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Promoting insight into algebraic formulas through graphing by hand

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ABSTRACT

Student insight into algebraic formulas, including the ability to identify the structure of a formula and its components and to reason with and about formulas, is an issue in mathematics education. In this study, we investigated how 16- and 17-year-old pre-university students' insight into algebraic formulas can be promoted through graphing formulas by hand. In an intervention of five 90-min lessons, 21 grade 11 students were taught to graph formulas by hand. The intervention's design was based on experts' strategies in graphing formulas, that is, using a combination of recognition and qualitative reasoning, and on principles of teaching complex skills. To assess the effect of this intervention, pre-, post-, and retention tests were administered, as well as a post-intervention questionnaire. Six students were asked to think aloud during the pre- and posttests. The results show that all students improved their abilities to graph formulas by hand. The think-aloud data suggest that the students improved both on recognition and reasoning, and give a detailed picture of how students used recognition and qualitative reasoning in combination. We conclude that graphing formulas by hand, based on the interplay of recognition and qualitative reasoning, might be a means to promote students' insight into algebraic formulas.

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Introduction

Research has shown that students in grades 11 and 12, and even beyond secondary school, have persistent difficulties with algebra in general, and with dealing algebraic formulas and making sense of them in particular (Arcavi, Drijvers & Stacey, 2017; Arcavi, 1994; Ayalon, Watson, & Lerman, 2015; Chazan & Yerushalmy, 2003; Drijvers, Goddijn, & Kindt, 2011; Hoch & Dreyfus, 2005, 2010; Kieran, 2006; Oehrtman, Carlson, & Thomp, 2008). The students lack symbol sense, which is defined as the very general notion of "when and how" to use symbols (Arcavi, 1994). Symbol sense has several aspects, such as the ability to read through algebraic expressions, to see the expression as a whole rather than as a concatenation of letters, and to make rough estimates of the pattern that would emerge in a graphical representation (Arcavi, 1994; Pierce & Stacey, 2001). Drijvers et al. (2011) describe symbol sense as complementary to basic skills. Symbol sense involves strategic work with a global view and an emphasis on algebraic reasoning, whereas basic skills involve procedural work with a local focus and an emphasis on algebraic calculations. Pierce and Stacey (2001) use algebraic insight to capture the symbol sense involved in using computer algebra software. This algebraic insight concerns identifying structure through the recognition of objects, key features, dominant terms, and simple factors, knowing the meaning of symbols, and the ability to link representations (Pierce & Stacey, 2001).

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In this study, we aimed at this one aspect of symbol sense, namely, insight into algebraic formulas, that is, the ability to "look through a formula." More specifically, we viewed insight into algebraic formulas as including the abilities to recognize the structure of a formula and its components and to reason with and about a formula. Structure in algebra has been defined by Hoch and Dreyfus (2010) as a broad analysis of the way an entity is made up by its parts. Structure sense includes abilities such as seeing an algebraic expression as an entity, recognizing the expression as a previously met structure, dividing the entity into sub-structures, and recognizing the connection between structures. In this study, we focused on functions of one variable and their Cartesian graphs. We chose to use graphing formulas by hand, without technology, as a means to promote students' insight into formulas. In this article, this graphing formulas by hand will be called *graphing formulas*.

Many studies about symbol sense and graphing are about the role of technology like graphic calculators to promote students' symbol sense (Arcavi et al., 2017; Drijvers, 2003; Heid et al., 2013; Hennessy, Fung, & Scanlon, 2001; Kieran & Drijvers, 2006; Philipp, Martin, & Richgels, 1993; Yerushalmy & Gafni, 1992). In some of these studies, the need for by hand activities has been stressed (Arcavi et al., 2017; Kieran & Drijvers, 2006), but to our knowledge, there are no recent studies that investigate effects of graphing by hand on students' symbol sense and this study might fill this gap. We investigated how graphing formulas might be learned by students and designed an intervention consisting of a series of lessons on graphing formulas, in grade 11 (16- and 17-year-old pre-university students) to enhance students' insight into algebraic formulas. In this way, the current study contributes to the understanding of how recognition, reasoning, and its interplay involved in graphing formulas may foster students' insight into formulas.

Theoretical framework

Graphing formulas is a complex task for students. In this section, we elaborate on the theoretical principles underlying our educational design. First, the literature about symbol sense and graphing is discussed. Next, we describe the nature and content of the knowledge base students need for graphing formulas. Finally, we discuss how this knowledge base might be addressed in student tasks, using the literature on teaching complex skills.

Symbol sense and graphing

To promote insight into formulas, we had two arguments for focusing on graphing formulas. First, we targeted insight into formulas that are often used in grade 11 textbooks, like $y = 4\sqrt{10 - x}$, $y = 2(x-3)^2(x+3)$, $y = (x+3)^4 - 9$, $y = (4x+2)/(x+3)^2$, $y = xe^{-x}$, $y = \ln(x-3)$, so, we needed a general domain, in which many different formulas could be addressed. Second, in literature, it has been recommended to use realistic contexts and multiple representations to give meaning to algebraic formulas (Kieran, 2006; Radford, 2004), and to learn about functions (Arcavi et al., 2017; Janvier, 1987; Kieran, 2006; Leinhardt, Zaslavsky, & Stein, 1990; Moschkovich, Schoenfeld, & Arcavi, 1993). However, besides linear and exponential functions, it is in general difficult to link formulas to realistic context, except in mathematical modeling. Therefore, we chose for using representations, in particular for linking formulas to their graphs.

Graphing tools such as graphic calculators are helpful for learning about functions and their multiple representations (Heid, Thomas, & Zbiek, 2013; Hennessy et al., 2001; Kieran & Drijvers, 2006; Philipp et al., 1993; Yerushalmy & Gafni, 1992). However, Goldenberg (1988) found that students established the connection between formula and graph more effectively when they did graphing by hand than when they only performed computer graphing. Therefore, we chose the context of graphing formulas by hand to promote students' insight.

In linking formulas to graphs, covariational reasoning comes into play. Covariational reasoning concerns coordinating two varying quantities while attending to how they change in relation to each other (Carlson, Jacobs, Coe, Larsen,, & Hsu, 2002; Thompson, 2013). While the focus often is on quantities in real-life situations, algebraic functions with "imagining running through all input-output pairs simultaneously and so reason about how a function is acting on an entire interval of input values" are also included (Carlson et al., 2002). Covariational reasoning often focuses on the global graph and five levels of development have been described: from the idea that change in one variable depends on change in another variable, to paying attention to the direction of change, to paying attention to the amount of change, to considering average rate with uniform increments of the input variable, to the instantaneous rate of change for entire domain (Carlson et al., 2002; Oehrtman et al., 2008). It has been argued that such covariational reasoning is critical in supporting student learning of functions in secondary and undergraduate mathematics (Carlson et al., 2002; Confrey & Smith, 1995; Oehrtman et al., 2008; Thompson & Carlson, 2017). Students have difficulties with this reasoning. This was shown by Carlson et al. (2015), who found that students were not able to select the correct graph (out of five alternatives) of $f(x) = 1/(x-2)^2$, indicating, according to the authors, that students were not able to reason "as the value of x gets larger the value of y decreases, and as the value of x approaches 2, the value of y increases." Such reasoning about functions requires a global perspective on a function, that is, seeing the function as an entity or object (Confrey & Smith, 1995; Even, 1998; Gray & Tall, 1994; Oehrtman et al., 2008). This may be hindered by another commonly used perspective, namely, seeing a function as an input-output machine (a given x–value is linked to a certain y-value). The latter view is considered a pointwise, process, or correspondence perspective. A global perspective is more powerful and gives a better understanding of the relation between formula and graph, but a pointwise approach is needed to construct initial meaning (Even, 1998). To learn graphing formulas, students have to learn to take a global perspective on functions and to use the first three levels of covariational reasoning, in particular paying attention to the direction of change and the global amount of change of a function (concavity).

