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Unstable Periodic Homotopy Theory

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Onstabiele Periodieke Homotopietheorie
(met een samenvatting in het Nederlands)

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Introduction

In this thesis we study unstable v_h-periodic homotopy theory, where $h \in \mathbb{N}$; here "unstable" refers to the homotopy theory of topological spaces (instead of spectra). The work consists of two parts. In Part I we give a detailed exposition of the foundations of unstable v_h-periodic homotopy theory, sharpen an existing result about v_h-periodic equivalences of H-spaces, and pose concrete questions and conjectures for future studies. The expository part follows [Bou94; Bou96; Bou97; Bou01; DroCS; Heu20b; Heu21] and aims to assemble in one place the central notions and theorems of unstable localisations with a focus on unstable periodic homotopy theory. The goal of Part II is to understand unstable v_h-periodic phenomena from the point of view of Lie algebras in the stable v_h-periodic homotopy category. We analyse the costabilisation of v_h-periodic homotopy types and obtain a universal property of the Bousfield–Kuhn functor. Below we provide an introduction for each part.

Our understanding of unstable v_h-periodic homotopy theory takes a lot of inspiration from rational homotopy theory of topological spaces, which we recall now. One goal of homotopy theory is to distinguish two given topological spaces up to weak homotopy equivalence, i.e. whether there exists a continuous map between them that induces an isomorphism of their homotopy groups in all degrees. For example, every topological space is weakly homotopy equivalent to a CW-complex by CW-approximation, see [HatAT, Proposition 4.13]. So, understanding the homotopy groups of a CW-complex, both the structural properties and concrete computations, is of great importance, but this is a very hard problem in general. Thus, we content ourselves with studying certain more approachable algebraic invariants which induce coarser equivalence relations of topological spaces. For pointed simply-connected CW-complexes X and Y, a *rational homotopy equivalence* between X and Y is a pointed continuous map $f\colon X \to Y$ which induces an isomorphism $f_*\colon \pi_\bullet(X) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\simeq} \pi_\bullet(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$ of graded rational homotopy groups. One of the goals of rational homotopy theory is to distinguish topological spaces up to rational homotopy equivalence. The rational homotopy groups of a CW-complex are more computable than its homotopy groups. For example, in [Ser53] the rational homotopy groups of spheres are completely determined, summarised in the following theorem.

***Theorem* (Serre).** — *For a natural number $n \geq 1$, denote the n-dimensional sphere by S^n. We have isomorphisms of graded rational vector spaces*

$$\pi_\bullet(\mathrm{S}^n) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \begin{cases} \mathbb{Q}[-n], & \text{if } n \text{ is odd,} \\ \mathbb{Q}[-n] \oplus \mathbb{Q}[-2n+1], & \text{if } n \text{ is even.} \end{cases}$$

Here $\mathbb{Q}[-n]$ denotes the $(-n)$-fold shifts of the graded vector space $\mathbb{Q}$ concentrated in degree 0, i.e. $\mathbb{Q}[-n]_k = \mathbb{Q}_{k-n}$.

A pointed simply-connected CW-complex Z is *rational* if its graded homotopy group $\pi_\bullet(Z)$ regarded as a graded abelian group underlies a graded rational vector space. In the seminal work [Qui69] Quillen shows that the rational homotopy theory for simply-connected topological spaces is determined by the homotopy theory of rational spaces, and the latter can be analysed in a purely algebraic manner. A *differential graded Lie algebra* over $\mathbb{Q}$ is a rational chain complex $((L_i)_{i\in\mathbb{Z}}, d)$ together with a Lie bracket $[-,-]\colon L_\bullet \otimes_{\mathbb{Q}} L_\bullet \to L_\bullet$ of degree 0 satisfying the graded anti-symmetry, the graded Jacobi identity and the graded Leibniz rule, see [FHT, §21]. A differential graded Lie algebra is *connected* if the homology groups of its underlying chain complex vanish in non-positive degrees.

***Theorem* (Quillen).** —

(i) *For every pointed simply-connected CW-complex Z, there exists a rational CW-complex $Z_{\mathbb{Q}}$ together with a continuous map $Z \to Z_{\mathbb{Q}}$ which induces an isomorphism on rational homotopy groups.*

(ii) *There is a functorial one-to-one correspondence between simply-connected rational CW-complexes and connected rational differential graded Lie algebras, up to homotopy equivalence of CW-complexes and quasi-isomorphisms of differential graded Lie algebras.*

The topological space $Z_{\mathbb{Q}}$ together with the map $Z \to Z_{\mathbb{Q}}$ in (i) is known as a *rationalisation* of Z. Furthermore, the graded rational homotopy group of a pointed simply-connected CW-complex Z is isomorphic (up to a shift of degrees) to the graded homology group of the differential graded Lie algebras corresponding to a rationalisation $Z_{\mathbb{Q}}$, and the latter is much more computable. In this short summary we only included those aspects of rational homotopy theory that are directly relevant to our later applications in the thesis. For an overview of the history of rational homotopy theory, see [Hes99]. A detailed study of this subject can be found in [FHT].

From now on we fix a prime number p. An abelian group is *p-local* if multiplication by each prime number $\ell \neq p$ is an automorphism of the group. A pointed simply-connected CW-complex is *p-local* if its homotopy groups are p-local abelian groups in degrees at least 1. Unstable v_h-periodic homotopy theory with $h \in \mathbb{N}$ (and p elided from the notation) is concerned with the set $[V, Z]$ of pointed homotopy class of maps of pointed simply-connected p-local CW-complexes where V is a finite CW-complex. Our approach to studying this theory is a generalisation of the methods used in rational homotopy theory. We will explain this below in the introduction of the first part of the thesis.

Since we treat everything in this work up to coherent homotopy, it is important for us to work in a framework where we can perform our constructions and state our theorems in a homotopy invariant way. Classically, one may choose to work in topologically or simplicially enriched model categories, where homotopy invariance is guaranteed by using fibrant and/or cofibrant replacements. We choose to use the language of $(\infty, 1)$-categories, abbreviated as ∞-categories and modelled by quasi-categories, as introduced in [Joy02; HTT]. This framework allows us to present mathematical statements in an elegant way and it eases certain categorical constructions, such as limits and colimits of categories. Since much of the theory of ∞-categories we use in this work is a natural generalisation of ordinary category theory, the reader who is unfamiliar with the ∞-categorical language may ignore the "∞" symbol and understand statements using the logic of ordinary category theory, while keeping in mind that everything is defined up to some suitable notion of equivalences. For example, a colimit in an ∞-category enjoys a similar universal property as ordinary categorical colimit, while the existence of the colimit and the commutative diagram exhibiting its universal property hold only up to coherent homotopy. In other words, one can regard an ∞-categorical colimit equivalently as a homotopy colimit in a topologically enriched model category.

The fundamental ∞-category we consider is *the ∞-category $\mathcal{H}\mathrm{o}$ of homotopy types*, which is the ∞-categorical ground for doing homotopy theory of topological spaces. An object of $\mathcal{H}\mathrm{o}$ is called a *homotopy type* (also known as *∞-groupoids* or *animas* or *spaces*), which one shall consider as the homotopy type of a CW-complex or of a simplicial set. Given two pointed homotopy types X and Y, one can construct a homotopy type $\mathcal{M}\mathrm{ap}_*(X, Y)$, called the *pointed mapping space* from X to Y. Representing X and Y by pointed CW-complexes, $\mathcal{M}\mathrm{ap}_*(X, Y)$ encodes the homotopy type of the topological space of pointed maps from X to Y endowed with the compact-open topology. Finally $\mathcal{M}\mathrm{ap}_*(X, Y)$ becomes a pointed homotopy type whose basepoint is the constant map sending X to the basepoint of Y.

Introduction for Part I

Fix a prime number p. We explain now the idea of p-local unstable v_h-periodic homotopy theory for $h \in \mathbb{N}$, starting with v_h-periodic homotopy groups. Let Z be a pointed simply-connected p-local homotopy type. In the case $h = 0$, the *v_0-periodic homotopy groups* are just the rational homotopy groups of Z. We summarise the key ingredients in this case in order to motivate the $h \geq 1$ cases. Let V_0 denote the 0-dimensional sphere S^0. Define the *graded V_0-homotopy group* $\pi_\bullet(Z; V_0)$ of Z as the graded homotopy group $\pi_\bullet(\mathcal{M}\mathrm{ap}_*(V_0, Z))$ of the pointed mapping space $\mathcal{M}\mathrm{ap}_*(V_0, Z)$, which is isomorphic to the graded homotopy group $\pi_\bullet(Z)$ of Z. Since Z is p-local, the only torsion information present consists of p-primary torsion. The multiplication-by-p map on $\pi_k(Z) \cong \pi_k(Z; V_0)$, with $k \geq 1$, is induced by the degree p self-map $v_0 \colon \mathrm{S}^k \to \mathrm{S}^k$ representing the element $p \in \mathbb{Z} \cong \pi_k(\mathrm{S}^k)$. The graded rational homotopy group $\pi_\bullet(Z) \otimes_{\mathbb{Z}} \mathbb{Q}$ of Z is obtained from $\pi_\bullet(Z)$ by inverting the multiplication-by-p action, i.e.

$$\pi_\bullet(Z) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \varinjlim \left(\pi_\bullet(Z) \xrightarrow{v_0} \pi_\bullet(Z) \xrightarrow{v_0} \cdots \right),$$

where the colimit is taken in the category of graded abelian groups.

A finite homotopy type is an object in $\mathcal{H}\mathrm{o}$ that is equivalent to the homotopy type of a finite CW-complex, so we sometimes call a finite homotopy type a *finite complex*. For $h \geq 1$, the v_h-periodic homotopy groups of Z carry the "v_h-torsion free" information of the (graded) *V_h-homotopy groups*

$$\pi_\bullet(Z; V_h) \coloneqq \pi_\bullet(\mathcal{M}\mathrm{ap}_*(V_h, Z))$$

where V_h is a p-local finite complex of so-called "type h". We explain what these terminologies mean for $h = 1$ and say briefly how it works in the general case. Denote the complex K-theory spectrum by KU. Its coefficient group $\mathrm{KU}_\bullet$ is isomorphic to $\mathbb{Z}[\beta^{\pm}]$, where the element β, lying in degree 2, denotes the Bott periodicity class. A p-local finite complex V_1 is of *type 1* if its reduced rational homology $\mathrm{H}_\bullet(V_1; \mathbb{Q})$ vanishes and its reduced mod p KU-homology $\mathrm{KU}_\bullet(V_1)/p$ is non-trivial. For $k \geq 1$, the homotopy type S^k/p defined by the cofibre sequence

$$\mathrm{S}^k \xrightarrow{v_0} \mathrm{S}^k \to \mathrm{S}^k/p$$

is an example of a type 1 finite complex, which can be seen by using the long exact sequence of homology induced by the cofibre sequence and some knowledge about the K-theory of spheres. The set $[\mathrm{S}^k/p, Z]$ of pointed homotopy class of maps is a group for $k \geq 2$ (since $\mathrm{S}^k/p \simeq \Sigma(\mathrm{S}^{k-1}/p)$), and it is abelian for $k \geq 3$. However, the group $[\mathrm{S}^k/p, Z]$ is not easier to compute than the classical homotopy groups of Z. For suitable choices of natural numbers $n \geq 2$ and d_1 depending on the prime number p, it is shown in [Ada66] that S^n/p admits a self-map $v_1 \colon \Sigma^{d_1}\mathrm{S}^n/p \to \mathrm{S}^n/p$ which induces an isomorphism on KU-homology (strictly speaking, v_1 is not a self-map. However, we embrace this traditional abuse of notation, cf. [MRW77; HS98].) For instance, for $p = 2$, the map on KU-homology induced by v_1 is given by the multiplication by β^4. In particular, this implies that the map v_1 is non-nilpotent, i.e. for $m \geq 1$, *no* iterated composition

$$v_1^{\circ m}\colon \Sigma^{md_1}\mathrm{S}^n/p \xrightarrow{\Sigma^{(m-1)d_1}(v_1)} \Sigma^{(m-1)d_1}\mathrm{S}^n/p \xrightarrow{\Sigma^{(m-2)d_1}(v_1)} \cdots \xrightarrow{v_1} \mathrm{S}^n/p$$

is null-homotopic. The self-map v_1 of S^n/p induces an endomorphism

$$v_1\colon \pi_\bullet(Z; \mathrm{S}^n/p) \to \pi_\bullet(Z; \mathrm{S}^n/p)$$

of the graded S^n/p-homotopy group of Z. A v_1*-torsion* element in $\pi_\bullet(Z; \mathrm{S}^n/p)$ is an element which is mapped to zero under v_1. The v_1*-periodic homotopy group* $v_1^{-1}\pi_\bullet(Z; \mathrm{S}^n/p)$ of Z with coefficient in the type 1 finite complex S^n/p is defined by inverting the v_1-action on the graded S^n/p-homotopy group of Z, i.e.

$$v_1^{-1}\pi_\bullet(Z; \mathrm{S}^n/p) := \varinjlim \left(\pi_\bullet(Z; \mathrm{S}^n/p) \xrightarrow{v_1} \pi_\bullet(Z; \mathrm{S}^n/p) \xrightarrow{v_1} \cdots\right),$$

which captures the v_1-torsion free information of $\pi_\bullet(Z; \mathrm{S}^n/p)$.

For the general cases, we need to use the so-called p-local Morava K-theory spectrum $\mathrm{K}(h)$ of *height* h with $h \in \mathbb{N}$. We know that $\mathrm{K}(0)$ can be identified with the rational Eilenberg–MacLane spectrum $\mathrm{H}\mathbb{Q}$, and $\mathrm{K}(1)$ is a summand of the mod-p K-theory spectrum KU/p (defined as the cofibre of the multiplication-by-p self-map on KU). The reader may view $\mathrm{K}(h)$, for $h \geq 1$, as a generalisation of KU/p. For example, the coefficient group $\mathrm{K}(h)_\bullet$ is isomorphic to $\mathbb{F}_p[v_h^{\pm}]$, where the element v_h, lying in degree $2(p^h - 1)$, can be regarded as a generalised Bott periodicity class. A p-local finite complex V_h is of *type* h if its reduced $\mathrm{K}(h)$-homology $\widetilde{\mathrm{K}(h)}_\bullet(V_h)$ is non-trivial and $\widetilde{\mathrm{K}(i)}_\bullet(V_h) = 0$ for all $0 \leq i \leq h - 1$. The following deep results from [Mit85] and [HS98] are fundamental for periodic homotopy theory.

***Theorem* (Mitchell, Hopkins–Smith).** — *Let p be a prime number.*

(i) *For every $h \in \mathbb{N}$, there exists a p-local finite complex of type h.*

(ii) *Every non-trivial p-local finite complex is of type h for a unique $h \in \mathbb{N}$.*

(iii) *Let V_h be a finite complex of type h. There exists an $n \in \mathbb{N}$ such that $\Sigma^n V_h$ (which is also a type h finite complex) admits a self-map $v_h\colon \Sigma^{d_h}(\Sigma^n V_h) \to \Sigma^n V_h$ inducing an isomorphism on* $\mathrm{K}(h)$*-homology. In particular, the self-map v_h is non-nilpotent.*[1]

For a natural number $h \geq 1$ let V_h be a finite complex of type h together with a v_h self-map $v_h\colon \Sigma^{d_h} V_h \to V_h$. The map v_h induces an endomorphism of the

[1] By abuse of notation, v_h denotes both the self-map v_h and the generator of the coefficient group of $\mathrm{K}(h)$. The map on $\mathrm{K}(h)_\bullet$-homology induced by the map v_h is given by the multiplication by a power of the element v_h, which depends on the prime number p and the height h.

graded V_h-homotopy groups $\pi_\bullet(X;V_h) \coloneqq \mathcal{M}\mathrm{ap}_*(V_h, X)$. With these ingredients we can define the (graded) *v_h-periodic homotopy group*

$$v_h^{-1}\pi_\bullet(Z;V_h) \coloneqq \varinjlim \left(\pi_\bullet(Z;V_h) \xrightarrow{v_h} \pi_\bullet(Z;V_h) \xrightarrow{v_h} \cdots\right)$$

of Z with coefficient in V_h, by inverting the v_h-“multiplication” on $\pi_\bullet(Z;V_h)$. In contrast to rational homotopy groups, for $h \geq 1$, the v_h-periodic homotopy groups are much less amenable to computation. There exist some works computing the v_1-periodic homotopy groups of spheres, compact Lie groups and H-spaces, see for example [Mah82], [Dav91; Dav03] and [Bou99b], respectively. For an overview of the methods used in those computations, see [Dav95] and [Bou05]. In his thesis [Wan15] Wang computes the v_2-periodic homotopy groups of the 3-dimensional p-local sphere in the case of prime numbers $p \geq 5$. There is no general computational results when $h \geq 3$.

A morphism $f\colon X \to Y$ of pointed simply-connected p-local homotopy types is a *v_h-periodic equivalence* if it induces an isomorphism on v_h-periodic homotopy groups with coefficient in V_h. By the milestone work of [HS98], in particular the *Thick Subcategory Theorem*, we know that the notion of v_h-periodic equivalence of homotopy types is independent of the choice of the finite complex V_h and the self-map v_h. This notion of equivalence relates closely to the so-called notion of $\mathrm{T}(h)$-homology equivalence of homotopy types that we introduce next. A spectrum is p-local if its stable homotopy groups are p-local abelian groups. We can construct the p-local *telescope spectrum of height h*

$$\mathrm{T}(h) \coloneqq \varinjlim \left(\Sigma^\infty V_h \xrightarrow{v_h} \Sigma^{-d_h}\Sigma^\infty V_h \xrightarrow{v_h} \Sigma^{-2d_h}\Sigma^\infty V_h \xrightarrow{v_h} \cdots\right)$$

using V_h, where the colimit is taken in the ∞-category $\mathcal{S}\mathrm{p}$ of spectra. Again by the Thick Subcategory Theorem, the notion of $\mathrm{T}(h)$-homology equivalence of homotopy types does not depend on the choice of V_h and the self-map v_h. The p-local sphere spectrum $\mathbb{S}_{(p)}$ is a p-local spectrum together with a morphism $\mathbb{S} \to \mathbb{S}_{(p)}$ such that the induced map on stable homotopy groups $\pi_\bullet^{\mathrm{st}}(-)$ is the p-localisation of the abelian group $\pi_\bullet^{\mathrm{st}}(\mathbb{S})$. The reader shall view the construction of $\mathrm{T}(h)$ as an analogue of a construction of the rational Eilenberg–MacLane spectrum $\mathrm{H}\mathbb{Q}$ which is given by inverting the degree p self-map v_0 of $\mathbb{S}_{(p)}$. Because of this we set $\mathrm{T}(0) \coloneqq \mathrm{H}\mathbb{Q}$.

By the Whitehead Theorem, a morphism of simply-connected homotopy types is a rational homotopy equivalence if and only if it is a rational homology equivalence, see [DK, Theorem 10.6]. For $h \geq 1$ it is in general not true that a v_h-periodic equivalence of pointed simply-connected p-local homotopy types is a $\mathrm{T}(h)$-homology equivalence, nor vice versa. See [LS01] for concrete examples in the case of $h = 1$. In the case of morphisms between H-spaces, considered as objects in the homotopy category $\mathrm{ho}(\mathcal{H}\mathrm{o})$ of homotopy types, we prove the following theorem.

Theorem 1 **(Theorem 4.4.0.4). —** *Let h be a natural number and let $f\colon X \to Y$ be a morphism of connected H-spaces. If the map f is a $\mathrm{T}(n)$-homology equivalence for all $0 \le n \le h$ and f induces an isomorphism $f_*\colon \pi_\bullet(X) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] \xrightarrow{\simeq} \pi_\bullet(Y) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$, then f is a v_n-periodic equivalence for all $0 \le n \le h$.*

This is an improvement of [Bou94, Theorem 13.5], which concludes that f is a v_n-periodic equivalence after some finite number of suspensions. By the *Class Invariance Theorem* of [HS98], a morphism of p-local *finite* homotopy types is a $\mathrm{T}(h)$-homology equivalence if and only if it is a $\mathrm{K}(h)$-homology equivalence. Based on these results and the non-trivial fact that a $\mathrm{K}(h)$-homology equivalence of p-local homotopy types is also a $\mathrm{K}(h-1)$-homology equivalence for $h \ge 2$, see [Bou99a], we expect a slightly stronger version of Theorem 1 and make the following conjecture.

Conjecture 2 **(Conjecture 4.4.0.8). —** *Fix a natural number $h \ge 1$. Let $f\colon X \to Y$ be a morphism of pointed simply-connected p-local homotopy types whose rational homotopy groups are trivial. The following statements are equivalent:*

(i) *The map Ωf is a $\mathrm{T}(h)$-homology equivalence.*
(ii) *The map Ωf is a $\mathrm{T}(n)$-homology equivalence for all $0 \le n \le h$.*
(iii) *The map f is a v_n-periodic equivalence for all $0 \le n \le h$ and f induces an isomorphism $\pi_k(f)$ on homotopy groups in all degree $k \le h+1$.*

The second important part of rational homotopy theory, namely rationalisation, can also be generalised to unstable v_h-periodic homotopy theory and it is called *unstable v_h-periodic localisation.* For this purpose we need to use the theory of localisations of ∞-categories, which is the ∞-categorical analogue of localisations of ordinary categories (also known as *the calculus of fractions*). A *localisation of an ∞-category $\mathcal{C}$ at a set W of morphisms of $\mathcal{C}$* is an ∞-category $\mathcal{D}$ together with a functor $F\colon \mathcal{C} \to \mathcal{D}$ such that F sends morphisms in W to equivalences in $\mathcal{D}$ and is universal with respect to this property. In many situations of our interest, the functor F admits a fully faithful right adjoint, making $\mathcal{D}$ equivalent to a full ∞-subcategory of $\mathcal{C}$. Conversely, if the canonical inclusion functor of a full ∞-subcategory $\mathcal{C}_0$ in $\mathcal{C}$ admits a left adjoint F_0, then F_0 exhibits $\mathcal{C}_0$ as a localisation of $\mathcal{C}$, where W is the collection of morphisms in $\mathcal{C}$ that are sent to equivalences in $\mathcal{C}_0$.

The ∞-category $\mathcal{H}\mathrm{o}_{\mathbb{Q}}^{>1}$ of pointed simply-connected rational homotopy types is a full ∞-subcategory of the ∞-category $\mathcal{H}\mathrm{o}_{(p)}^{>1}$ of pointed simply-connected p-local homotopy types. Moreover, the inclusion $\mathcal{H}\mathrm{o}_{\mathbb{Q}}^{>1} \hookrightarrow \mathcal{H}\mathrm{o}_{(p)}^{>1}$ admits a left adjoint $\mathrm{L}_{\mathbb{Q}}$, which exhibits $\mathcal{H}\mathrm{o}_{\mathbb{Q}}^{>1}$ as the localisation of $\mathcal{H}\mathrm{o}_{(p)}^{>1}$ at the set of rational homotopy equivalences. For $Z \in \mathcal{H}\mathrm{o}_{(p)}^{>1}$ the rational homotopy type $\mathrm{L}_{\mathbb{Q}}(Z)$ together with the morphism $Z \to \mathrm{L}_{\mathbb{Q}}(Z)$ in $\mathcal{H}\mathrm{o}_{(p)}^{>1}$ induced by the adjunction-unit natural transformation is a model for the rationalisation of Z.

To explain the generalisation from rationalisation to v_h-periodic localisations, we begin with an alternative description of rational p-local homotopy types. Recall the cofibre sequence $\mathrm{S}^2 \xrightarrow{v_0=p} \mathrm{S}^2 \to \mathrm{S}^2/p$, which induces a fibre sequence

$$\mathcal{M}\mathrm{ap}_*(\mathrm{S}^2/p, Z) \to \mathcal{M}\mathrm{ap}_*(\mathrm{S}^2, Z) \xrightarrow{v_0} \mathcal{M}\mathrm{ap}_*(\mathrm{S}^2, Z).$$

for $Z \in \mathcal{H}\mathrm{o}_{(p)}^{>1}$. The following three statements are equivalent:

(i) The homotopy type Z is rational.

(ii) The map $v_0\colon \mathcal{M}\mathrm{ap}_*(\mathrm{S}^2, Z) \to \mathcal{M}\mathrm{ap}_*(\mathrm{S}^2, Z)$ is an equivalence.

(iii) The mapping space $\mathcal{M}\mathrm{ap}_*(\mathrm{S}^1/p, Z)$ is contractible.

Recall that S^1/p is a p-local finite complex of type 1. The condition (iii) has an immediate generalisation to any p-local finite complex V_h of type h, for $h \geq 1$.

We say that a pointed simply-connected p-local homotopy type Z is *V_h-less* if the pointed mapping space $\mathcal{M}\mathrm{ap}_*(V_h, Z)$ is contractible, that is, any pointed map from V_h to Z is homotopic to the unique pointed map from the one-point space to Z.(2) Because of technical requirements, from now on we assume without loss of generality that every type h finite complex V_h we consider is equivalent to a suspension $\Sigma V_h'$ of some finite complex V_h'.

