ORIGINAL PAPER

How to Fit a Tree in a Box

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Abstract

We study compact straight-line embeddings of trees. We show that perfect binary trees can be embedded optimally: a tree with n nodes can be drawn on a $\sqrt{n}$ by $\sqrt{n}$ grid. We also show that testing whether a given rooted binary tree has an upward embedding with a given combinatorial embedding in a given grid is NP-hard.

Keywords Binary trees · Graph drawing · Upward drawing · Area requirement

1 Introduction

Let $T = (V, E)$ be a combinatorial tree; that is, a connected graph without cycles. A *straight-line embedding* of T in a grid is an injective map $f : V \to \mathbb{Z}^2$. An embedding f is *planar* if for every pair of edges $(v_1, v_2), (w_1, w_2) \in E$ the line segments $f(v_1)f(v_2)$ and $f(w_1)f(w_2)$ do not intersect except, possibly, at common endpoints. In this paper all the embeddings considered are planar. The *size* of an embedding (or, with slight abuse of terminology, the size of the grid) is the width and height of the portion of $\mathbb{Z}^2$ used by f; that is,

$$\text{size}_f(T) = \left(\max_{v \in V} f(v).x - \min_{v \in V} f(v).x + 1, \quad \max_{v \in V} f(v).y - \min_{v \in V} f(v).y + 1 \right).$$

We are interested in finding embeddings with as small a size as possible. We say an embedding of a graph with n vertices is *packed* if it is embedded in a $w \times h$ grid with $n = wh$. We say it is *almost packed* if it cannot be embedded in a strictly smaller rectangular grid; that is, if it is embedded in a $w \times h$ grid with $n < wh$ but $n > (w-1)h$ and $n > w(h-1)$. In the following, we may also write a $w \times h$ rectangle to refer to a rectangle of size (w, h).

A *rooted* tree is a tree T with a special vertex $r \in V$ marked as root. Because a tree has no cycles, a rooted tree has an induced partial order on its vertices: for two

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vertices $v, w \in V$, we say $v \prec w$ if and only if v lies on the path from r to w. An embedding is *strictly upward* if, for all $v, w \in V$ with $v \prec w$, we have $f(v).y > f(w).y$. An embedding is *upward* if, for all $v, w \in V$ with $v \prec w$, we have $f(v).y \geq f(w).y$. A rooted tree together with a combinatorial embedding is called an *ordered* tree. A drawing or an ordered tree is *order-preserving* if it has the same combinatorial embedding.

Related Work. Drawing graphs with small area has a long and rich history [7]. However, despite being studied for more than 50 years, we still do not know even whether simple classes of graphs, such as trees and outerplanar graphs, admit linear-area embeddings (a graph with n nodes can be embedded in a $w \times h$ grid with $wh \in O(n)$). Chan [4] showed that every tree admits an embedding with $n2^{O(\sqrt{\log\log n \log\log\log n})}$ area, improving the long-standing $O(n \log n)$ bound one obtains by a simple divide-and-conquer layout algorithm.

Furthermore, not much is known about the exact minimum area requirements for graphs that do admit linear-area embeddings. It is clear that not every tree admits a packed embedding in a grid with exactly n points: for instance, when the graph is a star, some grid points are "blocked" and cannot be used. The star graph *can* be drawn on a linear-area grid: Euler already showed that the fraction of points visible from the center of a square grid tends to $\frac{6}{\pi^2}$ more than 300 years ago [8]. For graphs of bounded degree, there is hope that we can do better. Clearly, every path admits a packed embedding. Garg and Rusu [11, 12] showed that trees of degree $d = O(n^\delta)$ with $\delta < 1/2$, and in particular of degree 3, have linear-area embeddings in a square grid, and even in grids of different aspect ratio; their main concern is studying the relation between the aspect ratio and the area, but they do not give concrete bounds on the constant factor.

We conjecture that every tree with maximum degree 3 admits an (almost) packed embedding in a square grid, and we prove here that this is the case for perfect binary trees.

For general graphs, when edges are additionally required to be on the grid as well, this problem is known to be NP-hard [13], using a reduction from the Logic Engine by Eades and Whitesides [9]. Later, Krug and Wagner showed that it is also NP-hard to minimize the area of a straight-line grid drawing of a general graph (order-preserving or not) [15]. The reduction produces a disconnected graph.

