines the two presymplectic forms.

Corollary 8 (presymplectic embedding)**.** Let $n\in\mathbb{N}$ and (M,ω) be a geometrically bounded symplectic manifold, such that

$$\int_{S^2} u^*\omega\in\pi\mathbb{Z},\quad \forall u\in C^\infty(S^2,M). \tag{10}$$

Let (M',ω') be a nonempty closed regular presymplectic manifold, such that every isotropic leaf of M' is simply connected, and

$$\begin{aligned}\dim M' + \operatorname{corank}\omega' &= 2n+\dim M,\\ \dim M' &< 2n.\end{aligned} \tag{11}$$

Then (M',ω') does not presymplectically embed into the symplectic manifold $(\mathbb{C}\mathrm{P}^n\times M,\omega_{\mathrm{FS}}\oplus\omega)$. □

Example. Let F be a simply connected closed manifold of positive dimension and (X,σ) a closed symplectic manifold. By Corollary 8 the presymplectic manifold $(X\times F,\sigma\oplus 0)$ does not embed into $(\mathbb{C}\mathrm{P}^n,\omega_{\mathrm{FS}})$, where $n:=\frac{1}{2}\dim X+\dim F$. □

Corollary 8 will be proved in Section 4. It has the following immediate application.

Corollary 9 (Lagrangian embedding)**.** Let $n\in\mathbb{N}$ and (M,ω) be a geometrically bounded symplectic manifold such that (10) holds and $\dim M<2n$. Then no simply connected closed manifold embeds into $\mathbb{C}\mathrm{P}^n\times M$ in a Lagrangian way. □

1.3 Related work

Arnol'd observed in [4] that the strict chord property for (S^3,α_0) follows from an elementary argument. In [18, Corollary 1], Givental' proved that there exists a Reeb chord between every pair of Legendrian submanifolds of $\mathbb{R}\mathrm{P}^{2n-1}$ with the standard contact form, if they are isotopic via Legendrian submanifolds to the standard $\mathbb{R}\mathrm{P}^{n-1}$.

In [8], Chekanov provided lower bounds on the number of critical points of a quasi-function, that is, a Legendrian submanifold of the 1-jet bundle of a manifold, that is smoothly homotopic (via Legendrians) to the zero section. These points correspond to Reeb chords between the zero section and the Legendrian.

Abbas [1–3] proved the strict chord property for certain Legendrian knots in tight closed contact 3-manifolds.

We say that a contact form α on a manifold M has the *chord property* iff every closed Legendrian intersects some characteristic for α at least twice or it intersects some closed characteristic (i.e., periodic Reeb orbit). Note that this property is trivially satisfied iff all characteristics are closed.

Consider now a contact manifold (M, ξ) that arises as the boundary of a compact Stein manifold, and α a contact form on M inducing ξ. In [23, Theorem 2], Mohnke proved that α has the chord property. It follows that a nonempty closed Legendrian submanifold of M admits a *strict* Reeb chord if it does not intersect any closed characteristic for α. Intuitively, such Legendrian submanifolds are generic, provided that $\dim M \geq 3$ and that α has only countably many closed Reeb orbits.

In [11], Cieliebak proved that Legendrian spheres in the boundaries of certain subcritical Weinstein domains intersect some characteristic for α at least twice.

Let $U \subseteq \mathbb{R}^{2n}$ be a bounded star-shaped domain with smooth boundary and $\emptyset \neq L \subseteq \partial U$ a closed Legendrian submanifold of nonpositive curvature. The last condition means that L that admits a Riemannian metric of nonpositive sectional curvature. In the recent preprint [12], Cieliebak and Mohnke proved that L possesses a Reeb chord of length bounded above by the (toroidal) Lagrangian capacity of U, see [12, Corollary 1.12]. Using [12, Corollary 1.3], they deduced that L admits a Reeb chord of length bounded above by π/n, if $n \geq 2$, and $U = B_1^{2n}$, that is, ∂U is the unit sphere.

As explained in [12] after Corollary 1.13, it follows that there exists no exact Lagrangian embedding into $\mathbb{C}\mathrm{P}^n$ of a closed manifold $\emptyset \neq X$ of nonpositive curvature. (Corollary 7 is a stabilized version of this without the nonpositive curvature assumption.)

Cieliebak and Mohnke also proved that for $S \subseteq \mathbb{R}^{2n}$ sufficiently C^1-close to the unit sphere, every closed Legendrian submanifold $\emptyset \neq L \subseteq S$ of nonpositive curvature possesses a strict Reeb chord, see [12, Corollary 1.15].

A powerful tool for finding Reeb chords is Legendrian contact homology. Based on work by Eliashberg *et al.* [16] and Chekanov [10], this homology was developed by Bourgeois *et al.*, see [7, 13–15] and references therein.

Using embedded contact homology, Hutchings and Taubes [19, 20] proved that every contact form on a closed three-manifold has the chord property. Further results are contained in [22, 24–26].

In [27, Theorem 3.1], Seidel proved that if a closed manifold X embeds into $\mathbb{C}\mathrm{P}^n$ in a Lagrangian way then $H^1(X, \mathbb{Z}/(2n+2)) \neq 0$. In particular, X is not simply connected. Corollary 8 extends the latter statement to presymplectic embeddings into $\mathbb{C}\mathrm{P}^n \times M$.

Biran and Cieliebak [5, 6] generalized Seidel's result in various ways. In the case $\int_{S^2} u^*\omega = 0$, for every $u \in C^\infty(S^2, M)$, Corollary 9 follows from their results. Further references about results on the topology of Lagrangian embeddings are provided in [5, 6].

2 Proof of Theorem 1 (strict chord property)

The proof of Theorem 1 is based on the following construction. Let $M, \alpha, T, W, \lambda, n, \varphi$ be as in the hypothesis of Theorem 1, and $L \subseteq M$ a nonempty closed Legendrian submanifold. We construct a Lagrangian immersion in $\mathbb{R}^{2n} \times W$ by flowing L with the Reeb flow. It will follow from Theorem 11 and Lemma 12 that this immersion is not injective. This means that L admits a strict Reeb chord.

We identify

$$S^1 \cong \mathbb{R}/T\mathbb{Z},$$

and denote by R the Reeb vector field on M w.r.t. α, and by

$$\psi : S^1 \times M \to M \tag{12}$$

its flow. This map is welldefined, since by hypothesis all Reeb orbits of α are closed and of period T. We write

$$\tilde{W} := \mathbb{R}^{2n} \times W, \quad \tilde{\lambda} := \lambda_0 \oplus \lambda, \quad \tilde{\omega} := d\tilde{\lambda}, \tag{13}$$

and denote by $\iota : S^1 \times L \to S^1 \times M$ the inclusion. We define

$$f := \varphi \circ \psi \circ \iota : S^1 \times L \to \tilde{W}. \tag{14}$$

Lemma 10. The map f is a Lagrangian immersion w.r.t. $\tilde{\omega}$. □

Proof of Lemma 10. Since ψ is the flow of R, we have

$$d\psi(z, x)(T_z S^1 \times \{0\}) = \mathbb{R} R_{\psi(z,x)}, \quad \forall (z, x) \in S^1 \times M. \tag{15}$$

We show that f *is an immersion*. Since L is Legendrian, we have $TL \subseteq \ker\alpha$. Since the Reeb flow $\psi_z := \psi(z, \cdot)$ preserves α, it preserves $\ker\alpha$, for every $z \in S^1$. It follows that $d\psi(\{0\} \times TL) \subseteq \ker\alpha$. Let $(z, x) \in S^1 \times L$. Using (15) and the fact $\alpha(R) \equiv 1$, it follows that

$$d\psi(z, x)(\{0\} \times T_x L) \cap d\psi(z, x)(T_z S^1 \times \{0\}) = \{0\}. \tag{16}$$

Since ψ is a flow, $d\psi_z(x)$ is injective. It follows from (15) and the fact $R \neq 0$ that $d(\psi(\cdot, x))(z)$ is injective. Combining this with (16), it follows that

$$d\psi(z, x) : T_{(z,x)}(S^1 \times L) \to T_{\psi(z,x)} M$$

is injective. Using (14) and that φ is an immersion, it follows that the same holds for f, as claimed.