Denote by $\mathcal{H}\mathrm{o}_{(p),V_h}^{>1}$ the full ∞-subcategory of $\mathcal{H}\mathrm{o}_{(p)}^{>1}$ whose objects are V_h-less homotopy types. Using formal arguments, one can show that the fully faithful inclusion $i\colon \mathcal{H}\mathrm{o}_{(p),V_h}^{>1} \hookrightarrow \mathcal{H}\mathrm{o}_{(p)}^{>1}$ admits a left adjoint P_{V_h}. We call the functor P_{V_h} *contraction of* V_h, which can be interpreted as a "quotient of $\mathcal{H}\mathrm{o}_{(p)}^{>1}$ modulo V_h"; in particular, it sends V_h to a point and preserves V_h-less homotopy types.(3) The functor P_{V_h} exhibits $\mathcal{H}\mathrm{o}_{(p),V_h}^{>1}$ as the localisation of $\mathcal{H}\mathrm{o}_{(p)}^{>1}$ where the so-called *V_h-equivalences* in $\mathcal{H}\mathrm{o}_{(p)}^{>1}$ are inverted. For example, for $h = 1$ and $V_h = \mathrm{S}^1/p$, an S^1/p-equivalence is a rational homotopy equivalence. More generally, one can show that a V_h-equivalence of simply connected p-local homotopy types, e.g. $Z \to \mathrm{P}_{V_h}(Z)$, is a v_n-periodic equivalence for all $0 \leq n \leq h-1$ (see Theorem 3.3.2.2); the converse is true after imposing a connectivity assumption on Z depending on V_h (see Theorem 3.3.2.6). Moreover, the v_m-periodic homotopy groups of $\mathrm{P}_{V_h}(Z)$ vanish for all $m \geq h$ (see Proposition 3.3.2.4).

By abuse of notation we denote the composition $i \circ \mathrm{P}_{V_h}\colon \mathcal{H}\mathrm{o}_{(p)}^{>1} \to \mathcal{H}\mathrm{o}_{(p),V_h}^{>1} \hookrightarrow \mathcal{H}\mathrm{o}_{(p)}^{>1}$ by P_{V_h}, where V_h is a finite complex of type $h \geq 1$. We can choose a suitable sequence $(\widetilde{V}_h)_{h\geq 1}$ of finite complexes $\widetilde{V}_h$ of type h such that an object Z is $\widetilde{V}_{h+1}$-less if it is $\widetilde{V}_h$-less. Under this assumption we obtain a tower

$$\cdots \to \mathrm{P}_{\widetilde{V}_{h+1}} \xrightarrow{\widetilde{\mathrm{P}}_h} \mathrm{P}_{\widetilde{V}_h} \to \cdots \to \mathrm{P}_{\widetilde{V}_2} \xrightarrow{\widetilde{\mathrm{P}}_1} \mathrm{P}_{\widetilde{V}_1} \tag{0.1}$$

of functors (viewed as objects in the ∞-category $\mathcal{F}\mathrm{un}(\mathcal{H}\mathrm{o}_{(p)}^{>1}, \mathcal{H}\mathrm{o}_{(p)}^{>1})$ of functors), where the natural transformation $\widetilde{\mathrm{P}}_h$ is given by contraction of $\widetilde{V}_h$.

(2)This definition is also known as Z being "V_h-null", or "V_h-local", or "V_h-periodic".

(3)The functor P_{V_h} is also known as "V_h-nullification", or "V_h-localisation" or "V_h-periodisation".

Define the functor $\mathrm{F}_{\widetilde{V}_{h+1},\widetilde{V}_h}\colon \mathcal{H}\mathrm{o}^{>1}_{(p)} \to \mathcal{H}\mathrm{o}^{>1}_{(p)}$ by the fibre sequence

$$\mathrm{F}_{\widetilde{V}_{h+1},\widetilde{V}_h} \to \mathrm{P}_{\widetilde{V}_{h+1}} \xrightarrow{\widetilde{\mathrm{P}}_h} \mathrm{P}_{\widetilde{V}_h}$$

in $\mathcal{F}\mathrm{un}(\mathcal{H}\mathrm{o}^{>1}_{(p)}, \mathcal{H}\mathrm{o}^{>1}_{(p)})$. In particular, for $Z \in \mathcal{H}\mathrm{o}^{>1}_{(p)}$, we have a fibre sequence

$$\mathrm{F}_{\widetilde{V}_{h+1},\widetilde{V}_h}(Z) \to \mathrm{P}_{\widetilde{V}_{h+1}}(Z) \xrightarrow{\widetilde{\mathrm{P}}_h} \mathrm{P}_{\widetilde{V}_h}(Z)$$

of pointed simply-connected p-local homotopy types. By the universal property of a fibre $\mathrm{F}_{\widetilde{V}_{h+1},\widetilde{V}_h}(Z)$ and Z have isomorphic v_h-periodic homotopy groups, and for $m \neq h$ the v_m-periodic homotopy groups of $\mathrm{F}_{\widetilde{V}_{h+1},\widetilde{V}_h}(Z)$ vanish. Let f be a morphism of simply connected p-local homotopy types. If $\mathrm{F}_{\widetilde{V}_{h+1},\widetilde{V}_h}(f)$ is an equivalence of homotopy types, then f is a v_h-periodic equivalence; the converse is true if the source and target of f are suitably highly connected (depending on the connectivity of $\widetilde{V}_{h+1}$, see Proposition 3.4.0.5).

Finally, let us conclude the above discussion by the following theorem about a concrete model of the localisation of $\mathcal{H}\mathrm{o}^{>1}_{(p)}$, where v_h-periodic equivalences are inverted. Denote the natural number $1 + \mathrm{conn}(\widetilde{V}_{h+1})$ by c_{h+1}, where $\mathrm{conn}(\widetilde{V}_{h+1})$ denotes the connectivity of $\widetilde{V}_{h+1}$. The c_{h+1}-*connected cover* of an object $Z \in \mathcal{H}\mathrm{o}^{>1}_{(p)}$ is a homotopy type $\tau_{>c_{h+1}}(Z)$ together with a morphism $f\colon \tau_{>c_{h+1}}(Z) \to Z$ such that the homotopy groups of $\tau_{>c_{h+1}}(Z)$ in all degrees $j \leq c_{h+1}$ vanish and f induces an isomorphism of homotopy groups in all degrees $k > c_{h+1}$.

***Theorem* (Bousfield [Bou01]).** — *Let $h \geq 1$ be a natural number. The localisation $\mathcal{H}\mathrm{o}_{v_h}$ of the ∞-category $\mathcal{H}\mathrm{o}^{>1}_{(p)}$, where v_h-periodic equivalences are inverted, is given by the full ∞-subcategory of $\mathcal{H}\mathrm{o}^{>1}_{(p)}$ whose objects are of the form $\tau_{>c_{h+1}}(\mathrm{F}_{\widetilde{V}_{h+1},\widetilde{V}_h}(Z))$ for a $Z \in \mathcal{H}\mathrm{o}^{>1}_{(p)}$. The localisation functor $\mathcal{H}\mathrm{o}^{>1}_{(p)} \to \mathcal{H}\mathrm{o}_{v_h}$ is given by $\tau_{>c_{h+1}} \circ \mathrm{F}_{\widetilde{V}_{h+1},\widetilde{V}_h}$.*

This localisation is known as *unstable v_h-periodic localisation*. See Theorem 3.4.0.7 for a precise formulation of the theorem. Although the construction of this model for $\mathcal{H}\mathrm{o}_{v_h}$ depends on concrete choices of finite complexes $\widetilde{V}_{h+1}$ and $\widetilde{V}_h$, the resulting ∞-category is independent of those choices, by uniqueness (up to contractible choice) of localisations.

We consider the tower (0.1) of localisations as a tower of "truncations" of the ∞-category $\mathcal{H}\mathrm{o}^{>1}_{(p)}$ with respect to v_h-periodic homotopy groups, in the following sense: For every $Z \in \mathcal{H}\mathrm{o}^{>1}_{(p)}$, the evaluation of (0.1) at Z gives a tower

$$\cdots \to \mathrm{P}_{\widetilde{V}_{h+1}}(Z) \to \mathrm{P}_{\widetilde{V}_h}(Z) \to \cdots \to \mathrm{P}_{\widetilde{V}_1}(Z)$$

under Z, where the canonical morphism $Z \to \mathrm{P}_{\widetilde{V}_{h+1}}(Z)$ induces an isomorphism on the v_n-periodic homotopy groups for all $0 \leq n \leq h$, and the v_m-periodic homotopy groups of $\mathrm{P}_{\widetilde{V}_{h+1}}(Z)$ are zero for all $m \geq h+1$. From this point of view, one can interpret the unstable v_h-periodic localisation $\mathcal{H}\mathrm{o}_{v_h}$ as the "associated graded" of the tower (0.1), called the *unstable monochromatic layer* of height h. Therefore, it is also important to ask how to use our knowledge of the associated graded pieces to improve our understanding of p-local homotopy types or p-local $\widetilde{V_h}$-less homotopy types. Questions in this direction are studied under the keyword *chromatic assembly*, and they are awaiting further exploration.

Introduction for Part II

A *Lie algebra* over a field k is a k-vector space together with a k-bilinear operation, called the *Lie bracket*, satisfying the so-called anti-symmetry and Jacobi-identity relations. For example, an associative algebra over k becomes a Lie algebra where the Lie bracket is given by the commutator bracket. The universal enveloping algebra $\mathrm{U}(L)$ of a Lie algebra L is an associative algebra over k together with a Lie algebra morphism $L \to \mathrm{U}(L)$ which satisfies the following universal property: For every Lie algebra morphism $f\colon L \to A$, where A is an associative algebra endowed with the commutator Lie bracket, there exists a unique factorisation

$$\begin{array}{ccc} L & \xrightarrow{\quad f \quad} & A \\ & \searrow & \nearrow \;\exists!\ f' \\ & \mathrm{U}(L), & \end{array}$$

where f' is a morphism of associative algebras. For a k-vector space V, denote by $\mathrm{T}(V)$ the free associative algebra, i.e. the tensor algebra, generated by V. An explicit construction of $\mathrm{U}(L)$ is given as the quotient of $\mathrm{T}(L)$ by the two-sided ideal generated by the elements $a \otimes b - b \otimes a - [a, b]$ for $a, b \in L$. The free Lie algebra functor is the left adjoint to the forgetful functor from the category of Lie algebras to the category of k-vector spaces. The universal enveloping algebra of a free Lie algebra generated by a k-vector space V is isomorphic to the free associative algebra $\mathrm{T}(V)$, see [ReuFLA, Theorem 0.5].

There are many applications of universal enveloping algebras in representation theory, algebra and topology; for some expositions, see [DixEA], [MM65] and [NeiAMU]. We discuss one example in rational homotopy theory. Let Z be a pointed simply-connected homotopy type. We recall now briefly the construction of the *Samelson product* on the graded homotopy group $\pi_\bullet(\Omega Z)$ of the loop space ΩZ of Z. Let $f\colon \mathrm{S}^n \to \Omega Z$ and $g\colon \mathrm{S}^m \to \Omega Z$ be morphisms of pointed homotopy types representing elements of $\pi_\bullet(\Omega Z)$ in degree n and m, respectively. We define the following composition

$$\phi\colon \mathrm{S}^n \times \mathrm{S}^m \xrightarrow{f\times g} \Omega Z \times \Omega Z \to \Omega Z$$
$$(l_1, l_2) \mapsto l_1 \star l_2 \star l_1^{-1} \star l_2^{-1}$$

where $\star$ denotes the concatenation of loops and $(-)^{-1}$ denotes the reversed loop. One can check that the composition $\mathrm{S}^n \vee \mathrm{S}^m \xhookrightarrow{i} \mathrm{S}^n \times \mathrm{S}^m \xrightarrow{\phi} \Omega Z$ is null-homotopic, where $\mathrm{S}^n \vee \mathrm{S}^m$ denotes the wedge of spheres. The cofibre of i is equivalent to the smash product $\mathrm{S}^n \wedge \mathrm{S}^m$. Now the Samelson product $[f, g] \in \pi_{n+m}(\Omega Z)$ of f and g is represented by the map $\mathrm{S}^{n+m} \simeq \mathrm{S}^n \wedge \mathrm{S}^m \to \Omega Z$ induced by ϕ.

***Theorem* (Cartan–Serre, Milnor–Moore).** —

(i) *The Samelson product endows the graded rational homotopy group $\pi_\bullet(\Omega Z) \otimes_{\mathbb{Z}} \mathbb{Q}$ of ΩZ with the structure of a graded Lie algebra over $\mathbb{Q}$.*

(ii) *There exists an isomorphism*

$$\mathrm{U}\left(\pi_\bullet(\Omega Z) \otimes_{\mathbb{Z}} \mathbb{Q}\right) \cong \mathrm{H}_\bullet\left(\Omega Z; \mathbb{Q}\right) \tag{0.2}$$

of graded associative algebras over $\mathbb{Q}$, where the algebra structure on the singular homology $\mathrm{H}_\bullet(\Omega Z; \mathbb{Q})$ is induced by concatenating loops in Z.

See [FHT, Theorem 21.5] for a proof of the above theorem. For shadowing later considerations let us remark that the rational homology group $\mathrm{H}_\bullet(\Omega Z; \mathbb{Q})$ is isomorphic to the stable homotopy group $\pi_\bullet^{\mathrm{st}}(\Sigma_+^\infty(\Omega Z) \otimes \mathrm{H}\mathbb{Q})$ of the tensor product $\Sigma_+^\infty(\Omega Z) \otimes \mathrm{H}\mathbb{Q}$ of the suspension spectrum $\Sigma_+^\infty(\Omega Z)$ with the rational Eilenberg–MacLane spectrum $\mathrm{H}\mathbb{Q}$.

The construction of the universal enveloping algebra can be generalised in two directions, see [Knu18], using the theory of enriched ∞-operads. Let us first give a brief introduction to the role played by ∞-operads, which generalises the classical categorical theory of (coloured) operads [MayGIL; FreHO; LV]. Operads were invented to abstract over algebraic structures themselves, by expressing the axioms of an algebraic structure in terms of objects and morphisms in a suitable symmetric monoidal category. For example, an associative algebra over $\mathbb{Z}$ can be defined as an algebra over the associative operad in the category of $\mathbb{Z}$-modules. Using the framework of operads one is able to define algebraic structures on objects of an arbitrary symmetric monoidal category, not just on $\mathbb{Z}$-modules.

In the setting of ∞-categories, the theory of enriched ∞-operads [HTT; CH20; Hau22] allows us to formalise algebraic structures compatible with coherent homotopy. We work with the model for one-coloured enriched ∞-operads with values in a presentable symmetric monoidal ∞-category given by certain *symmetric sequences*, following the ideas of [Bra17; Hei18; Tri]. Thus, a Lie algebra in a presentable stable symmetric monoidal ∞-category $\mathcal{C}$ is an algebra over the *spectral Lie ∞-operad* $\mathcal{L}\mathrm{ie}$ (see Example 5.3.4.10) and an associative algebra in $\mathcal{C}$ is an algebra over the associative ∞-operad $\mathcal{E}_1$. Using this, the construction of the universal enveloping algebra can be generalised to a functor assigning to a Lie algebra in $\mathcal{C}$ an $\mathcal{E}_1$-algebra in $\mathcal{C}$.

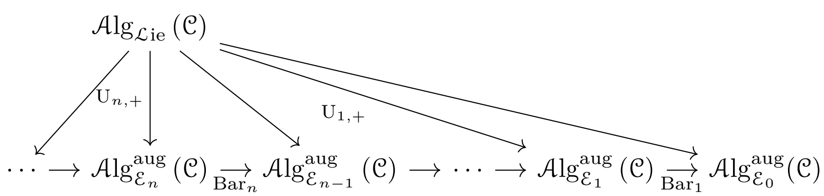
The second direction of generalisation is topological in nature: Unlike $\mathbb{Z}$-modules, a homotopy type can have more sophisticated multiplicative structures than just the associative or commutative one. In particular, one can consider algebras over the so-called $\mathcal{E}_n$ ∞-operad for $n \in \mathbb{N}$, generalising the associative ∞-operad $\mathcal{E}_1$. Thus, one may ask whether it is possible to assign to a Lie algebra in $\mathcal{C}$ an $\mathcal{E}_n$-algebra for every $n \in \mathbb{N}$. Informally speaking, an algebra over the $\mathcal{E}_n$ ∞-operad in a symmetric monoidal ∞-category $\mathcal{D}$ is an object $X \in \mathcal{D}$ together with n-many binary operations, each of which endows X with the structure of an associative algebra (i.e. an $\mathcal{E}_1$-algebra), and each pair of these associative multiplications satisfies an Eckmann–Hilton-like compatibility condition up to coherent homotopy. A standard example of an $\mathcal{E}_n$-algebra is the n-fold loop space $\Omega^n Y$ of a homotopy type Y. For example, for $n = 2$, the double loop space $\Omega^2 X$ is equivalent to the pointed mapping space $\mathcal{M}\mathrm{ap}_*\left(\mathrm{S}^1, \mathcal{M}\mathrm{ap}_*(\mathrm{S}^1, X)\right)$; the two associative multiplications are given by concatenation of loops in X and of loops in $\mathcal{M}\mathrm{ap}_*(\mathrm{S}^1, X)$, respectively. We leave the verification of the compatibility of these multiplications to the interested reader.

An *augmented* $\mathcal{E}_n$-algebra in $\mathcal{C}$ is an $\mathcal{E}_n$-algebra A together with a morphism from A to the unit of the symmetric monoidal structure of $\mathcal{C}$; note that the universal enveloping algebra of a Lie algebra L over a field k is an augmented associative algebra. For each $n \in \mathbb{N}$, there exists a generalisation of the universal enveloping algebra construction provided by a cocontinuous functor (i.e. preserving small colimits)

$$\mathrm{U}_{n,+} \colon \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{C}) \to \mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{\mathcal{E}_n}(\mathcal{C})$$

from the ∞-category of Lie algebras in $\mathcal{C}$ to the ∞-category of augmented $\mathcal{E}_n$-algebras in $\mathcal{C}$, which sends a free Lie algebra generated by an object $X \in \mathcal{C}$ to the augmented $\mathcal{E}_n$-algebra freely generated by the n-fold suspension $\Sigma^n_{\mathcal{C}} X$ of X. Note that the evaluation of the functor $\mathrm{U}_{n,+}$ on a Lie algebra is uniquely determined by its value on morphisms between free Lie algebras, since every Lie algebra is equivalent to a (∞-categorical) colimit of free Lie algebras. It is shown in [Knu18] that $\mathrm{U}_{n,+}(L)$ is equivalent to $\Sigma^\infty_+(\Omega^n_{\mathcal{L}\mathrm{ie}} L)$ where $\Omega_{\mathcal{L}\mathrm{ie}}$ denotes the loop functor of $\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{C})$ and Σ^∞_+ denotes the stabilisation functor of $\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{C})$, which one shall consider as the suspension spectrum functor, as the notation suggests. Denote by $\mathcal{S}\mathrm{p}_{\mathbb{Q}}$ the ∞-category of rational spectra, i.e. the localisation of the ∞-category $\mathcal{S}\mathrm{p}$ at the set of rational homology equivalences of spectra. In the case $n = 1$ and $\mathcal{C} = \mathcal{S}\mathrm{p}_{\mathbb{Q}}$ the equivalence $\mathrm{U}_{1,+}(L) \simeq \Sigma^\infty_+ \Omega_{\mathcal{L}\mathrm{ie}} L$ enhances the isomorphism (0.2) of homotopy groups to the spectral level.

The family of functors $\{\mathrm{U}_{n,+}\}_{n \in \mathbb{N}}$ fits into a commutative diagram (up to coherent homotopy)



of ∞-categories and cocontinuous functors (the horizontal functor Bar_n is the so-called Bar construction). In §6.1 we construct the above diagram using the recent result about Koszul duality of the (spectral) $\mathcal{E}_n$ ∞-operad [CS22], which is different from the original approach in [Knu18]. By the universal property of (∞-categorical) limits, there exists an induced cocontinuous functor

$$\mathrm{U}_{\infty,+}\colon \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{C}) \to \varprojlim_n \mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{\mathcal{E}_n}(\mathcal{C}).$$

By formality of the $\mathcal{E}_n$ ∞-operad over $\mathbb{Q}$ (see Theorem 6.2.1.11) we obtain the following theorem.

***Theorem 3* (Theorem 6.2.1.17). —** *The cocontinuous functor*

$$\mathrm{U}_{\infty,+}\colon \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}\left(\mathcal{S}\mathrm{p}_{\mathbb{Q}}\right) \to \varprojlim \mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{\mathcal{E}_n}\left(\mathcal{S}\mathrm{p}_{\mathbb{Q}}\right)$$

is an equivalence of ∞-categories.

As before we fix a prime number p and let $h \in \mathbb{N}$. Recall from the introduction of Part I that the p-local telescope spectrum $\mathrm{T}(h)$ of height h is an analogue of the rational Eilenberg–MacLane spectrum $\mathrm{H}\mathbb{Q}$ (with $\mathrm{T}(0) \simeq \mathrm{H}\mathbb{Q}$). Denote by $\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$ the localisation of the ∞-category $\mathcal{S}\mathrm{p}_{(p)}$ of p-local spectra at the set of $\mathrm{T}(h)$-homology equivalences. We prove the following theorem.

***Theorem 4* (Theorem 6.3.0.1). —** *Let $\mathcal{C} = \mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$ for a natural number $h \geq 1$. Then the functor $\mathrm{U}_{\infty,+}$ is fully faithful.*

This gives a hint towards the following conjecture, generalising Theorem 3.

***Conjecture 5*. —** *Let $\mathcal{C} = \mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$ for a natural number $h \geq 1$. Then the functor $\mathrm{U}_{\infty,+}$ is an equivalence of ∞-categories.*

Recall the ∞-category $\mathcal{H}\mathrm{o}_{v_h}$ from the end of the introduction of Part I. It is the unstable monochromatic layer of height h and is defined as the localisation of the ∞-category $\mathcal{H}\mathrm{o}^{>1}_{(p)}$ of pointed simply-connected p-local homotopy types at the set of v_h-periodic equivalences. The following theorem generalises Quillen's rational differential graded Lie algebra model for simply-connected rational homotopy types.

***Theorem* (Heuts). —** *For every natural number $h \geq 1$, there exists an ∞-categorical equivalence*

$$\mathcal{H}\mathrm{o}_{v_h} \simeq \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}).$$

Thus, Lie algebras provides a different viewpoint for studying $\mathcal{H}\mathrm{o}_{v_h}$, which is our main motivation for investigating the properties of the ∞-category $\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})$. There exist two adjunctions

$$\mathcal{S}\mathrm{p}_{\mathrm{T}(h)} \underset{\mathrm{forg}}{\overset{\mathrm{free}}{\rightleftarrows}} \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}) \underset{\mathrm{triv}}{\overset{\mathrm{indec}}{\rightleftarrows}} \mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$$

associated with $\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})$; the first one is the free–forgetful adjunction and the second is the indecomposable–trivial adjunction. The functor triv endows every object $E \in \mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$ with the structure of a trivial Lie algebra: The Lie bracket is defined as the zero map. Under the equivalence $\mathcal{H}\mathrm{o}_{v_h} \simeq \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})$, the above adjunctions correspond to the following adjunctions

$$\mathcal{S}\mathrm{p}_{\mathrm{T}(h)} \underset{\Phi_h}{\overset{\Theta_h}{\rightleftarrows}} \mathcal{H}\mathrm{o}_{v_h} \underset{\Omega^\infty_{v_h}}{\overset{\Sigma^\infty_{v_h}}{\rightleftarrows}} \mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$$

associated to $\mathcal{H}\mathrm{o}_{v_h}$, where Φ_h is the so-called *Bousfield–Kuhn functor* (see Theorem 3.4.1.5) and $\Sigma^\infty_{v_h} \dashv \Omega^\infty_{v_h}$ exhibits $\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$ as the stabilisation of $\mathcal{H}\mathrm{o}_{v_h}$ (see ¶6.2.2.6). Since the suspension functor is an auto-equivalence of the *stable* ∞-category $\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$, the left adjoint functor Θ_h supplies the ∞-category $\mathcal{H}\mathrm{o}_{v_h}$ with non-trivial objects X admitting infinite desuspensions, i.e. there exists an infinite sequence $(X_i)_{i\geq 0}$ of objects in $\mathcal{H}\mathrm{o}_{v_h}$ with $X \simeq X_0$ and $X_i \simeq \Sigma_{v_h} X_{i+1}$ for $i \in \mathbb{N}$, where Σ_{v_h} denotes the suspension functor of $\mathcal{H}\mathrm{o}_{v_h}$.

The general theory of the study of objects of an ∞-category $\mathcal{D}$ that admit infinite desuspensions is called *costabilisation*; i.e. it is the opposite of stabilisation. The costabilisation of an ∞-category $\mathcal{D}$ admitting finite colimits is a stable ∞-category $\mathrm{co}\mathcal{S}\mathrm{p}(\mathcal{D})$ together with a functor $\Sigma_\infty\colon \mathrm{co}\mathcal{S}\mathrm{p}(\mathcal{D}) \to \mathcal{D}$ (defined uniquely up to coherent homotopy) such that every finite colimit preserving functor from a stable ∞-category to $\mathcal{D}$ factors through Σ_∞. The costabilisation of the ∞-category $\mathcal{H}\mathrm{o}_*$ of pointed homotopy types is trivial, since a pointed homotopy type admitting infinite desuspensions is contractible. However, as we indicated earlier, the costabilisation of the v_h-periodic localisation $\mathcal{H}\mathrm{o}_{v_h}$ of homotopy types is not trivial.

***Theorem 6* (Theorem 7.3.0.4). —** *Let $h \geq 1$ be a natural number. The adjunction $\Theta_h \dashv \Phi_h$ exhibits the ∞-category $\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$ as the costabilisation of $\mathcal{H}\mathrm{o}_{v_h}$.*

In other words, the functor Θ_h plays the role of Σ_∞, which gives a universal property for the Bousfield–Kuhn functor Φ_h.