When drawing rooted trees, a natural restriction is to require drawings to be *upward*. Clearly, packed strictly upward embeddings are impossible unless the tree is a path, but we may still investigate almost packed embeddings that leave only few grid points unused. It is known that, for strictly upward embeddings, we cannot do better than $\Omega(n \log n)$ area with [10] or without [5] a prescribed combinatorial embedding. There is a matching upper bound for non-order-preserving embedding [10], but the best known upper bound when the combinatorial embedding is given is $O(n4^{\sqrt{2\log n}})$ [3]. For upward embeddings, Chan [4] also gave an $o(n \log n)$ bound. In the special case of binary trees, it is known that complete binary trees admit linear area upward embeddings [5], even with extra constraints requiring that edges are axis-aligned and that the bounding boxes of disjoint subtrees are disjoint [6].

Biedl and Mondal [2] proved NP-hardness for deciding whether a strictly upward embedding exists for a straight-line high-degree tree in a given grid. Their reduction uses the flexibility of determining the combinatorial embedding in order to obtain the NP-hardness.

Contribution. We have the following results.

– It is NP-hard to test whether binary trees with fixed combinatorial embedding admit strictly upward embeddings in a given grid.
– Perfect binary trees of odd height with n vertices admit almost packed embeddings in a $\sqrt{n} \times \sqrt{n}$ grid. For even height there are almost packed embeddings in a rectangular grid. In both cases, a single grid point is left unused.

2 Packed Embeddings of Perfect Binary Trees

We consider the following problem. Given a $\lceil \sqrt{n} \rceil \times \lceil \sqrt{n} \rceil$ grid and a tree T with n vertices, can we draw it with straight non-crossing edges?

Conjecture 1 *If T has max degree 3, it is always possible.*

In particular, if $n = 2^{k+1}$, a perfect binary tree of odd height k with additional parent of the root (to make the number of vertices exactly n) can be drawn on the $\sqrt{n} \times \sqrt{n}$ grid. We use a recursive strategy to show it. Similar approaches recursively embedding trees have been previously used to show asymptotic bounds (but disregarding smaller order terms); in particular, to prove that perfect binary trees and Fibonacci trees can be upward drawn in linear area [5] and to bound the area of complete ternary and 7-ary trees on the 8-grid [1].

Theorem 1 *The perfect binary tree of odd height k with an additional root parent, with $n = 2^{k+1}$ vertices, has a packed embedding in the $\sqrt{n} \times \sqrt{n}$ square grid.*

Proof We will recursively argue that perfect binary trees can be embedded in square grids in two ways. Let T_k be the perfect binary tree on $n = 2^{k+1} - 1$ vertices. We will recursively define two straight-line embeddings, F_k and G_k, of T_k, both on a $\lceil \sqrt{n} \rceil \times \lceil \sqrt{n} \rceil$ grid and leaving a single spot empty. The vertices in these embeddings are placed in the grid points $\{(x, y) \in \mathbb{Z}^2 : 1 \leq x \leq 2^{(k+1)/2}, 1 \leq y \leq 2^{(k+1)/2}\}$.

We first list the required properties of F_k and G_k, also illustrated in Fig. 1a:

(i)   both F_k and G_k map the root of T_k to the point $(2^{(k-1)/2} + 1, 2^{(k-1)/2})$;
(ii) both F_k and G_k do not place any edges in the vertical strip between $x = 2^{(k-1)/2}$ and $x = 2^{(k-1)/2} + 1$, except for the edges incident to the root of T_k;
(iii) F_k leaves the point $(2^{(k-1)/2}, 1)$ unused; and
(iv) G_k leaves the point $(1, 1)$ unused.

Observe that $F_1 = G_1$ is trivial to draw: both are drawings of a path of length 2, drawn by connecting the point $(1, 2)$ to the point $(2, 1)$ to the point $(2, 2)$.

What remains is to argue that we can recursively draw F_k and G_k using drawings

of F_{k-2} and G_{k-2}. The argument is illustrated in Fig. 1.