Expertise in graphing formulas: recognition and reasoning

To investigate what is needed to master a complex skill, it has been recommended to examine expert behavior (Kirschner & Van Merriënboer, 2008; Schoenfeld, 1978). In expertise research, it has been established that for effective and efficient problem solving one needs recognition, and reasoning when recognition falls short (Berliner & Ebeling, 1989; Chi, Feltovich, & Glaser, 1981). In our previous studies, we described experts' recognition and strategies involved in graphing formulas (Kop, Janssen, Drijvers, Van Driel, & Veenman, 2015, Kop, Janssen, Drijvers, Van Driel, 2017). Five experts from different backgrounds, but all holding a master's or PhD in mathematics, were selected to investigate expertise in graphing formulas: three mathematicians who worked at Dutch universities, one mathematical textbook writer who was also a mathematics teacher in upper secondary school, and one who worked at the Dutch Institute for Testing and Assessment. Because all had more than 10 years of experience in work which often required them to graph formulas, we considered them experts in graphing formulas (Kop et al., 2015/2017).

To describe experts' thinking processes for graphing formulas, different levels of recognition were formulated: the formula can be instantly visualized as a graph or is recognized as a member of a function family of which the global graph is known; the formula can be decomposed into sub-formulas of function families; some characteristics of the graph are instantly recognized but not the whole graph; there is no recognition at all (Kop et al., 2015). These levels of recognition can be linked to Mason's (2003) levels of attention, in which he described how attention can shift from seeing essential structure to gazing at the whole and not knowing how to proceed. For recognition, a repertoire of basic function families that can be instantly visualized by a graph (Eisenberg & Dreyfus, 1994) and knowledge of features to describe graphs are needed (Slavit, 1997). Kop et al. (2015) found that experts' repertoires of basic function families resembled the basic function families taught in secondary school, like exponential, logarithmic, and polynomial functions. Experts seem to have linked prototypes of these function families to a set of critical graph features. For instance, a prototypical logarithmic graph has a vertical asymptote, only positive x–values as a domain, and is concave down. Experts use their repertoire of basic function families as building blocks in working with formulas to decompose complex functions into simpler basic ones and to read characteristic graph features from formulas (Kop et al., 2015).

When experts graph more complex formulas and instant recognition falls short, they start reasoning about, for instance, infinity behavior, in/decreasing of a function, and weaker/stronger components of a function, but they hardly use calculation of points and/or derivatives. In short, our previous studies suggest an interplay of recognition and reasoning being the backbone of the expertise at stake. We give five examples to illustrate experts' recognition and reasoning. (1) Sketching $y = 2\sqrt{x+6}$: "It is a root-function translated to the left." (2) Sketching $y = -2x(x-3)(x-6)$: "It is a polynomial function of degree 3, reversed because of $-x^3$, and zeroes at 0, 3, 6." (3) Sketching $y = 100 - 50 \cdot 0.75^x$: "It has $y = 100$ as an horizontal asymptote, 100 minus …, so, it comes from beneath to the asymptote; when x is very negative, it is 100 minus very large outcomes, so y-values will be very negative." (4) Sketching $y = x - 4/x$: "I can sketch $y = x$ and $y = 4/x$, now I have to subtract the graphs, here (with large values of x) it is almost $y = x$, when x is a little bit larger than 0 y is very negative, etc. (sketch the graph). (5) Sketching $y = 500/(2 + 0.75^x)$: "When x goes to infinity then it is 500/2 = 250; 500 dividing by a decreasing number, so outcomes increase; it is always positive, and when x goes to minus infinity it is almost 0." (Kop et al., 2015/2017)

The interplay between recognition and reasoning is visible when experts use prototypical graphs of function families. For example, "a root-function translated" in (1); "a polynomial of degree 3, reversed" in (2); "decomposing a formula, graphing both sub-formulas, and compose these sub-graphs" in (4) (Kop et al., 2015; Schwarz & Hershkowitz, 1999). These examples show that experts can start with prototypical graphs and use reasoning about transformations, about characteristics, about composing sub-formulas to finish the graph. Sometimes, experts only recognize a key graph feature and have to use more reasoning to complete the graph. For example, in (3), the horizontal asymptote was instantly recognized. It also possible that there is no recognition, then experts start strategic exploration of the graph. For example, in (5) the expert started reasoning about infinity behavior of the function. Experts' reasoning is often qualitative of character, that is, global reasoning, using global descriptions without strict proofs, and ignoring what is not relevant. We illustrated this experts' qualitative reasoning in the five examples above. In their reasoning experts ignored the factor 2 when sketching $y = 2\sqrt{x+6}$ (1) and $y = -2x(x-3)(x-6)$ (2), the factor 50 in $y = 100 - 50 \cdot 0.75^x$ (3). Ignoring what is not relevant is an aspect of adaptive reasoning and an indication of expertise (Chi, 2011; Chi et al., 1981). Global reasoning is found when exploring parts of a graph, for instance, infinity behavior of the function in (3) "x is very negative, it is 100 minus very large outcomes, so y-values will be very negative" and in (5) "when x goes to minus infinity it is almost 0." Global descriptions were used in (2) "reversed" and in (3) "it comes from beneath to the asymptote."

In literature, the importance of qualitative reasoning with its focus on the global shape of the graph and ignoring what is not relevant has been addressed. Leinhardt et al. (1990) spoke about qualitative interpretation of graphs to gain meaning about the relationship between the two variables, and their pattern of covariation. In physics and physics education, qualitative reasoning is used to describe essential entities and processes and to provide the necessary grounding for a deep and robust understanding of quantitative models (Bredeweg & Forbus, 2003; Forbus, 1996). Friedlander and Arcavi (2012) used the term qualitative thinking in their framework for cognitive processes involved in algebraic skills. Qualitative thinking is about predicting and interpreting results without calculation and/or manipulation skills and strict proofs. Experts use this qualitative reasoning also in their communication with students. For example, Thompson (2013) described how an experienced teacher added two sub-graphs using blank axes to keep students away from calculations, focusing on an estimation of the sum-graph, and using qualitative reasoning in the discussion with the class, with descriptions like "it is less negative," "how negative," "it will get lower." However, this qualitative reasoning, with its ignoring what is not relevant and its focus on the global shape of the graph, is often used implicitly and hardly taught explicitly in school (Duval, 2006; Leinhardt et al., 1990).

Experts' recognition and reasoning in graphing formulas inform us about "what to teach": students should learn a repertoire of basic function families, with prototypes and key features, for recognition and students should learn experts' reasoning, with its qualitative character, using global descriptions, ignoring what is not relevant, and without strict proofs. In the next section, we address the literature on complex skills to formulate design principles (DP) about how to teach graphing formulas, based on recognition and reasoning.

