Recognizing DAGs with Page-Number 2 Is NP-complete

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Abstract. The page-number of a directed acyclic graph (a DAG, for short) is the minimum k for which the DAG has a topological order and a k-coloring of its edges such that no two edges of the same color cross, i.e., have alternating endpoints along the topological order. In 1999, Heath and Pemmaraju conjectured that the recognition of DAGs with page-number 2 is NP-complete and proved that recognizing DAGs with page-number 6 is NP-complete [*SIAM J. Computing*, 1999]. Binucci et al. recently strengthened this result by proving that recognizing DAGs with page-number k is NP-complete, for every $k \geq 3$ [*SoCG* 2019]. In this paper, we finally resolve Heath and Pemmaraju's conjecture in the affirmative. In particular, our NP-completeness result holds even for *st*-planar graphs and planar posets.

Keywords: Page-number · Directed acyclic graphs · Planar posets

1 Introduction

The problem of embedding graphs in books [27] has a long history of research with early results dating back to the 1970's. Such embeddings are specified by a linear order of the vertices along a line, called *spine*, and by a partition of the edges into sets, called *pages*, such that the edges in each page are drawn crossing-free in a half-plane delimited by the spine. The *page-number* of a graph is the minimum number of pages over all its book embeddings, while the page-number of a graph family is the maximum page-number over its members.

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An important branch of literature focuses on the page-number of planar graphs. An upper bound of 4 was known since 1986 [30], while a matching lower bound was only recently proposed [4,31]. Better bounds are known for several subfamilies [14,15]. A special attention has been devoted to the planar graphs with page-number 2 [3,8,11,13,19,21,25,28]. These have been characterized as the subgraphs of Hamiltonian planar graphs [17] and hence are called *subhamiltonian*. Recognizing subhamiltonian graphs turns out to be NP-complete [29].

If the input graph is directed and acyclic (a DAG, for short), then the linear vertex order of a book embedding is required to be a *topological order* [26]. Heath and Pemmaraju [16] showed that there exist planar DAGs whose page-number is linear in the input size. Certain subfamilies of planar DAGs, however, have bounded page-number [1,6,10,18], while recently it was shown that upward planar graphs have sublinear page-number [20], improving previous bounds [12]. From an algorithmic point of view, testing whether a DAG has page-number k is NP-complete for every fixed value of $k \geq 3$ [7], linear-time solvable for $k = 1$ [16], and fixed-parameter tractable with respect to the vertex cover number for every k [6] and with respect to the treewidth for st-graphs when $k = 2$ [7]. In contrast to the undirected setting, however, for $k = 2$ the complexity question has remained open since 1999, when Heath and Pemmaraju posed the following conjecture.

Conjecture 1 (Heath and Pemmaraju [16]). *Deciding whether a DAG has page-number 2 is NP-complete.*

Our Contribution. In this work, we settle Conjecture 1 in the positive. More precisely, we show that testing st-planar graphs for 2-page embeddability is NP-complete. In [2], we further show that the problem remains NP-complete for *planar posets*, i.e., upward-planar graphs with no transitive edges.

2 Preliminaries

A *plane embedding* of a connected graph is an equivalence class of planar drawings of the graph, where two drawings are equivalent if they define the same clockwise order of the incident edges at each vertex and the same clockwise order of the vertices along the outer face. The *flip* of a plane embedding produces a plane embedding in which the clockwise order of the incident edges at each vertex and the clockwise order of the vertices along the outer face is the reverse of the original one. A drawing of a DAG is *upward* if each edge is represented by a curve whose y-coordinates monotonically increase from the source to the sink, and it is *upward planar* if it is both upward and planar. An *upward planar embedding* is an equivalence class of upward planar drawings of a DAG, where two drawings are equivalent if they define the same plane embedding and the same left-to-right order of the outgoing (and incoming) edges at each vertex. A *plane DAG* is a DAG together with an upward planar embedding. A DAG is *st-planar* if it has a single source s and a single sink t, and admits a planar drawing with s and t on the outer face. It is known that every st-planar graph is upward planar [9,22]. An