***Corollary 7* (Corollary 7.3.0.7). —** *Let $h \geq 1$ be a natural number. For a presentable stable ∞-category $\mathcal{D}$, composition with the Bousfield–Kuhn functor Φ_h induces an equivalence*

$$\mathcal{F}\mathrm{un}^{\mathrm{R}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}, \mathcal{D}) \xrightarrow{\sim} \mathcal{F}\mathrm{un}^{\mathrm{R}}(\mathcal{H}\mathrm{o}_{v_h}, \mathcal{D})$$

where $\mathcal{F}\mathrm{un}^{\mathrm{R}}$ denotes the ∞-category of functors that are accessible and preserve small limits, i.e. functors that admits a left adjoint.

Recall the tower (0.1) of localisations of the ∞-category $\mathcal{H}\mathrm{o}^{>1}_{(p)}$ of p-local pointed simply-connected homotopy types given by $\widetilde{V}_h$-contractions for $h \geq 1$. We consider Theorem 6 as our first step towards understanding chromatic assembly of homotopy types. For a natural number $h \geq 2$, the finite truncation $\mathcal{H}\mathrm{o}^{>1}_{(p),\widetilde{V}_h}$ of this tower also admits a non-trivial costabilisation; for $h = 1$ the costabilisation of $\mathcal{H}\mathrm{o}^{>1}_{(p),\widetilde{V}_1}$ (for example $\widetilde{V}_1 = \mathrm{S}^2/p$) of pointed simply-connected rational homotopy types is trivial, for the same reason as for the ∞-category $\mathcal{H}\mathrm{o}_*$. We would like to pose the following questions.

Question 8. —

(i) What is the costabilisation of the ∞-category $\mathcal{H}\mathrm{o}^{>1}_{(p),\widetilde{V}_h}$?

(ii) What is the relationship between the Bousfield–Kuhn functor Φ_h and the right adjoint $\Phi_{\leq h}$ to the natural functor

$$\Sigma_\infty\colon\ \mathrm{co}\mathcal{S}\mathrm{p}(\mathcal{H}\mathrm{o}^{>1}_{(p),\widetilde{V}_h}) \to \mathcal{H}\mathrm{o}^{>1}_{(p),\widetilde{V}_h}\ ?$$

This is related to Heuts's question [Heu20b, Question C.4]:

(iii) Let V be a pointed finite p-local homotopy type. If the v_h-periodic homotopy groups of V vanish, are the v_n-periodic homotopy groups of V also trivial for every $0 \leq n \leq h$?

Question (iii) has an affirmative answer if we replace finite homotopy types by finite spectra. Indeed, a spectrum has vanishing v_h-periodic homotopy groups if and only if it has vanishing $\mathrm{T}(h)$-homology groups (recall that this is not true for p-local homotopy types). Furthermore, a $\mathrm{T}(h)$-homology equivalence of finite spectra is also a $\mathrm{K}(h)$-homology equivalence, and vice versa. Let F be a finite p-local spectrum such that its $\mathrm{T}(h)$-homology $\mathrm{T}(h)_\bullet(F)$ is trivial. Then we have $\mathrm{K}(h)_\bullet(F) = 0$, which implies that $\mathrm{K}(h-1)_\bullet(F) = 0$ and $\mathrm{T}(h-1)_\bullet(F) = 0$.

Conventions. — We follow the size conventions in [HTT, §1.2.15] to deal with set-theoretic technicalities; see [LanII, §1.1] for a more precise explanation. In particular, we work in the set-theoretic framework of the ZFC axioms. In addition we assume the large cardinal axiom which guarantees the existence of a tower $\mathcal{U}_0 \subsetneq \mathcal{U}_1 \subsetneq \cdots$ of Grothendieck universes [BorHCA, Definition 1.1.2]. We call a mathematical object *small* if it is an element of $\mathcal{U}_0$. Then, for example, the ∞-category $\mathcal{H}\mathrm{o}$ of small homotopy types is well-defined and it is itself an element of $\mathcal{U}_1$. Since we are not writing a foundational text, this will be the last comment about set theory that we make.

CHAPTER 1

Background

1.1. ∞-categories and ∞-categorical localisations

Throughout the text we use the theory of $(\infty, 1)$-categories, following [HTT]. For abbreviation, we call an $(\infty, 1)$-category an *∞-category*. In this expositional section we recall some prerequisites of the theory of ∞-categories and ∞-categorical localisations.

1.1.1. ∞-categorical prerequisites. — We use *quasi-categories* as our model for ∞-categories. In other words, an ∞-category is a simplicial set satisfying "inner horn filling conditions" [HTT, Definition 1.1.2.4], and a functor between ∞-categories is a morphism of the underlying simplicial sets. We refer the reader to [HTT, Chapter 1] for a more detailed motivation and introduction on the theory of ∞-categories. For a more compact textbook on this subject, see [LanII].

1.1.1.1. Example. — We list some methods of constructing of ∞-categories.

(i) The *nerve* $\mathrm{N}(\mathbf{C})$ of a 1-category $\mathbf{C}$ is an ∞-category [HTT, Proposition 1.1.2.2].

(ii) Let $\mathbf{C}$ be a simplicially enriched category whose mapping simplicial sets are Kan complexes [HTT, Definition 1.1.2.1].[1] The *simplicial nerve* $\mathcal{N}(\mathbf{C})$ of $\mathbf{C}$ is an ∞-category [HTT, Proposition 1.1.5.10].

(iii) Every simplicial model category gives rise to an ∞-category by taking the simplicial nerve of the full subcategory of fibrant and cofibrant objects [HTT, Appendix A.2].

1.1.1.2. Example. — Here are some examples of ∞-categories.

(i) The *simplex category* $\mathbf{\Delta}$ is a 1-category whose objects are totally ordered sets

$$[n] \coloneqq \{0 \leq 1 \leq \cdots \leq n\} \text{ for } n \in \mathbb{N}$$

[1] In other words, $\mathbf{C}$ is a fibrant simplicially enriched category with respect to the Quillen model structure of the category of simplicially enriched categories [QuiHA].

and morphisms are order preserving maps. Fix a natural number n, the category $\mathbf{\Delta}^n$ is the full subcategory of $\mathbf{\Delta}$ whose objects are $[m]$ for $0 \leq m \leq n$. Define the ∞-*simplex category* $\Delta := \mathrm{N}(\mathbf{\Delta})$ and $\Delta^n := \mathrm{N}(\mathbf{\Delta}^n)$ for $n \in \mathbb{N}$.

(ii) A Kan complex is an ∞-category. Furthermore, it is an ∞-groupoid, i.e. an ∞-category where every morphism is an equivalence, see [HTT, §1.2.5].

(iii) The ∞-category $\mathcal{H}\mathrm{o}$ of homotopy types is defined as the simplicial nerve of the simplicially enriched category of Kan complexes [HTT, Definition 1.2.16.2]. It can be equivalently defined as the simplicial nerve of the simplicially enriched category of CW-complexes [HTT, Remark 1.2.16.3].

(iv) The ∞-category $\mathcal{S}\mathrm{p}$ of spectra can be constructed from a simplicial model category of spectra, using the method in Example 1.1.1.1.iii). Alternatively, one can consider $\mathcal{S}\mathrm{p}$ as the stabilisation of the ∞-category $\mathcal{H}\mathrm{o}$ of homotopy types, see ¶6.2.2.6 and [HA, §1.4].

(v) The ∞-category $\mathcal{C}\mathrm{at}_\infty$ of small ∞-categories has objects small ∞-categories, and a morphism in $\mathcal{C}\mathrm{at}_\infty$ is a functor between small ∞-categories, see [HTT, Definition 3.0.0.1] for a construction of $\mathcal{C}\mathrm{at}_\infty$. Denote the ∞-category of (not necessarily small) ∞-categories by $\mathcal{CAT}_\infty$.

1.1.1.3. Convention. — We use the following conventions throughout the text.

(i) As we already demonstrated in the above examples, we denote ∞-categories by calligraphy letters and 1-categories by boldface letters.

(ii) For an ∞-category $\mathcal{C}$, we denote by $\mathrm{ho}(\mathcal{C})$ the homotopy category of $\mathcal{C}$ [HTT, §1.2.3]. The homotopy category $\mathrm{ho}(\mathcal{H}\mathrm{o})$ of the ∞-category $\mathcal{H}\mathrm{o}$ of homotopy types is abbreviated as Ho.

(iii) For two objects X and Y in an ∞-category $\mathcal{C}$, let $\mathcal{M}\mathrm{ap}_{\mathcal{C}}(X, Y) \in \mathcal{H}\mathrm{o}$ denote the ∞-groupoid of morphisms from X to Y, see [HTT, §1.2.2]. Denote the image of $\mathcal{M}\mathrm{ap}_{\mathcal{C}}(X, Y)$ in the homotopy category Ho of by $\mathrm{Map}_{\mathcal{C}}(X, Y)$.

(iv) For ∞-categories $\mathcal{C}$ and $\mathcal{D}$, let $\mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{D})$ denote the ∞-category of functors from $\mathcal{C}$ to $\mathcal{D}$, see [HTT, §1.2.7].

1.1.2. Localisations of ∞-categories. — We review the necessary background of the theory of localisations of (presentable) ∞-categories, which plays an important role in this thesis. The main references for this section are [HTT, Chapter 5] and [HA, §1.3.4]. Since ∞-categorical localisations work analogously to ordinary categorical localisations, we also refer the reader to corresponding statements in the context of ordinary categories for comparison and motivation, following [BorHCA, Chapter 5]. For an exposition of the applications of localisations to homotopy theory containing many interesting examples, see [Law20b].

1.1.2.1. Definition. — Let $\mathcal{C}$ be an ∞-category and W be a set of morphisms in $\mathcal{C}$. A *localisation of* $\mathcal{C}$ *at* W is an ∞-category $\mathcal{D}$ together with a functor $L\colon \mathcal{C} \to \mathcal{D}$ such that L satisfies the following universal property: For every ∞-category $\mathcal{E}$, composing with L induces a fully faithful functor

$$\mathcal{F}\mathrm{un}\,(\mathcal{D},\mathcal{E}) \xrightarrow{-\circ L} \mathcal{F}\mathrm{un}(\mathcal{C},\mathcal{E})$$

whose essential images are functors sending morphisms in W to equivalences in $\mathcal{E}$. In particular, L sends morphisms in W to equivalences in $\mathcal{D}$.

1.1.2.2. Remark. — The definition is a natural generalisation of the ordinary categorical localisations, known as *category of fractions*, see [BorHCA, Definition 5.2.1]. The localisation of an ordinary small category $\mathbf{C}$ at a *small* set W of morphisms exists; one can explicitly construct it by formally adding morphisms to $\mathbf{C}$ which serve as inverses of morphisms in W, see [BorHCA, Proposition 5.2.2]. The localisation of an ∞-category always exists up to contractible choice [HA, §1.3.4].

1.1.2.3. Proposition. — *Let $\mathcal{C}$ be an ∞-category and let W be a set of morphisms in $\mathcal{C}$. Then the localisation of $\mathcal{C}$ at W exists.*

Sketch. — A marked simplicial set is a pair of a simplicial set and a set of edges of the simplicial set which contains every degenerate edge [HTT, Definition 3.1.0.1]. The idea is to consider $(\mathcal{C}, W)$ as a marked simplicial sets and construct the localisation as a fibrant replacement of $(\mathcal{C}, W)$ in the category of marked simplicial sets endowed with the cartesian model structure [HTT, Proposition 3.1.3.7]. By [HTT, Proposition 3.1.4.1] a fibrant object under the above model structure is equivalent to a pair $(\mathcal{D}, E)$ where $\mathcal{D}$ is an ∞-category and E is the set of equivalences in $\mathcal{D}$. A fibrant replacement of the pair $(\mathcal{C}, W)$ is a functor

$$F\colon (\mathcal{C}, W) \to (\mathcal{D}, E)$$

of marked simplicial sets such that $(\mathcal{D}, E)$ is fibrant and F is a cartesian equivalence. It follows from the definition of cartesian equivalences that F exhibits $\mathcal{D}$ as the localisation of $\mathcal{C}$ at W [HTT, Proposition 3.1.3.3]. □

1.1.2.4. Proposition. — *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories admitting a fully faithful right adjoint. Denote the set of morphisms of $\mathcal{C}$ that are sent to equivalences in $\mathcal{D}$ under F by W_F. Then F exhibits $\mathcal{D}$ as the localisation of $\mathcal{C}$ at W_F.*

Proof. — See [HTT, Proposition 5.2.7.12]. □

1.1.2.5. Remark. — In the situation of Proposition 1.1.2.4 we say F is a *reflective localisation*, i.e. a localisation functor that admits a fully faithful right adjoint, see [HTT, §5.2.7] and [BorHCA, §5.3]. Reflective localisations have nice closure properties, e.g. the set W_F is closed under small colimits in the ∞-category $\mathcal{F}\mathrm{un}(\Delta^1, \mathcal{C})$ of morphisms in $\mathcal{C}$. Given a set W of morphisms in $\mathcal{C}$, we would like to know whether the reflective localisation of $\mathcal{C}$ at W exists. Ordinary categorically, one construction of reflective localisations is given with the help of orthogonal pairs or factorisation systems of morphisms in cocomplete categories whose objects are presentable; in particular, locally presentable categories, see [BorHCA, §§5.4–5.5] for more details. This approach generalises naturally to ∞-categorical settings [HTT, §§5.2.8, 5.5.4, and 5.5.5].

1.1.2.6. Proposition. — *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor exhibiting the ∞-category $\mathcal{D}$ as a reflective localisation of the ∞-category $\mathcal{C}$. Denote the right adjoint of F by G. The unit natural transformation $\lambda_{\mathcal{C}}\colon \mathrm{id} \to G \circ F$ satisfies the following property: For every object $X \in \mathcal{C}$, the morphisms $(G \circ F)(\lambda(X))$ and $\lambda((G \circ F)(X))$ are homotopic to each other and are both equivalences in $\mathcal{C}$.*

Proof. — For every object $X \in \mathcal{C}$, the evaluation of λ on the morphism $\lambda(X)$ gives the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\lambda(X)} & (G\circ F)(X) \\ {\scriptstyle \mathrm{id}(\lambda(X))=\lambda(X)}\downarrow & & \downarrow{\scriptstyle (G\circ F)(\lambda(X))} \\ (G\circ F)(X) & \xrightarrow[\lambda((G\circ F)(X))]{} & (G\circ F)\left((G\circ F)(X)\right), \end{array}$$

that is, the upper-right composition $((G\circ F)(\lambda(X)))\circ\lambda(X)$ of morphisms is homotopic to the lower-left composition $\lambda((G \circ F)(X)) \circ \lambda(X)$. For an object $Y \in \mathcal{D}$, composing with $\lambda(X)$ induces an equivalence

$$\mathcal{M}\mathrm{ap}_{\mathcal{C}}((G \circ F)(X), G(Y)) \simeq \mathcal{M}\mathrm{ap}_{\mathcal{C}}(X, G(Y)),$$

on mapping spaces, since G is fully faithful. Therefore, the morphisms $(G \circ F)(\lambda(X))$ and $\lambda((G \circ F)(X))$ are homotopic. Since G is fully faithful, the counit natural transformation $F \circ G \to \mathrm{id}$ is an equivalence in the ∞-category $\mathcal{F}\mathrm{un}(\mathcal{D}, \mathcal{D})$. Thus, the morphism $\lambda((G \circ F)(X))$ is an equivalence in $\mathcal{C}$, which implies that $(G \circ F)(\lambda(X))$ is also an equivalence in $\mathcal{C}$ since the two maps are homotopic. See also [HTT, Proposition 5.2.7.4] for a stronger statement. □

1.1.2.7. Corollary. — *In the situation of Proposition 1.1.2.6 let L denote the composition $G \circ F$. The induced natural transformation $L \to L \circ L$ is an equivalence in the ∞-category $\mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{C})$.* □

1.1.2.8. Definition. — Let $\mathcal{C}$ be an ∞-category and let W be a set of morphisms in $\mathcal{C}$.

(i) An object X of $\mathcal{C}$ is W-*local* if for every morphism $f\colon A \to B$ in W the induced morphism $\mathcal{M}\mathrm{ap}_{\mathcal{C}}(B, X) \to \mathcal{M}\mathrm{ap}_{\mathcal{C}}(A, X)$ of mapping spaces is an equivalence in $\mathcal{H}\mathrm{o}$.[2]

[2] Our definition is equivalent to [HTT, Definition 5.5.4.1], since an equivalence in an ∞-category is defined as a morphism that becomes an isomorphism in the homotopy category [HTT, §1.2.4].

(ii) A morphism $f\colon Y \to Z$ in $\mathcal{C}$ is a *W-equivalence* if for every W-local object X the induced morphism $\mathcal{M}\mathrm{ap}_{\mathcal{C}}(Z, X) \to \mathcal{M}\mathrm{ap}_{\mathcal{C}}(Y, X)$ of mapping spaces an equivalence in $\mathcal{H}\mathrm{o}$.

1.1.2.9. Notation. — In the situation of Definition 1.1.2.8, if the set W consists of single morphism f, we abbreviate "W-local" and "W-equivalence" by "*f-local*" and "*f-equivalence*", respectively.

1.1.2.10. Definition. — Let $\mathcal{C}$ be an ∞-category admitting small colimits. A set S of morphisms of $\mathcal{C}$ is *strongly saturated* if it satisfies the following conditions:

(i) Given a pushout diagram

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow{\scriptstyle f} & \ulcorner & \downarrow{\scriptstyle f'} \\ Y & \longrightarrow & Y' \end{array}$$

in $\mathcal{C}$, the map f' is in S if f is in S.

(ii) The full ∞-subcategory of $\mathcal{F}\mathrm{un}(\Delta^1, \mathcal{C})$ whose objects are morphisms in S is closed under small colimits.

(iii) Let f, g and h be morphisms in $\mathcal{C}$ such that $f \circ g \simeq h$. If any two of the three maps are in S, so is the remaining one.

1.1.2.11. Example. — Let $\mathcal{C}$ be an ∞-category admitting small colimits.

(i) The set of equivalences in $\mathcal{C}$ are strongly saturated.

(ii) In the situation of Definition 1.1.2.8 the set of W-equivalences is strongly saturated, see [HTT, Lemma 5.5.4.11].

1.1.2.12. Definition. — A strongly saturated class S of morphisms is *generated* by a set $S_0 \subseteq S$ if S is the smallest strongly saturated class of morphisms containing S_0. We say that S is *of small generation* if S_0 is a small set. See also [HTT, Remark 5.5.4.7].

1.1.2.13. Definition. — An ∞-category $\mathcal{C}$ is *presentable* if there exists a regular cardinal κ and a small ∞-subcategory $\mathcal{C}_0$ such that

(i) $\mathcal{C}$ admits small colimits,

(ii) for every object $X \in \mathcal{C}_0$ the functor $\mathcal{M}\mathrm{ap}_{\mathcal{C}}(X, -)$ commutes with κ-filtered colimits, and

(iii) every object of $\mathcal{C}$ is equivalent to a small colimit in $\mathcal{C}$ of objects in $\mathcal{C}_0$.

1.1.2.14. Definition. — A functor between presentable ∞-categories is

(i) *continuous* if it preserves small limits;

(ii) *cocontinuous* if it preserves small colimits;

(iii) *accessible* if it preserves κ-filtered colimits for some regular cardinal κ, see [HTT, Definitions 5.4.2.5 and 5.3.4.5].

1.1.2.15. Notation. — Let $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ denote the ∞-category of presentable ∞-categories whose morphisms are *cocontinuous functors*. Let $\mathcal{P}\mathrm{r}^{\mathrm{R}}$ denote the ∞-category of presentable ∞-categories whose morphisms are accessible and continuous functors. See [HTT, Definition 5.5.3.1].

1.1.2.16. Remark. — By the Adjoint Functor Theorem a functor is an morphism in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ (respectively $\mathcal{P}\mathrm{r}^{\mathrm{R}}$) if it admits a right adjoint (respectively left adjoint), see [HTT, Corollary 5.5.2.9].

1.1.2.17. Theorem. — *Let $\mathcal{C}$ be a presentable ∞-category and S be a strongly saturated class of morphisms of $\mathcal{C}$ generated by a small set $S_0 \subseteq S$. The following statements holds:*

(i) *The ∞-category $\mathcal{C}_{S_0}$ of S_0-local objects is presentable.*
(ii) *The inclusion $\mathcal{C}_{S_0} \hookrightarrow \mathcal{C}$ admits a left adjoint* L.
(iii) *A morphism f in $\mathcal{C}$ is an S_0-equivalence if and only if it is in S.*
(iv) *The functor $L\colon \mathcal{C} \to \mathcal{C}_{S_0}$ exhibits $\mathcal{C}_{S_0}$ as the localisation of $\mathcal{C}$ at S. In particular, it is a reflective localisation.*

Proof. — See [HTT, Proposition 5.5.4.15]. □

1.1.2.18. Example. — Let $\mathcal{C}$ be a presentable ∞-category and let f be a morphism in $\mathcal{C}$. The ∞-category $\mathcal{C}_f$ of f-local objects is the localisation of $\mathcal{C}$ at the set of f-equivalences. In Part I we are concerned with such localisations of the ∞-category $\mathcal{H}\mathrm{o}$ of homotopy types. We refer interested readers to [Law20b, §4] to more concrete examples of such localisations.

1.2. Stable chromatic homotopy theory

Many ideas and methods from stable chromatic homotopy theory are indispensable for the development of unstable periodic homotopy theory. Fix a prime number p. In this section we give a very brief exposition of the stable chromatic localisations of the ∞-category $\mathcal{S}\mathrm{p}_{(p)}$ of p-local spectra and include necessary prerequisites from this subject for later applications.

Denote the suspension of a spectrum by Σ, and the sphere spectrum by $\mathbb{S}$. We begin by explaining the role of "self-maps" of finite spectra played in studying the stable homotopy groups $\pi_\bullet^{\mathrm{st}}(\mathbb{S})$ of the sphere spectrum. Let F be a finite spectrum, i.e. a spectrum that is equivalent to, up to desuspension, the suspension spectrum of a finite CW-complex. Given a morphism $f\colon \Sigma^d F \to F$ of spectra, where $d \in \mathbb{N}$, one can define the following composition

$$\psi_f^t\colon \mathbb{S}^n \hookrightarrow \Sigma^{dt} F \xrightarrow{f^{\circ t}} F \to \mathbb{S}^k \in \pi_n^{\mathrm{st}}(\mathbb{S}^k) \cong \pi_{n-k}(\mathbb{S})$$

for any $t \in \mathbb{N}$, where

(i) the first map (from the left) is induced by the inclusion of a bottom cell into $\Sigma^{dt}F$,

(ii) the middle map denotes the composition

$$f^{\circ t}\colon\ \Sigma^{dt} F \xrightarrow{\Sigma^{(d-1)t} f} \cdots \to \Sigma^{2d} F \xrightarrow{\Sigma^d f} \Sigma^d F \xrightarrow{f} F, \text{ and}$$

(iii) the last map $F \to \mathbb{S}^k$ is induced by the collapse map onto a top cell of dimension k.

Thus the family $\{\psi_f^t\}$ of maps supplies us with an infinite family of elements in $\pi_\bullet^{\mathrm{st}}(\mathbb{S})$. It would be good to know that, with a suitable choice of f, the above procedure produces some *non-trivial* elements in $\pi_\bullet^{\mathrm{st}}(\mathbb{S})$. An obvious candidate for f is the degree n self-map $\mathbb{S} \xrightarrow{\times n} \mathbb{S}$ for $n \in \mathbb{Z}$, which gives us every element in $\pi_0^{\mathrm{st}}(\mathbb{S}) \cong \mathbb{Z}$. The next example of f is constructed by Adams [Ada66] using complex K-theory KU. Fix a prime number p. Consider the following cofibre sequence

$$\mathbb{S} \xrightarrow{\times p} \mathbb{S} \to \mathbb{S}/p$$

It is shown in [Ada66] that for $p \geq 3$ there exists a map

$$v_1\colon\ \Sigma^{2(p-1)} \mathbb{S}/p \to \mathbb{S}/p,$$

called a v_1 *self-map* of $\mathbb{S}/p$, which induces an isomorphism on their complex K-theory homology.[(3)] Furthermore, it was observed one can construct more such self-maps by taking the cofibre of the existing self-maps; we clarify this by giving two more examples. Denote the cofibre of the self-map v_1 by $\mathbb{S}/(p, v_1)$. It is shown in Smith [Smi70] that

(3) There is also similar construction in the case $p = 2$ [Ada66], which we omit for the simplicity of this exposition.

for $p > 3$ there exists a non-trivial map

$$v_2\colon\ \Sigma^{2(p^2-1)}\mathbb{S}/(p,v_1) \to \mathbb{S}/(p,v_1),$$

which is called a v_2 self-map of the finite spectrum $\mathbb{S}/(p,v_1)$. Let $\mathbb{S}/(p,v_1,v_2)$ denote the cofibre of the v_2 self map. Toda proves that for $p > 5$ there exists a non-trivial self-map

$$v_3\colon\ \Sigma^{2(p^3-1)}\mathbb{S}/(p,v_1,v_2) \to \mathbb{S}/(p,v_1,v_2)$$

of the finite spectrum $\mathbb{S}/(p,v_1,v_2)$.