We show that f is *isotropic*. We define $\omega := d\alpha$. The equalities (3,14) and $\tilde{\omega} = d\tilde{\lambda}$ imply that

$$f^*\tilde{\omega} = d\iota^*\psi^*\varphi^*\tilde{\lambda} = d\iota^*\psi^*\alpha = \iota^*\psi^*\omega.$$

Therefore, it suffices to show that $\psi^*\omega$ vanishes on pairs of vectors in $T(S^1 \times L)$ (over the same point). To see this, note that for every $z \in S^1$ the Reeb flow $\psi_z : M \to M$ preserves ω, since it preserves α. Since L is Legendrian, it is isotropic w.r.t. ω. It follows that $\psi^*\omega$ vanishes on pairs of vectors in $\{0\} \times TL$.

The equalities (15) and $\mathbb{R}R = \ker\omega$ imply that $\psi^*\omega$ vanishes on each pair of vectors in $T(S^1 \times L)$, of which at least one lies in $TS^1 \times \{0\}$. It follows that $\psi^*\omega$ vanishes on all pairs of vectors in $T(S^1 \times L)$. This shows that f is isotropic.

Equality (1) implies that the domain of f has dimension equal to $\frac{1}{2}\dim\tilde{W}$. It follows that f is a Lagrangian immersion, as claimed. This proves Lemma 10. ■

The proof that the map f is not injective, is based on the next result, which is due to Chekanov. Let (M, ω) be a symplectic manifold. We denote by $\mathcal{H}(M, \omega)$ the set of all functions $H \in C^\infty([0, 1] \times M, \mathbb{R})$ whose Hamiltonian time t flow $\varphi_H^t : M \to M$ exists and is surjective, for every $t \in [0, 1]$. (The time t flow of a time-dependent vector field on a manifold M is always an injective smooth immersion on its domain of definition. Hence, if it is everywhere well defined and surjective then it is a diffeomorphism of M.) We define

$$\|\cdot\| : \mathcal{H}(M, \omega) \to [0, \infty], \quad \|H\| := \int_0^1 \left(\sup_M H^t - \inf_M H^t\right) dt,$$

and the *displacement energy* of a subset $X \subseteq M$ to be

$$\begin{aligned} e(X) &:= e(M, X) := e(M, \omega, X) \\ &:= \inf\{\|H\| \mid H \in \mathcal{H}(M, \omega) : \varphi_H^1(X) \cap X = \emptyset\}. \end{aligned}$$

(Alternatively, one can define a displacement energy, using only functions H with compact support. However, it seems more natural to allow for all functions in $\mathcal{H}(M, \omega)$.) Let $L \subseteq M$ be a Lagrangian submanifold. We denote by $\mathbb{D} \subseteq \mathbb{R}^2$ the closed unit disk. The *minimal symplectic action (or area) of* L is defined to be

$$A(L) := A(M, \omega, L) := \inf\left(\left\{\int_{\mathbb{D}} u^*\omega \,\middle|\, u \in C^\infty(\mathbb{D}, M) : u(S^1) \subseteq L\right\} \cap (0, \infty)\right) \in [0, \infty].$$

Theorem 11 (displacement energy of a Lagrangian)**.** If (M,ω) is geometrically bounded and L is closed then

$$e(L) \geq A(L). \qquad \square$$

Proof. This follows from the main theorem in [9]. ■

Another key ingredient in the proof that f (defined as in (14)) is not injective, is the following lemma. Let $n \in \mathbb{N}$. We denote by ω_0 the standard symplectic form on $\mathbb{R}^{2n}$.

Lemma 12 (bound on displacement energy)**.** Let $a > 0$ and X be a compact subset of the closed ball $\bar{B}^{2n}(a)$, which does not contain $S^{2n-1}(a)$, the sphere in $\mathbb{R}^{2n}$ of radius $\sqrt{a/\pi}$. Then

$$e(\mathbb{R}^{2n}, X) < a. \tag{17}$$

$\square$

Proof of Lemma 12. W.l.o.g. we may assume that $a = \pi$. Since X does not contain S^{2n-1}, there exists an orthogonal linear symplectic map $\Psi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, such that $(1, 0, \ldots, 0) \notin \Psi(X)$. We denote

$$Y_c := \{(q, p) \in \mathbb{D} \mid q \leq c\}.$$

Since $\Psi(X)$ is compact and contained in $\bar{B}^{2n}$, there exists $c < 1$, such that

$$\Psi(X) \subseteq Y_c \times \mathbb{R}^{2n-2}. \tag{18}$$

We have

$$\begin{aligned} e(\mathbb{R}^{2n}, X) &= e(\mathbb{R}^{2n}, \Psi(X)) \\ &\leq e(\mathbb{R}^{2n}, Y_c \times \mathbb{R}^{2n-2}) \\ &\leq e(\mathbb{R}^2, Y_c) \\ &= \text{area}(Y_c) \\ &< \pi \\ &= a. \end{aligned}$$

The fourth step follows from a concrete construction of a Hamiltonian diffeomorphism that displaces Y_c or from a Moser-type argument. This proves (17) and hence Lemma 12. ■

Proof of Theorem 1. Let $M, \alpha, T, W, \lambda, n, \varphi$ be as in the hypothesis of this theorem, and $L \subseteq M$ a nonempty closed Legendrian submanifold. By hypothesis all characteristics for α are closed, that is, all Reeb orbits are periodic. Furthermore, their periods are all equal to T. We identify $S^1 \cong \mathbb{R}/T\mathbb{Z}$ and define ψ, $\tilde{W}$, $\tilde{\lambda}$, $\tilde{\omega}$, f as in (12–14).

Claim 1. The map f is not injective. □

Proof of Claim 1. We denote

$$\tilde{L} := f(S^1 \times L),$$

and by

$$\mathrm{pr}_1 : \tilde{W} \to \mathbb{R}^{2n}$$

the projection onto the first factor. The hypotheses (1) and $n \geq \frac{1}{2}\dim W + 2$ imply that

$$\dim(S^1 \times L) = 1 + \frac{\dim M - 1}{2} = n + \frac{1}{2}\dim W \leq 2n - 2.$$

Hence by Sard's theorem, it follows that

$$S^{2n-1}(T) \not\subseteq \mathrm{pr}_1 \circ f(S^1 \times L) = \mathrm{pr}_1(\tilde{L}).$$

On the other hand, hypothesis (2) implies that

$$\mathrm{pr}_1(\tilde{L}) \subseteq \bar{B}^{2n}(T).$$

Therefore, applying Lemma 12, we have

$$e(\tilde{W}, \tilde{L}) \leq e(\mathbb{R}^{2n}, \mathrm{pr}_1(\tilde{L})) < T. \tag{19}$$

Assume now by contradiction that f was injective. This map is proper, since its domain is compact. Hence, it follows from Lemma 10 that f is a Lagrangian embedding. Since $(\mathbb{R}^{2n}, \omega_0)$ and (W, ω) are geometrically bounded, the same holds for $(\tilde{W}, \tilde{\omega})$. Therefore, Theorem 11 implies that

$$e(\tilde{W}, \tilde{L}) \geq A(\tilde{W}, \tilde{L}).$$

Combining this inequality with (19) and the next claim, we arrive at a contradiction. ■