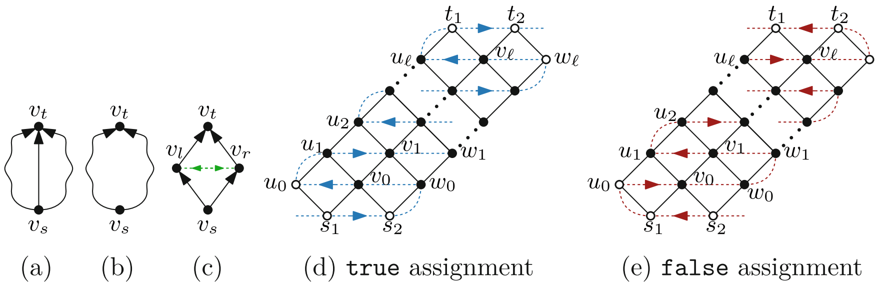


Fig. 1. Curly curves represent paths and straight-lines represent edges. Edges with no arrow are directed upward, also in subsequent figures. (a) Generalized diamond, (b) non-transitive face, (c) rhombus, and (d)-(e) the two subhamiltonian paths of a double ladder of even length ℓ.

st-plane graph is an st-planar graph together with an upward planar embedding in which s and t are incident to the outer face. As in the undirected case, a DAG G has page-number 2 if it is *subhamiltonian*, i.e., it is a spanning subgraph of an st-planar graph $\overline{G}$ that has a directed Hamiltonian st-path P [24]. In the previous definition, if G has a prescribed plane embedding, we additionally require that the plane embedding of $\overline{G}$ restricted to G coincides with the one of G. We say that P is a *subhamiltonian path* for G, and we refer to the edges of P that are not in G as *augmenting edges*. Further, $\overline{G}$ is an *HP-completion* of G.

A *generalized diamond* is an st-plane graph consisting of three directed paths from v_s to v_t, one of which is the edge v_sv_t and appears between the other two paths in the upward planar embedding; see Fig. 1a. A face (by *face* of a plane DAG we always mean an *internal face*, unless otherwise specified) of a plane DAG whose boundary consists of two directed paths is an *st-face*. An st-face is *transitive* if one of these paths is an edge; *non-transitive*, otherwise (see Fig. 1b). A *rhombus* is a non-transitive st-face whose boundary paths have length 2; see Fig. 1c. From [24, Theorem 1], we obtain Property 1 which implies Property 2.

Property 1. *A Hamiltonian st-plane graph contains only transitive faces and no generalized diamond.*

Property 2. *Let G be a plane DAG and P be a subhamiltonian path for G. If G contains a rhombus (v_s, v_l, v_r, v_t) with source v_s and sink v_t, then P contains either the edge v_lv_r or the edge v_rv_l, i.e., v_l and v_r are consecutive in P.*

The next property follows directly from Theorem 1 in [23] and Property 1. We provide a full proof in [2].

Property 3 (⋆). *Let G be a plane DAG and P be a subhamiltonian path for G. If G contains a non-transitive face f with boundaries (v_s, w, v_t) and $(v_s, v_1, \ldots, v_r, v_t)$, then the augmenting edges of P inside f are either (i) the edge wv_1, or (ii) the edge v_rw, or (iii) edges v_iw and wv_{i+1} for some $1 \leq i < r$.*

3 NP-completeness

Let ϕ be a Boolean 3-SAT formula with n variables $x_1, \ldots, x_n$ and m clauses $c_1, \ldots, c_m$. A clause of ϕ is *positive* (*negative*) if it has only positive (negative) literals. The *incidence graph* G_ϕ of ϕ is the graph that has *variable vertices* $x_1, \ldots, x_n$, *clause vertices* $c_1, \ldots, c_m$, and has an edge (c_j, x_i) for each clause c_j containing x_i or $\overline{x}_i$. Note that we use the same notation for variables (clauses) in ϕ and variable vertices (clause vertices) in G_ϕ. If ϕ has clauses with less than three literals, we introduce parallel edges in G_ϕ so that all clause vertices have degree 3 in G_ϕ; see, e.g., the dotted edge in Fig. 4. The formula ϕ is an instance of the NP-complete Planar Monotone 3-SAT problem [5], if each clause of ϕ is positive or negative, and G_ϕ has a plane embedding $\mathcal{E}_\phi$ to which the edges of a cycle $\mathcal{C}_\phi := x_1, \ldots, x_n$ can be added that separates positive and negative clause vertices. The problem asks whether ϕ is satisfiable. Next, we present our gadgets.