Teaching complex skills

Although graphing formulas is a well-described task, it can also be considered a complex task, because functions may vary from basic functions to very complex ones. In this section, we outline a social constructivist approach to teaching graphing formulas as a complex skill. In this approach, students learn component knowledge and skills in the context of more complex whole tasks, with adaptive support and students are invited to articulate and reflect on their own problem-solving processes (De Corte, 2010).

Complex cognitive skills consist of many constituent skills, which have to be integrated and coordinated. In education, a part-task approach is often used: all constituent skills are taught separately and in succession, and only at the end are students confronted with the complexity of the whole task. This results in students having difficulties in integrating and coordinating all the constituent skills (Kirschner & Van Merriënboer, 2008).

Instead of the part-task strategy, a whole-task-first approach is recommended: students learn skills and knowledge in the context of entire tasks (Collins, 2006; Kirschner & Van Merriënboer, 2008; Merrill, 2013; Van Merriënboer Clark, & De Croock, 2002). Of course, students cannot immediately perform an entire task without help. Therefore, it is recommended to support student learning processes in different ways (Kirschner & Van Merriënboer, 2008; Merrill, 2013; Van Merriënboer et al., 2002). In the context of graphing formulas, the whole-task-first approach means that students are confronted from the start with the full complexity of graphing formulas; that is, they have to deal with different kinds of functions and strategies (DP 1). In order to support students, help is provided in different ways: through modeling (that is, showing expert thinking processes to students), examples, overviews, sub-questions, and reflection questions (DP 2) (Kirschner & Van Merriënboer, 2008; Merrill, 2013; Van Merriënboer et al., 2002).

Landa (1983) described the importance of general thinking methods or meta-heuristics that are needed to use one's skills and knowledge in problem situations. Pierce and Stacey (2007) indicated the importance of teaching students the habit of starting with the question "What do I notice about this expression which may be important?" We call this "questioning the formula," which can be considered a meta-heuristic (Arievitch & Haenen, 2005; Landa, 1983). In graphing formulas, students should learn to internalize and automatize the habit of questioning the formula (DP 3).

In the current study, we used these three design principles to design an intervention on graphing formulas, with the aim to promote insight into formulas of functions of one variable. The following main research question guides the study:

How can 16- and 17-year-old pre-university students' insight into algebraic formulas be promoted through graphing formulas?

Methods

In this section, we subsequently describe the intervention, including the tasks that were used in the teaching, the participants, the instruments used in the pretest, posttest, and retention test, and the data analysis.

Intervention

The intervention took five lessons of 90 minutes. Each day, the lesson started with a short plenary discussion (max 10 minutes) with general feedback on the students' work, reflection on the tasks, and modeling of expert thinking processes. After the plenary, the students worked in pairs or groups of three, studied their personal feedback and the written elaborations on the tasks given by the teacher, and discussed strategies and solutions for the whole tasks. The teacher visited each group at least once during a lesson to give further explanations and coaching. At the end of a lesson, all pairs and groups handed in their work for personal feedback which focused on the reflection questions, for which students had to construct their own examples.

The intervention started with a whole-class discussion about the levels of recognition; this was to introduce the meta-heuristic "questioning the formula" (DP 3). The aim was that students would develop the habit of asking themselves questions like: "Do I instantly know the graph?", "Do I recognize a function family?", "Can I decompose the formula?", "Do I recognize graph features?", "Can I do some strategic search for, for instance, infinity behavior?" At the end of the intervention, but before the posttest, 18 of the 21 students voluntarily attended a longer plenary discussion of 30 minutes in which they discussed strategies for graphing several formulas.

The tasks used in the teaching were formulated as whole tasks, reflecting the levels of recognition and the meta-heuristic "questioning the formula": task 1 and 2 concerned recognition of basic functions and aimed to develop a knowledge base of function families with their characteristic features and to deal with simple transformations; task 3 concerned the decomposition of formulas and the composition of sub-graphs through qualitative reasoning; task 4 concerned the instant recognition of key graph features; and task 5 was about strategic exploration of parts of a graph, through qualitative reasoning. We now give some examples of the tasks.

Task 1 required students to match formulas of basic function $y = \sqrt{x}, y = x^3,\ y = 0,5^x, y = \ln(x), y = |x|$ to their graphs. Task 2 was based on Swan (2005): Describe the differences and similarities between the graphs of the pairs of functions like $y = 2\sqrt{x} - 4$, $y = 2\sqrt{x-4}$ and $y = -3^x$, $y = 3^{-x}$. In task 3, the function $y = \sqrt{x}(3x-6)$ had to be graphed by multiplying the graphs of the sub-functions $y = \sqrt{x}$ and $y = 3x - 6$. Task 4 was inspired by Burkhardt and Swan (2013) and Swan (2005), and concerned the recognition of graph features: What features of the given graph can be instantly read from the given two equivalent formulas $y = (x-4)^2 - 1$ and $y = (x-5)(x-3)$?

Task 5 concerned reasoning about parts of a graph (part-graph exploration). For instance, what happens to the y-values of the functions $y = 0,6^x \cdot x^{60}$, $y = 52.7/(1 + 62,9 \cdot 0,692^x)$, when$x \rightarrow +\infty$? Choose $y \rightarrow +\infty$; $y \rightarrow a \neq 0$; $y \rightarrow 0$; $y \rightarrow -\infty$

For each task, help was provided, and a reflection question was added. For instance, in task 2 (about recognizing transformations of basic functions) students could choose to use GeoGebra, and/or to study worked-out examples for help. After each whole task, a reflection question was posed, in which students were asked to construct three new examples to demonstrate the principles of the whole task. Constructing examples are a means to stimulate students to reflect (Watson & Mason, 2002).

Participants

The intervention was held in the first author's grade 11 mathematics B class, a regular class of 21 pre-university students, who were 16 or 17 years old. Mathematics B is a course that prepares students in the Netherlands for university studies in mathematics, science, and engineering. In regular education in the Netherlands, students learn about linear, quadratic, and exponential functions in grades 8 and 9. In grade 10, the graphic calculator is introduced and power, rational, logarithmic functions, and the derivative are the most important topics. In grade 11, further exploration of derivatives and rules for differentiation are taught, together with solving calculus problems (e.g., optimization, tangent, and parameter problems) using algebraic manipulation and the graphic calculator. In this school, students were used to working together on tasks in an open space, as there was only one small room for plenary instruction available, which could be used once a week.

Data collection

We collected all individual student responses to three written tests: the pre-test, the post-test, and the retention test, all of which had two similar tasks: a graphing task and a multiple-choice task that focused on recognition (indication of the total time: 25 min). The formulas used in the three tests, though not the same, were comparable in structure and difficulty. To avoid a learning effect from the tests, the students' work was not returned to them. The period of 4 months between the post-test and the retention test, including a holiday period, was considered long enough to prevent learning effects.

The three written tests demonstrated the students' competencies to graph formulas but gave only limited information about their recognition and reasoning. Therefore, more detailed information about the students' thinking processes was needed: six students were asked to think aloud during the pre-test and post-test, when working on the graphing task. These interviews were videotaped and transcribed. Thinking aloud is not expected to disturb the thinking process and should give reliable information about problem-solving activities (Ericsson, 2006). As it was possible that the effect of the intervention would depend on students' previous mathematics performance, the six students were selected on the basis of their earlier mathematics performances during the school year: two high-achieving (S and K), two more than average-achieving (A and M), and two average-achieving students (Y and I).