To draw F_k we place two (possibly mirrored) copies of F_{k-2} and two (possibly mirrored) copies of G_{k-2} in the four quadrants of the $2^{(k+1)/2}$ by $2^{(k+1)/2}$ grid: one identical copy of F_{k-2} in the top right quadrant, one horizontally mirrored copy of F_{k-2} in the top left quadrant, one vertically mirrored copy of G_{k-2} in the bottom right quadrant, and one horizontally mirrored copy of G_{k-2} in the bottom right quadrant. Because, except for edges incident to the respective roots, each copy leaves a vertical strip empty, we can connect the roots of the top right and bottom right subtrees to a new node at $(2^{(k-1)/2} + 2^{(k-3)/2}, 2^{(k-1)/2} + 1)$ (which was left empty by definition of F). Note that, since the new edges are also incident to the roots of the subtrees, they don't cross with the edges incident to the respective roots that cross the strip. Moreover, the new top edge lies completely above the edge crossing the strip in the bottom right quadrant and the new bottom edge lies completely below the edge crossing the strip in the top right quadrant. Thus, the two new edges are drawn without crossings. Similarly, we connect the roots of the top left and bottom left subtrees to a new node at $(2^{(k-3)/2} + 1, 2^{(k-1)/2} + 1)$. Finally, we connect both of these new nodes to the new root of T_k, drawn at $(2^{(k-1)/2} + 1, 2^{(k-1)/2})$ (which was left empty by definition of G), as required. The point $(2^{(k-1)/2}, 1)$ remains unused,
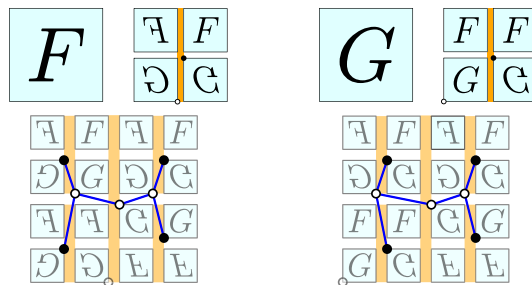
(a) (Top) Tiles F and G, and their recursive definition. The root is shown as a black dot and the unused point is show as a white dot. (Bottom) New edges used to connect the root of T_k to the four roots (grandchildren of the root of T_k) of the subtrees defined recursively. Note that we place the two children of the root of T_k in points left unused by the recursive subtrees.
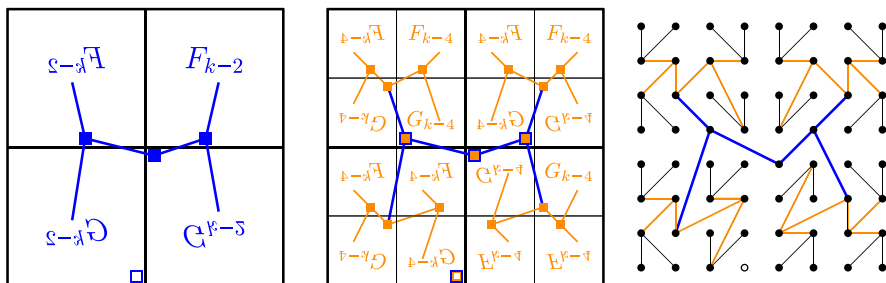
(b) Left and center: recursive definition of F_k. Right: F_5.

Fig. 1 Recursive embedding of perfect binary trees

and the central vertical strip is empty except for one edge connecting the root of T_k to the left subtree.

To draw G_k we also place two copies of F_{k-2} and two copies of G_{k-2} in the four quadrants of the $2^{(k+1)/2}$ by $2^{(k+1)/2}$ grid: one identical copy of F_{k-2} in the top right quadrant, another identical copy of F_{k-2} in the top left quadrant, one identical copy of G_{k-2} in the bottom left quadrant, and one vertically mirrored copy of G_{k-2} in the bottom right quadrant. As in the case of F, we can connect the roots of the top right and bottom right subtrees to a new node at $(2^{(k-1)/2} + 2^{(k-3)/2}, 2^{(k-1)/2} + 1)$, and the roots of the top left and bottom left subtrees to a new node at $(2^{(k-3)/2}, 2^{(k-1)/2} + 1)$. We connect the two new nodes to the new root of G_k at $(2^{(k-1)/2} + 1, 2^{(k-1)/2})$. The point $(1, 1)$ remains unused, and the central vertical strip is again empty except for the edge connecting the root of G_k to the left subtree.   □