In [MRW77] Miller, Ravenel and Wilson develop a program to show that the infinite families $\psi^t_{v_1}$, $\psi^t_{v_2}$ and $\psi^t_{v_3}$ of elements of $\pi^{\mathrm{st}}_\bullet(\mathbb{S})$ constructed using the above v_1, v_2 and v_3 self-maps are non-trivial. Their methods is to reduce the questions purely algebraic computations. We list the ingredients of this algebraic approach and give references to the definitions and details for the interested reader. Recall that we work with a fixed prime number p. Denote the p-local *Brown–Peterson* spectrum by BP, which was constructed in [BP66]. The BP-homology $\mathrm{BP}_\bullet(\mathbb{S}_{(p)})$ of the p-local sphere spectrum admits the structure of a (left) $\mathrm{BP}_\bullet(\mathrm{BP})$-comodule of the so-called *Hopf algebroid* $(\mathrm{BP}_\bullet, \mathrm{BP}_\bullet\mathrm{BP})$; see [Hov04] for an introduction of Hopf algebroids. The Ext group $\mathrm{Ext}_{\mathrm{BP}_\bullet(\mathrm{BP})}\left(\mathrm{BP}_\bullet, \mathrm{BP}_\bullet\left(\mathbb{S}_{(p)}\right)\right)$ of $(\mathrm{BP}_\bullet, \mathrm{BP}_\bullet\mathrm{BP})$-comodules is isomorphic, as a bi-graded abelian group, to the E_2-page of the $\mathrm{BP}_\bullet$-based *Adams–Novikov spectral sequence* converging to the stable homotopy groups $\pi^{\mathrm{st}}_\bullet(\mathbb{S}_{(p)})$ of the p-local sphere spectrum. It is explained in [MRW77] how to detect non-trivial elements in $\mathrm{Ext}_{\mathrm{BP}_\bullet(\mathrm{BP})}\left(\mathrm{BP}_\bullet, \mathrm{BP}_\bullet\left(\mathbb{S}_{(p)}\right)\right)$ which *survives* to the E_∞-page of the spectral sequence and their images on the E_∞-page represent elements of $\pi^{\mathrm{st}}_\bullet(\mathbb{S}_{(p)})$ constructed from the v_1, v_2 or v_3 self-maps (that we mentioned at the beginning of this paragraph).

In order to compute the group $\mathrm{Ext}_{\mathrm{BP}_\bullet(\mathrm{BP})}\left(\mathrm{BP}_\bullet, \mathrm{BP}_\bullet\left(\mathbb{S}_{(p)}\right)\right)$, an algebraic *chromatic* resolution

$$\mathrm{BP}_\bullet(\mathbb{S}) \to M^0 \to M^1 \to \cdots \to M^h \to \cdots \tag{1.2.0.1}$$

of $\mathrm{BP}_\bullet\mathrm{BP}$-comodules was constructed in [MRW77]. This resolution induces the so-called *chromatic spectral sequence* converging to the bi-graded group $\mathrm{Ext}_{\mathrm{BP}_\bullet(\mathrm{BP})}(\mathrm{BP}_\bullet, \mathrm{BP}_\bullet(\mathbb{S}_{(p)}))$. Recall the p-local Morava K-theory spectrum $\mathrm{K}(h)$ of height h, where $h \in \mathbb{N}$. The E_1-page of the spectral sequence, given by the Ext group $\mathrm{Ext}_{\mathrm{BP}_\bullet(\mathrm{BP})}\left(\mathrm{BP}_\bullet, M^h\right)$, can be computed using the Ext groups $\mathrm{Ext}_{\mathrm{K}(h)_\bullet\mathrm{K}(h)}(\mathrm{K}(h), \mathrm{K}(\bullet))$ of $\mathrm{K}(h)_\bullet\mathrm{K}(h)$-comodules. Finally, the spectrum $\mathrm{K}(h)$ relates closely to the so-called *Morava stabiliser groups* of automorphisms of the so-called *Honda formal group* $\Gamma_{h,p}$ of height h over the algebraic closure $\overline{\mathbb{F}_p}$ of $\mathbb{F}_p$. This gives the possibility to obtain partial information of the group $\mathrm{Ext}_{\mathrm{K}(h)_\bullet\mathrm{K}(h)}(\mathrm{K}(h), \mathrm{K}(h))$ from the continuous group cohomology of the Morava stabiliser group; the latter involves very complicated computations.

Denote the ∞-category of p-local spectra by $\mathcal{S}\mathrm{p}_{(p)}$. The p-local Lubin–Tate spectrum E_h of height h is a spectrum associated with the universal deformation ring of the Lubin–Tate formal group law over the p-adic integers; we recommend the non-expert reader to view E_h as a auxiliary spectrum with certain properties that we will explain later, since several pages prerequisites from chromatic homotopy theory are needed to make full sense of this definition of E_h. Let L_h denote the localisation functor of $\mathcal{S}\mathrm{p}_{(p)}$ at E_h-homology equivalences. By ¶1.2.0.15 we obtain the following tower

$$\cdots \to \mathrm{L}_{h+1} \to \mathrm{L}_h \to \cdots \to \mathrm{L}_1 \to \mathrm{L}_0 \tag{1.2.0.2}$$

of localisations of $\mathcal{S}\mathrm{p}_{(p)}$. Evaluation this tower at the p-local sphere spectrum $\mathbb{S}_{(p)}$, we obtain the commutative diagram

$$\begin{array}{c} \mathbb{S}_{(p)} \\ \swarrow \quad \swarrow \quad \downarrow \quad \searrow \quad \searrow \quad \searrow \\ \cdots \longrightarrow \mathrm{L}_{h+1}\left(\mathbb{S}_{(p)}\right) \longrightarrow \mathrm{L}_h\left(\mathbb{S}_{(p)}\right) \longrightarrow \cdots \longrightarrow \mathrm{L}_1\left(\mathbb{S}_{(p)}\right) \longrightarrow \mathrm{L}_0\left(\mathbb{S}_{(p)}\right). \end{array} \tag{1.2.0.3}$$

In [Rav84] Ravenel observed that the lower horizontal tower in (1.2.0.2) relates to the algebraic chromatic resolution (1.2.0.1) as follows: The truncated chromatic spectral sequence induced by the truncated resolution

$$\mathrm{BP}_\bullet(\mathbb{S}) \to M^0 \to M^1 \to \cdots \to M^h$$

converges to the the bi-graded Ext group $\mathrm{Ext}_{\mathrm{BP}_\bullet(\mathrm{BP})}\left(\mathrm{BP}_\bullet, \mathrm{BP}_\bullet\left(\mathrm{L}_h\left(\mathbb{S}_{(p)}\right)\right)\right)$, which is isomorphic to the E_2-page of the $\mathrm{BP}_\bullet$-based Adams–Novikov spectral sequence converging to the stable homotopy groups of $\pi_\bullet^{\mathrm{st}}\left(\mathrm{L}_h\left(\mathbb{S}_{(p)}\right)\right)$. Because of this relationship, we view the tower (1.2.0.2) as a realisation of the resolution (1.2.0.1) on the spectral level. Thus, the tower (1.2.0.2) is known as the *chromatic tower* of p-local spectra.

We have already mentioned the concept of formal group here and there in the exposition above. Let us make a side remark that there is a very close tie between stable chromatic homotopy theory and the theory of the moduli stacks of formal groups. See [HopCOC; LurCHT; PetFG; PstFHC] for more detailed explanations of this algebro-geometric viewpoint of chromatic homotopy theory.

In his seminal paper [Rav84] Ravenel made conjectures saying that the problems about finding non-trivial self-maps $\Sigma^d F \to F$ of a finite spectrum F can be answered using chromatic homotopy theory. The related conjectures were proven in [HS98], which we recall in the following. Denote by $\otimes$ the smash product of spectra, which makes $\mathcal{S}\mathrm{p}$ a symmetric monoidal ∞-category (see Definition 5.2.1.30). Recall that we work with a fixed prime number p. We set $\mathrm{K}(0) \coloneqq \mathrm{H}\mathbb{Q}$ and $\mathrm{K}(\infty) \coloneqq \mathrm{H}\mathbb{F}_p$.

1.2.0.1. Definition. — Let E be a spectrum and let $d \in \mathbb{N}$. A map $f\colon \Sigma^d E \to E$ is *nilpotent* if there exists a $n \in \mathbb{N}$ such that the composition

$$f^{\circ n}\colon \Sigma^{nd} E \xrightarrow{\Sigma^{(n-1)d} f} \Sigma^{(n-1)d} E \xrightarrow{\Sigma^{(n-2)d} f} \cdots \to \Sigma^d E \xrightarrow{f} E$$

is the zero map.

***1.2.0.2. Theorem* (Nilpotence Theorem).** — *Let F be a p-local finite spectrum and let $d \in \mathbb{N}$. A map $f\colon \Sigma^d F \to F$ is nilpotent if and only if, for all $h \in \mathbb{N} \cup \{\infty\}$, the induced map $\mathrm{K}(h)_\bullet(f)$ on $\mathrm{K}(h)$-homology is nilpotent.*

Proof. — See [HS98, Theorem 3]. □

1.2.0.3. Definition. — Let h be a natural number. A p-local finite spectrum F is of *type at least* h if its $\mathrm{K}(n)$-homology $\mathrm{K}(n)_\bullet(F)$ is trivial for every $0 \leq n \leq h-1$. If in addition its $\mathrm{K}(h)$-homology $\mathrm{K}(h)_\bullet(F)$ is non-trivial, we say F is of *type* h.

***1.2.0.4. Theorem* (Ravenel).** — *Let F be a p-local finite spectrum and let h be a natural number. If $\mathrm{K}(h)_\bullet(F) = 0$, then F is of type at least h.*

Proof. — See [Rav84, Theorem 2.11]. □

***1.2.0.5. Theorem* (Mitchell).** — *For every $h \in \mathbb{N}$, there exists a finite p-local spectrum of type h.*

Proof. — See [Mit85]. □

***1.2.0.6. Theorem* (Periodicity Theorem).** — *Let F be a finite spectrum of type at least h for a natural number h. Then there exists a v_h* self-map *$\Sigma^{d_h} F \to F$ for some natural number d_h which induces an isomorphism of $\mathrm{K}(h)$-homology and induces the zero map of $\mathrm{K}(j)$-homology for all $j \neq h$.*

Proof. — See [HS98, Theorem 9]. □

1.2.0.7. Definition. — Let $\mathcal{C}$ be a stable ∞-category. A full ∞-subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ is *thick* if the homotopy category $\mathrm{ho}(\mathcal{C}_0)$ is a thick subcategory of the triangulated category $\mathrm{ho}(\mathcal{C})$. In other words $\mathcal{C}_0$ is thick if it is closed under equivalences, cofibre sequences and retracts. By closed under cofibre sequences we mean that if two of the three objects of a cofibre sequence is contained in $\mathcal{C}_0$, then so is the remaining one.

1.2.0.8. Example. — Denote the ∞-category of p-local finite spectra by $\mathcal{S}\mathrm{p}^{\mathrm{fin}}_{(p)}$. The full ∞-subcategory $\mathcal{P}_{\geq h}$, whose objects are p-local finite spectra of type at least h, is a thick subcategory of $\mathcal{S}\mathrm{p}^{\mathrm{fin}}_{(p)}$. By convention we denote the full ∞-subcategory whose objects are finite contractible p-local spectra by $\mathcal{P}_{\geq\infty}$.

***1.2.0.9. Theorem* (Thick Subcategory Theorem).** — *If $\mathcal{T}$ is a thick subcategory of the ∞-category $\mathcal{S}\mathrm{p}^{\mathrm{fin}}_{(p)}$, then there exists a unique $h \in \mathbb{N} \cup \{\infty\}$ such that $\mathcal{T}$ is equivalent to the ∞-category $\mathcal{P}_{\geq h}$.*

Proof. — See [HS98, Theorem 7]. □

1.2.0.10. Remark. — In the situation of Theorem 1.2.0.9, one can regard the full ∞-category $\mathcal{P}_{\geq h}$ as the "prime ideals" of $\mathcal{S}\mathrm{p}^{\mathrm{fin}}_{(p)}$, using the theory of tensor-triangulated geometry, see [Bal20].

1.2.0.11. Corollary. — *Let F be a non-trivial finite p-local spectrum. There exists a unique natural number h such that F is of type h.* □

Using the family of thick subcategories $\mathcal{P}_{\geq h}$ of $\mathcal{S}\mathrm{p}^{\mathrm{fin}}_{(p)}$, we can construct another tower of localisations of $\mathcal{S}\mathrm{p}^{\mathrm{fin}}_{(p)}$, known as the p-local *finite chromatic localisation tower*. This is the stable analogue of the *unstable chromatic localisation tower* (3.4.0.1) that will be introduced in §3.4.

***1.2.0.12.* The finite chromatic localisation tower of $\mathcal{S}\mathrm{p}_{(p)}$.** — Consider the nested sequence

$$\cdots \subseteq \mathcal{P}_{\geq h+1} \subseteq \mathcal{P}_{\geq h} \subseteq \cdots \subseteq \mathcal{P}_{\geq 1} \subseteq \mathcal{P}_{\geq 0}$$

of thick ∞-subcategories of $\mathcal{S}\mathrm{p}^{\mathrm{fin}}_{(p)}$. For $h \in \mathbb{N}$, denote by $\mathcal{S}\mathrm{p}_{\geq h}$ the full ∞-subcategory whose objects are equivalent to small colimits of objects of $\mathrm{P}_{\geq h}$. Thus, this gives a filtration

$$\cdots \subseteq \mathcal{S}\mathrm{p}_{\geq h+1} \subseteq \mathcal{S}\mathrm{p}_{\geq h} \subseteq \cdots \subseteq \mathcal{S}\mathrm{p}_{\geq 1} \subseteq \mathcal{S}\mathrm{p}_{\geq 0} = \mathcal{S}\mathrm{p}_{(p)}$$

of the ∞-category $\mathcal{S}\mathrm{p}_{(p)}$ of p-local spectra. The *Verdier quotient*

$$\mathrm{L}^{\mathrm{f}}_h \colon \mathcal{S}\mathrm{p}_{(p)} \to \mathcal{L}^{\mathrm{f}}_h\left(\mathcal{S}\mathrm{p}_{(p)}\right) \coloneqq \mathcal{S}\mathrm{p}_{(p)} \Big/ \mathcal{S}\mathrm{p}_{\geq h+1}$$

is a reflective localisation of $\mathcal{S}\mathrm{p}_{(p)}$, characterised by the property that it is the initial functor from $\mathcal{S}\mathrm{p}_{(p)}$ to a stable ∞-category sending every object of $\mathcal{S}\mathrm{p}_{\geq h+1}$ to the zero object. The *finite chromatic localisation tower* denotes the following sequence

$$\cdots \to \mathrm{L}^{\mathrm{f}}_h \to \mathrm{L}^{\mathrm{f}}_{h-1} \to \cdots \to \mathrm{L}^{\mathrm{f}}_0 \simeq \mathrm{L}_{\mathbb{Q}} \to \mathrm{pt} \tag{1.2.0.4}$$

of localisations of the $\mathcal{S}\mathrm{p}_{(p)}$; here $\mathrm{L}^{\mathrm{f}}_0 \simeq \mathrm{L}_{\mathbb{Q}}$ is the localisation of $\mathcal{S}\mathrm{p}_{(p)}$ at the set of rational equivalences. The comparison between the tower (1.2.0.2) and (1.2.0.4) is known as the *telescope conjecture*.

***1.2.0.13.* The spectra $\mathrm{S}(h)$ and $\mathrm{T}(h)$.** — There are two p-local spectra related closely to the finite chromatic localisations, which we introduce now. They will also be used in later applications. Let $h \in \mathbb{N}$.

(i) The p-local spectrum $\mathrm{S}(h)$ is the evaluation $\mathrm{L}^{\mathrm{f}}_h\left(\mathbb{S}_{(p)}\right)$ of the localisation functor $\mathrm{L}^{\mathrm{f}}_h$ at the p-local sphere spectrum $\mathbb{S}_{(p)}$.

(ii) Let F_h be a finite spectrum of type h together with a v_h self-map $\Sigma^d F_h \to F_h$. The *p-local telescope spectrum* $\mathrm{T}(h)$ *of height* h is defined as the colimit

$$\varinjlim \left(F_h \xrightarrow{\Sigma^{-d} v_h} \Sigma^{-d} F_h \to \cdots \xrightarrow{\Sigma^{-nd} v_h} \Sigma^{-nd} F_h \to \cdots \right)$$

in the ∞-category $\mathcal{S}\mathrm{p}_{(p)}$.

The construction of $\mathrm{T}(h)$ depends on the choices of F_h and the v_h self-map. However, these choices are elided from the notation by the following reason: By the Thick Subcategory Theorem and the asymptotically uniqueness of the v_h-self maps [HS98, Corollaries 3.7 and 3.8], the notion of $\mathrm{T}(h)$-homology equivalence does not depend on these choices, see [Bou01, §3]. In this thesis, we are only concerned with $\mathrm{T}(h)$-homology equivalences, and not with any specific $\mathrm{T}(h)$-spectrum.

Moreover, an $\mathrm{S}(h)$-homology equivalence is a $\mathrm{T}(n)$-homology equivalence for every $0 \leq n \leq h$, and vice versa. The localisation $\mathrm{L}_h^{\mathrm{f}}$ is equivalent to the localisation of the ∞-category $\mathcal{S}\mathrm{p}_{(p)}^{\geq 1}$ at the set of $\mathrm{S}(h)$-homology equivalences, see [Bou01, §3].

1.2.0.14. **Homology localisations of spectra.** — We recall some basic notions of homological localisation of spectra, which we have been using throughout the section.

Let E be a spectrum. A spectrum Y is $E_\bullet$*-acyclic* if the smash product $E \otimes Y$ is the zero spectrum, i.e. the E-homology $E_\bullet(Y)$ of Y is zero. A spectrum X is $E_\bullet$*-local* if the mapping spectrum $\mathbb{M}\mathrm{ap}_*(Y, X) \simeq \mathrm{pt}$ for every $E_\bullet$-acyclic spectrum Y. A morphism $Z_1 \to Z_2$ of spectra is a E*-homology equivalence*, or $E_\bullet$-equivalence, if it induces an isomorphism on E-homology. Denote the ∞-category of $E_\bullet$-local spectra by $\mathcal{S}\mathrm{p}_E$. It is shown in [Bou79b] that $\mathcal{S}\mathrm{p}_E$ is a reflective ∞-subcategory of $\mathcal{S}\mathrm{p}$, i.e. there exists a left adjoint L_E to the fully faithful inclusion $\mathcal{S}\mathrm{p}_E \hookrightarrow \mathcal{S}\mathrm{p}$. Furthermore, the functor L_E exhibits the ∞-category $\mathcal{S}\mathrm{p}_E$ as the localisation of $\mathcal{S}\mathrm{p}$ at the set of $E_\bullet$-homology equivalences. Thus the functor L_E is called the E*-homology localisation*, and is also denoted by $E_\bullet$*-localisation*.

Fix a prime number p. The *p-local Moore spectrum* $\mathbb{S}_{(p)}$ is characterised by the property that $\mathrm{H}\mathbb{Z}_\bullet(\mathbb{S}_{(p)}) \cong \mathbb{Z}_{(p)}$, as an isomorphism of graded abelian groups. One can construct $\mathbb{S}_{(p)}$ by formally inverting the degree ℓ self-map on the sphere spectrum $\mathbb{S}$, for all prime number $\ell \neq p$. A spectrum is *p-local* if it is $(\mathbb{S}_{(p)})_\bullet$-local. There is the following characterisation of p-local spectra: A spectrum is p-local if and only if its stable homotopy groups are p-local abelian groups in all degrees, see [Bou79b, Proposition 2.4]. Furthermore, a bounded below spectrum is p-local if and only if its $\mathrm{H}\mathbb{Z}$-homology groups are p-local abelian groups in all degrees. Let $\mathcal{S}\mathrm{p}_{(p)}$ denote the ∞-category of p-local spectra. The $(\mathbb{S}_{(p)})_\bullet$-localisation $\mathrm{L}_{\mathbb{S}_{(p)}} \colon \mathcal{S}\mathrm{p} \to \mathcal{S}\mathrm{p}_{(p)}$ is called the *p-localisation*. For a spectrum E, we denote

$$E_{(p)} \coloneqq \mathrm{L}_{\mathbb{S}_{(p)}}(E).$$

1.2.0.15. **The Bousfield class of spectra.** — Closely related to homological localisations of spectra is the notion of Bousfield class of spectra. For $E_1, E_2 \in \mathcal{S}\mathrm{p}$, we say E_1 and E_2 are *Bousfield equivalent* if a spectrum X is $(E_1)_\bullet$-acyclic exactly when it is $(E_2)_\bullet$-acyclic. This gives an equivalence relation of spectra. For a spectrum E, we denote the equivalence class of E under the above Bousfield equivalence relation by $\langle E\rangle$, called the *Bousfield class of* E. We write

$$\langle E_1\rangle \geq \langle E_2\rangle$$

if every $(E_1)_\bullet$-acyclic spectrum is also $(E_2)_\bullet$-acyclic. For example, fix a prime number p, we have the following comparisons of Bousfield classes: Let h be a natural number.

(i) $\langle \mathrm{S}(h)\rangle = \langle \mathrm{T}(0) \vee \mathrm{T}(1) \vee \cdots \vee \mathrm{T}(h)\rangle$;

(ii) $\langle \mathrm{E}_h\rangle = \langle \mathrm{E}(h)\rangle = \langle \mathrm{K}(0) \vee \mathrm{K}(1) \vee \cdots \vee \mathrm{K}(h)\rangle$, where $\mathrm{E}(h)$ denotes the so-called *Johnson–Wilson spectrum*;

(iii) $\langle \mathrm{S}(h)\rangle \geq \langle \mathrm{E}(h)\rangle$;

(iv) $\langle \mathrm{T}(h)\rangle \geq \langle \mathrm{K}(h)\rangle$;

We refer the reader to [Rav84] and [Bou01] for the proofs of above comparisons.

Let E_1 and E_2 be two spectra such that $\langle E_1\rangle \geq \langle E_2\rangle$. Then there exists a natural transformation

$$\mathrm{L}_{E_1} \to \mathrm{L}_{E_2},$$

given on object by E_2-homology localisation. Using this and the above comparisons of the Bousfield classes, we obtain a natural transformation $\mathrm{L}_h^{\mathrm{f}} \to \mathrm{L}_h$ from the finite chromatic localisation of height h to the $(\mathrm{E}_h)_\bullet$-localisation L_h.

1.3. Prerequisites on unstable homotopy theory

In this section we recall some prerequisites, conventions and notations concerning homotopy types.

1.3.0.1. Convention. — Let X and Y be pointed homotopy types. We denote the ∞-groupoid of pointed maps from X to Y by $\mathcal{M}\mathrm{ap}_*(X,Y)$, called the *pointed mapping space* from X to Y. Choosing the constant morphism sending X to the basepoint of Y as the basepoint, $\mathcal{M}\mathrm{ap}_*(X,Y)$ becomes a pointed homotopy type. Let $[X,Y]$ denote the set of pointed homotopy classes of pointed maps from X to Y.

1.3.0.2. **Adding a disjoint basepoint.** — Let $\mathcal{H}\mathrm{o}_*$ denote the ∞-category of pointed homotopy types. There exists an adjoint pair

$$(-)_+ \colon \mathcal{H}\mathrm{o} \rightleftarrows \mathcal{H}\mathrm{o}_* : \underline{(-)}$$

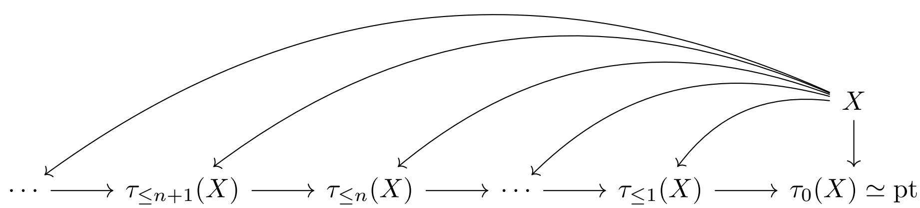
given on objects as follows:

(i) The functor $(-)_+$ assigns to a homotopy type W the pointed homotopy type $(W_+, +)$ given by the disjoint union of W and a one-point space $\{+\}$.
(ii) The functor $\underline{(-)}$ assigns to a pointed homotopy types (X, x_0) its underlying homotopy type X.

1.3.0.3. Convention. — Let (X, x_0) be a pointed homotopy type. By abuse of notation we abbreviate most of the time homotopy groups $\pi_\bullet(X, x_0)$ of (X, x_0) by $\pi_\bullet(X)$; it will be either clear from the context which basepoint we choose, or the choice of the basepoint does not matter.

1.3.0.4. Definition. — Let $n \in \mathbb{N}$. A homotopy type X is *n-connected* if the homotopy groups $\pi_i(X, x_0)$ are zero for all $i \leq n$ and for any choice of basepoint $x_0 \in X$. The connectivity of X is a natural number $\mathrm{conn}(X)$ such that X is $\mathrm{conn}(X)$-connected and the homotopy group $\pi_{\mathrm{conn}(X)+1}(X)$ of X in degree $\mathrm{conn}(X)+1$ is non-trivial. We say that a homotopy type is *connected* if it is 0-connected, and a homotopy type is *simply-connected* if it is 1-connected.

1.3.0.5. **The Postnikov tower.** — Let X be a pointed connected homotopy type. There exists the following commutative diagram



in ∞-category $\mathcal{H}\mathrm{o}_*$ of pointed homotopy types satisfying the following properties:

(i) For every $n \in \mathbb{N}$, the homotopy groups of $\tau_{\leq n}(X)$ in all degrees above n are trivial.
(ii) For every $n \in \mathbb{N}$, the map $X \to \tau_{\leq n}(X)$ induces an isomorphism of homotopy groups in every degree $0 \leq i \leq n$.
(iii) The inverse limit of the lower horizontal tower in $\mathcal{H}\mathrm{o}_*$ is equivalent to X.