Claim 2. We have

$$A(\tilde{W}, \tilde{L}) \geq T. \tag{20}$$

□

Proof of Claim 2. Let $\tilde{u} \in C^\infty(\mathbb{D}, \tilde{W})$ be such that

$$\tilde{u}(S^1 = \partial\mathbb{D}) \subseteq \tilde{L} = f((\mathbb{R}/T\mathbb{Z}) \times L).$$

We show that

$$\int_{\mathbb{D}} \tilde{u}^*\tilde{\omega} \in T\mathbb{Z}. \tag{21}$$

We define

$$x := \varphi^{-1} \circ \tilde{u} : S^1 \to \psi((\mathbb{R}/T\mathbb{Z}) \times L) \subseteq M.$$

(Recall that $\varphi : M \to \tilde{W}$ is the given embedding.) The equality $\tilde{\omega} = d\tilde{\lambda}$, Stokes' theorem, and the hypothesis (3) imply that

$$\int_{\mathbb{D}} \tilde{u}^*\tilde{\omega} = \int_{S^1} (\varphi \circ x)^*\tilde{\lambda} = \int_{S^1} x^*\alpha. \tag{22}$$

We define

$$(z, y) := \psi^{-1} \circ x : S^1 \to (\mathbb{R}/T\mathbb{Z}) \times L.$$

This makes sense, since the restriction of ψ to $(\mathbb{R}/T\mathbb{Z}) \times L$ is injective. (Here we use our assumption that f is injective.) Since $x = \psi \circ (z, y)$, we have

$$x^*\alpha = \alpha(R \circ \psi \circ (z, y)dz + d\psi_z(y)dy) = dz + \alpha dy = dz.$$

Here we view dz as a real-valued one-form on S^1. In the second equality, we used that $\alpha(R) \equiv 1$ and that the Reeb flow ψ preserves α. In the last equality, we used that L is Legendrian, and hence $\alpha|_{TL} = 0$. It follows that

$$\int_{S^1} x^*\alpha = \int_{S^1} \mathrm{d}z = T \deg(z).$$

Using (22), this proves (20), that is, Claim 2, and therefore Claim 1. ■

By Claim 1 there exist distinct points $(z_i, x_i) \in S^1 \times L = (\mathbb{R}/T\mathbb{Z}) \times L$, $i = 0, 1$, such that

$$f(z_0, x_0) = f(z_1, x_1). \tag{23}$$

Recalling the definition (14), our hypothesis that the period of every characteristic equals T, implies that the map $f(\cdot, x_0)$ is injective. It follows that $x_0 \neq x_1$. Using (14, 23), it follows that these two points lie on the same characteristic for α. Hence, L intersects this characteristic at least twice. This proves Theorem 1. ■

Remark. The above proof relies on the sharp bound for the displacement energy of a closed Lagrangian submanifold due to Chekanov [9]. The same result was used by Mohnke [23] and later by Cieliebak and Mohnke [12, Corollaries 1.4 and 1.5] to find Reeb chords. The construction of the closed Lagrangian submanifold in the proof of Theorem 1 is a variation on the construction in [12, 23].

A new feature is that here the Reeb flow alone is used to produce a Lagrangian submanifold, whereas in [12, 23] both the Reeb flow and the Liouville flow are used. The new approach works because of the upper bound on the displacement energy of a compact subset of a ball given in Lemma 12. □

3 Proof of Proposition 3 (contact form)

The proof of Proposition 3 is based on the following result. Let (W, ρ, λ) be an exact Hamiltonian S^1-manifold, and $c \in \mathbb{R} \setminus \{0\}$. We denote by X the vector field generated by ρ and define

$$H := \lambda(X) : W \to \mathbb{R}, \quad M := H^{-1}(c) \subseteq W.$$

We denote by

$$\iota : M \to W$$

the inclusion.

Proposition 13. The set M is a hypersurface in W, $\alpha := \iota^* \lambda$ is a contact form on M, and all characteristics of α are closed. Their periods are equal to $2\pi c$ if the restriction of the action ρ to M is free. □

Proof of Proposition 13. By hypothesis the form

$$\omega := d\lambda$$

is nondegenerate, that is, symplectic. We denote by V the Liouville vector field on W w.r.t. λ. This is the unique vector field satisfying

$$\iota_V \omega = \lambda.$$

We have

$$\begin{aligned} dH \cdot V &= \iota_V d\iota_X \lambda \\ &= \iota_V \mathcal{L}_X \lambda - \iota_V \iota_X d\lambda \\ &= 0 + \iota_X \iota_V \omega \\ &= \iota_X \lambda \\ &= H. \end{aligned}$$

Here in the second line we used Cartan's formula, and in the third line we used our hypothesis that λ is ρ-invariant. It follows that $dH \cdot V \equiv c \neq 0$ along $M = H^{-1}(c)$. Hence c is a regular value for H, M is a hypersurface in W, and the Liouville vector field V is transverse to M. It follows that $\alpha := \iota^* \lambda$ is a contact form on M. By the next claim its characteristics are closed.

Claim 1. The characteristics of α are the orbits of the restriction of ρ to M. □

Proof of Claim 1. It suffices to show that X is c times the Reeb vector field of α. To see this, note that X is tangent to M, since

$$dH \cdot X = \iota_X d\iota_X \lambda = \iota_X \mathcal{L}_X \lambda - \iota_X \iota_X d\lambda = 0 - 0.$$

By definition, we have

$$\alpha(X) = \lambda(X) \equiv c \text{ on } M = H^{-1}(c).$$

Finally,

$$\iota_X d\alpha = \iota_X d\lambda = \mathcal{L}_X \lambda - d\iota_X \lambda = 0 - dH = 0 \text{ on } TM.$$

It follows that X equals c times the Reeb vector field of α. This proves Claim 1. ■

Assume now that the restriction of ρ to M is free. Let C be a characteristic for α. We choose $x_0 \in C$ and define

$$\varphi : S^1 \to C, \quad \varphi(z) := \rho(z, x_0). \tag{24}$$

This is a diffeomorphism, since the restriction of ρ to C is free. We denote by $\iota : C \to M$ the inclusion and by γ the standard angular one-form on S^1, whose integral equals 2π.

We have

$$\varphi^*\iota^*\alpha = (\iota\circ\varphi)^*\alpha = (\lambda(X)\circ\varphi)\gamma = (H\circ\varphi)\gamma = c\gamma.$$

Here in the second step we used the fact that X generates the action ρ, and (24). It follows that

$$\int_C \iota^*\alpha = \int_{S^1}\varphi^*\iota^*\alpha = 2\pi c.$$

Here we equipped C with the orientation induced by the standard orientation on S^1 and the map φ. This proves Proposition 13. ■

Proof of Proposition 3. We denote by ρ_0 the standard diagonal S^1-action on $\mathbb{R}^{2n} = \mathbb{C}^n$, given by

$$\rho_0(z)z_0 := zz_0 = (zz_0^1, \dots, zz_0^n).$$

By $\rho_0\times\rho$ we denote the product S^1-action on $\mathbb{R}^{2n}\times W$. We define H_0, H as in (4, 5). The triple

$$(\tilde{W}, \tilde{\rho}, \tilde{\lambda}) := ((\bar{B}^{2n}(2\pi c)\setminus\{0\})\times W, (\rho_0\times\rho)|_{\tilde{W}}, \lambda_0\oplus\lambda)$$

is an exact Hamiltonian S^1-manifold, and

$$\tilde{H} := H_0\oplus H = \iota_{\tilde{X}}\tilde{\lambda} : \tilde{W}\to\mathbb{R},$$

where $\tilde{X}$ denotes the vector field generated by $\tilde{\rho}$. The set M defined in (6) is given by

$$M = \tilde{H}^{-1}(c).$$

Since the restriction of ρ_0 to $\bar{B}^{2n}(2\pi c)\setminus\{0\}$ is free, the action $\tilde{\rho}$ is free. Therefore, by Proposition 13 M is a hypersurface in $\tilde{W}$, and $\alpha := \iota^*\tilde{\lambda}$ is a contact form on M all of whose characteristics are closed and of period $2\pi c$. This proves Proposition 3. ■

4 Proof of Theorem 6 (minimal action) and of Corollary 8 (presymplectic embedding)

In this section we denote by

$$B_r^n, \quad \bar{B}_r^n, \quad S_r^{n-1}$$

the open and closed balls around 0 in $\mathbb{R}^n$ of radius r, and the sphere around 0 in $\mathbb{R}^n$ of radius r.