Perfect binary trees with even height (with an additional root parent) trivially do not admit a packed embedding in a square grid, but we can use the construction in the proof of Theorem 1 to obtain packed embeddings in rectangular grids. If the even height is k, a packed embedding of the perfect binary tree with an additional root parent and $n = 2^{k+1}$ vertices is given by the subdrawing in the two right quadrants of F_{k+1}.

Corollary 1 *The perfect binary tree of even height k with an additional root parent, with $n = 2^{k+1}$ vertices, has a packed embedding in the $2^{k/2+1} \times 2^{k/2}$ rectangular grid.*

3 Upward Embedding of Trees in a Given Grid is NP-Hard

Recall that an embedding of a rooted tree is *upward* if the y-coordinate of a node is strictly greater than the y-coordinate of its children. A *combinatorial embedding* is given by a circular order of incident edges around each vertex. In this section we show that deciding if a rooted binary tree with a fixed combinatorial embedding can be drawn upward and without crossings in a given square grid is NP-complete.

Theorem 2 *Deciding whether an upward planar straight-line drawing of a fixed combinatorial embedding of a rooted binary tree on a grid of given size $(w \times h)$ exists is NP-complete.*

Proof The problem is in NP since a grid drawing of a tree with k vertices in the grid can be expressed in $O(k)$ size by assigning vertices to grid points. Checking whether the drawing is an embedding can trivially be done in $O(k^2)$ time by checking pairwise edge crossings. Checking whether the drawing preserves the given rotation system takes $O(k^2)$ time and checking whether it is upward can be done in $O(k)$ time.

We prove NP-hardness by a reduction from 3SAT which is an NP-complete problem [14]. An instance of 3SAT is given by a set $\{x_1, \ldots, x_n\}$ of n variables and a set $\{c_1, \ldots, c_m\}$ of m clauses. Each variable can assume one of two values in $\{\texttt{true}, \texttt{false}\}$. Each clause is defined by 3 literals, i.e., positive or negative copies of a variable. A clause is *satisfied* if at least one of its literals is $\texttt{true}$. The problem

3SAT asks for an assignment from the variables to $\{\texttt{true}, \texttt{false}\}$ that satisfies all clauses. We give an arbitrary order for the variables and say that x_i appears *before* x_j if $i < j$. The first (resp., second, resp., third) literal of a clause is the literal (among the 3 literals that define the clause) of the variable that appears first (resp., second, resp., third) in the order assigned to variables. Given an instance of 3SAT we build a rooted tree with $O(m^2 + mn)$ vertices and set $w = 4m + 4$ and $h = \lceil \log_2(4m + 4) \rceil + 5n + 4m + 1$.