In particular, for every $n \in \mathbb{N}$, we obtain a fibre sequence

$$\tau_{>n}(X) \to X \to \tau_{\leq n}(X)$$

in $\mathcal{H}\mathrm{o}_*$. The fibre $\tau_{>n}(X)$ is n-connected, and the pointed map $\tau_{>n}(X) \to X$ induces an isomorphism of homotopy groups in all degrees larger than n. The pointed homotopy type $\tau_{>n}(X)$ is called the *n-connected cover* of X; sometimes we denote it equivalently by $\tau_{\geq n+1}(X)$.

1.3.0.6. **Fiber sequences.** — Let $F \to X \xrightarrow{f} B$ be a fibre sequence in the ∞-category $\mathcal{H}\mathrm{o}_*$ of pointed homotopy types, and assume that B is connected. We introduce two ways to construct X using F and B, up to equivalence.

(i) Consider $F \to X \xrightarrow{f} B$ as morphisms in the ∞-category $\mathcal{H}\mathrm{o}$ of unpointed homotopy types. Let $\mathcal{H}\mathrm{o}_{/X}$ denote the over-∞-category of objects in $\mathcal{H}\mathrm{o}$ together with a morphism to X (see [HTT, §§1.2.9 and 4.2.1]). The morphism f induces a functor $f_!\colon \mathcal{H}\mathrm{o}_{/X} \to \mathcal{H}\mathrm{o}_{/B}$ by composing with f. By the proof of [HTT, Lemma 6.1.3.14] the functor $f_!$ admits a right adjoint f^*. For $(Y \to B) \in \mathcal{H}\mathrm{o}_{/B}$, we have $f^*(Y \to B) \simeq X \times_Y B$. For a point $b \in B$, consider the pullback diagram

$$\begin{array}{ccc} F_b & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow f \\ \mathrm{pt} & \xrightarrow{b} & X. \end{array}$$

in $\mathcal{H}\mathrm{o}$. Thus we have $f^*(\mathrm{pt} \xrightarrow{b} B) = (F_b \to X)$. Furthermore, the functor f^* preserves colimits. This implies that f^* sends

$$B \simeq \varinjlim_B (\mathrm{pt} \xrightarrow{b} B) \in \mathcal{H}\mathrm{o}_{/B}$$

to the object

$$\varinjlim_B (F_b \to X) \in \mathcal{H}\mathrm{o}_{/X}\,.$$

Since the continuous functor f^* preserves terminal objects, $\varinjlim_B (F_b \to X)$ is equivalent to the terminal object X of $\mathcal{H}\mathrm{o}_{/X}$. Therefore, we have

$$X \simeq \varinjlim_B F_b \in \mathcal{H}\mathrm{o},$$

where the limit is taken over the diagram $B \to \mathcal{H}\mathrm{o}$ sending a point b to the fibre F_b over B.

(ii) The pointed homotopy type ΩB is a group object in the ∞-category $\mathcal{H}\mathrm{o}_*$ of pointed homotopy types. Thus we can consider the induced fibre sequence $\Omega B \to F \to X$ as a (ΩB)-principal ∞-bundle in $\mathcal{H}\mathrm{o}_*$, classified by the map $X \to B \simeq \mathrm{Bar}(\Omega B)$, where Bar denotes the classifying space Bar construction. By [NSS15, §3] we have the following commutative diagram

$$\begin{array}{ccccccccc}
\cdots & \substack{\rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow} & F & \substack{\rightarrow\\ \rightarrow\\ \rightarrow} & F & \rightrightarrows & F & \longrightarrow & F \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \substack{\rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow} & F \times \Omega B \times \Omega B & \substack{\rightarrow\\ \rightarrow\\ \rightarrow} & F \times \Omega B & \rightrightarrows & F & \longrightarrow & X \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f} \\
\cdots & \substack{\rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow} & \Omega B \times \Omega B & \substack{\rightarrow\\ \rightarrow\\ \rightarrow} & \Omega B & \rightrightarrows & \mathrm{pt} & \longrightarrow & \mathrm{Bar}(\Omega B) \simeq B
\end{array} \tag{1.3.0.1}$$

in $\mathcal{H}\mathrm{o}_*$ where

(a) each column is a fibre sequence,

(b) every vertical fibre sequences except for the right most one is a trivial fibration, and

(c) the right most object of each horizontal row is equivalent to the geometric realisation of the simplicial objects (where we only drew the face maps) to its left.

In particular, we obtain an equivalence

$$X \simeq \varinjlim \left(\cdots \substack{\rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow} F \times \Omega B \times \Omega B \substack{\rightarrow\\ \rightarrow\\ \rightarrow} F \times \Omega B \rightrightarrows F \right)$$

of pointed homotopy types.

1.3.0.7. Definition. — Let E be a spectrum and let X be a pointed homotopy type. The *reduced E-homology* of X is defined as the graded abelian group

$$\widetilde{E}_\bullet(X) \coloneqq \pi_\bullet^{\mathrm{st}}(E \otimes \Sigma^\infty X).$$

The *E-homology* of X is defined as the graded abelian group

$$E_\bullet(X) \coloneqq \pi_\bullet^{\mathrm{st}}\left(E \otimes \Sigma^\infty(X_+)\right).$$

***1.3.0.8.* Nilpotent groups and nilpotent homotopy types.** — Let G be a group. The *lower central series* $(\Gamma_i(G))_{i \geq 1}$ of G is defined inductively as follows:

$$\Gamma_1 \coloneqq G$$
$$\Gamma_{i+1} \coloneqq [G, \Gamma_i(G)], \text{ for all } i \geq 1,$$

where $[-,-]$ denotes the commutator bracket. The group G is *nilpotent* if there exists a natural number n such that $\Gamma_n(G)$ is the trivial group (and thus $\Gamma_m(G)$ are trivial for all $m \geq n$). Abelian groups are nilpotent.

Let H be a group and a G-module. The *G-lower central series* $(\Gamma_i^G(H))_{i \geq 1}$ of H is defined inductively as follows: $\Gamma_1^G(H) \coloneqq H$, and for all $i \geq 1$ the group $\Gamma_{i+1}^G(H)$ is the normal subgroup of $\Gamma_i^G(H)$ generated by elements of the form $h(g.a)h^{-1}a^{-1}$ for all $h \in H$, all $g \in G$ and all $a \in \Gamma_i^G(H)$. We say H is *G-nilpotent* if there exists a natural number n such that $\Gamma_n^G(H)$ is the trivial group (and thus $\Gamma_m^G(H)$ are trivial for all $m \geq n$).

Let X be a pointed connected homotopy type. The homotopy group $\pi_n(X)$ in every degree $n \geq 2$ admits an action by the fundamental group $\pi_1(X)$, see [HatAT, p.341] for a detailed construction of this action. The homotopy type X is *nilpotent* if

(i) $\pi_1(X)$ is nilpotent, and

(ii) $\pi_n(X)$ is $\pi_1(X)$-nilpotent, for all natural numbers $n \geq 2$.

Examples of nilpotent homotopy types are simple homotopy types (the action of $\pi_1(-)$ on $\pi_n(-)$ are trivial for all $n \geq 2$), in particular simply connected homotopy types, and connected H-spaces.

Let R be a subring of the rational numbers $\mathbb{Q}$. For a nilpotent group G, the notation $G \otimes R$ denotes the R-Malcev completion of G, see [BK, §V.2.3] for the definitions. If G is abelian, then $G \otimes R$ is isomorphic to the tensor product $G \otimes_{\mathbb{Z}} R$ of $\mathbb{Z}$-modules.

See [Dro71; Hil82; BK] for more details about these topics.

***1.3.0.9.* Homological localisations of homotopy types.** — Let E be a spectrum. A morphism $f\colon X \to Y \in \mathcal{H}o_*$ is an *E-homology equivalence*, or *$E_\bullet$-equivalence*, if it induces an isomorphism $f_*\colon E_\bullet(X) \xrightarrow{\simeq} E_\bullet(Y)$. A pointed connected homotopy type Z is *$E_\bullet$-acyclic* if the map $Z \to \mathrm{pt}$ is an $E_\bullet$-equivalence. Equivalently, Z is $E_\bullet$-acyclic if $\widetilde{E}_\bullet(X) = 0$. The localisation of $\mathcal{H}o_*$ at the set of $E_\bullet$-equivalences exists and is denoted by $\mathrm{L}_E\colon \mathcal{H}o_* \to \mathcal{H}o_*$, see [Bou75]. This is a reflective localisation and it assigns to every pointed homotopy type X a pointed homotopy type $\mathrm{L}_E(X)$ together with a morphism $\lambda_E(X)\colon X \to \mathrm{L}_E(X)$ in $\mathcal{H}o_*$ such that

(i) the map $\lambda_E(X)$ is an $E_\bullet$-equivalence, and

(ii) every $E_\bullet$-equivalence $g\colon X \to Y \in \mathcal{H}o_*$ admits an extension $g'\colon Y \to \mathrm{L}_E X$ with $g' \circ g \simeq \lambda_E(X)$.

A pointed homotopy type is *$E_\bullet$-local* if it is the map $X \to \mathrm{L}_E(X)$ is an equivalence.

Fix a prime number p and recall the mod-p Moore spectrum $\mathbb{S}_{(p)}$ from ¶1.2.0.14. A pointed homotopy type is *p-local* if it is $(\mathbb{S}_{(p)})_\bullet$-local.(4) For a nilpotent group G, we say G is *p-local* if there exists a group isomorphism $G \cong \mathbb{Z}_{(p)} \otimes G$. For a pointed nilpotent homotopy type Z, there are the following characterisations of being p-local:

(i) Z is p-local if and only if its $\mathrm{H}\mathbb{Z}$-homology groups are p-local abelian groups in every degrees.

(ii) Z is p-local if and only if its homotopy groups are p-local in all degrees.

(4) By [Bou75, Proposition 4.1] it is equivalently $(\mathrm{H}\mathbb{Z}_{(p)})_\bullet$-local.

In general, homological localisations of non-nilpotent homotopy types are complicated, see [BK, Chapter VII]. Let $\mathcal{H}\mathrm{o}_{(p)}$ denote the ∞-category of pointed p-local homotopy types. The $(\mathbb{S}_{(p)})_\bullet$-localisation $\mathrm{L}_{\mathbb{S}_{(p)}}\colon \mathcal{H}\mathrm{o}_* \to \mathcal{H}\mathrm{o}_{(p)}$ is called the *p-localisation* of homotopy types, as constructed in [Bou75]. For $X \in \mathcal{H}\mathrm{o}_*$, we denote

$$X_{(p)} \coloneqq \mathrm{L}_{\mathbb{S}_{(p)}}(X).$$

When restricted to nilpotent homotopy types, the above p-localisation $\mathrm{L}_{\mathbb{S}_{(p)}}$ coincide with other constructions of p-localisations of homotopy types, see [BK; CP93; SulGT] for alternative constructions. However, these constructions become inequivalent for non-nilpotent homotopy types, see [MP, Remark 19.3.11]. In our later applications, we are only concerned with p-localisation of nilpotent homotopy types.

1.3.0.10. Definition. — A finite homotopy type is an object of $\mathcal{H}\mathrm{o}$ which is equivalent the homotopy type of a finite CW-complex, i.e. a CW-complex having only finitely many cells. Thus, we also often call a finite homotopy type a *finite complex*. For a fixed prime number p, a *finite p-local homotopy type (complex)*, or equivalently denoted as *p-local finite homotopy type (complex)*, is a p-local homotopy type that is equivalent to the p-localisation of a finite homotopy type (complex).

1.3.0.11. Remark. — A finite p-local homotopy type doesn't have to be a finite homotopy type, for example, consider the p-local spheres. However, let F be a nilpotent finite p-local homotopy type such that its singular homology with rational coefficient is trivial, then F is equivalent to a finite complex: The hypotheses on F imply that the integral singular homologies of F is bounded and are degree-wise finitely generated p-primary abelian groups.

Part I

Unstable Periodic Homotopy Theory

CHAPTER 2

The contraction of a homotopy type

2.1. The W-contraction

We introduce the theory of W-contraction of a homotopy type W (also known as the *W-nullification* or the *W-localisation*), following [Bou94] and [DroCS]. This section is expository, where we supplement [Bou94, §2] with more details of the proofs of certain theorems. In particular, we discuss the W-contraction for pointed and unpointed homotopy types in detail (see for example Proposition 2.1.1.15).

We begin with the basic definitions and properties in §2.1.1, and proof the existence of the W-contraction functor in §2.1.2 and discuss some closure properties in the last subsection §2.1.3.

2.1.1. W-less homotopy types and W-equivalences. —

2.1.1.1. Definition. — Let W be a homotopy type. We say a homotopy type X is *W-less* if the map $W \to \mathrm{pt}$ induces an equivalence

$$X \simeq \mathcal{M}\mathrm{ap}(\mathrm{pt}, X) \to \mathcal{M}\mathrm{ap}(W, X)$$

on the mapping spaces.

2.1.1.2. Example. — Let us first consider some trivial examples.

(i) If W is contractible, then every homotopy type is W-less.

(ii) If X is contractible, then X is W-less for every homotopy type W.

(iii) For a non-connected homotopy type W, then a homotopy type X is W-less if and only if X is contractible. A section of the map $W \to \pi_0(W)$ makes the discrete homotopy type $\pi_0 W$ a retract of W. Assuming X is W-less, we have that $\prod_{i \in \pi_0(W)} X$ is a retract of $X \simeq \mathcal{M}\mathrm{ap}(W, X)$, which holds if and only if X is contractible.

2.1.1.3. Proposition. — *Let W be a connected homotopy type. A homotopy type X is W-less if and only if each connected component of X is W-less.*

Proof. — Write X as the disjoint union $\sqcup_{\alpha\in\pi_0(X)} X_\alpha$ of its connected components. Denote the map $W \to \mathrm{pt}$ by t. Since W is connected, we obtain the following commutative diagram

$$\begin{array}{ccc} \mathcal{M}\mathrm{ap}(\mathrm{pt}, X) & \xrightarrow{\quad t^* \quad} & \mathcal{M}\mathrm{ap}(W, X) \\ \simeq\downarrow & & \downarrow\simeq \\ \coprod_{\alpha\in\pi_0(X)} \mathcal{M}\mathrm{ap}(\mathrm{pt}, X_\alpha) & \xrightarrow{\coprod_{\alpha\in\pi_0(X)} t^*} & \coprod_{\alpha\in\pi_0(X)} \mathcal{M}\mathrm{ap}(W, X_\alpha), \end{array}$$

where the lower horizontal map is the disjoint union of maps

$$t^* \colon \mathcal{M}\mathrm{ap}(\mathrm{pt}, X_\alpha) \to \mathcal{M}\mathrm{ap}(W, X_\alpha)$$

for every $\alpha \in \pi_0(X)$. Thus the upper horizontal arrow is an equivalence if and only if the lower horizontal arrow is an equivalence. In other words, X is W-less if and only if the connected component X_α is W-less for all $\alpha \in \pi_0(X)$. □

2.1.1.4. Definition. — Let W be a homotopy type and let $f \colon A \to B$ be a map of homotopy types. We say $f \in \mathcal{H}\mathrm{o}$ is an *W-equivalence* if for every W-less homotopy type Y the induced map $f^* \colon \mathcal{M}\mathrm{ap}(B, Y) \to \mathcal{M}\mathrm{ap}(A, Y)$ is an equivalence. In this case, we say A and B are *W-equivalent.*

2.1.1.5. Example. — Here are some trivial examples of W-equivalences.

(i) By Definition 2.1.1.1 the unique map $W \to \mathrm{pt}$ is a W-equivalence.

(ii) If W is a non-connected homotopy type, then every morphism of homotopy types is a W-equivalence, by Example 2.1.1.2.(iii).

2.1.1.6. Notation. — Let $f \colon A \to B$ be a map of homotopy types. Write A as the disjoint union $\coprod_{\alpha\in\pi_0(A)} A_\alpha$ of its connected components. For $\alpha \in \pi_0(A)$, denote the restriction of f to A_α by $f_\alpha \colon A_\alpha \to B_\alpha$, where B_α is the connected component of B such that $f(A_\alpha) \subseteq B_\alpha$.

2.1.1.7. Proposition. — *Let W be a connected pointed homotopy type. A morphism $f \colon A \to B$ of homotopy types is a W-equivalence if and only if the following two conditions hold:*

(i) *The induced map $f_* \colon \pi_0(A) \to \pi_0(B)$ is an isomorphism of the set of connected components.*

(ii) *For every $\alpha \in \pi_0(A)$ and for every W-less connected homotopy type Z, the map f_α induces an equivalence $\mathcal{M}\mathrm{ap}(B_\alpha, Z) \xrightarrow{\sim} \mathcal{M}\mathrm{ap}(A_\alpha, Z)$.*

Proof. — Assume f is a W-equivalence. Since W is connected, every set S with discrete topology is W-less. Thus the induced map $f^* \colon \mathcal{M}\mathrm{ap}(B, S) \to \mathcal{M}\mathrm{ap}(A, S)$ is an equivalence. Furthermore, note that we have $\mathcal{M}\mathrm{ap}(B, S) \simeq \mathrm{Hom}_{\mathbf{Set}}(\pi_0(B), S)$ and $\mathcal{M}\mathrm{ap}(A, S) \simeq \mathrm{Hom}_{\mathbf{Set}}(\pi_0(A), S)$ where the set $\mathrm{Hom}_{\mathbf{Set}}(-, -)$ is regarded as a discrete homotopy type. By Yoneda's lemma f induces an isomorphism $\pi_0(A) \cong \pi_0(B)$. Since Z is connected, we obtain the following commutative diagram

$$\begin{array}{ccc} \mathcal{M}\mathrm{ap}(B, Z) & \xrightarrow{f^*} & \mathcal{M}\mathrm{ap}(A, Z) \\ \simeq \downarrow & & \downarrow \simeq \\ \prod_{\alpha \in \pi_0(A)} \mathcal{M}\mathrm{ap}(B_\alpha, Z) & \xrightarrow[\prod_{\alpha \in \pi_0(X)} f^*_\alpha]{} & \prod_{\alpha \in \pi_0(A)} \mathcal{M}\mathrm{ap}(A_\alpha, Z), \end{array}$$

where we identified the set $\pi_0(B)$ with $\pi_0(A)$ under the isomorphism $\pi_0(A) \cong \pi_0(B)$. By assumption f^* is an equivalence. Thus, the map $\prod_{\alpha \in \pi_0(X)} f^*_\alpha$ is an equivalence.

Conversely, assume (i) and (ii) hold. Let Y be a W-less homotopy type, which we write as the disjoint union $\sqcup_{\upsilon \in \pi_0(Y)} Y_\upsilon$ of its connected components. By Proposition 2.1.1.3 each connected component Y_υ is W-less. Thus, for every $\upsilon \in \pi_0(Y)$ and for every $\alpha \in \pi_0(A)$ we obtain an induced equivalence

$$f^*_\alpha \colon \mathcal{M}\mathrm{ap}(B_\alpha, Y_\upsilon) \to \mathcal{M}\mathrm{ap}(A_\alpha, Y_\upsilon)$$

by assumption (ii). Therefore, the induced map $f^* \colon \mathcal{M}\mathrm{ap}(B, Y) \to \mathcal{M}\mathrm{ap}(A, Y)$, equivalent to the following induced map

$$\coprod_{\upsilon \in \pi_0(Y)} \prod_{\alpha \in \pi_0(A)} \mathcal{M}\mathrm{ap}(B_\alpha, Y_\upsilon) \to \coprod_{\upsilon \in \pi_0(Y)} \prod_{\alpha \in \pi_0(A)} \mathcal{M}\mathrm{ap}(A_\alpha, Y_\upsilon),$$

is an equivalence. In other words, the map f is a W-equivalence. □

2.1.1.8. Proposition. — *Let (W, w_0) and (X, x_0) pointed homotopy types. Then the (underlying homotopy type) X is W-less if and only if the pointed map $W_+ \to \mathrm{pt}_+$ induces an equivalence*

$$\mathcal{M}\mathrm{ap}_*(\mathrm{pt}_+, X) \xrightarrow{\sim} \mathcal{M}\mathrm{ap}_*(W_+, X)$$

of pointed mapping spaces.

Proof. — The adjunction in ¶1.3.0.2 gives the following commutative diagram

$$\begin{array}{ccc} \mathcal{M}\mathrm{ap}_*(\mathrm{pt}_+, X) & \longrightarrow & \mathcal{M}\mathrm{ap}_*(W_+, X) \\ \simeq \downarrow & & \downarrow \simeq \\ \mathcal{M}\mathrm{ap}(\mathrm{pt}, X) & \longrightarrow & \mathcal{M}\mathrm{ap}(W, X) \end{array}$$

in $\mathcal{H}\mathrm{o}$ (or $\mathcal{H}\mathrm{o}_*$). Thus, the upper horizontal arrow is an equivalence if and only if the lower horizontal arrow is an equivalence. □

2.1.1.9. Proposition. — *Let (W, w_0) and (X, x_0) be pointed homotopy types, and assume that X is connected. Then X is W-less if and only if the pointed mapping space $\mathcal{M}\mathrm{ap}_*(W, X) \in \mathcal{H}\mathrm{o}_*$ is contractible.*

Proof. — The mapping space $\mathcal{M}\mathrm{ap}(W, X)$ becomes a pointed homotopy type by choosing the basepoint to be the morphism $W \to \{x_0\} \hookrightarrow X$. There is a fibre sequence

$$\mathcal{M}\mathrm{ap}_*(W, X) \to \mathcal{M}\mathrm{ap}(W, X) \xrightarrow{\mathrm{ev}} X \tag{2.1.1.1}$$

in the ∞-category $\mathcal{H}\mathrm{o}_*$ of pointed homotopy types where ev is given by evaluating a map $W \to X$ at w_0. If X is W-less, we have $\mathcal{M}\mathrm{ap}(W, X) \simeq X$. Thus the homotopy fibre $\mathcal{M}\mathrm{ap}_*(W, X)$ is contractible. Note that in this implication we don't need to assume that X is connected.

If $\mathcal{M}\mathrm{ap}_*(W, X) \simeq \mathrm{pt}$, the mapping space $\mathcal{M}\mathrm{ap}(W, X)$ is equivalent to the connected component of X where the basepoint x_0 lies. Since X is connected, we have the equivalence $\mathcal{M}\mathrm{ap}(W, X) \simeq X$. □

2.1.1.10. Corollary. — *Let (W, w_0) be a connected pointed homotopy type. A homotopy type $X \in \mathcal{H}\mathrm{o}$ is W-less if and only if $\mathcal{M}\mathrm{ap}_*(W, X)$ is contractible for every choice of basepoint in X.*

Proof. — We can apply the proof of Proposition 2.1.1.9 to each connected component of X. Then the statement follows from Proposition 2.1.1.3. □

2.1.1.11. Proposition. — *Let (W, w_0) and (X, x_0) be pointed homotopy types and assume that X is connected. Then X is W-less if and only if the set $[\Sigma^k W, X]$ of pointed homotopy class of maps is the one-point set for every $k \geq 0$.*

Proof. — By Proposition 2.1.1.9 the homotopy type X is W-less if and only if $\mathcal{M}\mathrm{ap}_*(W, X)$ is contractible. The latter condition is equivalent to

$$\pi_k\left(\mathcal{M}\mathrm{ap}_*(W, X)\right) \cong [\Sigma^k W, X] = 0, \text{ for every } k \in \mathbb{N}. \qquad \square$$

2.1.1.12. Corollary. — *Let (W, w_0) be a pointed connected homotopy type. If a homotopy type X is W-less, then X is also ΣW-less.*

Proof. — By Corollary 2.1.1.10 the pointed mapping space $\mathcal{M}\mathrm{ap}_*(W, X)$ is contractible for every choice of basepoint in X. Thus we have

$$\mathcal{M}\mathrm{ap}_*(\Sigma W, X) \simeq \Omega \mathcal{M}\mathrm{ap}_*(W, X) \simeq \mathrm{pt}$$

for any choice of basepoint in X, which implies that X is ΣW-less. □

2.1.1.13. Example. — Let n be a natural number.

(i) Considering the n-dimensional sphere S^n. A pointed connected homotopy type X is S^n-less if and only if its homotopy groups $\pi_i(X)$ vanishes in every degrees $i \geq n$.

(ii) A pointed homotopy type X is n-connected if and only if $\mathrm{K}(G, i)$ is X-less for every abelian group G and for every i with $0 \leq i \leq n$ where $\mathrm{K}(G, 0)$ is the discrete homotopy type G. For every $k \in \mathbb{N}$, the set $[\Sigma^k X, \mathrm{K}(G, i)]$ is isomorphic, as an abelian group, to the reduced singular cohomology group $\widetilde{\mathrm{H}}^{i-k}(X;G)$ of X with coefficient in G. Then the statement follows from Proposition 2.1.1.11 and the Hurewicz Theorem.

2.1.1.14. Definition. — Let (W, w_0) be a pointed homotopy type. Given a morphism $f\colon (Y, y_0) \to (Z, z_0)$ in $\mathcal{H}o_*$, we say f is a *W-equivalence* (of pointed homotopy types) if for every pointed W-less homotopy type (X, x_0) the induced map

$$f^*\colon \mathcal{M}ap_*(Z, X) \to \mathcal{M}ap_*(Y, X)$$

is an equivalence of *pointed* mapping spaces.