The proof of Theorem 6 is based on Lemma 12 (bound on displacement energy) and the following. Let (M, ω) be a symplectic manifold.

Theorem 14. Assume that (M, ω) is geometrically bounded. Let $N \subseteq M$ be a closed, regular coisotropic submanifold. Then

$$e(N) \geq A(N). \qquad \square$$

Proof. This is an immediate consequence of [29, Theorem 1.1]. ∎

Remark. This theorem generalizes Chekanov's Theorem 11. □

In the proof of Theorem 6, we will also use the following lemma. Let (M, ω) be a presymplectic manifold. Recall that this means that ω is closed two-form on M, and $\mathrm{corank}\omega_x := \dim(T_xM)^\omega$ does not depend on $x \in M$. Let $N \subseteq M$ be a coisotropic submanifold. This means that for every $x \in N$ the space $(T_xN)^\omega$ is contained in T_xN. We denote by $\iota : N \to M$ the inclusion.

Remark 15. The form $\iota^*\omega$ is presymplectic. That its corank is constant, follows from Lemma A.4 and Remark A.6 in appendix. □

By Remark 15 the distribution $(TN)^\omega$ defines a foliation on N. We denote by N_ω the set of its leaves and define the *action spectrum* $S(N) = S(M, N) = S(M, \omega, N)$ and the *minimal action (or area)*

$$A(N) = A(M, N) = A(M, \omega, N)$$

of such a submanifold as in (8,9).

Lemma 16 (lift of coisotropic submanifold)**.** Let (M, ω) and (M', ω') be presymplectic manifolds, $f : M' \to M$ a surjective proper presymplectic submersion, and $N \subseteq M$ a coisotropic submanifold. (That f is presymplectic means that $f^*\omega = \omega'$.) Then the following statements hold:

(i) The set $N' := f^{-1}(N)$ is a coisotropic submanifold of M'.

(ii) $$A(M, N) \leq A(M', N'). \qquad (25)$$

(iii) Assume that N is regular and, for all $x', y' \in M'$,

$$f(x') = f(y') \Rightarrow x' \text{ and } y' \text{ lie on the same isotropic leaf of } M'. \qquad (26)$$

Then N' is regular. □

Remark. In fact equality in (25) holds. However, this will not be used here. □

In the proof of Lemma 16 we will use the following. By a presymplectic vector space we mean a vector space together with a skew-symmetric bilinear form.

Lemma 17. Let (V,ω) and (V',ω') be presymplectic vector spaces, $\Phi: V'\to V$ a linear presymplectic map (This means that $\Phi^*\omega=\omega'$.), and $W\subseteq V$ a linear subspace. (That Φ is presymplectic means that $\Phi^*\omega=\omega'$.) Then the following statements hold:

(i)
$$\Phi^{-1}(W^{\omega})\subseteq(\Phi^{-1}(W))^{\omega'}. \tag{27}$$

(ii) If Φ is surjective then the inclusion "$\supseteq$" in (27) holds. □

Proof of Lemma 17. This follows from the definitions. ■

The proof of Lemma 16(iii) is based on the following. Let M be a (smooth finite-dimensional) manifold and $\mathcal{F}$ a (smooth) foliation on M, that is, a maximal atlas of foliation charts. We denote by $R^{\mathcal{F}}$ its *leaf relation*. This is the subset of $M\times M$ consisting of pairs of points lying in the same leaf. We call $\mathcal{F}$ *regular* iff there exists a manifold structure on the set of leaves $M/R^{\mathcal{F}}$, such that the canonical projection $\pi^{\mathcal{F}}: M\to M/R^{\mathcal{F}}$ is a (smooth) submersion. (The induced topology on $M/R^{\mathcal{F}}$ is by definition Hausdorff and second countable.)

Lemma 18. Let $(M,\mathcal{F})$ and $(M',\mathcal{F}')$ be foliated manifolds, such that $\mathcal{F}$ is regular. Let $f: M'\to M$ be a smooth surjective submersion such that

$$x' R^{\mathcal{F}'} y' \iff f(x') R^{\mathcal{F}} f(y'), \quad \forall x', y'\in M'. \tag{28}$$

Then $\mathcal{F}'$ is regular. □

Proof of Lemma 18. We define the map

$$\varphi: M'/R^{\mathcal{F}'}\to M/R^{\mathcal{F}}, \quad \varphi(F'):=[f(x')],$$

where $x'\in F'$ is an arbitrary point. It follows from (28) that this map is well defined and injective. Our hypothesis that f is surjective implies that φ is surjective, as well. By our assumption that $\mathcal{F}$ is regular there exists a manifold structure $\mathcal{A}$ on $M/R^{\mathcal{F}}$, for which the canonical projection $\pi^{\mathcal{F}}: M\to M/R^{\mathcal{F}}$ is a smooth submersion. Since f is a smooth submersion and

$$\pi^{\mathcal{F}'} = \varphi^{-1} \circ \pi^{\mathcal{F}} \circ f,$$

the map $\pi^{\mathcal{F}'}$ is a smooth submersion w.r.t. the pullback of $\mathcal{A}$ under φ. Hence $\mathcal{F}'$ is regular. This proves Lemma 18. ■

Proof of Lemma 16. (i) Since f is a submersion, N' is a submanifold of M'. It follows from Lemma 17(ii) that it is coisotropic. This proves (i).

To prove (ii, iii), we denote by $R^{N,\omega}$ the isotropic leaf relation on N. This is the subset of $N \times N$ consisting of pairs of points that lie in the same isotropic leaf of N.

Claim 1.

(a) If $(x'_0, x'_1) \in R^{N',\omega'}$ then $(f(x'_0), f(x'_1)) \in R^{N,\omega}$.

(b) If $x'_0, x'_1 \in N'$ are such that $(f(x'_0), f(x'_1)) \in R^{N,\omega}$ then

$$N'_{x'_0} \cap f^{-1}(f(x'_1)) \neq \emptyset. \tag{29}$$

Here $N'_{x'_0}$ denotes the isotropic leaf of N' through x'_0. □

Proof of Claim 1. Let $x' \in N'$. Since f is a submersion, we have

$$T_{x'}N' = df(x')^{-1}(T_{f(x')}N).$$

Using that f is presymplectic, Lemma 17 therefore implies that

$$(T_{x'}N')^{\omega'} = df(x')^{-1}(T_{f(x')}N)^{\omega}. \tag{30}$$

It follows that $f(N'_{x'}) \subseteq N_{f(x')}$. This proves (a).