Overview. Refer to Fig. 2. This paragraph gives a brief informal overview of the reduction. The following paragraphs will give a full proof. In this section, all logarithms are base 2. The reduction is divided into 3 parts. The top part (spanning the top $\lceil \log(4m + 4) \rceil$ rows in Fig. 2) is a perfect binary tree with $2^{\lceil \log(4m+4) \rceil - 1}$ leaves. The middle part (spanning the next $5n$ rows in Fig. 3) is where the variables are assigned a Boolean value. The bottom part (spanning the last $4m + 1$ rows in Fig. 3) enforces that every clause of the original instance of 3SAT is satisfied. Each variable is represented by a red subtree with two long paths that have to span all but one row below the least common ancestor. The left (resp., right) path represents a positive (resp., negative) literal of the variable. The construction forces one of the paths to be drawn one unit above the other and that encodes the Boolean assignment. If the left path does not span the last row, then the variable is set to $\texttt{true}$. The variable is set to $\texttt{false}$ otherwise. In Fig. 2, x_2 and x_3 (resp., x_1 and x_4) are set to $\texttt{true}$ (resp., $\texttt{false}$). The blue subtrees encode the clauses by allowing the rest of the construction to occupy specific extra grid positions. The incidence of a variable in a clause is encoded by an extra leaf child in one of the paths that represent the incident literal corresponding to the variable. If none of the incident literals of a clause are set to $\texttt{true}$, the drawing would require the use of an extra row or column.
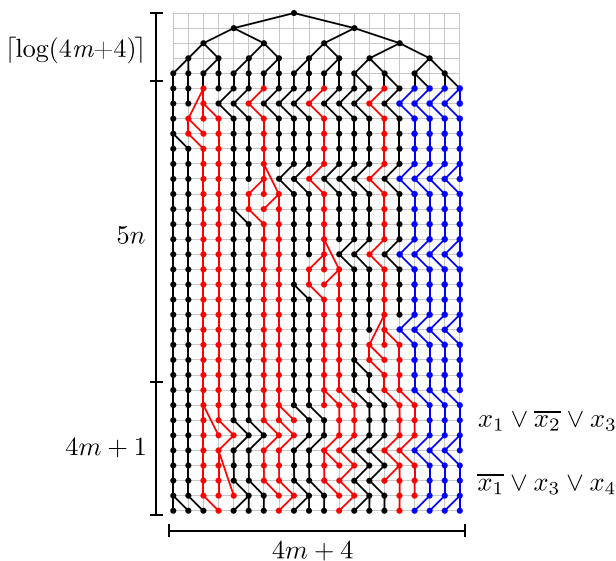

Fig. 2 Reduction from 3-SAT

Otherwise, the extra leaves can be accommodated exactly by the space provided by the blue subtrees.

Construction. There are exactly $4m + 4$ subtrees attached to the perfect binary tree on the top of the construction. The fixed combinatorial embedding prescribes a left-to-right order of such subtrees. For each variable x_i, do the following. Set the $4(i - 1) + 1$-th subtree to be a path p of length $5n + 4m$; attach another path of length $5n + 4m - 5(i - 1) - 4$ to the right of the $5(i - 1) + 4$-th vertex of p. Attach a right child to the second to last vertex of p. Set the $4(i - 1) + 2$-th subtree to be a path of length $5(i - 1) + 1$. Set the $4(i - 1) + 3$-th subtree to be a path of length $5(i - 1)$. At the end of the path, attach two paths p_t and p_f of length $5(n - i + 1) + 4m - 2$ each as left and right subtrees respectively. Attach a right (resp., left) child to the first vertex of p_t (resp., p_f). We now describe the position of the vertices that encode the incidence of a variable in a clause. We call such vertices *literal leaves*. If x_i (resp., $\overline{x_i}$) is the first or second literal of c_j, then add a right child $\ell_{i,j}$ to the $5(n - i + 1) + 4(j - 1)$-th vertex of p_t (resp., p_f). If x_i (resp., $\overline{x_i}$) is the third literal of c_j, then add a left child $\ell_{i,j}$ to the $5(n - i + 1) + 4(j - 1)$-th vertex of p_t (resp., p_f). The $4(i - 1) + 3$-th subtree is shown in red in Fig. 2. Set the $4(i - 1) + 4$-th subtree to be a path of length $5(i - 1)$. Finally, we describe the four last subtrees (shown in blue in Fig. 2). Set the $4m + 1$-th, $4m + 2$-th, and $4m + 3$-th subtrees to be paths of length $5n + 4m$ each. For every clause c_j, attach a right leaf child to the $5n + 4j$-th vertex of the $4m + 3$-th subtree. Set the last subtree to be a path of length $5n$. For each variable x_i, attach a right leaf child to the $5i - 4$-th vertex of the path. This finalizes the construction.

Correctness. We argue that the construction is correct, by showing that every satisfiable 3SAT instance can indeed be embedded in a $w \times h$ grid, and that every drawing that fits in a $w \times h$ grid must correspond to a satisfiable 3SAT instance.