In a post-intervention questionnaire, the students were asked to report their ideas about the series of lessons. Six questions were posed: whether they had improved their skills in graphing formulas (1), whether they had learned to use more strategies (2), whether their recognition of graph features had improved (3), whether their recognition of formulas that could be instantly graphed had improved (4), whether they could switch their strategy more often (5), and whether they used the meta-heuristic "questioning the formula" more often when graphing formulas (6). In each question, the students were asked to indicate, on a scale of 1 to 4, to what extent they agreed with the statement. In two open questions, the students were invited to make remarks about the series of lessons and their learning during these lessons. The first author (teacher) kept a logbook with lesson plans, and short descriptions of the plenary discussions and other aspects of the student's learning.

Graphing task and multiple-choice task in the tests

The first task used to investigate the students' insight into formulas was "Draw a rough sketch of the following functions … ." We selected seven simple and seven more complex functions, all of which could appear in the students' mathematics textbooks. The simple functions aimed to assess the students' repertoire of basic function families and their reasoning using prototypical graphs, transformations and/or function family characteristics. Examples of these simple functions are $y = \sqrt{6-2x}$ and $y = (x^2-4)(x^2-6)$. The more complex functions, like $y = \sqrt{x}(x-2)(x-6)$ and $y = 3x\sqrt{x+2}$, aimed to assess the students' recognition of graph features and their part-graph exploration.

To assess the students' recognition abilities, a multiple-choice task with 21 alternatives (20 graphs and one "none of these"; see Figure 1) was also used. For 16 functions, the students were asked to match the formula to the global shape of the graph. The following are examples of functions that were used: $y = 2x(x-9)$, $y = x^2(6-x^2)$, $y = 4^x - 5$, $y = 2\sqrt{8-x}$, $y = -2\sqrt{x}$. Figure 1 shows four of the 20 graphs that were used as alternatives.

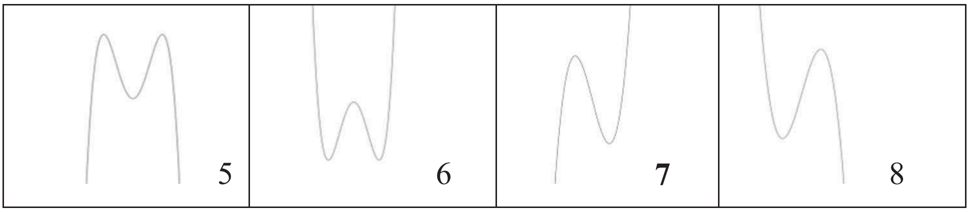

Figure 1. Some alternatives in the multiple-choice task.

Data analysis

For the analysis of the graphing task, the graphs in all tests were graded as correct, partly correct, or not correct, resulting in a score of 1, 0.5, or 0. We graded a graph as partly correct when many but not all aspects of (the construction of) the graph were correct. For example, when the graph of $y = -2x(x-2)(x-5)$ had zeroes at $x = 2$ and $x = 5$, and the direction of the "oscillation" was correct, but the graph failed to show the zero at $x = 0$, or when the sub-graphs of $y = x^2e^x$ ($y = x^2$ and $y = e^x$) were correctly graphed but mistakes were made in the composition of the sub-graphs. For each student, the total score, the score on simple functions, and the score on complex functions were calculated. For the multiple-choice task in all tests, each item was graded as correct (score 1) or incorrect (score 0), resulting in a total score on the multiple-choice task.

To compare the scores on the graphing task and multiple-choice task of the pretest, posttest, and retention test, the mean scores, and standard deviations were calculated. A paired t-test with the effect size (Cohen's d) for each task was calculated to determine differences between pre-test and post-test results (short-term effect) and differences between pre-test and retention test (long-term effect).

The thinking-aloud protocols of the graphing task of the six students were transcribed and time was recorded. The transcripts were cut into units of analysis which contained crucial steps of students' recognition and reasoning (Schwarz & Hershkowitz, 1999). To analyze students' insight, we used categories with descriptions of the experts' strategies in graphing formulas (see examples in theory section): combinations of recognizing and reasoning involving function families, involving key graph features, and part-graph exploration. The encoding of the thinking-aloud protocols was done according to the instructions in Table 1.

The units of analysis were encoded by the first author and checked by another researcher, which resulted in recoding of 10% of the transcripts. When the student succeeded in making a correct (rough) sketch of the formula (score of 1), using P1, P2, F, PG, we interpreted this as a sign of insight into this formula, resulting in an insight-score of 1. However, if the student used the calculation of more than two points and/or the derivative (C), we said that the student had no insight in this formula, resulting in an insight-score of 0. If the graph was partly correct and sketched via P1, P2, F, PG, we considered this as showing "some insight," resulting in an insight-score of 0.5.

Below, we illustrate these encodings with two examples (other examples in the result section).

Student A sketching $y = -(x-3)^4 - 5$ correctly with insight-score 1, used a prototypical graph: *Looks like parabola; turning point is in* (3,-5) (P1); *parabola with a maximum* (P1)

Student A sketching $y = (x^2+6)/(x^2-4)$ correctly with insight-score 1, decomposed the formula into two parabola, graphed both sub-graphs, then used graph features (vertical asymptotes and symmetry) and part-graph exploration about infinity behavior of the function and the graph's behavior in the neighborhood of the vertical asymptotes. Only two points of the graph were calculated, so the insight-score was 1.

(tries to manipulate the function $(x-2)(x+2)$); no, this does not work; first decomposing; (graphed both parabolas) (P2); *when x is very large than y is close to* 1 *(PG); when x = 2 no outcomes, so a vertical asymptote (F); and also at x = − 2; when x is a little bit larger than 2 than the denominator is very small and the dominator relative large (PG); the larger x will be the smaller the outcomes will be*

Table 1. Codebook for thinking-aloud protocols.

Encoding	Description	Example
P1 (a prototypical graph)	If a function family has been recognized (mentioned) and a prototypical graph and/or (qualitative) reasoning (with transformations and/or characteristics) are used to sketch the graph.	Sketching $y = 2\sqrt{x+8}$: "it is a root-function, translated to the left." (factor 2 can be ignored); Sketching $y = \sqrt{8-x}$: "it is a reversed root-function, and edgepoint is (8,0)"
P2 (two prototypical graphs)	If two function families have been recognized (mentioned), two sub-formulas are graphed and the two sub-graphs are combined using qualitative reasoning.	Sketching $y = xe^{-x}$: "decompose it into $y = x$ and $y = e^{-x}$, and multiplying the two sub-graphs, when x is very large e^{-x} is almost 0 and stronger than x, so $y \approx 0$; when x is very negative y will be very negative."
F (key graph feature)	If the graph has not been recognized but a key graph feature has been recognized.	"It has a vertical asymptote at x = 3" or "it has zeroes at … "; but not when calculating the y-intercept.
PG (part-graph)	If the graph has not been recognized and parts of the graph are explored using qualitative reasoning.	"When x is large, y is … (infinity behavior)," or "in the neighborhood of x = 3 … "
C (calculation)	If more than two points of the graph are calculated, or if a derivative is calculated, or brackets in a formula are expanded.	

(PG); when $x = -2$ *this will be the same (F); when* $x = 1$ y *is* $7/-3$ *is about* -2; *when* $x = -1$ *I get the same; when* $x = 0$ *it is* -1.5; *when* x *is just under* 2, *the denominator is negative and the dominator is very large, so is goes to minus infinity (PG); at* $x = -2$ *the same, because of symmetry (F).*

To analyze the post-intervention questionnaire, the mean scores were calculated for each question and an inventory of the remarks about the series of lessons was made.