Proof of (b): We choose a path $x \in C^\infty([0,1], N)$, that is tangent to $(TN)^\omega$ and satisfies $x(i) = f(x'_i)$ for $i = 0, 1$. Since f is a proper submersion, by Proposition A.1 in the appendix there exists a path $x' \in C^\infty([0,1], M')$ satisfying $x'(0) = x'_0$ and $f \circ x' = x$. It follows that $x'([0,1]) \subseteq N'$. Since $\dot{x}(t) \in (T_{x(t)}N)^\omega$, equality (30) implies that $\dot{x}'(t) \in (T_{x'(t)}N')^{\omega'}$, for every $t \in [0,1]$. It follows that $x'(1) \in N'_{x'(0)}$. Since $x'(0) = x'_0$ and $f(x'(1)) = x(1) = f(x'_1)$, condition (29) follows. This proves (b) and completes the proof of Claim 1. ■

Proof of (ii): Let $u' \in C^\infty(\mathbb{D}, M')$ be such that $u'(S^1)$ is contained in some isotropic leaf of N'. Claim 1(a) implies that $f \circ u'(S^1)$ is contained in some isotropic leaf of N. Since

f is presymplectic, we have

$$\int_{\mathbb{D}} u'^*\omega' = \int_{\mathbb{D}} (f \circ u')^*\omega.$$

It follows that $S(M', N') \subseteq S(M, N)$, and therefore,

$$A(M', N') \geq A(M, N).$$

This proves (ii).

Proof of (iii): By Claim 1(a) the implication "$\Rightarrow$" in condition (28) with M, M' replaced by N, N', and $\mathcal{F} = \mathcal{F}^{N,\omega}$, $\mathcal{F}' = \mathcal{F}^{N',\omega'}$, is satisfied. Here $\mathcal{F}^{N,\omega}$ denotes the isotropic foliation on N w.r.t. ω.

To see the opposite implication, let $x'_0, x'_1 \in N'$ be such that the relation $f(x'_0) R^{N,\omega} f(x'_1)$ holds. By Claim 1(b) there exists $y'_1 \in N'_{x'_0} \cap f^{-1}(f(x'_1))$. Since $f(x'_1) = f(y'_1)$, our hypothesis (26) implies that

$$(x'_1, y'_1) \in R^{M',\omega'} \cap (N' \times N') \subseteq R^{N',\omega'}.$$

Since $(x'_0, y'_1) \in R^{N',\omega'}$, it follows that $(x'_0, x'_1) \in R^{N',\omega'}$. This shows the implication "$\Leftarrow$" in (28) with M, M' replaced by N, N', and $\mathcal{F} = \mathcal{F}^{N,\omega}$, $\mathcal{F}' = \mathcal{F}^{N',\omega'}$. Hence (28) is satisfied. Therefore, applying Lemma 18, it follows that N' is regular. This proves (iii) and completes the proof of Lemma 16. ■

In the proof of Theorem 6, we will also use the following lemma.

Lemma 19. Let (M, ω) be a presymplectic manifold, M' a coisotropic submanifold of M and M'' a coisotropic submanifold of M'. Then the following holds.

(i) M'' is a coisotropic submanifold of M.

(ii) If M strongly smoothly deformation retracts onto M' then

$$A(M', M'') \leq A(M, M''). \tag{31}$$

□

Remarks 20.

(i) That M strongly smoothly deformation retracts onto M' means that there exists a smooth map $h: [0, 1] \times M \to M$ such that

$$h(0, \cdot) = \mathrm{id}, \quad h(\{1\} \times M) \subseteq M', \quad h(t, x) = x, \quad \forall t \in [0, 1], \ x \in M'.$$

(ii) The inequality "$\geq$" in (31) is true without the retraction condition. However, this will not be used here. □

In the proof of Lemma 19, we will use the following lemma.

Lemma 21. Let (V, ω) be a finite-dimensional presymplectic vector space, V' a coisotropic subspace of (V, ω), and V'' a coisotropic subspace of $(V', \omega' := \omega|_{V' \times V'})$. Then V'' is a coisotropic subspace of (V, ω). □

Proof of Lemma 21. This follows from Lemma A.4 in appendix. ■

Proof of Lemma 19. (i) This follows from Lemma 21.

We prove (ii). It suffices to show that

$$S(M, M'') \subseteq S(M', M''). \tag{32}$$

Let $u \in C^\infty(\mathbb{D}, M)$ be such that $u(S^1)$ is contained in some isotropic leaf of M''. We choose a map h as in Remark 20(i). We denote $h_t := h(t, \cdot)$ and define

$$f : [0,1] \times \mathbb{D} \to M, \quad f(t,z) := h_t \circ u(z).$$

Using that $d\omega = 0$ and Stokes' theorem, we have

$$\begin{aligned} 0 &= \int_{[0,1]\times\mathbb{D}} \mathrm{d} f^*\omega \\ &= \int_{\partial([0,1]\times\mathbb{D})} f^*\omega \\ &= \int_{\mathbb{D}} (h_1 \circ u)^*\omega - \int_{\mathbb{D}} (h_0 \circ u)^*\omega + \int_{[0,1]\times S^1} f^*\omega \\ &= \int_{\mathbb{D}} (h_1 \circ u)^*\omega - \int_{\mathbb{D}} u^*\omega + 0. \end{aligned} \tag{33}$$

(We used a version of Stokes' theorem that allows the manifold to have corners. See e.g. [21, Theorem 16.25].) Here in the last equality we used the fact $h_0 = \mathrm{id}$ and that $u(S^1) \subseteq M'' \subseteq M'$, and therefore $h_t \circ u|_{S^1}$ is constant in t. Since $h_1(M) = h(\{1\} \times M) \subseteq M'$, the map $h_1 \circ u$ takes values in M'. It is of the sort occurring in the definition of $S(M', M'')$. Hence (33) implies (32). This proves (ii) and completes the proof of Lemma 19. ■

In the proof of Theorem 6, we will also use the following lemma.

Lemma 22. Let (M, ω) and (M', ω') be presymplectic manifolds, $N \subseteq M \times M'$ a coisotropic submanifold, and $x \in M$, such that $\dim M > 2$ and $N \cap (\{x\} \times M') = \emptyset$. Then

$$A((M \setminus \{x\}) \times M', N) \leq A(M \times M', N). \tag{34}$$

□

Remark. In fact equality in (34) holds. However, this will not be used here. □

Proof of Lemma 22. It suffices to prove that

$$S(M \times M', N) \subseteq S((M \setminus \{x\}) \times M', N). \tag{35}$$

Let

$$\tilde{u} = (u, u') \in C^\infty(\mathbb{D}, M \times M')$$

be a map that sends S^1 to some isotropic leaf of N. Since $\dim M > 2$, by Sard's theorem $M \setminus u(B_1^2)$ is dense in M. Hence, an argument in a chart shows that there exists a smooth map $h: [0,1] \times M \to M$, such that

$$h(0, \cdot) = \mathrm{id}, \quad x \notin h(\{1\} \times u(B_1^2)),$$

$$h(t, \cdot) = \mathrm{id} \text{ in some neighbourhood of } \mathrm{pr}_1(N) \subseteq M.$$

Here we denoted by $\mathrm{pr}_1 : M \times M' \to M$ the canonical projection, and we used the hypothesis that $N \cap (\{x\} \times M') = \emptyset$. We denote $h_t := h(t, \cdot)$ and define

$$f : [0,1] \times \mathbb{D} \to M, \quad f(t, z) := h_t \circ u(z).$$

We have, as in (33),

$$\int_{\mathbb{D}} (h_1 \circ u)^* \omega = \int_{\mathbb{D}} u^* \omega.$$

Here we used the facts $h_0 = \mathrm{id}$, $u(S^1) = \mathrm{pr}_1 \circ \tilde{u}(S^1) \subseteq \mathrm{pr}_1(N)$, and $h_t = \mathrm{id}$ in a neighbourhood of $\mathrm{pr}_1(N)$. It follows that

$$\int_{\mathbb{D}} (h_1 \circ u, u')^* \tilde{\omega} = \int_{\mathbb{D}} \tilde{u}^* \tilde{\omega}. \tag{36}$$

Since $x \notin h_1(u(B_1^2))$, the map

$$(h_1 \circ u, u') : \mathbb{D} \to (M \setminus \{0\}) \times M'$$

is of the sort occurring in the definition of $S((M \setminus \{x\}) \times M', N)$. Hence (36) implies (35). This proves Lemma 22. ■

In the proof of Theorem 6, we will also use the following.