Correctness ($\Rightarrow$). First assume that the 3SAT instance has a positive solution. Then we can upward embed the constructed tree in a $(w \times h)$ grid as follows. Use the first $\lceil \log(4m + 4) \rceil$ rows to embed the top part of the tree (perfect binary tree). We describe the embedding of the bottom part of the tree by assigning grid points to vertices of the subtrees from left to right. Let the bottom left grid point be $(0, 0)$. Assign the root of the s-th subtree to $(s, 5n + 4m)$. We describe the embedding from left to right with s starting from 1 to $4m + 4$.

General Rule. Apart from the roots of the paths p_t and p_f of the red subtrees, and the literal leaves, recursively assign the left child to the leftmost free grid point in the row immediately below its parent, then assign the right child to the leftmost free grid point in the same row.

Encoding Truth Assignment. For the i-th red subtree (i.e., $s = 4i - 1$), let v_i be the least common ancestor of p_t and p_f. Denote by $(v_i.x, v_i.y)$ the coordinate of the grid point assigned to v_i. If x_i is assigned `true` (resp., `false`) in the solution of the 3SAT instance, respectively assign the roots of p_t and p_f to $(v_i.x, v_i.y - 1)$ and $(v_i.x + 1, v_i.y - 2)$ (resp., $(v_i.x - 1, v_i.y - 2)$ and $(v_i.x, v_i.y - 1)$).

Position of Literal Leaves $\ell_{i,j}$. Let a be the parent of $\ell_{i,j}$. If x_i or $\overline{x_i}$ is the first literal of a clause c_j, we place $\ell_{i,j}$ as follows. If it is the last (and therefore the only) literal of c_j with `true` value, assign it to $(a.x + 1, a.y - 1)$ (see Fig. 3a). Else, if its value is
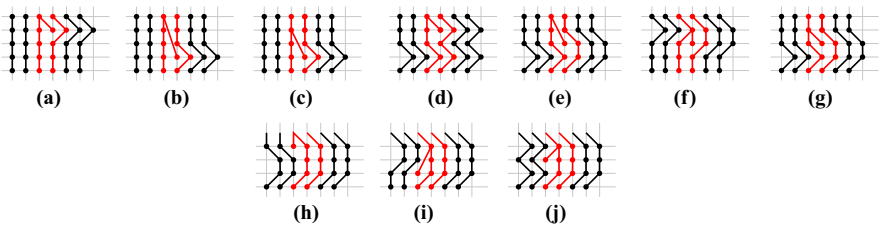
Fig. 3 Possible embeddings for the literal leaves of a clause

`true`, assign it to $(a.x + 1, a.y - 3)$ (see Fig. 3b). Else, assign it to $(a.x + 1, a.y - 2)$ (see Fig. 3c). If x_i or $\overline{x}_i$ is the second literal of a clause c_j, we proceed as follows. If it is the last `true` literal of c_j, assign $\ell_{i,j}$ to $(a.x + 1, a.y - 1)$ (see Fig. 3d). Else, if its value is `true`, assign it to $(a.x + 1, a.y - 2)$ (see Fig. 3e). If the first literal of c_j was embedded as in Fig. 3a, then assign $\ell_{i,j}$ to $(a.x, a.y - 1)$ (see Fig. 3f). Else, assign $\ell_{i,j}$ to $(a.x + 1, a.y - 1)$ (see Fig. 3g). If x_i or $\overline{x}_i$ is the third literal of a clause c_j, we proceed as follows. If its value is `true`, assign $\ell_{i,j}$ to $(a.x, a.y - 1)$ (see Fig. 3h). Else, if the second literal of c_j was embedded as in Fig. 3f (which happens when the first and second literals are `true` and `false` respectively), then assign $\ell_{i,j}$ to $(a.x - 1, a.y - 2)$ (see Fig. 3i). Else, assign $\ell_{i,j}$ to $(a.x - 1, a.y - 1)$ (see Fig. 3j). By construction, the blue subtrees will occupy one less grid point in each of the 3 rows of y-coordinate $1 + 4(m - j)$, $2 + 4(m - j)$ and $3 + 4(m - j)$ for each c_j. Since the SAT instance has a positive solution, these will be exactly filled by the literal leaves of the corresponding variable gadgets.

Correctness ($\Leftarrow$). Now, assume that the constructed tree can be upward embedded in the $(w \times h)$ grid. We show that the 3SAT instance has a positive solution.