Results

The results of the graphing and multiple-choice tasks of the pretest, posttest, and retention test are described first, then we report the results of the six thinking-aloud students on the graphing task, and finally the results of the post-intervention questionnaire and fragments of the teacher's logbook.

Graphing tasks

The results of the graphing tasks gave information about the students' abilities to graph formulas. For a first impression of the effect of the intervention, we compared the mean scores in the pre-test, post-test, and retention test. We distinguished between the simple and the complex functions. Table 2 shows that the mean total score in the pretest was 2.95 out of 14, with a standard deviation of 2.42. The post-test scores were higher, with a mean total score of 9.21. In the retention test, the mean score dropped to 6.97. A similar pattern was found for basic functions and complex functions.

The paired *t*-tests that were used to calculate the differences between the scores in the pre-test and post-test and between the pre-test and retention test showed that all score differences were significant with $p < .01$. Cohen's *d*, used to quantify these differences were rather large. See Table 2.

Table 2. Results of graphing task in pre-, post-, and retention test compared.

	Pretest mean (SD)	Posttest mean (SD)	*t*-value & *p*-value pre-post test	Cohen's *d*	Retention test Mean (SD)	*t*-value & *p*-value pre-retention test	Cohen's *d*
Total	2.95 (2.42)	9.21 (2.58)	$t(20) = 13.00$; $p < .001$	2.50	6.97 (3.35)	$t(14) = 3.48$; $p = .001$	1.38
Simple	2.42 (1.42)	6.19 (1.26)	$t(20) = 10.73$; $p < .001$	2.87	4.87 (1.80)	$t(14) = 5.40$; $p < .001$	1.56
Complex	0.74 (1.41)	3.02 (1.71)	$t(20) = 9.51$; $p < .001$	1.45	2.10 (1.85)	$t(14) = 2.46$; $p = .028$	0.83

Multiple-choice tasks

The results of the multiple-choice tasks gave information about the students' recognition of basic function families and graph features. The results on these tasks showed the same pattern as on the graphing tasks: the scores on the 14 items were low in the pre-test, increased substantially in the post-test, and decreased slightly in the retention test. Table 3 shows that the differences were significant and that the effect sizes were rather large.

Thinking-aloud protocols on the graphing task

First, we give an overview of the results of the six students who thought aloud during the graphing task, then, we portray the recognition and reasoning of two representative students (student M and the high-achieving student K) in the pre-test and post-test. These examples illustrate their insight into formulas, that is, their recognition and reasoning when graphing formulas. We end this section with some remarks about the results of the four other students.

Table 4 shows the scores of the six thinking-aloud students on the graphing task in the pre-test and post-test (on simple and complex functions) and the time they needed to finish these tasks. In addition, their scores on the retention test are indicated. For instance, in the pretest, student K had a score of 4 out of 7 on simple functions and a score of 4 out of 7 on complex functions; their insight-score was 3 on both simple formulas and complex formulas; K used in 6 graphs prototypical graphs and in 5 graphs part-graph exploration.

Table 4 confirmed the higher scores in the post-test and retention test in comparison with the pre-test and the differences in scores between simple and complex formulas, as found in Table 2. Table 4 shows that the time needed in the posttest was much shorter than in the pre-test that the scores and insight-scores in the post-test were higher than in the pre-test, and that the scores and insight-scores were closely related. The latter indicates that calculations were hardly used in successfully graphing formulas. Table 4 also shows that the high-achieving students did relatively well on complex formulas in the pre-test and only missed one graph in the post-test. The results show that most students used more prototypes of function families in the post-test. In the retention test, only student S graphed all formulas correctly, but the scores of the other five students were still higher than in the pre-test.

To illustrate the student's insight, we portray the recognition and reasoning of two representative thinking-aloud students: student K as a high-achieving student, and student M as one of the other four students. In the pre-test, student M had great difficulties graphing formulas: M only recognized the graphs of root-functions and features like zeroes of polynomial-functions and vertical asymptotes, but had a limited repertoire of reasoning. This resulted in a score of only 2 correct graphs out of 14 (only $y = 3\sqrt[4]{x} + 2$ and $y = \sqrt{6 - 2x}$). Some citations illustrate their thinking processes and insight.

M sketching $y = (x - 3)^2 - 9$ (insight-score 0); after calculating a point and part-graph reasoning with "when x is increasing then y …," M could not sketch the graph:

> … *At $x = 3$ $y = -9$. (After some time) the larger x is, the larger y, so it increases (PG). It is a parabola. (M stopped talking for a while; after a couple of minutes) I do not know how to proceed.* Encoding: PG.

M also had problems with sketching $y = \ln(x - 3)$(insight-score 0); after recognizing a translation, M did not know the shape of the $\ln(x)$-graph, and tried to construct the graph via the inverse function (but did not succeed):

Table 3. Results of multiple-choice task in pre-, post-, and retention-test compared.

	Pre-test mean (SD)	Post-test mean (SD)	*t*-value & *p*-value pre-post test	Cohen's *d*	Retention test Mean (SD)	*t*-value & *p*-value pre-retention test	Cohen's *d*
Total	2.95 (2.29)	10.01 (2.79)	$t(20) = 10.17$; $p < .001$	2.22	8.07 (3.33)	$t(14) = 4.65$; $p < .001$	1.20

Table 4. Results from thinking-aloud protocols: scores, insight-scores, encodings, time.

	Pre-test				Post-test				Retention test
Student	Score (s + c)	Insight score (s + c)	Kinds of reasoning	Time (min)	Score (s + c)	Insight-score (s + c)	Kinds of reasoning	Time (min)	Score (s + c)
[1]S	3.5 + 4	3 + 4	P12:6 F:3 C:0 PG:4	>16:42[1]	7 + 6	7 + 6	P12:8 F:2 C:0 PG:5	17:48	7 + 7
K	4 + 4	3 + 3	P12:6 F:1 C:5 PG:5	38:25	7 + 6	7 + 6	P12:12 F:2 C:0 PG:2	16:00	5 + 2
M	2 + 0	2 + 0	P12:2 F:6 C:0 PG:1	34:20	7 + 3	7 + 3.5	P12:10 F:4 C:0 PG:1	16:00	5.5 + 2
A	5 + 0	4 + 0	P12:4 F:1 C:3 PG:5	16:25	7 + 2	7 + 2.5	P12:12 F:1 C:0 PG:1	17:12	6 + 2
I	2 + 0.5	1 + 0.5	P12:4 F:6 C:0 PG:9	27:50	7 + 3	5 + 3	P12:5 F:6 C:2 PG:9	21:00	5 + 1.5
[2]Y	2 + 1	2 + 0	P12:5 F:0 C:3 PG:7	>30:55[2]	5.5 + 3.5	5.5 + 3.5	P12:10 F:2 C:0 PG:3	18:22	4 + 1

Simple (s) and complex (c); P12 = using prototypical graphs and/or composition of 2 sub-graphs; F = recognizing key graph feature; PG = part-graph exploration.
[1]For 11 graphs in pretest.
[2]For 12 graphs in pretest.