Lemma 23. Let (M,ω) be a connected symplectic manifold and $N\subseteq M$ coisotropic submanifold. Then

$$S(N)+\left\{\int_{S^2}u^*\omega\,\middle|\,u\in C^\infty(S^2,M)\right\}\subseteq S(N). \tag{37}$$

□

Proof of Lemma 23. Let $u\in C^\infty(\mathbb{D},M)$ be such that $u(S^1)$ is contained in some isotropic leaf of N, and $v\in C^\infty(S^2,M)$. We choose a point $z_0\in S^2$.

Claim 1. There exist maps $\tilde{u}\in C^\infty(\mathbb{D}\setminus B^2_{\frac{1}{2}},M)$ and $\tilde{v}\in C^\infty(\bar{B}^2_{\frac{1}{3}},M)$ such that

$$\int_{\mathbb{D}\setminus B^2_{\frac{1}{2}}}\tilde{u}^*\omega=\int_{\mathbb{D}}u^*\omega,\qquad \int_{\bar{B}^2_{\frac{1}{3}}}\tilde{v}^*\omega=\int_{S^2}v^*\omega, \tag{38}$$

$\tilde{u}=u$ in some neighbourhood of S^1, $\tilde{u}\equiv u(0)$ in some neighbourhood of $S^1_{\frac{1}{2}}$, and $\tilde{v}\equiv v(z_0)$ in some neighbourhood of $S^1_{\frac{1}{3}}$. □

Proof of Claim 1. We choose a map $f\in C^\infty(\mathbb{D}\setminus B^2_{\frac{1}{2}},\mathbb{D})$ that restricts to an orientation preserving diffeomorphism from $\mathbb{D}\setminus\bar{B}^2_{\frac{3}{4}}$ to $\mathbb{D}\setminus\{0\}$, equals identity in a neighbourhood of S^1, and sends $\bar{B}^2_{\frac{3}{4}}\setminus B^2_{\frac{1}{2}}$ to 0. We define

$$\tilde{u}:=u\circ f:\mathbb{D}\setminus B^2_{\frac{1}{2}}\to M.$$

This map has the required properties.

To construct $\tilde{v}$, we choose a map $g\in C^\infty(\bar{B}^2_{\frac{1}{3}},S^2)$ that restricts to an orientation preserving diffeomorphism from $B^2_{\frac{1}{4}}$ to $S^2\setminus\{z_0\}$ and sends $\bar{B}^2_{\frac{1}{3}}\setminus B^2_{\frac{1}{4}}$ to z_0. The map $\tilde{v}:=v\circ g$ has the required properties. This proves Claim 1. ■

We choose $\tilde{u},\tilde{v}$ as in this claim. Since M is connected, there exists a path $x\in C^\infty\left(\left[\frac{1}{3},\frac{1}{2}\right],M\right)$, such that $x(0)=v(z_0)$ and $x(1)=u(0)$. We may modify x, such that it is constant in some neighbourhoods of $\frac{1}{3}$ and $\frac{1}{2}$. We define

$$w(z):=\begin{cases}\tilde{v}(z), & \text{if } z\in\bar{B}^2_{\frac{1}{3}},\\ x(|z|), & \text{if } z\in B^2_{\frac{1}{2}}\setminus\bar{B}^2_{\frac{1}{3}},\\ \tilde{u}(z), & \text{if } z\in\mathbb{D}\setminus B^2_{\frac{1}{2}}.\end{cases}$$

This map is smooth. It follows from (38) that

$$\int_{\mathbb{D}}w^*\omega=\int_{S^2}v^*\omega+0+\int_{\mathbb{D}}u^*\omega. \tag{39}$$

Since $w=u$ in some neighbourhood of S^1, the image $w(S^1)$ is contained in some isotropic leaf of N. It follows that $\int_{\mathbb{D}} w^*\omega \in S(N)$. Combining this with (39), the inclusion (37) follows. This proves Lemma 23. ■

In the proof of Theorem 6, we will also use the following.

Remark 24. Let (M,ω) be a symplectic manifold and $N\subseteq M$ a coisotropic submanifold. Then

$$S(N) = -S(N) = \{-a \mid a\in S(N)\}.$$

This follows from the fact that for every $u\in C^\infty(\mathbb{D},M)$ we have

$$\int_{\mathbb{D}} \bar{u}^*\omega = -\int_{\mathbb{D}} u^*\omega,$$

where $\bar{u}(z) := u(\bar{z})$, for every $z\in\mathbb{D}\subseteq\mathbb{C}$. □

Proof of Theorem 6. We denote by

$$\pi : S^{2n+1}\times M \to \mathbb{C}\mathrm{P}^n\times M$$

the canonical projection, by $\iota : S^{2n+1}\to\mathbb{R}^{2n}$ the inclusion, and by ω_0 the standard symplectic form on $\mathbb{R}^{2n}$. We equip $S^{2n+1}\times M$ with the presymplectic form $\iota^*\omega_0\oplus\omega$. It follows from Lemma 16(i,ii) that $N'=\pi^{-1}(N)$ is a coisotropic submanifold of $S^{2n+1}\times M$, and

$$A(\mathbb{C}\mathrm{P}^n\times M, N) \le A(S^{2n+1}\times M, N'). \tag{40}$$

Since π is proper and N is compact, N' is compact. The manifold $(\mathbb{R}^{2n+2}\setminus\{0\})\times M$ strongly smoothly deformation retracts onto $S^{2n+1}\times M$. Hence by Lemma 19(ii), we have

$$A(S^{2n+1}\times M, N') \le A((\mathbb{R}^{2n+2}\setminus\{0\})\times M, N'). \tag{41}$$

Since $n\ge 1$, by Lemma 22 we have

$$A((\mathbb{R}^{2n+2}\setminus\{0\})\times M, N') \le A(\mathbb{R}^{2n+2}\times M, N'). \tag{42}$$

The symplectic manifold $\mathbb{R}^{2n+2}$ is geometrically bounded. Using our hypothesis that M is geometrically bounded, it follows that $\mathbb{R}^{2n+2}\times M$ has the same property. Since by hypothesis N is regular, by Lemma 16(iii) with $f=\pi$ the same holds for N'. (Condition

(26) with $M' := S^{2n+1} \times M$ is satisfied, since

$$R^{S^{2n+1}\times M} = \{((x, y), (zx, y)) \mid (x, y) \in S^{2n+1} \times M,\ z \in S^1\},$$

where we consider S^{2n+1} as a subset of $\mathbb{C}^{n+1}$ and $S^1 \subseteq \mathbb{C}$.) Hence applying Theorem 14, we obtain

$$A(\mathbb{R}^{2n+2} \times M, N') \leq e(\mathbb{R}^{2n+2} \times M, N'). \tag{43}$$