Double Ladder. A double ladder of even length ℓ is defined as follows. Its vertex set consists of two sources, s_1 and s_2, two sinks, t_1 and t_2, and vertices in $\cup_{i=0}^{\ell}\{u_i, v_i, w_i\}$. Its edge set consists of edges s_1u_0, s_1v_0, s_2v_0, s_2w_0, $u_\ell t_1$, $v_\ell t_1$, $v_\ell t_2$, $w_\ell t_2$, and $\cup_{i=0}^{\ell-1}\{u_iu_{i+1}, v_iu_{i+1}, v_iv_{i+1}, w_iv_{i+1}, w_iw_{i+1}\}$.

Property 4. *The double ladder has a unique upward planar embedding (up to a flip), shown in Figs. 1d and 1e.*

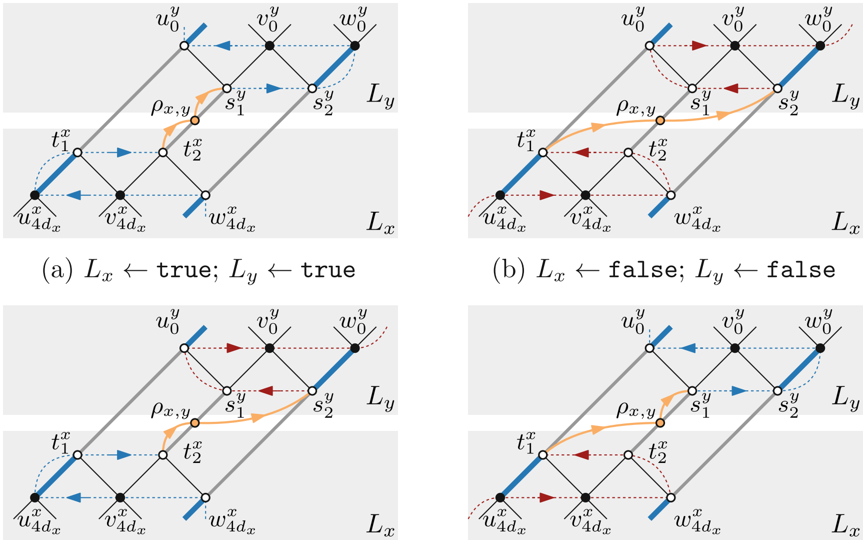
Proof. The embedding shown in Figs. 1d and 1e clearly is an upward planar embedding. The underlying graph of the double ladder has four combinatorial embeddings, which are obtained from the embedding in Figs. 1d and 1e, by possibly flipping the path $u_1u_0s_1v_0$ along u_1v_0 and the path $w_{\ell-1}w_\ell t_2v_\ell$ along $w_{\ell-1}v_\ell$. However, such flips respectively force s_1v_0 and $v_\ell t_2$ to point downward. Finally, since the outer face of the embedding in Figs. 1d and 1e is the only face containing at least one source and one sink, the claim follows. □

Property 5. *Let G be a plane DAG with a subhamiltonian path P. If G contains a double ladder of length ℓ, then P contains the pattern $[\ldots u_iv_iw_i \ldots w_{i+1} v_{i+1}u_{i+1} \ldots]$ or $[\ldots w_iv_iu_i \ldots u_{i+1}v_{i+1}w_{i+1} \ldots]$ for $i = 0, \ldots, \ell-1$.*

Proof. By Properties 2 and 4, u_i, v_i, w_i are consecutive along P, for $i = 0, \ldots, \ell$. The edge u_iu_{i+1} implies that u_i, v_i, w_i precede $u_{i+1}, v_{i+1}, w_{i+1}$. So, it remains to rule out patterns $[\ldots u_iv_iw_i \ldots u_{i+1}v_{i+1}w_{i+1} \ldots]$ and $[\ldots w_iv_iu_i \ldots w_{i+1}v_{i+1} u_{i+1} \ldots]$. If P contains one of them, then edges u_iu_{i+1}, v_iv_{i+1} and w_iw_{i+1} pairwise cross, implying that G has page-number at least 3; a contradiction. □

Corollary 1. *There exist two subhamiltonian paths for the double ladder, shown in Figs. 1d and 1e.*

Variable Gadget: Let $x \in \{x_1, \ldots, x_n\}$. The variable gadget L_x for x is the double ladder of length $4d_x$, where d_x is the degree of x in G_ϕ. To distinguish between vertices of different variable gadgets, we denote the vertices of L_x with



(a) $L_x \leftarrow$ `true`; $L_y \leftarrow$ `true` (b) $L_x \leftarrow$ `false`; $L_y \leftarrow$ `false`