> *… graph of* $\ln(x)$, *that is translated 3 to the right (F)* (M *did not use this, instead writes* $\log_e(x-3) = \log(e)/\log(x-3); e^y = x-3$*). This is an asymptote (F); x cannot be* 3; *… .; when* $y = 0, x - 3 = 1$ *(drew point* (4,0) *and stopped).* Encoding: F.

In the post-test, M's insight had improved, resulting in a score of 10 out of 14. Some citations to illustrate these improvements:

M sketching correctly $y = 30 \cdot 0.92^x + 40$ (insight-score 1), used a prototypical decreasing exponential graph and a translation, and described globally the function's infinity behavior:

> *Decreasing exponential function (sketched a prototypical graph)(P1); 40 above (P1); when x* = 0, *y* = 70; *later approximately* 40 *(PG).* Encoding: P1,PG.

When sketching $y = -2x(x-2)(x-5)$ correctly (insight-score 1), M recognized the zeroes, and used qualitative reasoning when exploring the function's behavior at $x = 1$ and when ignoring the factor 2 in $-2x$:

> *… goes downwards (F); zeroes at* 0, 2, 5 *(F). At x*=1, *it is negative (PG).* Encoding: F,PG.

M showed "some insight" into $y = (x^2+6)/(x^2-4)$(insight-score 0.5), as M did not indicate the horizontal asymptote; the function's behavior in the neighborhood of $x = 2$ was explored:

> *… asymptotes at* 2 *and* −2 (*F*); *zero at* $\sqrt{6}$; *no, no zeroes, because* x^2 *cannot be negative; when x is smaller than* 2, *then it is positive here, and negative here, so it is negative (PG); when x is a bit larger than* 2, *positive here, positive here, so positive (PG); the same for* − 2 (*F*). Encoding: F,PG.

Although the high-achieving student K scored 8 correct graphs out of 14 in the pre-test, K then had problems with recognizing basic function graphs. However, K was able to compensate this lack of recognition through her reasoning abilities and the calculation of many points. We give two examples to illustrate this: when K did not know the $\ln(x)$ graph and when K could not read the zeroes from $y = (x-2)(x-6)$.

K, sketching $y = \ln(x-3)$ (insight-score 1), did not recognize the shape of a logarithmic function, but used qualitative reasoning about the inverse function to sketch the graph correctly:

> *I do not know the ln-graph anymore. When* $x - 3 = 0$, *then … … When* $x - 3 = 1$, *then* $y = 0$, *so* $x = 4$. *At x-as the x-axis is intersected. When x is increasing then y increases, so the graph increases (PG). When x is negative … (thinking). Because something to the power of* e (e $^{\cdots}$) *does not give negative y-values (PG). So,* $x - 3$ *cannot be negative; the graph only exists from x* = 3, *larger than* 3 (*PG*). *So, at x* = 3 *a tangent (means asymptote) and outcomes smaller when x is in the neighborhood of* 3 *(PG).* Encoding: PG.

K sketching $y = \sqrt{x}(x-2)(x-6)$ correctly but with insight-score 0; K decomposed the formula, but was then unable to sketch the graph of the parabola $y = (x-2)(x-6)$ using recognition and reasoning, as K did not recognize the zeroes and needed the calculation of more than two points of this parabola; however, K showed their reasoning abilities when constructing a correct graph by multiplying the two sub-graph using qualitative reasoning:

> *First expanding the brackets*: $y = \sqrt{x}(x^2 - 8x + 12)$, *… .sub-function is parabola with minimum and root function,* $\sqrt{x}$ *goes like this* (P1); *when x is negative, this part remains empty (left y-axis)*(P1); *at x = 0, parabola gives* = +12; *(sketched an incorrect parabola through (0,0)); … ; (calculation of points*, (1,5) and (4, − 8) (C)*(noticed that parabola is incorrect and calculated more points of parabola;* (2,0), (4, − 4), (6,0)); *so, parabola goes like this (correct parabola); between* 2 *and* 6 *(parabola) negative, so, positive (root) times negative gives negative* (P2), *and more negative than* − 4; *… ; it goes through* (1,5); *… so, I expect that the graph progressively increases because of* $\sqrt{x}$ (P2) *and that is looks like a parabola; at x* = 0, *y* = 0, *that means that between 0 and 1 something strange happens; it goes like …..*$\sqrt{x}$; *… .; (sketched a correct graph).* Encoding: P1,P2,C

In the post-test, K's recognition of basic functions had improved and K still used their reasoning abilities, resulting in a score of 13 out of 14. Some citations to portray their insight into formulas:

When sketching $y = 2\sqrt{5-x}$ correctly (insight-score 1), K ignored the factor 2:

> "*times* -1; *exists for* $x \leq 5$ (P1); *so, starts at* $x = 5$ (P1), *and from there is goes like this*". Encoding: P1

When sketching $y = 2x\sqrt{x+6}$ correctly (insight-score 1), K decomposed the formula and used part-graph reasoning in the composition of the two sub-graphs:

> $2x$ *goes like this* (P1); $\sqrt{x+6}$ *goes like this (sketch)*(P1); *here it is* 0; *here negative, here* 0, *and after this it is steeper* (P2). Encoding: P1,P2.

K sketching $y = 30/(2 + 6 \cdot 0.9^x)$ correctly with insight-score 1, gesturing the sub-graph of the denominator, ignoring the factors 2 and 6 and reasoning about infinity behavior of the function:

> 0.9^x *goes like this* (P1); $6 \cdot 0.9^x$ *and* $2 + 6 \cdot 0.9^x$ (P1) *like this (gestured correct graph); … 30/ … .; 30/2 goes to* 15; *that means a horizontal asymptote (F); 30 divided by an ever increasing number (looks at the negative x-axis) becomes smaller, goes to* 0 (*PG*). Encoding: P1,F,PG.

These examples of student K illustrate how the two high-achieving students (K and S) were already able to reason with and about formulas in the pre-test, but had problems with the recognition of the prototypical graphs and characteristics of basic function families. In the post-test, their recognition had improved, and they were able to combine their recognition and reasoning more effectively and efficiently, resulting in more insight into formulas. Also, the four other students had problems with recognizing basic function families and their characteristics in the pre-test, but their reasoning then was very limited, as was illustrated by citations of student M and Table 4. In the post-test, these four students showed insight in almost all the simple formulas. However, they still had problems with complex formulas. Although they showed more insight as they were able to make the first steps (e.g., decomposing into sub-formulas and graphing correct sub-graphs), they had difficulties composing the two sub-graphs and/or finding and combining all relevant graph information. Two examples to illustrate these problems:

Student Y graphed $y = -x^4 + 2x^2$ partly correct (insight-score 0.5); Y missed that the graph of x^4 is "flatter" than the one of x^2 in the neighborhood of $x = 0$:

> *Adding both* (P2); *I split the function; a parabola 'to the power of* 4*' will run like this* (P1)*(sketches the graph of* $y = -x^4$); $2x^2$ *goes like this* (P1); *this is not nice, you have to add them; the 'to the power of* 4*' is stronger than ' to the power of* 2*', so, … .; it goes through* 0; *then adding; this one* $(-x^4)$ *is stronger, thus it goes under this one" (sketched a parabola-shaped graph with maximum)*(P2). Encoding: P1,P2

Student I graphed $y = x + e^{-x}$ partly correctly (insight-score 0.5); the sub-graphs were correct, but the composition was incorrect:

> e^x *goes like this; -x, so* $(y = e^x)$ *is reversed over y-axis* (P1)*: it becomes smaller and is not negative; the larger x, the smaller y;* e^{-x} *is stronger;* $y = x$ *goes like this* (P1) *(sketched two correct sub-graphs); when* $x = -1$ *it is positive; when x is more positive, than* e^{-x} *becomes smaller and x larger, but* e^{-x}*is stronger, so, the outcomes are smaller and negative* (P2)*(sketched a graph beneath the x-axis for large values of x)*. Encoding: P1,P2

Post-intervention questionnaire and teacher's logbook

In the post-intervention questionnaire, the students indicated, on a scale of 1 to 4, whether they thought they had improved their skills in graphing formulas (mean score 3.1), that they used more strategies (mean score 3.2), that they had improved their recognition of graph features (mean score 3.3), and that they used "questioning the formula" more often (mean score 3.0). However, the scores on "more formulas could be instantly graphed" and "being able to switch strategy" were lower: 2.8 and 2.4, respectively. In their answers on the open questions about the series of lessons and their learning during these lessons, the students indicated they thought their recognition of basic functions and graphs had improved, that they could visualize formulas (of basic functions) faster, and that they "understood" formulas better.

Also, the teacher's logbook confirmed the progress in the students' insight during the intervention. To illustrate this, we provide some quotations from the teacher's logbook. During the first lesson: "The pre-test was not motivating for the students, but after some time they are working on the teaching tasks." During the second lesson: "The task about transformations is hard for these students and costs more time than needed." During the fourth lesson: "In the groups, I heard their reasoning with 'this one goes like this (with gesturing)' and 'when x is very large, then … '." During the last lesson: "The high-achieving students show more interest in the plenary discussions than usual. They seem to be challenged by these tasks. One of the students indicated that they thought these lessons (in the intervention) are different from regular lessons: 'we now use global reasoning (referring to qualitative reasoning); it is fun this kind of reasoning.' In a discussion, students show their abilities to reason qualitatively when discussing the graph of $y = 10\sqrt{6-x} + 3$. One of the students had drawn a global graph on the whiteboard, using (6,3) as starting point, and sketched a reversed root-graph (i.e., a root graph to the left). Another student wondered what had to be done with the 10 in the formula. The first student responded 'hardly anything, only when one wants to compare the graph with 10 and the graph with, for instance, 8. However, the graph with -10 is reversed, so very different. The same is true for the 3 in the formula: 3, 4, 5 does not matter, but -3 does matter.' A third student then explained this fact by referring to the scaling of the vertical axis."

Discussion and conclusion

The current research aimed at promoting insight into algebraic formulas, an important aspect of symbol sense. To foster grade 11 students' insight, we chose to teach experts' strategies in graphing formulas, which could be described through a combination of recognition and reasoning (Kop et al., 2015/2017). In this study, we designed an intervention of five lessons of 90 minutes, focusing on the recognition of basic function families and of graph features, and on qualitative reasoning, and investigated whether students' insight was enhanced. The pre-test results of the written tests showed that the students had problems with graphing formulas and the thinking-aloud protocols suggested a lack of recognition and reasoning skills, which resulted in time-consuming calculations and many incorrect graphs. The lack of recognition was confirmed by the results of the multiple-choice test.

In the post-test, the results of the written tests showed large improvements. The thinking-aloud protocols of six students showed how their recognition and reasoning skills had improved. All six students showed insight into formulas, as they could now recognize function families and use these in their reasoning. However, unlike the two high-achieving students S and K, the other four students still had problems in graphing the complex functions. Although these four students showed more insight into complex functions, using decompositions into sub-functions and graphing these sub-functions correctly, they often made mistakes in the composition of the two sub-graphs and/or in finding and combining all relevant information. The results of the post-intervention questionnaire suggest that the students themselves thought that their skills in graphing formulas had improved, that they used more strategies and more recognition, and that they had more insight into formulas, as they indicated that they understood formulas better.

In the retention test, the scores on the graphing task and multiple-choice task were, as expected, lower than in the post-test. Still, the scores were higher than in the pre-test. This suggests a long-lasting effect of the intervention, in particular on simple functions.

The findings of the current study suggest that through this intervention in which students were taught to graph formulas using recognition and qualitative reasoning, students improved their insight into formulas, that is, the ability to identify the structure of a formula and to reason with and about formulas.

Before we address the study's limitations and reflect on the intervention, we discuss the findings. In the current study, we chose to use graphing formulas to foster students' insight into formulas, in contrast to other approaches that focus on manipulations and/or structures of expressions. Graphing formulas is a small domain in algebra, which makes it more possible for students to learn experts' strategies. However, graphing formulas is also a rich domain, as it can involve all kinds of functions and involves aspects which are important in learning about functions: the relation between two major representations of functions, formulas, and graphs, allowing students to give meaning to abstract algebraic formulas (Kieran, 2006), and the need of both a global and a local perspective on functions to learn about the process and object duality of functions. The results of the thinking-aloud protocols reveal that all students started to use experts' strategies, although only high-achieving students were able to correctly graph complex formulas. Students used insight into formulas to graph formulas, but hardly used algebraic manipulations even if these would be more convenient, for example, when graphing $y = -x^4 + 2x^2$. The results of the questionnaire and the logbook suggested that the graphing tasks in the intervention were challenging and encouraged students to engage in algebraic reasoning. We believe that our strategy to select a small domain in algebra and to focus on just reading through formulas and making sense of formulas might explain a part of the positive students' results in this study.

Our approach differs from regular approaches as well as from innovative approaches to learn about algebraic formulas as it was based on a systematical analysis of experts' strategies in which the two elements, recognition of function families and key graph features and qualitative reasoning, both play an important role. Regular approaches often focus on manipulation of algebraic expressions (Arcavi et al., 2017; Schwartz & Yerushalmy, 1992), and use graphing tools, for example, the graphic calculator, to explore function families and to work on calculus problems. In comparison to regular approaches, our intervention paid more attention to the recognition of function families and graph features, to part-graph exploration, and to the reasoning with and about formulas. In innovative approaches, graphing tools have been used to learn to reason about functions using the structure of the formula, for instance, the composition and translation of graphs (Schwartz & Yerushalmy, 1992; Yerushalmy, 1997; Yerushalmy & Gafni, 1992), about the role of parameters (Drijvers, 2003; Heid et al., 2013), and about special function families (Heid et al., 2013). Pierce and Stacey (2007) suggested highlighting the formula's structure and key features when considering graphs in classroom discussions. Friedlander and Arcavi (2012) developed a framework comprising different cognitive processes and activities, including qualitative thinking and global comprehension, and formulated small tasks in which components of their framework had been worked out. In comparison to these innovative approaches, our intervention paid more attention to the systematical teaching of thinking tools: a repertoire of basic function families, the recognition of function families and key graph features, and qualitative reasoning. In the designed intervention these aspects were taught in an integrated way via a task-centered approach with adaptive support.