We denote by $\mathrm{pr}_1 : S^{2n+1} \times M \to S^{2n+1}$ the projection onto the first factor. We have

$$\begin{aligned} e(\mathbb{R}^{2n+2} \times M, N') &\leq e(\mathbb{R}^{2n+2} \times M, \mathrm{pr}_1(N') \times M) \\ &\leq e(\mathbb{R}^{2n+2}, \mathrm{pr}_1(N')). \end{aligned} \tag{44}$$

Our hypothesis $\dim M < 2n$ implies that $\dim N' = \dim N + 1 \leq 2n$. Hence, the restriction $\mathrm{pr}_1|_{N'} : N' \to S^{2n+1}$ is not submersive at any point, and therefore the set of its regular values is the complement of its image. Hence by Sard's Theorem $\mathrm{pr}_1(N') \neq S^{2n+1}$. Therefore by Lemma 12 we have

$$e(\mathbb{R}^{2n+2}, \mathrm{pr}_1(N')) < \pi.$$

Combining this with (40–44), it follows that

$$A(N) = A(\mathbb{C}\mathrm{P}^n \times M, N) < \pi.$$

Hence there exists $a \in S(N) \cap (0, \pi)$. If $a \leq \frac{\pi}{2}$ then it follows that $A(N) \leq \frac{\pi}{2}$, as claimed. Otherwise $-a + \pi < \frac{\pi}{2}$. By Remark 24 we have $-a \in S(N)$. Since there exists $u \in C^\infty(S^2, \mathbb{C}\mathrm{P}^n)$, such that $\int_{S^2} u^*\omega_{\mathrm{FS}} = \pi$, Lemma 23 implies that $-a + \pi \in S(N)$. Since $-a + \pi < \frac{\pi}{2}$, it follows that $A(N) < \frac{\pi}{2}$. Hence in every case we have $A(N) \leq \frac{\pi}{2}$. This proves Theorem 6. ■

Proof of Corollary 8. We denote

$$\tilde{M} := \mathbb{C}\mathrm{P}^n \times M, \quad \tilde{\omega} := \omega_{\mathrm{FS}} \oplus \omega.$$

Assume by contradiction that there exists a presymplectic embedding $\varphi : M' \to \tilde{M}$. We denote $N := \varphi(M')$. It follows from our hypothesis (11) and Lemma A.4 in appendix that N is coisotropic. It is regular, since M' is regular.

Claim 2. We have

$$A(\tilde{M}, N) \geq \pi. \tag{45}$$

□

Proof of Claim 2. It follows from our hypothesis (10) that

$$\int_{S^2} \tilde{w}^*\tilde{\omega} \in \pi\mathbb{Z}, \quad \forall \tilde{w} \in C^\infty(S^2, \tilde{M}). \tag{46}$$

Let $\tilde{u} \in C^\infty(\mathbb{D}, \tilde{M})$ be such that $\tilde{u}(S^1)$ is contained in some isotropic leaf F of N. We choose a map $f \in C^\infty(\mathbb{D}, \mathbb{D})$ that restricts to an orientation preserving diffeomorphism from $B^2_{\frac{1}{2}}$ to B^2_1 and satisfies $f(z) = z/|z|$ on $\mathbb{D} \setminus B^2_{\frac{1}{2}}$.

The pre-image $\varphi^{-1}(F)$ is an isotropic leaf of M'. By hypothesis it is simply connected. Hence, the same holds for F. It follows that there exists a map $\tilde{v} \in C^\infty(\mathbb{D}, F)$ satisfying $\tilde{v} = \tilde{u}$ on S^1. Modifying $\tilde{v}$, we may assume that $\tilde{v}(z) = \tilde{v}(z/|z|)$ for every $z \in \mathbb{D} \setminus B^2_{\frac{1}{2}}$.

We denote by $\bar{\mathbb{D}}$ the disk with the reversed orientation and by $\mathbb{D}\#\bar{\mathbb{D}}$ the smooth oriented manifold obtained by concatenating the two disks along their boundary. We define $\tilde{w} : \mathbb{D}\#\bar{\mathbb{D}} \to \mathbb{C}\mathrm{P}^n \times M$ to be the concatenation of $\tilde{u} \circ f$ and $\tilde{v}$. This is a smooth map. It follows that

$$\int_{\mathbb{D}\#\bar{\mathbb{D}}} \tilde{w}^*\tilde{\omega} = \int_{\mathbb{D}} (\tilde{u} \circ f)^*\tilde{\omega} - \int_{\mathbb{D}} \tilde{v}^*\tilde{\omega} = \int_{\mathbb{D}} \tilde{u}^*\tilde{\omega} - 0. \tag{47}$$

Since $\mathbb{D} \cup \bar{\mathbb{D}}$ is diffeomorphic to S^2, (46) implies that $\int_{\mathbb{D}\#\bar{\mathbb{D}}} \tilde{w}^*\tilde{\omega} \in \pi\mathbb{Z}$. Combining this with (47), inequality (45) follows. This proves Claim 2. ■

This claim and the hypothesis $\dim M' < 2n$ contradict Theorem 6. Hence the presymplectic embedding $\varphi : M' \to \tilde{M}$ does not exist. This proves Corollary 8. ■

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Appendix 1. Lifting Paths

The following result was used in the proof of Lemma 16. Let M', M be smooth manifolds, $f : M' \to M$ a smooth proper submersion, $p' \in M'$, and $x \in C^\infty([0, 1], M)$.

Proposition A.1 (lifting a path). If $f(p') = x(0)$ then there exists a path $x' \in C^\infty([0, 1], M')$, such that

$$f \circ x' = x, \quad x'(0) = p'. \qquad \square$$

The proof of this lemma is based on the following.

Lemma A.2 (locally lifting a path)**.** Let $t_0 \in [0,1]$ and $H \subseteq TM'$ be a (smooth) subbundle, that is complementary to $\ker df$, that is, it satisfies $TM' = H \oplus \ker df$.

(i) (local existence) If $f(p') = x(t_0)$ then there exists a (relatively) open neighbourhood V of t_0 in $[0,1]$ and a path $x' \in C^\infty(V, M')$, satisfying

$$\dot{x}'(t) \in H_{x'(t)}, \qquad f \circ x'(t) = x(t), \qquad \forall t \in V, \tag{A.1}$$

$$x'(t_0) = p'. \tag{A.2}$$

(ii) (local uniqueness) If V_0, V_1 are open neighbourhoods of t_0 in $[0,1]$ and x'_0, $x'_1 \in C^\infty([0,1], M')$ are paths, satisfying (A.1,A.2) then there exists an open neighbourhood $V \subseteq V_0 \cap V_1$ of t_0 in $[0,1]$, such that $x'_0 = x'_1$ on V. □

Remark A.3 (global uniqueness)**.** For $i = 0, 1$ let $t_i \in [0,1]$ and $x'_i \in C^\infty([0, t_i], M')$ be a path satisfying (A.1) and $x'_i(0) = p'$. Then $x'_0 = x'_1$ on $[0, \min\{t_0, t_1\}]$. This follows from Lemma A.2(ii). □

Proof of Lemma A.2. By using a chart in M we may assume w.l.o.g. that $M = \mathbb{R}^n$. Using the Implicit Function Theorem and our hypothesis that f is a smooth submersion, we may further assume w.l.o.g. that $M' = \mathbb{R}^m \times \mathbb{R}^n$ and $f = \mathrm{pr}_2 : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$, the canonical projection.