(c) $L_x \leftarrow$ `true`; $L_y \leftarrow$ `false` (d) $L_x \leftarrow$ `false`; $L_y \leftarrow$ `true`

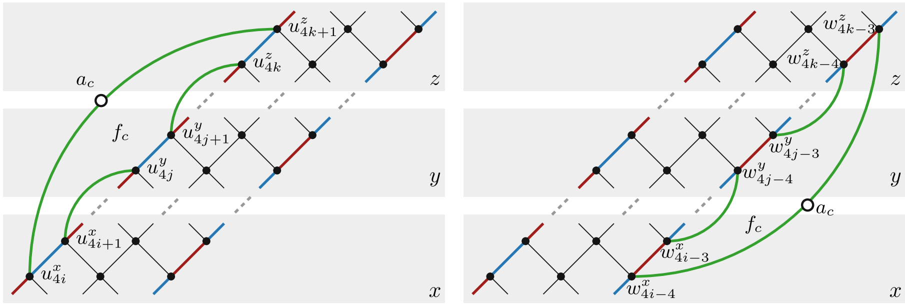
Fig. 2. The connector gadget for two variables having (a)-(b) the same truth assignment, and (c)-(d) the opposite truth assignment.

the superscript x, as in Fig. 2. Vertices s_1^x, s_2^x, u_0^x are the *bottom connectors* and $w_{4d_x}^x$, t_1^x, t_2^x are the *top connectors* of L_x. The two subhamiltonian paths of Corollary. 1 correspond to the truth assignments of x; Fig. 1d corresponds to `true`, while Fig. 1e to `false`. Also, we refer to the edges of L_x that are part of the subhamiltonian path of Fig. 1d (of Fig. 1e) as *true edges* (*false edges*, respectively). In particular, $u_{2j}^x u_{2j+1}^x$ and $w_{2j+1}^x w_{2j+2}^x$ are true edges of L_x, while $u_{2j+1}^x u_{2j+2}^x$ and $w_{2j}^x w_{2j+1}^x$ are false edges of L_x, for $j = 0, \ldots, 2d_x - 1$.

Connector Gadget: A connector gadget connects two variable gadgets L_x and L_y by means of three paths from the top connectors of L_x to the bottom connectors of L_y; see Fig. 2. These paths are: the edge $t_1^x u_0^y$, the length-2 path $t_2^x \rho_{x,y} s_1^y$, where $\rho_{x,y}$ is a newly introduced vertex, and the edge $w_{4d_x}^x s_2^y$.

Property 6. *Given subhamiltonian paths P_x for L_x and P_y for L_y, there is a subhamiltonian path P containing P_x and P_y for the graph obtained by adding a vertex $\rho_{x,y}$ and edges $t_1^x u_0^y, t_2^x \rho_{x,y}, \rho_{x,y} s_1^y, w_{4d_x}^x s_2^y$ to $L_x \cup L_y$.*

Proof. Each of P_x and P_y is one of the two subhamiltonian paths of Corollary 1; see Fig. 1. In particular, the last vertex of P_x is t_1^x or t_2^x, and the first vertex of P_y is s_1^y or s_2^y, depending on the truth assignments for x and y, respectively, as shown in Fig. 2. We obtain P by adding directed edges from the last vertex of P_x to $\rho_{x,y}$ and from $\rho_{x,y}$ to the first vertex of P_y. □



(a) $(x \vee y \vee z)$ (b) $(\overline{x} \vee \overline{y} \vee \overline{z})$

Fig. 3. Clause gadgets for (a) a positive clause and (b) a negative clause. (Color figure online)

Clause Gadget: Let c be a positive (negative) clause. Assume that the variables x, y and z of c appear in this order along $\mathcal{C}_\phi$, when traversing $\mathcal{C}_\phi$ from x_1 towards x_n. In $\mathcal{E}_\phi$, the edges between x and the positive (negative) clause vertices of G_ϕ appear consecutively around x. Assume that the edge (c, x) is the $(i+1)$-th such edge in a clockwise (counter-clockwise) traversal of the edges around x starting at the edge of $\mathcal{C}_\phi$ incoming x. Similarly, define indices j and k for y and z, respectively. Let L_x, L_y and L_z be the three variable gadgets for x, y and z.