2.1.1.15. Proposition. — *Let (W, w_0) be a pointed homotopy types. A morphism $f\colon (Y, x_0) \to (Z, z_0)$ of pointed homotopy types is a W-equivalence if and only if the induced morphism $\underline{f}\colon Y \to Z$ in $\mathcal{H}o$ of the underlying homotopy types is a W-equivalence.*

Proof. — If W is non-connected, the proposition follows from Example 2.1.1.5. In the following we consider the case where W is connected.

Let (X, x_0) be a pointed W-less homotopy type. Assume that $\underline{f}\colon Y \to Z$ is a W-equivalence. It induces an equivalence $(\underline{f})^*\colon \mathcal{M}ap(Z, X) \xrightarrow{\sim} \mathcal{M}ap(Y, X)$ of (unpointed) mapping spaces. By the proof of Proposition 2.1.1.9 we see that f^* is an equivalence.

Assume that f^* is an equivalence for every W-less pointed homotopy type (X, x_0), we show that $\underline{f}$ is a W-equivalence by the following claims.

Claim. The morphism f induces an isomorphism $\pi_0(Y) \xrightarrow{\simeq} \pi_0(Z)$ on the sets of connected components.

The proof of this claim works similarly as the proof of Proposition 2.1.1.7.(i); we use the fact that every pointed sets with discrete topology is W-less and apply the Yoneda's lemma on the category of pointed sets.

Claim. Let C be a connected W-less homotopy type. Then $\underline{f}$ induces an equivalence

$$(\underline{f})^*\colon \mathcal{M}ap(Z, C) \xrightarrow{\sim} \mathcal{M}ap(Y, C).$$

Choose a point $c_0 \in C$ and consider the W-less pointed connected homotopy type (C, c_0). Setting $(X, x_0) = (C, c_0)$, the claim follows from Proposition 2.1.1.9, since the homotopy type C is connected.

Claim. If Y and Z are connected, then $\underline{f}$ is a W-equivalence.

Let U be a W-less homotopy type. If U is connected, we see from the previous claim that $\underline{f}$ induces an equivalence of mapping spaces. Assume that U is not connected, and write U as the disjoint union $\sqcup_{\mu \in \pi_0(U)} U_\mu$ of its connected components. By the previous claim, for each connected component U_μ, the map $\underline{f}$ induces an equivalence

$$(\underline{f}_\mu)^*\colon \mathcal{M}ap(Z, U_\mu) \xrightarrow{\sim} \mathcal{M}ap(Y, U_\mu).$$

Furthermore, we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{M}\mathrm{ap}(Z,U) & \xrightarrow{\underline{f}^*} & \mathcal{M}\mathrm{ap}(Y,U) \\ \simeq\downarrow & & \downarrow\simeq \\ \sqcup_{\mu\in\pi_0(U)}\mathcal{M}\mathrm{ap}(Z,U_\mu) & \xrightarrow{\sqcup_{\mu\in\pi_0(U)}\underline{f}^*_\mu} & \sqcup_{\mu\in\pi_0(U)}\mathcal{M}\mathrm{ap}(Y,U_\mu) \end{array}$$

in $\mathcal{H}\mathrm{o}$. The vertical arrows are equivalences since Z and Y are connected. Thus, the morphism $\underline{f}^*$ is an equivalence since the other three arrows in the above diagram are. In other words $\underline{f}$ is a W-equivalence.

Claim. The morphism $\underline{f}$ is a W-equivalence.

By the previous claim it remains to check the case where Y and Z are not connected. We write $Y = Y_0 \sqcup Y'$ and $Z = Z_0 \sqcup Z'$ where Y_0 is the connected component of Y containing the basepoint y_0 and Z_0 is the connected component of Z containing the basepoint z_0. Note that we have $f(Y_0) \subseteq Z_0$ and $f(Y') \subseteq Z'$ by the first claim. Let $U = \sqcup_{\mu\in\pi_0(U)}U_\mu$ be a W-less homotopy type where U_μ's are its connected components. If U is connected, we see in the second claim that $\underline{f}$ induces an equivalence on mapping spaces. Otherwise, we have

$$\begin{aligned} \mathcal{M}\mathrm{ap}(Z,U) &\simeq \mathcal{M}\mathrm{ap}\left(Z_0 \sqcup Z', U\right) \\ &\simeq \mathcal{M}\mathrm{ap}\left(Z_0, U\right) \times \mathcal{M}\mathrm{ap}\left(Z', U\right) \\ &\simeq \mathcal{M}\mathrm{ap}\left(Z_0, \sqcup_{\mu\in\pi_0(U)}U_\mu\right) \times \mathcal{M}\mathrm{ap}\left(Z', U\right) \\ &\simeq \left(\sqcup_{\mu\in\pi_0(U)}\mathcal{M}\mathrm{ap}\left(Z_0, U_\mu\right)\right) \times \mathcal{M}\mathrm{ap}\left(Z', U\right) \\ &\simeq \sqcup_{\mu\in\pi_0(U)}\left(\mathcal{M}\mathrm{ap}\left(Z_0, U_\mu\right) \times \mathcal{M}\mathrm{ap}\left(Z', U\right)\right) \end{aligned} \tag{2.1.1.2}$$

Replacing Z by Y, we obtain

$$\mathcal{M}\mathrm{ap}(Y,U) \simeq \sqcup_{\mu\in\pi_0(U)}\left(\mathcal{M}\mathrm{ap}\left(Y_0, U_\mu\right) \times \mathcal{M}\mathrm{ap}\left(Y', U\right)\right).$$

In particular, the above equivalences are compatible with induced map $(\underline{f})^*$ on mapping spaces. Choose a basepoint $u_\mu \in U_\mu$ of U and consider the W-less pointed homotopy type (U, u_μ). We have

$$\begin{aligned} \mathcal{M}\mathrm{ap}_*(Z,U) &\simeq \mathcal{M}\mathrm{ap}_*(Z_0 \sqcup Z', U_\mu \sqcup (U' \setminus U_\mu)) \\ &\simeq \mathcal{M}\mathrm{ap}_*(Z_0, U_\mu) \times \mathcal{M}\mathrm{ap}\left(Z', U\right) \end{aligned}$$

and similar equivalences of the pointed mapping spaces if replace Z by Y. Since f induces an equivalence on the pointed mapping spaces mapping into (U, u_μ), we have an equivalence

$$\mathcal{M}\mathrm{ap}_*(Z_0, U_\mu) \times \mathcal{M}\mathrm{ap}\left(Z', U\right) \to \mathcal{M}\mathrm{ap}_*(Y_0) \times \mathcal{M}\mathrm{ap}(Y', U)$$

in $\mathcal{H}\mathrm{o}$, which induces an equivalence

$$\mathcal{M}\mathrm{ap}(Z_0, U_\mu) \times \mathcal{M}\mathrm{ap}\left(Z', U\right) \to \mathcal{M}\mathrm{ap}(Y_0, U_\mu) \times \mathcal{M}\mathrm{ap}(Y', U) \tag{2.1.1.3}$$

for every $\mu \in \pi_0(U)$, since Z_0, Y_0 and U_μ are connected (2.1.1.1). Taking the disjoint union over the indexing set $\pi_0(U)$ of the equivalences (2.1.1.3) and using (2.1.1.2) give the induced equivalence

$$(\underline{f})^* \colon \mathcal{M}\mathrm{ap}(Z, U) \simeq \mathcal{M}\mathrm{ap}(Y, U),$$

for a non-connected W-less homotopy type U. Combining with the second claim, we conclude that $\underline{f}$ is a W-equivalence. □

2.1.1.16. Remark. — The proof of Proposition 2.1.1.15 provides an explanation for [Bou94, §2.5.(a)–(d)].

2.1.1.17. Remark. — Let (W, w_0) be a pointed homotopy type. Thus, a morphism of pointed homotopy types is a W-equivalence if and only if the induced morphism of the underlying homotopy types is a W-equivalence, cf. Definitions 2.1.1.4 and 2.1.1.14.

2.1.2. W-contraction. — In this section we show that the localisation of $\mathcal{H}\mathrm{o}$ (respectively $\mathcal{H}\mathrm{o}_*$) at the set of W-equivalences exists and it is a reflective localisation.

2.1.2.1. Theorem. — *Let W be a homotopy type. There is a functor $\mathrm{P}_W \colon \mathcal{H}\mathrm{o} \to \mathcal{H}\mathrm{o}$ together with a natural transformation $\lambda_W \colon \mathrm{id}_{\mathcal{H}\mathrm{o}} \to \mathrm{P}_W$ such that for every homotopy type X*

(i) *the homotopy type $\mathrm{P}_W(X)$ is W-less,*
(ii) *the map $\lambda_W(X) \colon X \to \mathrm{P}_W(X)$ is a W-equivalence*
(iii) *the induced natural transformation $\mathrm{P}_W \to \mathrm{P}_W \circ \mathrm{P}_W$ is an equivalence in the ∞-category $\mathcal{F}\mathrm{un}(\mathcal{H}\mathrm{o}, \mathcal{H}\mathrm{o})$.*

Proof. — Let $\mathcal{H}\mathrm{o}_W$ denote the full ∞-subcategory of $\mathcal{H}\mathrm{o}$ whose objects are W-less homotopy types. Denote the morphism $W \to \mathrm{pt}$ by t. By Definition 2.1.1.1 being W-less is the same as being t-local (see Definition 1.1.2.8). By Theorem 1.1.2.17 the reflective localisation $L_W \colon \mathcal{H}\mathrm{o} \to \mathcal{H}\mathrm{o}_W$ at W-equivalences exists and it is the left adjoint to the fully faithful inclusion $I \colon \mathcal{H}\mathrm{o}_W \hookrightarrow \mathcal{H}\mathrm{o}$.

Define the functor $\mathrm{P}_W := I \circ L_W$. The natural transformation $\lambda_W \colon \mathrm{id} \to \mathrm{P}_W$ is the unit of the adjunction $L_W \dashv I$. Thus, for every homotopy types X, we have $\mathrm{P}_W(X)$ is W-less by definition. The other two properties follows from Proposition 1.1.2.6 and Corollary 1.1.2.7. See also [Bou94, Theorem 2.10]. □

2.1.2.2. Theorem. — *Let (W, w_0) be a pointed homotopy type. There exists a functor $\mathrm{P}_{W,w_0} \colon \mathcal{H}\mathrm{o}_* \to \mathcal{H}\mathrm{o}_*$ together with a natural transformation $\lambda_{W,w_0} \colon \mathrm{id}_{\mathcal{H}\mathrm{o}_*} \to \mathrm{P}_{W,w_0}$ such that for every pointed homotopy type (X, x_0)*

(i) *the pointed homotopy type $\mathrm{P}_{W,w_0}(X)$ is W-less,*
(ii) *the pointed map $\lambda_{W,w_0}(X) \colon X \to \mathrm{P}_W(X)$ a W-equivalence in $\mathcal{H}\mathrm{o}_*$, and*
(iii) *the induced natural transformation $\mathrm{P}_{W,w_0} \to \mathrm{P}_{W,w_0} \circ \mathrm{P}_{W,w_0}$ is an equivalence in $\mathcal{F}\mathrm{un}(\mathcal{H}\mathrm{o}_*, \mathcal{H}\mathrm{o}_*)$*

Proof. — Recall from Proposition 2.1.1.8 that a pointed homotopy type is W-less if and only if it is t_+-local for the morphism $t_+ \colon W_+ \to \mathrm{pt}_+$ in $\mathcal{H}\mathrm{o}_*$ induced by the morphism $t \colon W \to \mathrm{pt}$. The construction of the functor P_{W,w_0} is the same as that for P_W. In other words, the natural transformation $\mathrm{id} \to \mathrm{P}_{W,w_0}$ is the unit natural transformation of the adjunction associated with the reflective localisation of $\mathcal{H}\mathrm{o}_*$ at the set of W-equivalences in $\mathcal{H}\mathrm{o}_*$. The properties (i), (ii) and (iii) also follows by using the similar arguments as in the proof of Theorem 2.1.2.1. □

2.1.2.3. Convention. — Let (W, w_0) be a pointed homotopy type. In the ∞-category $\mathcal{H}\mathrm{o}_*$ of homotopy types, the notion of being W-less is defined on the underlying (unpointed) homotopy types, see Proposition 2.1.1.8. From Proposition 2.1.1.15 we see that a W-equivalence in $\mathcal{H}\mathrm{o}_*$ is the same as a W-equivalence of the underlying homotopy types. Thus, we abbreviate by abuse of notation the functor $\mathrm{P}_{(W,w_0)}$ by P_W. When we use the functor P_W, we will be explicit about whether we are working in the pointed or unpointed settings.

2.1.2.4. Definition. — Let (W, w_0) be a homotopy type. We call the functors P_W, both the pointed and the unpointed version, the *contraction of* W or W*-contraction*. For a (pointed) homotopy type X, we call the (pointed) homotopy type $\mathrm{P}_W(X)$ together with the morphism $\lambda_W(X) \colon X \to \mathrm{P}_W(X)$ the W*-contraction of* X.

***2.1.2.5.* Explanation.** — In the situation of Definition 2.1.2.4 one can regard the contraction of W as the universal functorial way to "quotient out" the W-information in a (pointed) homotopy type. We include both the pointed and the unpointed version of this construction because of the following reasons: On the one hand, the notion of being W-less appears more naturally in the unpointed setting, and on the other hand, it is a bit more convenient to work with pointed homotopy types in certain situations, e.g. when we consider homotopy groups and fibre sequences.

One might consider the reflective localisation of $\mathcal{H}\mathrm{o}_*$ at $(W \xrightarrow{t} \mathrm{pt})$-equivalences as a candidate for W-contraction for pointed homotopy types. However, this localisation does not behave well on pointed non-connected homotopy types. The following example illustrates the problem. Let (X, x_0) be a W-less connected pointed homotopy type. The pointed homotopy type $(X \sqcup W, x_0)$ is t-local, but not W-less (it contains a connected component which is W!), since $\mathrm{Map}_*(W, W)$ is contractible if and only if W is contractible.

In the literature [Bou94; DroCS] the functor P_W is often called "W-nullification" or "W-localisation". We prefer the name W-contraction because under the construction P_W the homotopy type W becomes contractible instead of becoming "null".

2.1.2.6. Proposition. — *Let W be a homotopy type. The homotopy type $\mathrm{P}_W(X)$ together with the morphism $X \to \mathrm{P}_W(X)$ enjoys the following universal properties:*

(i) *For every W-less homotopy types Y, there is an induced equivalence*

$$\mathcal{M}\mathrm{ap}(\mathrm{P}_W(X), Y) \to \mathcal{M}\mathrm{ap}(X, Y);$$

(ii) *For every W-equivalence $f\colon X \to Z$, there exists a morphism $f'\colon Z \to \mathrm{P}_W(X)$, unique up to contractible choice, such that $f' \circ f \simeq \lambda_W(X)$.*

Replacing W by a pointed homotopy type (W, w_0) and mapping space by pointed mapping spaces, the statements hold for the pointed map $\lambda_W\colon X \to \mathrm{P}_W(X)$.

Proof. — The statement follows from properties (i) and (ii) of Theorem 2.1.2.1 and Theorem 2.1.2.2. □

2.1.2.7. Example. — Let W be a homotopy type. Recall the example of W-less homotopy types from Example 2.1.1.2.

(i) If W is contractible, then $\mathrm{P}_W(X) \simeq X$, for every homotopy type X.
(ii) If W is non-connected, then $\mathrm{P}_W(X) \simeq \mathrm{pt}$, for every homotopy type X.
(iii) If X is W-less, then we have an $\mathrm{P}_W(X) \simeq X$ in $\mathcal{H}\mathrm{o}$.
(iv) Let X be a connected homotopy type and let $n \in \mathbb{N}$. The natural map $X \to \mathrm{P}_{\mathrm{S}^{n+1}}(X)$ is the n-th Postnikov truncation $\tau_{\leq n}$ for X.

2.1.2.8. Definition. — Let W be a homotopy type. We say a connected homotopy type X is *W-full* if the morphism $X \to \mathrm{pt}$ is a W-equivalence, i.e. $\mathrm{P}_W(X) \simeq \mathrm{pt}$. A homotopy type is *$W$-full* if each of its connected component is W-full.

Replacing W and X by pointed homotopy types gives the notion of W-full pointed homotopy types.

2.1.2.9. Example. —

(i) A (pointed) homotopy type X is S^{n+1}-full if and and only if X is n-connected.
(ii) A (pointed) homotopy type W is W-full.

Recall the terminologies of homological localisations of homotopy types from ¶1.3.0.9.

2.1.2.10. Proposition. — *Let W be a pointed homotopy type and let E be a spectrum. If W is $E_\bullet$-acyclic, then each $E_\bullet$-local homotopy type is W-less, and each W-equivalence in $\mathcal{H}\mathrm{o}_*$ is an $E_\bullet$-equivalence. In particular, there is a natural transformation $\mathrm{P}_W \to \mathrm{L}_E$ of localisation functors.*

Proof. — We can verify the proposition on pointed connected homotopy types and the general case can be done by considering each connected components separately. Let X be a pointed connected $E_\bullet$-local homotopy type. Then $\mathrm{Map}_*(W, X) \simeq \mathrm{pt}$, because W is E_*-acyclic. Thus, X is W-less. Let $f\colon Y \to Z$ be a W-equivalence in $\mathcal{H}\mathrm{o}_*$. Since X is W-less, the morphism f induces an equivalence $\mathrm{Map}_*(Z, X) \xrightarrow{\sim} \mathrm{Map}_*(Y, X)$, that is, f is an $E_\bullet$-equivalence. See also [Bou94, Proposition 5.6]. □

2.1.3. Closure properties of W-equivalences and of being W-less. — From the constructions (see Theorems 2.1.2.1 and 2.1.2.2) of the W-contraction we obtain the following straightforward closure properties.

2.1.3.1. Proposition. — *Let W be a pointed homotopy type.*

(i) *The set of W-equivalences of homotopy types is closed under small colimits in the ∞-category $\mathcal{F}\mathrm{un}(\Delta^1, \mathcal{H}\mathrm{o})$ of morphisms in $\mathcal{H}\mathrm{o}$.*

(ii) *The set of W-equivalences of pointed homotopy types is closed under small colimits in the ∞-category $\mathcal{F}\mathrm{un}(\Delta^1, \mathcal{H}\mathrm{o}_*)$ of morphisms in $\mathcal{H}\mathrm{o}_*$.*

(iii) *The set of W-less (pointed) homotopy types is closed under small limits in the ∞-category $\mathcal{H}\mathrm{o}$ (respectively $\mathcal{H}\mathrm{o}_*$) of (pointed) homotopy types.*

Proof. — The proposition follows from the fact that the notion of W-equivalence and of W-less are defined via mapping spaces $\mathcal{M}\mathrm{ap}(-,-)$ (respectively $\mathcal{M}\mathrm{ap}_*(-,-)$), which commutes with small colimits of the source and small limits of the target. See also [Bou94, §§2.5–2.6]. □

2.1.3.2. Convention. — We call the colimit of a diagram in $\mathcal{H}\mathrm{o}$ an *unpointed colimit*, and the colimit of a diagram in $\mathcal{H}\mathrm{o}_*$ a *pointed colimit*.

2.1.3.3. Corollary. — *In particular, we have the following examples:*

(i) *Let $K \to \mathcal{H}\mathrm{o}_*$ be a small diagram mapping every vertex of the simplicial set K to W. Then its colimit in $\mathcal{H}\mathrm{o}_*$ is W-full.*

(ii) *Let X be a homotopy type and $f : Y \to Z$ be a W-equivalence in $\mathcal{H}\mathrm{o}$. Then the induced morphism $\mathrm{id}_X \times f : X \times Y \to X \times Z$ is a W-equivalence in $\mathcal{H}\mathrm{o}$. This also holds for pointed homotopy types.*

Proof. — (i) follows from Example 2.1.2.9 and Proposition 2.1.3.1.(ii). The cartesian product $X \times Y$ is equivalent to the colimit in $\mathcal{H}\mathrm{o}$ of the constant diagram $X \to \mathcal{H}\mathrm{o}$ sending each point in X to Y, and the same holds for $X \times Z$. Thus, (ii) is an application of Proposition 2.1.3.1.(i). The underlying homotopy type of the cartesian product of pointed homotopy types is the cartesian product of the underlying homotopy types. So, (ii) for pointed homotopy types holds by Proposition 2.1.1.15. □

2.1.3.4. Remark. — Let $K \to \mathcal{H}\mathrm{o}$ be a diagram in $\mathcal{H}\mathrm{o}$ mapping every vertex of the simplicial set K to W. An (unpointed) colimit of K in $\mathcal{H}\mathrm{o}$ doesn't have to be W-full. See for example [DroCS, Chapter 2, §D.3] for more explanation of the differences between unpointed and pointed colimits.

2.1.3.5. Proposition. — *Let W be a homotopy type and let $(X_1, X_2, \cdots, X_n)$ be a* finite *sequence of homotopy types. There exists an equivalence*

$$\mathrm{P}_W(X_1 \times X_2 \times \cdots X_n) \xrightarrow{\sim} \mathrm{P}_W(X_1) \times \mathrm{P}_W(X_2) \times \cdots \times \mathrm{P}_W(X_n).$$

The same statement holds for pointed homotopy types.

Proof. — By Proposition 2.1.3.1.(iii) the product $\mathrm{P}_W(X_1) \times \mathrm{P}_W(X_2) \times \cdots \times \mathrm{P}_W(X_n)$ is W-less. Thus, we have the following commutative diagram

$$\begin{array}{ccc} X_1 \times X_2 \times \cdots X_n & \xrightarrow{\prod_{i=1}^n \lambda_W(X_i)} & \mathrm{P}_W(X_1) \times \mathrm{P}_W(X_2) \times \cdots \times \mathrm{P}_W(X_n) \\ & {\scriptstyle \lambda_W} \searrow \quad\quad \nearrow {\scriptstyle (\text{dashed})} & \\ & \mathrm{P}_W(X_1 \times X_2 \times \cdots X_n) & \end{array}$$

from the universal property of W-contraction Proposition 2.1.2.6. The upper horizontal map is also a W-equivalence by Corollary 2.1.3.3.(ii): For example, take $n = 2$, the map is given by

$$X_1 \times X_2 \xrightarrow{\lambda(X_1)\times \mathrm{id}_{X_2}} \mathrm{P}_W(X_1) \times X_2 \xrightarrow{\mathrm{id}_{\mathrm{P}_W(X_1)} \times \lambda(X_2)} \mathrm{P}_W(X_1) \times \mathrm{P}_W(X_2).$$

Thus the dashed arrow is an equivalence in $\mathcal{H}\mathrm{o}$ by Proposition 2.1.2.6. The proof for pointed version works the same. See also [Bou94, Proposition 2.7]. □

2.1.3.6. Proposition. — *Let W be a pointed homotopy type and let $F \to X \to B$ be a fibre sequence in $\mathcal{H}\mathrm{o}_*$ where B is connected. If $\mathrm{P}_W F \simeq \mathrm{pt}$, then $X \to B$ is a W-equivalence.*

Proof. — For a point $b \in B$, denote the fibre of $X \to B$ at b by F_b. Recall from ¶1.3.0.6.(i) that the morphism $X \to B$ is equivalent to the following map

$$X \simeq \varinjlim_{b\in B} F_b \to \varinjlim_{B} \mathrm{pt} \simeq B,$$

since B is connected. Then the proposition follows from Proposition 2.1.3.1.(i). □

2.2. Commutations with loop spaces and fibration theorems

For a pointed homotopy type W the W-contraction functor does not always preserve small limits. In this section we summarise the interactions of the functor P_W with loop spaces (see §2.2.1) and fibre sequences (see §2.2.2), following [Bou94; Har18]. Here we rewrite some of the proofs in [Bou94, §4] using more modern languages and techniques; in particular Theorems 2.2.1.2 and 2.2.2.3. In the last subsection §2.2.3 we discuss some useful properties of the infinite symmetric product and $\Omega^\infty\Sigma^\infty X$ of a pointed homotopy type X (they are both infinite loop spaces), following [Har18, Lecture 3] and [Bou94, §6].

Results stated in this section are important technical tools for the later sections, where some of the proofs become rather technical. Therefore, the reader may also chose to skip this section first and come back later for specific theorems. We work in the ∞-category $\mathcal{H}\mathrm{o}_*$ of pointed homotopy types.