For $t \in [0,1]$ and $y \in \mathbb{R}^m$ we define $X_t(y) \in \mathbb{R}^m$ to be the unique vector, such that

$$(X_t(y), \dot{x}(t)) \in H_{(y, x(t))}. \tag{A.3}$$

This vector exists and is unique, since H is complementary to $\ker df = \ker \mathrm{pr}_2$. The family $(X_t)_{t \in [0,1]}$ is a smooth time-dependent vector field on $\mathbb{R}^m$. We write $p' = (y_0, x(t_0))$.

We prove (i). By the Picard-Lindelöf theorem there exist an open neighbourhood V of t_0 in $[0,1]$ and a smooth solution $y \in C^\infty(V, \mathbb{R}^m)$ of the ordinary differential equation

$$\dot{y} = X_t \circ y, \qquad y(t_0) = y_0.$$

Using (A.3), the path $x' := (y, x) : [0,1] \to M' = \mathbb{R}^m \times \mathbb{R}^n$ satisfies (A.1,A.2). This proves (i).

Statement (ii) follows from a similar argument. This proves Lemma A.2. ■

Proof of Proposition A.1. We choose a subbundle $H \subseteq TM'$, that is complementary to $\ker df$. (We may define H to be the normal bundle of $\ker df$ with respect to some Riemannian metric.) We define

$$y' := \bigcup \{x' \mid t_1 \in [0,1],\ x' \in C^\infty([0,t_1], M'): \\ (A.1) \text{ with } V = [0,t_1],\ x'(0) = p'\} \subseteq [0,1] \times M'. \tag{A.4}$$

It follows from Remark A.3 that there exists $t_0 \in [0,1]$ such that y' is a smooth map from $[0,t_0)$ or $[0,t_0]$ to M'. Proposition A.1 is a consequence of the following claim.

Claim 1. The domain of y' is $[0,1]$. □

Proof of Claim 1. We define

$$X := \{(x', v) \mid v \in T_{f(x')}M\}, \quad \Phi : X \to H,\ \Phi_{x'}v := \Phi(x', v) := v',$$

where $v' \in H_{x'}$ is the unique vector satisfying $df(x')v' = v$. Since $df(x')$ is surjective and $T_{x'}M' = H_{x'} \oplus \ker df(x')$, this vector exists and is unique, hence Φ is well-defined. This map is smooth, since H is smooth. We choose Riemannian metrics g on M and g' on M'. Since f is proper, the pre-image $K' := f^{-1}(x([0,1])) \subseteq M'$ is compact. Therefore,

$$C := \sup\{|\Phi_{x'}| \mid x' \in K'\} < \infty.$$

Here $|\Phi_{x'}|$ denotes the operator norm of the linear map $\Phi_{x'} : T_{f(x')}M \to T_{x'}M'$ w.r.t. the norms induced by g and g'.

Let $t \in [0,t_0)$. By (A.1) we have $\dot{y}'(t) \in H_{y'(t)}$ and $df(y'(t))\dot{y}'(t) = \dot{x}(t)$, and therefore

$$\dot{y}'(t) = \Phi_{y'(t)}\dot{x}(t).$$

Since $y'([0,t_0)) \subseteq K'$, it follows that

$$|\dot{y}'(t)| \le C|\dot{x}(t)| \le C \max_{t\in[0,1]} |\dot{x}(t)|.$$

It follows that $y'(t)$ converges to some point y'_0, as $t \uparrow t_0$.

Assume now by contradiction that the domain of y' is not equal to $[0,1]$. We choose V, x' as in Lemma 10(i), with p' replaced by y'_0. Concatenating y' with x', we obtain a solution z' of (A.1) with V replaced by an interval that strictly contains the domain of y', such that $z'(0) = p'$. (Here we use Lemma A.2(ii), which ensures that x' and y' agree on the intersection of V with the domain of y', if we shrink V.) By (A.4) we have $z' \subseteq y'$. This is a contradiction. It follows that the domain of y' is equal to [0, 1]. This proves Claim 1 and completes the proof of Proposition A.1. ■

Appendix 2. Coisotropic Subspaces of Presymplectic Vector Spaces

The following lemma was used in the proof of Lemma 21. Let (V, ω) be a finite-dimensional presymplectic vector space and $W \subseteq V$ a linear subspace. We denote by $i : W \to V$ the inclusion, and by

$$W^{\omega} := \{v \in V \mid \omega(v, w) = 0, \ \forall w \in W\}$$

the presymplectic complement of W in V.

Lemma A.4. The subspace W is coisotropic iff

$$\dim W + \dim W^{i^*\omega} \geq \dim V + \dim V^{\omega}. \tag{A.5}$$

□

(By definition, $W^{i^*\omega}$ is the presymplectic complement of W inside W.) The proof of this lemma is based on the following.

Lemma A.5. We have

$$\dim W + \dim W^{\omega} = \dim V + \dim(V^{\omega} \cap W). \tag{A.6}$$

□

Remark A.6. Since $W^{i^*\omega} \subseteq W^{\omega}$, Lemma A.5 implies that inequality "$\leq$" in (A.5) holds, for every linear subspace W. □

Proof of Lemma A.5. We define the linear map

$$\flat_{\omega} : V \to V^*, \quad \flat_{\omega} v := \omega(v, \cdot).$$

Then $W^{\omega} = \ker(i^* \flat_{\omega})$, and therefore,

$$\dim \mathrm{im}(i^* \flat_{\omega}) + \dim W^{\omega} = \dim V. \tag{A.7}$$

Consider the canonical isomorphism $\iota : V \to V^{**}$, $\iota(v)(\varphi) := \varphi(v)$. A direct calculation shows that $(\flat_{\omega})^* \iota = -\flat_{\omega}$. It follows that $(\flat_{\omega} i)^* \iota = -i^* \flat_{\omega}$, and therefore

$$\dim \mathrm{im}(\flat_{\omega} i) = \dim \mathrm{im}(\flat_{\omega} i)^* = \dim \mathrm{im}((\flat_{\omega} i)^* \iota) = \dim \mathrm{im}(i^* \flat_{\omega}).$$

Combining this with (A.7), we obtain

$$\begin{aligned}\dim W + \dim W^{\omega} &= \dim\ker(\flat_{\omega} i) + \dim\operatorname{im}(\flat_{\omega} i) + \dim W^{\omega}\\ &= \dim\ker(\flat_{\omega} i) + \dim V. \end{aligned} \tag{A.8}$$

Since $\ker(\flat_{\omega} i) = V^{\omega} \cap W$, equality (A.6) follows. This proves Lemma A.5. ∎

Proof of Lemma A.4. *We prove "⇒".* Assume that W is coisotropic. Then $W^{\omega} \subseteq W^{i^*\omega}$ and therefore, using Lemma A.5, we have

$$\begin{aligned}\dim W + \dim W^{i^*\omega} &\geq \dim W + \dim W^{\omega}\\ &= \dim V + \dim(V^{\omega} \cap W).\end{aligned}$$

Since $V^{\omega} \subseteq W^{\omega} \subseteq W$, inequality (A.5) follows. This proves "⇒."

To prove the opposite implication, assume that (A.5) holds. Using Lemma A.5, it follows that

$$\begin{aligned}\dim W^{\omega} &= \dim V - \dim W + \dim(V^{\omega} \cap W)\\ &\leq \dim W^{i^*\omega} - \dim V^{\omega} + \dim(V^{\omega} \cap W)\\ &\leq \dim W^{i^*\omega}.\end{aligned}$$

Since $W^{\omega} \supseteq W^{i^*\omega}$, it follows that $W^{\omega} = W^{i^*\omega} \subseteq W$. Therefore, W is coisotropic. This proves "⇐" and completes the proof of Lemma A.4. ∎

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