The clause gadget C_c for c consists of an *anchor vertex* a_c, and four edges. If c is positive, these edges are $u^x_{4i}a_c$, $a_c u^z_{4k+1}$, $u^x_{4i+1}u^y_{4j}$ and $u^y_{4j+1}u^z_{4k}$ (green in Fig. 3a); otherwise, they are $w^x_{4i-4}a_c$, $a_c w^z_{4k-3}$, $w^x_{4i-3}w^y_{4j-4}$ and $w^y_{4j-3}w^z_{4k-4}$ (green in Fig. 3b). Note that C_c creates a non-transitive face f_c, called *anchor face*, whose boundary is delimited by the two newly-introduced edges incident to a_c and by a directed path whose edges alternate between three true edges (if c is positive) or three false edges (if c is negative) and the two newly-introduced edges not incident to a_c; see Fig. 3. The three true (or false) edges on the boundary of f_c stem from L_x, L_y, and L_z. The length of the double ladders ensures that, if $x = y$ (which implies that $j = i+1$), then vertices u^x_{4i+1} and u^y_{4j} (w^x_{4i-3} and w^y_{4j-4}) are not adjacent in L_x and the edge $u^x_{4i+1}u^y_{4j}$ ($w^x_{4i-3}w^y_{4j-4}$) is well defined; this is the reason that we do not use vertices with indices $2, 3 \mod 4$.

Theorem 1. *Recognizing whether a DAG has page-number* 2 *is* NP*-complete, even if the input is an st-planar graph.*

Proof. The problem clearly belongs to NP, as a non-deterministic Turing machine can guess an order of the vertices of an input graph and a partition of its edges into two pages, and check in polynomial time whether the order is a topological order and if so, whether any two edges in the same page cross.

Given an instance ϕ of Planar Monotone 3-SAT, we construct in polynomial time an st-planar graph H that has page-number 2 if and only if ϕ is satisfiable; see Fig. 4. We consider the variable gadgets $L_{x_1}, \ldots, L_{x_n}$, where $x_1, \ldots, x_n$ is the order of the variables along the cycle $\mathcal{C}_\phi$; for $i = 1, \ldots, n-1$,

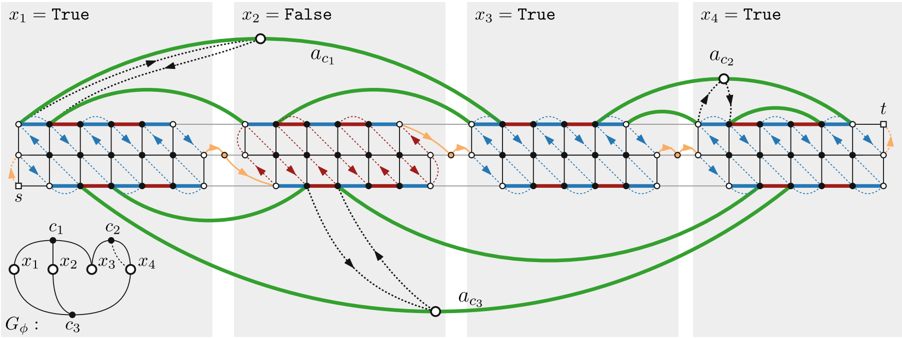


Fig. 4. The graph H obtained from $\phi = c_1 \wedge c_2 \wedge c_3$ with $c_1 = (x_1 \vee x_2 \vee x_3)$, $c_2 = (x_3 \vee x_4)$, and $c_3 = (\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_4)$. For space reasons, the variable gadgets have smaller length and the drawing is rotated by 45°.

we connect L_{x_i} with $L_{x_{i+1}}$ using a connector gadget. For each positive (negative) clause c of ϕ, we add a clause gadget C_c using the true (false) edges of the variable gadgets. This yields a plane DAG with two sources $s_1^{x_1}$ and $s_2^{x_1}$ and two sinks $t_1^{x_n}$ and $t_2^{x_n}$. We add a source s connected with outgoing edges to $s_1^{x_1}$ and $s_2^{x_1}$, and a sink t connected with incoming edges to $t_1^{x_n}$ and $t_2^{x_n}$. The constructed graph H is st-planar. Since the underlying graph of H is a subdivision of a triconnected planar graph and since only one face of H contains s and t, it follows that H has a unique upward planar embedding. We next prove that H is subhamiltonian (and therefore has page-number 2) if and only if ϕ is satisfiable.