In the current study, several levels of recognition and several aspects of qualitative reasoning were distinguished. Often recognition is treated as a dichotomous variable: there is recognition or there is no recognition. In our approach we use different levels of recognition: complete recognition and instantly knowing the graph, recognizing a member of a function family, decomposing the formula into manageable sub-formulas, perceiving key graph features, no recognition. These levels of recognition can be linked to Mason's (2003) levels of attention, in which has been described how attention can shift from seeing essential structure to gazing at the whole and not knowing how to proceed. An essential aspect in our approach was the explicit focus on qualitative reasoning. The importance of this kind of reasoning and its omission in mathematics curricula has been stressed by Leinhardt et al. (1990), Goldenberg, Lewis, and O'Keefe (1992), Yerushalmy (1997), and Duval (2006), who have indicated that qualitative reasoning could support the construction of meaning and understanding through its global focus. To our knowledge, this idea has never been applied in concrete and systematic teaching approaches. In our approach, students learned to use global descriptions and to ignore what is not relevant, when composing two sub-graphs (after decomposing a formula into two sub-formulas) and when exploring parts of a graph, for instance, infinity behavior. We recommend paying more attention to explicit teaching of qualitative reasoning in grades 11. We expect that in other domains of algebra, such as solving equations, qualitative reasoning might help students to become more proficient in algebra, as it might enable students to ignore what is not relevant and to focus on the structure of formulas/equations.

In the designed intervention not only attention has been paid to recognition and to qualitative reasoning, but also explicit attention is paid to the *interplay* between recognition and qualitative reasoning. In problem solving, recognition determines the problem space within which via certain heuristics can be searched for a solution (Berliner & Ebeling, 1989; Chi et al., 1981). In the intervention, each whole task was related to one of the levels of recognition (see intervention in method section), and attention was paid to the reasoning needed to sketch the graph, starting from this level of recognition. This approach enables students to use different ways to graph a function like $y = 30/(2 + 6 \cdot 0.9^x)$: in the post-test, we found students who decomposed this function into two sub-functions ($y = 30$ and the exponential function $y = 2 + 6 \cdot 0.9^x$), but also students who used part-graph exploration (infinity behavior and/or the function is increasing), and/or the calculation of the y-intercept. These examples illustrate how a correct graph can be produced via different levels of recognition in combination with different reasonings and that insight into formulas can be described as an interplay between recognition and reasoning. The analysis of the thinking-aloud protocols showed how students' insight into formulas could be described with the recognition of a function family and (qualitative) reasoning about transformations and/or family characteristics, the decomposition of a formula into two sub-functions and the composition of two sub-graphs through qualitative reasoning, the recognition of key graph features, and the qualitative reasoning about parts of a graph. Although in other domains of mathematics, like in modeling and solving equations, insight into formulas might consist of different aspects, our descriptions might be helpful in describing insight and in designing education to promote insight into formulas in these domains.

Insight in the interplay between recognition and reasoning can contribute to a better knowledge about covariational reasoning in the context of algebraic functions. Graphing formulas by hand is closely related to this kind of covariational reasoning. Both are about how a function is acting on an entire domain, have a focus on global graphs and use qualitative reasoning. The current study showed that the use of function families with their prototypical graphs and characteristics is crucial in graphing formulas. However, Moore and Thompson (2015) have problematized what they called static shape thinking, that is, seeing a graph-as-a-wire, and associating shapes with function properties. Previous studies about expert behavior in graphing formulas have showed that experts often use their repertoire of function families (Kop et al., 2015/2017). Eisenberg and Dreyfus (1994) and Slavit (1997) have indicated that students need such a repertoire of basic function families and function properties. The pre-test results of our study showed that before the intervention students lacked a repertoire of function families that could be instantly visualized by a graph. As a consequence, graphing formulas required too much reasoning of these students. Post-test results showed that students had improved their recognition of basic function families with their prototypical graphs and characteristics, which could be used as building blocks in their reasoning. The results of our study suggest that students' covariational reasoning might improve if they can use such repertoire of function families to reason with prototypes.

Limitations of the study

A limitation of the study is that only one class was involved, and no comparison group was included. As the results were positive, we would recommend involving more students and other teachers in a future study to provide stronger evidence that graphing formulas in this way does indeed promote students' insight into algebraic formulas. We suggest also including students and teachers from other countries in a future study, as we expect that difficulties with insight into algebraic formulas are not exclusive to students in The Netherlands. In the current study, insight into formulas was studied in the context of graphing formulas. We expect that there might be some transfer from insight into formulas from this domain of graphing formulas to other domains of algebra, such as solving algebraic problems and solving equations. More research is needed to explore whether students who have learned insight into formulas via graphing formulas will be able to use this insight when working on other algebraic problems that are related to graphs (e.g., discussing the number of solutions of a given equation).

In the pre-test, the students needed more time than expected for the graphing task. This might be the reason for the poor scores on the multiple-choice task in the pre-test, as many students did not have enough time to work on that task. From the thinking-aloud protocols, we conclude that some students had problems interpreting the graphs in the multiple-choice task, as they thought that the x-axis and y-axis were drawn instead of vertical and horizontal asymptotes. We suggest to explicitly indicate the asymptotes in the figures and illustrate this via an example in the task description. The whole task on transformations of basic functions (task 2) took much more time than planned, and the students often needed the help provided by the teaching material. The whole tasks on the composition of two sub-graphs and on part-graph exploration by qualitative reasoning (tasks 3 and 5) needed, as planned, extra modeling by the teacher, as this kind of reasoning was new to the students. The meta-heuristic of "questioning the formula" was at the core of this series of lessons and was demonstrated more than once. In the post-intervention questionnaire, the students indicated that they had started to question formulas. However, this was often very implicit, as the thinking-aloud protocols showed.

Some aspects of the series of lessons deserve more attention in the future. On each level of recognition, only one whole task with a reflection question was formulated, because of time constraints (5 lessons). With more time available, we would follow Kirschner and Van Merriënboer (2008) suggestion to use more variability in the whole tasks (more whole tasks on each level of recognition) with more practice of the integration and coordination of all sub-skills. To improve reflection, the implementation of cumulative reflection tasks, which promote reflection on the current task and all previous tasks, might be considered. In the current study students had problems with graphing polynomial functions, like with $y = -x^4 + 2x^2$, but not when zeroes could easily be read from the formulas, like with $y = -2x(x-3)(x-6)$. When graphing $y = -x^4 + 2x^2$, students used qualitative reasoning to compose two sub-graphs, after decomposing the formula into sub-formulas $y = -x^4$ and $y = 2x^2$, which gave them much trouble and incorrect graphs. These findings suggest to pay more attention to polynomial function families and to incorporate small manipulations of algebraic formulas, for instance, to rewrite $y = -x^4 + 2x^2$ into $y = x^2(-x^2 + 2)$, which would enable students to find zeroes of polynomial functions.

Conclusion

This study portrays how students might learn insight into formulas, that is, the ability to "look through a formula," to recognize the structure of a formula and its components, and to reason with and about a formula. Graphing formulas requires students to recognize the structure of formulas and to reason with and about formulas. Therefore, our teaching focused on using function families as meaningful building blocks and on using qualitative reasoning. Students often see formulas on an atomic level, that is, paying attention to every number and variable, which means that students cannot see the wood before the trees: they do not recognize any structure (Davis, 1983).

The current study showed how students learned to use function families as larger meaningful building blocks to recognize the structure of formulas and to graph formulas. The two ingredients, function families as larger building blocks and qualitative reasoning, are important thinking tools in the recognition of the structure of the formulas and so, in the reading of formulas, as they might relieve students' working memory. Our findings suggest that teaching graphing formulas to grade 11 students, based on recognition and qualitative reasoning, might be an efficient means to promote student insight into algebraic formulas in a meaningful and systematical way.

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