Top Part. Notice that the number of vertices in the subtrees of the descendants of the top perfect binary tree is $(5n + 4m + 1)(4m + 4)$. The uppermost y-coordinate that each root of a subtree can occupy is $5n + 4m$ because each has exactly $\lceil \log(4m + 4) \rceil$ ancestors. Then, the solution must place all $4m + 4$ roots of such subtrees on the row with y-coordinate $5n + 4m$ and

($\bigstar$) every point of the grid below this y-coordinate must be occupied.

Frame. The first subtree can only be embedded as shown in Fig. 2, i.e., each node occupying the uppermost leftmost grid point possible, or else a grid point would not be used contradicting ($\bigstar$). The same is valid for the blue subtrees as well switching leftmost for rightmost, by ($\bigstar$). The second tree also has only one possible embedding by ($\bigstar$): if a vertex is mapped to a point that is to the right of its leftmost possible position, then a grid point would be unused; else if it is mapped to a point lower than its uppermost possible position, then the edge between the vertex and its parent would go through a grid point, making it unusable by other vertices.

Variable assignment. Let T_1 be the first red subtree. We show that the top part of T_1 has only two possible embeddings. First, note that due to a "bump" created by a leaf of the last subtree, every grid point $(b, 4m + 5n - 1)$ for $b \geq 3$ is occupied by a

vertex of a tree to the right of T_1, or else ($\bigstar$) would be violated. Let v_l and v_r be the roots of p_l and p_r respectively (i.e., children of the root of T_1). Then, only one among v_l and v_r can be at $(2, 4m + 5n - 1)$ and those are the only vertices that can be assigned to that grid point. If we choose v_l (resp., v_r) to be at $(2, 4m + 5n - 1)$, then v_r (resp., v_l) must be at $(1, 4m + 5n - 2)$ (resp., $(3, 4m + 5n - 2)$). The remainder of the embedding of T_1 is fixed by ($\bigstar$) until row $4m$. Using similar arguments, we can show that, if ($\bigstar$) is satisfied, once we fix the embedding of the top part of T_1, the embeddings of all subtrees are fixed from their roots to row $4m + 5n - 5$. We can then apply induction to show that there are only two possible embeddings for the top part of i-th red subtree T_i and every non-red subtree has a fixed embedding until row $4m$.

Clause Satisfaction. Once the middle part of the construction is fixed, row 0 is also fixed because the $4m + 3$ leftmost vertices at row $4m$ have $4m + 4$ descendant leaves at distance $4m$. That implies that every vertex on a path of length $4m$ or $4m - 1$ in the bottom part of the construction has their y-coordinate fixed. Because of the fixed position of the leaves of the blue trees, the literal leaves of a clause c_j can only occupy the y-coordinates $4(m - j) + 1$, $4(m - j) + 2$, and $4(m - j) + 3$. Note that a literal leaf can only occupy a y-coordinate $4(m - j) + 3$ if the corresponding red path was embedded above the other red path of the same variable. Since every clause must have a literal leaf at y-coordinate $4(m - j) + 3$, assigning `true` (resp., `false`) to x_i if the corresponding p_l (resp., p_r) is embedded above p_r (resp., p_l) will result in a satisfying assignment. $\qquad\square$

4 Conclusions

We studied tree drawings in small areas.

For arbitrary drawings, we constructed (almost) packed embeddings of perfect binary trees. The main remaining open question here is whether every low-degree tree with wh vertices (or fewer) can be embedded in a $w \times h$ grid. We conjecture that this is the case when $w = h$. Another intriguing question is, for general trees, whether testing if they can be embedded in a given grid is computationally tractable.

For strictly upward drawings, we showed that even for bounded-degree trees, testing whether an order-preserving embedding of given tree in a $w \times h$ rectangle exists is already NP-hard. It would be interesting to know whether the same is true when one can freely choose the combinatorial embedding (Biedl and Mondal's proof [2] produces trees of degree $n^{\Omega(1)}$). Another question is whether the problem is also NP-hard for upward embeddings, where adjacent vertices may be embedded using the same y-coordinate.

Availability of Data and Material Not applicable.

Code Availability Not applicable.

Declarations

Conflict of interest Not applicable.

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