Assume first that ϕ is satisfiable. We show how to construct a subhamiltonian path P for H, by exploiting a satisfying truth assignment for ϕ. For $i = 1, \ldots, n$, we have that P contains the subhamiltonian path P_i for L_{x_i} shown in Fig. 1d if x_i is `true`, and the one shown in Fig. 1e otherwise. By Property 6, there is a subhamiltonian path P for the subgraph of H induced by the vertices of all variable and connector gadgets, containing $P_1, \ldots, P_n$ as subpaths. The path P starts from a source of L_{x_1} and ends at a sink of L_{x_n}; hence we can extend P to include s and t as its first and last vertices. We now extend P to a subhamiltonian path for H by including the anchor vertex of each clause gadget. Consider a positive clause $c = (x \vee y \vee z)$ with anchor vertex a_c; the case of a negative clause is similar. As ϕ is satisfied, at least one of x, y and z is `true`; assume w.l.o.g. that x is `true`. By construction, the anchor face f_c of C_c is non-transitive, with the anchor vertex a_c on its left boundary, and exactly one true edge of each of L_x, L_y, and L_z along its right boundary. Let $i \geq 0$ be such that $u_{4i}^x u_{4i+1}^x$ is the true edge of L_x on the right boundary of f_c. Since x is `true`, vertices u_{4i}^x and u_{4i+1}^x are consecutive in P. We extend P by visiting vertex a_c after u_{4i}^x and before u_{4i+1}^x. This corresponds to adding two augmenting edges $u_{4i}^x a_c$ and $a_c u_{4i+1}^x$ of P in the interior of f_c; see the black dashed edges of Fig. 4. At the end of this process, P is extended to a subhamiltonian path for H.

Assume now that there exists a subhamiltonian path P for H. For each variable gadget L_{x_i}, P induces a subhamiltonian path P_i for L_{x_i}. By Corollary 1, P_i is one of the two subhamiltonian paths of Fig. 1. We assign to x_i the value `true` if P_i is the path of Fig. 1d and `false` if P_i is the path of Fig. 1e. We claim that this truth assignment satisfies ϕ. Assume, for a contradiction, that there exists a clause c that is not satisfied. Assume that c is a positive clause $(x \vee y \vee z)$, where x, y and z are assigned `false`, as the other case is analogous. Also, assume that x, y, z appear in this order in C_ϕ, and that the right boundary of the anchor face f_c of the clause gadget C_c contains the true edges $u^x_{4i}u^x_{4i+1}$, $u^y_{4j}u^y_{4j+1}$ and $u^z_{4k}u^z_{4k+1}$ of L_x, L_y and L_z. As x, y and z are `false`, the corresponding subhamiltonian paths P_x, P_y and P_z of L_x, L_y and L_z are the ones of Fig. 1e. Hence, P contains the augmenting edges $u^x_{4i}v^x_{4i}$ and $v^x_{4i+1}u^x_{4i+1}$ of P_x, $u^y_{4j}v^y_{4j}$ and $v^y_{4j+1}u^y_{4j+1}$ of P_y and $u^z_{4k}v^z_{4k}$ and $v^z_{4k+1}u^z_{4k+1}$ of P_z. By Property 3 for the non-transitive face f_c, P contains either (i) the augmenting edge $a_c u^x_{4i+1}$, or (ii) the augmenting edge $u^z_{4k}a_c$, or (iii) for a pair of consecutive vertices, say u and u', along the right boundary of f_c, the augmenting edges ua_c and $a_c u'$. Cases (i) and (ii) contradict the existence of augmenting edges $v^x_{4i+1}u^x_{4i+1}$ and $u^z_{4k}v^z_{4k}$ of P respectively. Similarly, in case (iii) the augmenting edges of P that belong to P_x, P_y, and P_z imply that $u \notin \{u^x_{4i}, u^y_{4j}, u^z_{4k}\}$ and $u' \notin \{u^x_{4i+1}, u^y_{4j+1}, u^z_{4k+1}\}$. Hence $u = u^x_{4i+1}$ and $u' = u^y_{4j}$ holds, or $u = u^y_{4j+1}$ and $u' = u^z_{4k}$. In both cases, the HP-completion of H contains a generalized diamond with $v_s = u$ and $v_t = u'$, violating Property 1. Hence at least one of variables x, y and z must be `true`, contradicting our assumption that c is not satisfied. □

We conclude by mentioning that our NP-completeness proof can be adjusted so that the constructed graph is a planar poset; refer to [2] for details.

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