2.2.1. Interactions of the W-contraction with loop spaces. —

2.2.1.1. Proposition. — *Let W and X be pointed connected homotopy types. Then X is ΣW-less if and only if ΩX is W-less.*

Proof. — The homotopy type ΩX is W-less if and only if the induced map

$$\mathcal{M}\mathrm{ap}_*\left(\mathrm{pt}_+, \Omega X\right) \to \mathcal{M}\mathrm{ap}_*\left(W_+, \Omega X\right) \tag{2.2.1.1}$$

is an equivalence. By the $\Sigma \dashv \Omega$-adjunction, the map (2.2.1.1) is equivalent to the morphism

$$\mathcal{M}\mathrm{ap}_*\left(\Sigma(\mathrm{pt}_+), X\right) \to \mathcal{M}\mathrm{ap}_*\left(\Sigma(W_+), X\right). \tag{2.2.1.2}$$

Since $\Sigma(Y_+) \simeq \Sigma Y \vee \mathrm{S}^1$ for $Y \in \mathcal{H}\mathrm{o}_*$, the previous morphism (2.2.1.2) is equivalent to the one below

$$i \times \mathrm{id}_{\Omega X} \colon \mathrm{pt} \times \Omega X \to \mathcal{M}\mathrm{ap}_*\left(\Sigma W, X\right) \times \Omega X, \tag{2.2.1.3}$$

where i denotes the unique pointed map $\mathrm{pt} \to \mathcal{M}\mathrm{ap}_*(W, X)$. Finally, the homotopy type X is ΣW-less if and only if the last morphism (2.2.1.3) is an equivalence. □

2.2.1.2. Theorem. — *Let W be a connected pointed homotopy type. For a pointed connected homotopy type X, there exists a commutative diagram*

$$\begin{array}{ccc} \Omega X & \xrightarrow{\lambda_W(\Omega X)} & \mathrm{P}_W(\Omega X) \\ {\scriptstyle \Omega(\lambda_{\Sigma W}(X))}\downarrow & \nearrow{\scriptstyle \Omega f}\ \simeq & \\ \Omega(\mathrm{P}_{\Sigma W}(X)) & & \end{array}$$

in the ∞-category $\mathcal{A}\mathrm{lg}^{\mathrm{gpl}}_{\mathcal{E}_1}(\mathcal{H}\mathrm{o}_*)$ of grouplike $\mathcal{E}_1$-algebras in $\mathcal{H}\mathrm{o}_*$.

Proof. — We follow the proof idea of [Bou94, Theorem 3.1].

Claim. The homotopy type $\mathrm{P}_W(\Omega X)$ is a grouplike $\mathcal{E}_1$-algebra in the cartesian monoidal ∞-category $\mathcal{H}o_*$ of pointed homotopy types.

This is a corollary of Proposition 2.1.3.5. In particular, the localisation morphism $\lambda_W(\Omega X)$ is also a morphism of grouplike $\mathcal{E}_1$-algebras, since P_W preserves finite products.

The statement of the theorem follows by applying the loop functor Ω to the commutative diagram (2.2.1.4) in the following claim.

Claim. There is a commutative diagram

$$\begin{array}{ccc} X \simeq \mathrm{B}(\Omega X) & \xrightarrow{\mathrm{B}(\lambda_W(\Omega X))} & \mathrm{B}(\mathrm{P}_W(\Omega X)) \\ {\scriptstyle \lambda_{\Sigma W}(X)}\downarrow & \nearrow_{f}^{\simeq} & \\ \mathrm{P}_{\Sigma W}(X) & & \end{array} \tag{2.2.1.4}$$

in the ∞-category $\mathcal{H}o_*^{\geq 1}$ of connected homotopy types where B denotes the Bar construction(1) of augmented $\mathcal{E}_1$-algebras in the symmetric monoidal ∞-category $\mathcal{H}o_*$ with cartesian monoidal structure.

Applying the functor

$$\mathrm{B}\colon \mathcal{A}lg^{\mathrm{gpl}}_{\mathcal{E}_1}(\mathcal{H}o_*) \to \mathcal{H}o_*^{\geq 1}$$

to $\lambda_W(\Omega X)$ gives the horizontal arrow in the above commutative diagram. For the existence of the equivalence f, we need to show that $\mathrm{B}(\mathrm{P}_W(\Omega X))$ is ΣW-less and the morphism $\mathrm{B}\left(\lambda_W(\Omega X)\right)$ is a ΣW-equivalence. Note that we have equivalences

$$\mathcal{M}\mathrm{ap}_*(\Sigma W, \mathrm{B}(\mathrm{P}_W(\Omega X))) \simeq \mathcal{M}\mathrm{ap}_*(W, \Omega\mathrm{B}(\mathrm{P}_W(\Omega X))) \simeq \mathcal{M}\mathrm{ap}_*(W, \mathrm{P}_W(\Omega X)) \simeq \mathrm{pt}.$$

In other words, $\mathrm{B}(\mathrm{P}_W(\Omega X))$ is ΣW-less. Let Y be a connected ΣW-less homotopy type. The morphisms $\mathrm{B}\left(\lambda_W(\Omega X)\right)$ and $\lambda_W(\Omega X)$ induce the horizontal arrows in the commutative diagram

$$\begin{array}{ccc} \mathcal{M}\mathrm{ap}_*\left(\mathrm{B}(\mathrm{P}_W(\Omega X)), Y\right) & \longrightarrow & \mathcal{M}\mathrm{ap}_*\left(\mathrm{B}(\Omega X), Y\right) \\ \simeq\downarrow & & \downarrow\simeq \\ \mathcal{M}\mathrm{ap}_*\left(\mathrm{P}_W(\Omega X), \Omega Y\right) & \xrightarrow{\sim} & \mathcal{M}\mathrm{ap}_*\left(\Omega X, \Omega Y\right) \end{array}$$

which implies that the upper horizontal is a equivalence. In other words, it shows that the morphism $\mathrm{B}\left(\lambda_W(\Omega X)\right)$ is a W-equivalence. Note that it suffices to verify this on connected homotopy types Y, since the source and target of $\mathrm{B}\left(\lambda_W(\Omega X)\right)$ are connected. □

(1) In this special case it is also known as the classifying space functor

2.2.1.3. Corollary. — *Let W be a connected pointed homotopy type. For $X \in \mathcal{H}\mathrm{o}_*$ and for every $n \in \mathbb{N}$ the iterated loop map $\Omega^n(\lambda_{\Sigma^n W}(X)) : \Omega^n X \to \Omega^n(\mathrm{P}_{\Sigma^n W}(X))$ induces an equivalence*

$$\mathrm{P}_W(\Omega^n X) \xrightarrow{\sim} \Omega^n(\mathrm{P}_{\Sigma^n W}(X))$$

of pointed homotopy types. □

2.2.1.4. ***W*-contractions of infinite loop spaces.** — One can generalise the statement of Corollary 2.2.1.3 to infinite loop spaces. We sketch the argument now and refer the readers to [Bou96, §2] and [CG05] for more details.

Let W be a pointed connected homotopy type. A spectrum $E \in \mathcal{S}\mathrm{p}$ is $\Sigma^\infty W$-less if the *mapping space* $\mathcal{M}\mathrm{ap}(\Sigma^\infty W, E)$ is contractible. Let $\mathcal{S}\mathrm{p}_{\Sigma^\infty W}$ denotes the full ∞-subcategory of $\mathcal{S}\mathrm{p}$ of $\Sigma^\infty W$-less spectra. Similar as for homotopy types, one can define the reflective localisation functor $\mathrm{L}_{\Sigma^\infty W} \colon \mathcal{S}\mathrm{p} \to \mathcal{S}\mathrm{p}_{\Sigma^\infty W}$. Define the functor

$$\mathrm{P}_{\Sigma^\infty W} \colon \mathcal{S}\mathrm{p} \to \mathcal{S}\mathrm{p}_{\Sigma^\infty W} \hookrightarrow \mathcal{S}\mathrm{p}\,.$$

Using similar proof strategy as for Theorem 2.2.1.2, there exists an equivalence

$$\mathrm{P}_W\,(\Omega^\infty E) \simeq \Omega^\infty(\mathrm{P}_{\Sigma^\infty W}(E))$$

of pointed homotopy types for every spectrum E. Let R be a commutative ring spectrum. For a R-module spectrum M, the localisation $\mathrm{P}_{\Sigma^\infty W}(M)$ admits the structure of a R-module spectrum such that the natural map $M \to \mathrm{P}_{\Sigma^\infty W}(M)$ is a morphism of R-module spectra.

Let G be an abelian group and consider the Eilenberg–MacLane spectrum $\mathrm{H}G$ (characterised by the property that $\pi_\bullet^{\mathrm{st}}(\mathrm{H}G) \cong G$). For every $z \in \mathbb{Z}$, there exist abelian groups G_1 and G_2 such that

$$\mathrm{P}_{\Sigma^\infty W}\,(\Sigma^z(\mathrm{H}G)) \simeq \Sigma^z \mathrm{H}G_1 \wedge \Sigma^{z+1}\mathrm{H}G_2$$

In the special cases such as G is free or G is finitely generated, we have $G_2 = 0$. These statements are mentioned in [Bou96] and proven in [CG05, §5].

2.2.1.5. Corollary. — *Let W be a pointed connected homotopy type.*

(i) *The W-contraction of a product of Eilenberg–MacLane spaces is equivalent to a product of Eilenberg–MacLane spaces.*

(ii) *The W-contraction of a Eilenberg–MacLane space $\mathrm{K}(G, n)$ has trivial homotopy groups in all degrees other than n or $n+1$.*

Proof. — See [CG05, Proposition 5.6]. □

2.2.2. Fibration theorems. — In this subsection we work in the following situation.

2.2.2.1. Situation. — Let $F \to X \xrightarrow{f} B$ be a fibre sequence of pointed homotopy types where B is connected. Let W be a pointed connected homotopy type.

2.2.2.2. Proposition. —
(i) *If F and B are W-less, then X is W-less.*
(ii) *If F is W-less and X is ΣW-less, then B is ΣW-less.*
(iii) *If X is W-less and B is ΣW-less, then F is W-less.*

Proof. — For (i), consider the commutative diagram

$$\begin{array}{ccccc} \mathcal{M}\mathrm{ap}_*(\mathrm{pt}_+, F) & \longrightarrow & \mathcal{M}\mathrm{ap}_*(\mathrm{pt}_+, X) & \longrightarrow & \mathcal{M}\mathrm{ap}_*(\mathrm{pt}_+, B) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{M}\mathrm{ap}_*(W_+, F) & \longrightarrow & \mathcal{M}\mathrm{ap}_*(W_+, X) & \longrightarrow & \mathcal{M}\mathrm{ap}_*(W_+, B), \end{array} \tag{2.2.2.1}$$

induced by the fibre sequence $F \to X \to B$ and the pointed map $W_+ \to \mathrm{pt}_+$. The horizontal rows are induced fibre sequences. If F and B are W-less, then the first and the third vertical arrows are equivalences. Then the middle vertical arrow is also an equivalence, that is, X is W-less.

As for (ii), replace W_+ in the above diagram (2.2.2.1) by $(\Sigma W)_+$. Then the left and the middle vertical arrows are equivalences by assumptions and Corollary 2.1.1.12. Thus, the right arrow is also an equivalence, since B is connected. In other words, the homotopy type B is ΣW-less.

For (iii), consider the (ΩB)-principal ∞-bundle $\Omega B \to F \to X$ (see ¶1.3.0.6.(ii)). Since B is connected, the fibre over any point in X is equivalent to ΩB, see [NSS15, Lemma 3.9].[(2)] By our assumption and Proposition 2.2.1.1, the fibre ΩB is W-less. Therefore, we can apply (i) and Corollary 2.1.1.10 to obtain (iii). See also [Bou94, Proposition 4.2]. □

2.2.2.3. Theorem. — *There exists a pointed homotopy type $\overline{X} \in \mathcal{H}\mathrm{o}_*$ together with a W-equivalence $\overline{L}\colon X \to \overline{X}$ in $\mathcal{H}\mathrm{o}_*$ satisfying the following properties:*
(i) *The homotopy type $\overline{X}$ is ΣW-less,*
(ii) *There exists a fibre sequence $\mathrm{P}_W(F) \to \overline{X} \to \mathrm{P}_{\Sigma W}(B)$*
(iii) *The fibre sequence from (ii) fits in the commutative diagram*

$$\begin{array}{ccccc} F & \longrightarrow & X & \longrightarrow & B \\ \downarrow & & \downarrow{\scriptstyle \overline{L}} & & \downarrow \\ \mathrm{P}_W(F) & \longrightarrow & \overline{X} & \longrightarrow & \mathrm{P}_{\Sigma W}(B) \end{array}$$

in the ∞-category $\mathcal{H}\mathrm{o}_*$.

Proof. — Recall that we can consider the map $F \to X$ as a (ΩB)-principal ∞-bundle and the fibre sequence $F \to X \to B$ is equivalent to a geometric realisation of trivial

(2) This lemma is updated recently in the arxiv preprint of the paper [NSS15].

fibrations, see ¶1.3.0.6 Applying the functor P_W to each of the trivial fibre sequences, we obtain the following commutative diagram

$$\begin{array}{ccccccc}
\cdots \substack{\longrightarrow\\[-4pt]\longrightarrow\\[-4pt]\longrightarrow\\[-4pt]\longrightarrow} & \mathrm{P}_W(F) & \substack{\longrightarrow\\[-4pt]\longrightarrow\\[-4pt]\longrightarrow} & \mathrm{P}_W(F) & \substack{\longrightarrow\\[-4pt]\longrightarrow} & \mathrm{P}_W(F) \\
& \downarrow & & \downarrow & & \downarrow \\
\cdots \substack{\longrightarrow\\[-4pt]\longrightarrow\\[-4pt]\longrightarrow\\[-4pt]\longrightarrow} & \mathrm{P}_W(F \times \Omega B \times \Omega B) & \substack{\longrightarrow\\[-4pt]\longrightarrow\\[-4pt]\longrightarrow} & \mathrm{P}_W(F \times \Omega B) & \substack{\longrightarrow\\[-4pt]\longrightarrow} & \mathrm{P}_W(F) \\
& \downarrow & & \downarrow & & \downarrow \\
\cdots \substack{\longrightarrow\\[-4pt]\longrightarrow\\[-4pt]\longrightarrow\\[-4pt]\longrightarrow} & \mathrm{P}_W\,(\Omega B \times \Omega B) & \substack{\longrightarrow\\[-4pt]\longrightarrow\\[-4pt]\longrightarrow} & \mathrm{P}_W\,(\Omega B) & \substack{\longrightarrow\\[-4pt]\longrightarrow} & \mathrm{P}_W(\mathrm{pt}) \simeq \mathrm{pt}
\end{array} \tag{2.2.2.2}$$

in $\mathcal{H}\mathrm{o}_*$ induced by (1.3.0.1). Each column is a trivial fibre sequence since P_W preserves finite products (see Proposition 2.1.3.5).

Denote the geometric realisation of the simplicial object in the second row of the diagram (2.2.2.2) by $\overline{X}$, and note that the geometric realisation of the third row in the same diagram is equivalent to $\mathrm{P}_{\Sigma W}(B)$. The induced diagram of colimits

$$\begin{array}{ccccccc}
\cdots \substack{\longrightarrow\\[-4pt]\longrightarrow\\[-4pt]\longrightarrow} & \mathrm{P}_W(F \times \Omega B) & \substack{\longrightarrow\\[-4pt]\longrightarrow} & \mathrm{P}_W(F) & \longrightarrow & \overline{X} \\
\downarrow & \downarrow & & \downarrow & & \downarrow \\
\cdots \substack{\longrightarrow\\[-4pt]\longrightarrow\\[-4pt]\longrightarrow} & \mathrm{P}_W\,(\Omega B) & \substack{\longrightarrow\\[-4pt]\longrightarrow} & \mathrm{pt} & \longrightarrow & \mathrm{P}_{\Sigma W}(B)
\end{array} \tag{2.2.2.3}$$

exhibits $\mathrm{P}_W(F) \to \overline{X}$ as a $\Omega \mathrm{P}_{\Sigma W}(B) \simeq \mathrm{P}_W(\Omega B)$-principal ∞-bundle, by [NSS15, Definitions 3.1 and 3.4]. Moreover, the right most square is a pullback diagram in $\mathcal{H}\mathrm{o}_*$ by [NSS15, Proposition 3.13], and thus each square and each composite rectangle in the diagram (2.2.2.3) is a pullback in $\mathcal{H}\mathrm{o}_*$ by [NSS15, Proposition 2.3]. Therefore, we can take the take the fibre of each vertical arrow in (2.2.2.3) and obtain the following induced diagram

$$\begin{array}{ccccccc}
\cdots \substack{\longrightarrow\\[-4pt]\longrightarrow\\[-4pt]\longrightarrow} & \mathrm{P}_W(F) & \substack{\longrightarrow\\[-4pt]\longrightarrow} & \mathrm{P}_W(F) & \longrightarrow & \mathrm{P}_W(F) \\
\downarrow & \downarrow & & \downarrow & & \downarrow \\
\cdots \substack{\longrightarrow\\[-4pt]\longrightarrow\\[-4pt]\longrightarrow} & \mathrm{P}_W(F \times \Omega B) & \substack{\longrightarrow\\[-4pt]\longrightarrow} & \mathrm{P}_W(F) & \longrightarrow & \overline{X} \\
\downarrow & \downarrow & & \downarrow & & \downarrow \\
\cdots \substack{\longrightarrow\\[-4pt]\longrightarrow\\[-4pt]\longrightarrow} & \mathrm{P}_W\,(\Omega B) & \substack{\longrightarrow\\[-4pt]\longrightarrow} & \mathrm{P}_W(\mathrm{pt}) \simeq \mathrm{pt} & \longrightarrow & \mathrm{P}_{\Sigma W}(B),
\end{array}$$

where the first row is equivalent to the colimit diagram of the first row in (2.2.2.2). In the induced fibre sequence

$$\mathrm{P}_W(F) \to \overline{X} \to \mathrm{P}_{\Sigma W}(B),$$

of pointed homotopy types $\overline{X}$ is ΣW-less by Proposition 2.2.2.2, since both $\mathrm{P}_W(F)$ and $\mathrm{P}_{\Sigma W}(B)$ are. The natural transformation $\lambda_W \colon \mathrm{id} \to \mathrm{P}_W$ induces a map from the diagram (1.3.0.1) to the diagram (2.2.2.2) Therefore, we obtain the commutative diagram in (iii) by the universal property of colimits. See also [Bou94, Theorem 4.1]. □

2.2.2.4. Remark. — One can generalise the above theorem where the W-contraction is replaced by the localisation with respect to a morphism of pointed homotopy types [Bou97, Theorem 6.5]. Another variant of the theorem is [Bou97, Theorem 6.1] where the map $\overline{L}$ is constructed for the base space of a fibre sequence.

There are many works by Dror Farjoun and his coauthors on the interaction of fibrations with various homotopical localisations of homotopy types, see for example [DroCS; BD03; DS95; DD09].

2.2.2.5. Theorem. — *If* $\mathrm{P}_{\Sigma W}(B) \simeq \mathrm{P}_W(B)$, *then* $\mathrm{P}_W(F) \to \mathrm{P}_W(X) \to \mathrm{P}_W(B)$ *is a fibre sequence.*

Proof. — By Theorem 2.2.2.3 we obtain a commutative diagram

$$\begin{array}{ccccc} F & \longrightarrow & X & \longrightarrow & B \\ \downarrow & & \downarrow{\scriptstyle \overline{L}} & & \downarrow \\ \mathrm{P}_W(F) & \longrightarrow & \overline{X} & \longrightarrow & \mathrm{P}_W(B). \end{array}$$

in $\mathcal{H}\mathrm{o}_*$ where the rows are fibre sequences. If the base and the fibre of a fibre sequence is W-less, then the total space is W-less, by considering the induced fibre sequence on mapping spaces. Thus $\overline{X}$ is W-less. Since the map $\overline{L}$ is a W-equivalence whose target is W-less, it follows from Proposition 2.1.2.6 that $\overline{X} \simeq \mathrm{P}_W(X)$. Under the former identification the morphism $\overline{\mathrm{L}}$ is equivalent to $\lambda_W(X)\colon X \to \mathrm{P}_W X$. See also [Bou94, Theorem 4.3]. □

2.2.2.6. Proposition. — *Assume that W is n-connected for a natural number n. For a pointed homotopy type X, its W-contraction $\lambda_W(X)\colon X \to \mathrm{P}_W(X)$ induces an isomorphism on the homotopy groups in all degree less than $n+1$ and induces a surjection on the homotopy groups in degree $n+1$.*

Proof. — Since $\lambda_W(X)\colon X \to \mathrm{P}_W(X)$ is a W-equivalence, it induces an isomorphism on connected components, as we have shown in the second claim of the proof of Proposition 2.1.1.15.

Assume that X is connected; otherwise we treat each connected component separately. Let F be the fibre of $\lambda_W(X)$. According to Corollary 2.1.1.12 we know that $\mathrm{P}_W(\mathrm{P}_W(X)) \simeq \mathrm{P}_{\Sigma W}(\mathrm{P}_W(X))$. Thus, applying P_W we obtain a fibre sequence

$$\mathrm{P}_W(F) \to \mathrm{P}_W(X) \xrightarrow{\sim} \mathrm{P}_W(\mathrm{P}_W(X))$$

by Theorem 2.2.2.5. Thus, $\mathrm{P}_W(F) \simeq \mathrm{pt}$. Since W is n-connected, W is S^{n+1}-full. So, F is also S^{n+1}-full, which implies that F is also n-connected. The proposition follows by considering the long exact sequence of homotopy groups induced by the fibre sequence $F \to X \to \mathrm{P}_W(X)$. See also [Bou94, Proposition 2.9]. □

2.2.2.7. Corollary. — *Assume that W is n-connected for a natural number n. For a connected pointed homotopy type X, there is a natural equivalence*

$$\mathrm{P}_W\left(\tau_{>n}(X)\right) \xrightarrow{\sim} \tau_{>n}\left(\mathrm{P}_W X\right)$$

induced by the fibre sequence $\tau_{>n}(X) \to X \to \tau_{\leq n}(X)$.

Proof. — This is [Bou94, Corollary 4.4]. By connectivity we have an equivalence

$$\mathrm{P}_{\Sigma W}\left(\tau_{\leq n} Y\right) \simeq \mathrm{P}_W\left(\tau_{\leq n} Y\right)$$

Thus, we can apply Theorem 2.2.2.5. Note that the homotopy type $\mathrm{P}_W\left(\tau_{>n} X\right)$ is n-connected by Proposition 2.2.2.6. ☐

2.2.2.8. Proposition. — *Let X be a pointed connected homotopy type whose homotopy groups are non-trivial in finitely many degrees. The homotopy groups of the homotopy type $\mathrm{P}_{\Sigma W}(X)$ are non-trivial homotopy groups in finitely many degrees.*

Proof. — We prove the proposition inductively. First, we show that the classifying space $\mathrm{B}G$ for a discrete group G satisfies the proposition. Recall the canonical equivalence $\Omega \mathrm{B}G \simeq G$. Thus by the second claim in the proof of Theorem 2.2.1.2, we obtain

$$\Omega \mathrm{P}_{\Sigma W}(\mathrm{B}G) \simeq \mathrm{P}_W(\Omega \mathrm{B}G) \simeq \mathrm{P}_W(G) \simeq G$$

So we have

$$\mathrm{P}_{\Sigma W}(\mathrm{B}G) \simeq \mathrm{B}G.$$

Assume that the statement is true for all pointed connected homotopy types Y such that $\pi_i(Y) = 0$ for $i \geq n$ for a natural number $n \geq 2$. Let Z be pointed connected homotopy type such that $\pi_j(Z) = 0$ for $j \geq n+1$ and $\pi_n(Z) \neq 0$. We can apply Theorem 2.2.2.3 to the fibre sequence

$$\mathrm{K}(\pi_n(Z), n) \to Z \to \tau_{<n}(Z),$$

which produces a fibre sequence

$$\mathrm{P}_{\Sigma W}\left(\mathrm{K}(\pi_n(Z), n)\right) \to \overline{Z} \to \mathrm{P}_{\Sigma^2 W}\left(\tau_{<n}(Z)\right)$$

such that the first and the last terms has non-trivial homotopy groups in finitely many degrees by Corollary 2.2.1.5 and our inductive assumption. Thus, $\overline{Z} \simeq \mathrm{P}_{\Sigma W}(Z)$ also has non-trivial homotopy groups in finitely many degrees, by considering the long exact sequence of homotopy groups associated with fibre sequences. ☐

2.2.2.9. Remark. — In the situation of Proposition 2.2.2.8, if X is simply connected, or the fundamental group of X is abelian, then the proposition holds when we replace ΣW by W. The proof of the induction start is given by Corollary 2.2.1.5 and the rest of the proof still works if we replace W by ΣW.

2.2.3. Properties of $\mathrm{SP}^\infty(X)$ and $\Omega^\infty\Sigma^\infty X$. — Let X be a pointed connected homotopy type. We discuss several properties of the *infinite symmetric product* $\mathrm{SP}^\infty(X)$ and the underlying infinite loop space $\Omega^\infty\Sigma^\infty X$ of the suspension spectrum $\Sigma^\infty X$ of X, applying results from the previous two subsections.

2.2.3.1. Proposition. — *Let X be a pointed connected homotopy type. Then the homotopy type $\Omega^\infty\Sigma^\infty(X)$ is X-full.*

Proof. — By our observation ¶2.2.1.4 we have

$$\mathrm{P}_X\left(\Omega^\infty\Sigma^\infty X\right) \simeq \Omega^\infty \mathrm{P}_{\Sigma^\infty X}(\Sigma^\infty X) \simeq \mathrm{pt}.$$

We give another proof here, since we didn't prove the first equivalence in the above. Since X is connected, the homotopy type $\Omega^\infty\Sigma^\infty(X)$ is connected. It suffices to show that for a connected X-less homotopy type Y, $\mathcal{M}\mathrm{ap}_*(\Omega^\infty\Sigma^\infty(X), Y)$ is contractible. By the equivalence $\Omega^\infty\Sigma^\infty(X) \simeq \varinjlim_{n\geq 0} \Omega^n\Sigma^n X$ of pointed homotopy types we obtain

$$\begin{aligned}\mathcal{M}\mathrm{ap}_*(\Omega^\infty\Sigma^\infty(X), Y) &\simeq \mathcal{M}\mathrm{ap}_*\left(\varinjlim_{n\geq 0} \Omega^n\Sigma^n X, Y\right)\\ &\simeq \varprojlim_{n\geq 0} \mathcal{M}\mathrm{ap}_*\left(\Omega^n\Sigma^n X, Y\right).\end{aligned}$$

For every $n \in \mathbb{N}$, Corollary 2.2.1.3 gives an equivalence

$$\mathrm{P}_X\left(\Omega^n\Sigma^n X\right) \simeq \Omega^n(\mathrm{P}_{\Sigma^n X}(\Sigma^n X)) \simeq \mathrm{pt}$$

for $n \in \mathbb{N}$. Thus, the pointed mapping space $\mathcal{M}\mathrm{ap}_*\left(\Omega^n\Sigma^n X, Y\right)$ is contractible and so is $\mathcal{M}\mathrm{ap}_*(\Omega^\infty\Sigma^\infty(X), Y)$. □

2.2.3.2. Lemma. — *Let X and W be pointed connected homotopy types. If X is W-full, then $\Sigma^n X$ is $\Sigma^n W$-full, for all $n \in \mathbb{N}$.*

Proof. — Let Y be a pointed connected $\Sigma^n W$-less homotopy type. Since X is connected, we know that $\Sigma^n X$ is n-connected. The statement follows by considering the following sequence of equivalences

$$\mathcal{M}\mathrm{ap}_*\left(\Sigma^n X, Y\right) \simeq \mathcal{M}\mathrm{ap}_*\left(\Sigma^n X, \tau_{\geq n} Y\right) \simeq \mathcal{M}\mathrm{ap}_*\left(X, \Omega^n(Y)\right) \simeq \mathrm{pt},$$

where the last equivalence holds because $\Omega^n Y$ is W-less by Proposition 2.2.1.1. □

Recall that $-\otimes-$ denotes the tensor product of spectra.

2.2.3.3. Proposition. — *Let X and W be pointed connected homotopy types such that X is W-full. Let E be a $(n-1)$-connected spectrum for a natural number n. The homotopy type $\Omega^\infty(\Sigma^\infty X \otimes E)$ is $\Sigma^n W$-full.*

Proof. — We model E by a CW-spectrum and denote by $\mathrm{sk}_k\left(E\right)$ its k-th skeleton. We show inductively that $\Omega^\infty(\Sigma^\infty X \otimes \mathrm{sk}_k\left(E\right))$ is $\Sigma^n W$-full for every $k \geq n$, and use the equivalence $E \simeq \varinjlim_{k\geq n} \mathrm{sk}_k(E)$ and Proposition 2.1.3.1.(ii) to prove the statement. We also use the fact that the functor $\Omega^\infty\colon \tau_{\geq 0}(\mathcal{S}\mathrm{p}) \to \mathcal{H}\mathrm{o}_*$ from the ∞-category of (-1)-connected spectra to $\mathcal{H}\mathrm{o}_*$ preserves filtered colimts, see [HA, Proposition 1.4.3.9] and [HTT, Corollary 5.5.8.17].

Since E is $(n-1)$-connected, we have $\mathrm{sk}_n(E)$ is equivalent to a wedge sum of the n-fold suspensions $\mathbb{S}^n \simeq \Sigma^n\mathbb{S}$ of the sphere spectrum E. The homotopy type $\Omega^\infty(\Sigma^\infty X \otimes_{\mathcal{S}\mathrm{p}} \mathrm{sk}_n(E))$ is $\Sigma^n W$-full. Indeed, we have

$$\Omega^\infty(\Sigma^\infty X \otimes \mathrm{sk}_n(E)) \simeq \Omega^\infty\Sigma^\infty(\vee_{i\in I}\Sigma^n X)$$

and the latter is $\Sigma^n X$-full by Propositions 2.1.3.1 and 2.2.3.1. By Lemma 2.2.3.2 and Proposition 2.3.1.4 the homotopy type $\Omega^\infty(\Sigma^\infty X \otimes \mathrm{sk}_n(E))$ is $\Sigma^n W$-full.

For the induction step consider the following cofibre sequence

$$\Sigma^\infty X \otimes \mathrm{sk}_{k-1}(E) \to \Sigma^\infty X \otimes \mathrm{sk}_k(E) \to \Sigma^\infty X \otimes_{\mathcal{S}\mathrm{p}} \left(\vee_{i\in I}\mathbb{S}^k\right)$$

in spectra for an indexing set I, which is also a fibre sequence. Applying the Ω^∞ functor we obtain an induced fibre sequence

$$\Omega^\infty(\Sigma^\infty X \otimes \mathrm{sk}_{k-1}(E)) \to \Omega^\infty(\Sigma^\infty X \otimes \mathrm{sk}_k(E)) \to \Omega^\infty\Sigma^\infty\left(\vee_{i\in I}\Sigma^k X\right)$$

in $\mathcal{H}\mathrm{o}_*$. By Proposition 2.1.3.6 and our induction assumption it suffices to show that the right most homotopy type in the above fibre sequence is $\Sigma^n W$-full, which holds by similar arguments as we explained in induction begin. Furthermore, since $\Sigma^n X$ is $\Sigma^n W$-full, we have $\Sigma^m X$ is $\Sigma^n W$-full for all $m \geq n$. □

2.2.3.4. Corollary. — *The infinite symmetric product* $\mathrm{SP}^\infty(X) = \Omega^\infty(\Sigma^\infty X \otimes \mathrm{H}\mathbb{Z})$ *of X is X-full.* □

2.2.3.5. Proposition. — *Let X and W be pointed connected homotopy types such that X is W-full. The morphism*

$$\Omega^\infty\Sigma^\infty(X) = \Omega^\infty(\Sigma^\infty X) \to \mathrm{SP}^\infty(X) = \Omega^\infty(\Sigma^\infty X \otimes \mathrm{H}\mathbb{Z})$$

of pointed homotopy types induced by the Hurewicz map $\mathbb{S} \xrightarrow{h} \mathrm{H}\mathbb{Z}$ is a ΣW-equivalence. In particular, it is a ΣX-equivalence.

Proof. — We consider the fibre sequence

$$E \to \mathbb{S} \xrightarrow{h} \mathrm{H}\mathbb{Z}$$

of spectra. Since h is an isomorphism on π_0, the spectrum E is connected. In the induced fibre sequence

$$\Omega^\infty(\Sigma^\infty X \otimes E) \to \Omega^\infty\Sigma^\infty(X) \xrightarrow{h_*} \mathrm{SP}^\infty(X)$$

the fibre is ΣW-full by Proposition 2.2.3.3. Applying Proposition 2.1.3.6 we obtain that the induced map h_* is a ΣW-equivalence □

2.2.3.6. Lemma. — *Let X and W be pointed connected homotopy types where X is ΣW-full. For a pointed simply-connected homotopy type Y, the inclusion*

$$X \vee Y \to X \times Y$$

is a $\Sigma^2 W$-equivalence.

Proof. — Consider the fibre sequence

$$\Sigma(\Omega X \wedge \Omega Y) \to X \vee Y \to X \times Y.$$

in $\mathcal{H}o_*$. It suffices to show that the fibre $\Sigma(\Omega X \wedge \Omega Y) \simeq \Omega X \wedge (\Sigma\Omega Y)$ is $\Sigma^2 W$-full, by Proposition 2.1.3.6. Since $\mathrm{P}_{\Sigma W}(X) \simeq \mathrm{pt}$, we obtain

$$\mathrm{P}_W(\Omega X) \simeq \Omega \mathrm{P}_{\Sigma W}(X) \simeq \mathrm{pt}.$$

Since $\Sigma\Omega Y$ is simply connected, we can use induction along the skeleton of $\Sigma\Omega Y$ to show that $\Omega X \wedge (\Sigma\Omega Y)$ is $\Sigma^2 W$-full, as we did for the proof of Proposition 2.2.3.3. □

2.2.3.7. Proposition. — *Let X and W be pointed connected homotopy types where $\mathrm{P}_{\Sigma W}(X) \simeq \mathrm{pt}$. The induced map*

$$\mathrm{P}_{\Sigma^2 W}(X) \to \mathrm{P}_{\Sigma^2 W}(\Omega^\infty \Sigma^\infty(X))$$

admits a retract.

Proof. — It suffices to show that $\mathrm{P}_{\Sigma^2 W}(X)$ is an infinite loop space. Recall the commutative ∞-operad $\mathcal{C}\mathrm{om}^\otimes$ from Example 5.2.1.7. The pointed homotopy type X together with the fold map $X \vee X \to X$ equipps X the structure of a commutative monoid in symmetric monoidal ∞-category $(\mathcal{H}o_*, \vee, \mathrm{pt})$, which corresponds to a functor

$$\vee\colon \mathcal{C}\mathrm{om}^\otimes \to \mathcal{H}o_*, \langle n\rangle \mapsto \vee_n X,$$

see Definition 5.2.1.20. Define a functor

$$F\colon \mathcal{C}\mathrm{om}^\otimes \xrightarrow{\vee} \mathcal{H}o_* \xrightarrow{\mathrm{P}_{\Sigma^2 W}} \mathcal{H}o_*,\ \langle n\rangle \mapsto \mathrm{P}_{\Sigma^2 W}(\vee_n X).$$

Since $\mathrm{P}_{\Sigma W} X \simeq \mathrm{pt}$, the homotopy type X is simply connected by Proposition 2.2.2.6. Thus we can apply Lemma 2.2.3.6 and have

$$\mathrm{P}_{\Sigma^2 W}(\vee_n X) \simeq \mathrm{P}_{\Sigma^2 W}(X^{\times n}) \simeq \prod_n \mathrm{P}_{\Sigma^2 W}(X)$$

for every $n \in \mathbb{N}$ by Proposition 2.1.3.5. Thus the functor F exhibits $\mathrm{P}_{\Sigma^2 W}(X)$ as a commutative monoid in the *cartesian* symmetric monoidal ∞-category $(\mathcal{H}o_*, \times, \mathrm{pt})$. Moreover, the homotopy type $\mathrm{P}_{\Sigma^2 W}(X)$ is grouplike since X is simply-connected. So we have that $\mathrm{P}_{\Sigma^2 W}(X)$ is an infinite loop space. □

2.3. Unstable Bousfield classes and the W-Postnikov tower

Let W be a pointed homotopy type. Inspired by the case $W = \mathrm{S}^n$ (see Example 2.1.2.7), we can consider W-contraction as a generalisation of the Postnikov truncation of X. More precisely, the homotopy type $\mathrm{P}_{\Sigma^n W}(X)$ discards the information about the k-th *W-homotopy group* $[\Sigma^k W, X]$ of X for all $k \geq n$. Since a W-less homotopy type is also ΣW-less (see Corollary 2.1.1.12), we obtain the following commutative diagram

$$\begin{array}{ccccccc} & & & & & & X \\ & \swarrow & & \swarrow & & \swarrow & \downarrow \\ \cdots & \longrightarrow & \mathrm{P}_{\Sigma^2 W}\, X & \longrightarrow & \mathrm{P}_{\Sigma W}\, X & \longrightarrow & \mathrm{P}_W\, X. \end{array} \tag{2.3.0.1}$$

of (pointed) homotopy types. If $W = \mathrm{S}^0$, this recovers the Postnikov tower of X. Thus, we call the diagram the *W-Postnikov tower* of X. Recall that in the classical Postnikov tower ($W = \mathrm{S}^0$), the fibre of the map $\mathrm{P}_{\mathrm{S}^{n+1}}\, X \to \mathrm{P}_{\mathrm{S}^n}\, X$ is an Eilenberg–MacLane space for all $n \in \mathbb{N}$. Therefore, we may ask, whether the fibres in the tower (2.3.0.1) are also related to Eilenberg–MacLane spaces.

We begin this section with an introduction of unstable Bousfield classes (see §2.3.1), which provides the notational convenience to compare contraction of different homotopy types. Then we discuss the W-Postnikov tower and give a description of the layers of the tower using Eilenberg–MacLane spaces (see §2.3.2). This section is expository and mainly follows [Bou94; Bou96; DS95].

2.3.1. Unstable Bousfield classes. — Recall the notion of Bousfield class for spectra, see ¶1.2.0.15. One can similarly define this notion of equivalence for (pointed) homotopy types, which we discuss in this subsection.

2.3.1.1. Definition. — Let W and W' be homotopy types. We say W and W' are *unstably Bousfield equivalent* if every W-less homotopy type is W'-less and vice versa.

2.3.1.2. Remark. — It is straightforward to verify that the above definition gives an equivalence relation on the set of objects of $\mathcal{H}o$. We denote the equivalence class of W by $\langle W\rangle$, called the *(unstable) Bousfield class of W*.

2.3.1.3. Definition. — Let W and W' be homotopy types. We write

$$\langle W\rangle \leq \langle W'\rangle$$

if every W'-less homotopy type is W-less.

2.3.1.4. Proposition. — *Let W and W' be homotopy types. The following statements are equivalent:*

(i) $\langle W\rangle \leq \langle W'\rangle$.
(ii) *Each W-equivalence is a W'-equivalence.*
(iii) $\mathrm{P}_{W'}(W) \simeq \mathrm{pt}$.

Proof. — We will prove the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). Let $f\colon A \to B$ be a W-equivalence and $Y' \in \mathcal{H}\mathrm{o}$ be a W'-less homotopy type.

Assume (i), then Y' is also W-less. Thus, $\mathrm{Map}(B, Y') \to \mathrm{Map}(A, Y')$ is an equivalence. Hence, the map f is a W'-less equivalence.

Now, assume (ii). Recall that $W \to \mathrm{pt}$ is a W-equivalence. It is then also a W'-equivalence by assumption. Thus, $\mathrm{P}_{W'} W \simeq \mathrm{pt}$.

Assuming (iii), we show that Y' is W-less. Since $\mathrm{P}_{W'}(W) \simeq \mathrm{pt}$, the map $W \to \mathrm{pt}$ is an W'-equivalence. Thus, $\mathrm{Map}(\mathrm{pt}, Y) \to \mathrm{Map}(W, Y)$ is an equivalence by definition. Therefore, Y' is also W-less. □

2.3.1.5. Corollary. — *Let W and W' be homotopy types such that $\langle W\rangle \leq \langle W'\rangle$. Then there exists a natural transformation $\lambda_{W'}\colon \mathrm{P}_W \to \mathrm{P}_{W'}$ of endo-functors of $\mathcal{H}\mathrm{o}$ which fits in the commutative diagram*

$$\begin{array}{ccc} \mathrm{id} & & \\ {\scriptstyle \lambda_W}\downarrow & \overset{\lambda_{W'}}{\searrow} & \\ \mathrm{P}_W & \xrightarrow[\lambda_{W'}]{} & \mathrm{P}_{W'} \end{array}$$

in the ∞-category $\mathcal{F}\mathrm{un}(\mathcal{H}\mathrm{o}, \mathcal{H}\mathrm{o})$ of functors.

Proof. — By Proposition 2.3.1.4 we obtain the following commutative diagram

$$\begin{array}{ccc} \mathcal{H}\mathrm{o} & \xrightarrow{\mathrm{P}_W} & \mathcal{H}\mathrm{o} \\ & \underset{\mathrm{P}_{W'}}{\searrow} & \downarrow{\scriptstyle \mathrm{P}_{W'}} \\ & & \mathcal{H}\mathrm{o}\,. \end{array}$$

This gives the desired commutative diagram in the statement. In particular, for a homotopy type X, the evaluation $\mathrm{P}_W(X) \to \mathrm{P}_{W'}(X)$ of the natural transformation $\mathrm{P}_W \to \mathrm{P}_{W'}$ is given by $\lambda_{W'}(\mathrm{P}_W(X))\colon \mathrm{P}_W(X) \to \mathrm{P}_{W'}(\mathrm{P}_W(X))$. This justifies our choice (or abuse) of the notation $\lambda_{W'}$. □

***2.3.1.6.* Conjunction and disjunction of Bousfield classes.** — Consider a set $\{\langle W_i\rangle \mid W_i \in \mathcal{H}\mathrm{o}\}_{i\in I}$ of Bousfield classes of homotopy types. We can define the *conjunction*

$$\wedge_{i\in I}\langle W_i\rangle \coloneqq \langle \wedge_{i=I} W_i\rangle$$

and the *disjunction*

$$\vee_{i\in I}\langle W_i\rangle \coloneqq \langle \vee_{i=I} W_i\rangle$$

operations. The conjunction $\wedge_{i\in I}\langle W_i\rangle$ is a lower bound for $\{\langle W_i\rangle\}_{i\in I}$ in the sense that $\wedge_{i\in I}\langle W_i\rangle \leq \langle W_i\rangle$ for all $i \in I$. The disjunction is *the* least upper bound of $\{\langle W_i\rangle\}_{i\in I}$, i.e. $\langle W_i\rangle \leq \vee_{i\in I}\langle W_i\rangle$ for all $i \in I$, and $\vee_{i\in I}\langle W_i\rangle \leq \langle W\rangle$ if $\langle W_i\rangle \leq \langle W\rangle$ for every $i \in I$. Note that a homotopy type is W_i-less for every $i \in I$ if and only if it is $\vee_{i=I} W_i$-less. The naming of these operations comes from the theory of Boolean algebras. In [Bou79a] Boolean algebra structure of certain subsets of the set of Bousfield classes of spectra are discussed. In the unstable setting, it is yet unclear how to define complement Bousfield class of a homotopy type.

We list some properties about Bousfield classes [Bou94; Bou96] that are useful for our later applications.

2.3.1.7. Proposition. — *Let $A \to X \to C$ be a cofibre sequence in $\mathcal{H}o_*$. We have*

(i) $\langle X\rangle \leq \langle A\rangle \vee \langle C\rangle$,

(ii) $\langle C\rangle \leq \langle X\rangle \vee \langle \Sigma A\rangle \leq \langle X\rangle \vee \langle A\rangle$, *and*

(iii) $\langle \Sigma A\rangle \leq \langle C\rangle \vee \langle \Sigma X\rangle \leq \langle C\rangle \vee \langle X\rangle$

Proof. — Statement (ii) and (iii) follows from (i) by extending the cofibre sequence to the right using suspensions. (i) follows from the fact that the functor $\mathcal{M}\mathrm{ap}_*(-, Y)$ induces a fibre sequence

$$\mathcal{M}\mathrm{ap}_*(C, Y) \to \mathcal{M}\mathrm{ap}_*(X, Y) \to \mathcal{M}\mathrm{ap}_*(A, Y).$$

Thus if Y is C- and A-less it is X-less. See also [Bou94, Proposition 9.2]. □

2.3.1.8. Proposition. — *Let $F \to X \to B$ be a fibre sequence in $\mathcal{H}o_*$ where B is connected. We have*

(i) $\langle B\rangle \leq \langle F\rangle \vee \langle X\rangle$,

(ii) $\langle X\rangle \leq \langle B\rangle \vee \langle F\rangle$, *and*

(iii) $\langle F\rangle \leq \langle X\rangle \vee \langle \Omega B\rangle$.

Proof. — Recall that there exist equivalences

$$X \simeq \varinjlim_{b\in B} F_b \quad \text{and} \quad B \simeq \varinjlim_{B} \mathrm{pt}$$

of homotopy types where the colimits is taken in $\mathcal{H}o$, see ¶1.3.0.6.(i) Since B is connected, we have $F_b \simeq F$ for every point $b \in B$. Let Y be a homotopy type.

(i) We assume that Y is F-less and X-less. Then we have

$$\begin{aligned} Y \simeq \mathcal{M}\mathrm{ap}\,(X, Y) &\simeq \mathcal{M}\mathrm{ap}\left(\varinjlim_{b\in B}(F), Y\right) \\ &\simeq \varprojlim_{b\in B} \mathcal{M}\mathrm{ap}\,(F, Y) \simeq \varprojlim_{b\in B} \mathcal{M}\mathrm{ap}(\mathrm{pt}, Y) \simeq \mathcal{M}\mathrm{ap}(B, Y) \end{aligned}$$

where the first and the fourth equivalences are due to our assumption on Y and other equivalences are by universal property of limits and colimits.

(ii) Assuming that Y is F-less and B-less, we see that Y is X-less by reading the above sequence of equivalence starting from the second one.

(iii) If X is not connected, then the last inequality holds tautologically; since being X-less means being contractible. Otherwise, the last inequality holds by applying (i) to the fibre sequence $\Omega B \to F \to X$. □

2.3.1.9. Corollary. — *Let $B \in \mathcal{H}o_*$ be connected. Then $\langle B\rangle \leq \langle \Omega B\rangle$.*

Proof. — Apply Proposition 2.3.1.8 to the fibre sequence $\Omega B \to \mathrm{pt} \to B$. □

2.3.1.10. Example. — We can compare Bousfield classes of Eilenberg–MacLane spaces. Let G be an abelian group. Then

$$\langle \mathrm{K}(G,1)\rangle \geq \langle \mathrm{K}(G,2)\rangle \geq \cdots \geq \langle \mathrm{K}(G,m)\rangle \geq \langle \mathrm{K}(G,m+1)\rangle \geq \cdots \tag{2.3.1.1}$$

A short exact sequence $0 \to H \to G \to K \to 0$ of abelian groups induces a fibre sequence

$$\mathrm{K}(H,n) \to \mathrm{K}(G,n) \to \mathrm{K}(K,n)$$

for every $n \in \mathbb{N}$. Fix a prime number p. Consider the following short exact sequence

$$0 \to \mathbb{Z}/p^m\mathbb{Z} \xrightarrow{\times p} \mathbb{Z}/p^{m+1}\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$$

for every natural number $m \geq 1$. By Proposition 2.3.1.8 we have that

$$\langle \mathrm{K}(\mathbb{Z}/p^{m+1}\mathbb{Z},n)\rangle = \langle \mathrm{K}(\mathbb{Z}/p^m\mathbb{Z},n)\rangle \tag{2.3.1.2}$$

for natural numbers n and $m \geq 1$.

2.3.1.11. Example. — The *Prüfer group* $\mathbb{Z}/p^\infty$ is a p-primary torsion abelian group, which has several equivalent definitions. For example, we have

$$\mathbb{Z}/p^\infty \cong \varinjlim \left(\mathbb{Z}/p\mathbb{Z} \xrightarrow{\times p} \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{\times p} \cdots \to \mathbb{Z}/p^m\mathbb{Z} \xrightarrow{\times p} \mathbb{Z}/p^{m+1}\mathbb{Z} \xrightarrow{\times p} \cdots\right),$$

where the colimit is taken in the category of abelian groups. Alternatively, there is a group isomorphism

$$\mathbb{Z}/p^\infty \cong \mathbb{Z}[1/p]/\mathbb{Z},$$

induced by the canonical inclusion $\mathbb{Z} \hookrightarrow \mathbb{Z}[1/p]$, where $\mathbb{Z}[1/p]$ denotes the localisation of $\mathbb{Z}$ away from p. The Prüfer group is p-divisible, i.e. for every $x \in \mathbb{Z}/p^\infty$, we have $x = py$ for some $y \in \mathbb{Z}/p^\infty$. For more details and properties of the group $\mathbb{Z}/p^\infty$, see [RotITG, Chapter 10].

For $n \in \mathbb{N}$, there exists an equivalence

$$\mathrm{K}(\mathbb{Z}/p^\infty,n) \simeq \varinjlim_{m\geq 1} \mathrm{K}(\mathbb{Z}/p^m\mathbb{Z},n).$$

Therefore, by (2.3.1.2), we obtain

$$\langle \mathbb{Z}/p^\infty,n)\rangle = \langle \mathrm{K}(\mathbb{Z}/p^m\mathbb{Z},n)\rangle. \tag{2.3.1.3}$$

for every $n \in \mathbb{N}$ and every natural number $m \geq 1$.

2.3.1.12. Proposition. — *Let W and W' be pointed homotopy types where W is connected. Then $\langle W\rangle \leq \langle \Sigma W'\rangle$ if and only if $\langle \Omega W\rangle \leq \langle W'\rangle$.*

Proof. — This follows from Theorem 2.2.1.2. □

2.3.1.13. Corollary. — *Let $n \geq 1$ be a natural number. We have the following comparison of Bousfield classes:*

(i) $\langle \Omega^n\Sigma^n V\rangle \leq \langle V\rangle$ *for every* $V \in \mathcal{H}\mathrm{o}_*$.

(ii) $\langle W\rangle \leq \langle \Sigma^n\Omega^n W\rangle$ *for every pointed* $(n-1)$*-connected homotopy type.*

(iii) *Let W and W' be $(n-1)$-connected. If $\langle W\rangle \leq \langle W'\rangle$, then $\langle \Omega^n W\rangle \leq \langle \Omega^n W'\rangle$.*

2.3.1.14. Proposition. — *Let W and W' be pointed connected homotopy types. We have the following comparisons of Bousfield classes:*

(i) $\langle W\rangle \geq \langle \mathrm{SP}^\infty W\rangle$.

(ii) *If W is a product of Eilenberg–MacLane spaces, then $\langle W\rangle = \langle \mathrm{SP}^\infty W\rangle$*

(iii) *If $\langle W\rangle \geq \langle W'\rangle$, then $\langle \mathrm{SP}^\infty W\rangle \geq \langle \mathrm{SP}^\infty W'\rangle$.*

Proof. — (i) and (iii) hold by Proposition 2.2.3.3 and Corollary 2.2.3.4. (ii) holds because W is a retract of $\mathrm{SP}^\infty(W)$. □

2.3.2. The W-Postnikov tower. — Let W be a pointed homotopy type. By Corollary 2.1.1.12 we have that $\langle W\rangle \geq \langle \Sigma W\rangle$. Thus we obtain a tower

$$\cdots \to \mathrm{P}_{\Sigma^{n+1}W} \to \mathrm{P}_{\Sigma^n W} \to \cdots \to \mathrm{P}_{\Sigma W} \to \mathrm{P}_W$$

of contraction functors, generalising the classical Postnikov tower (take $W = \mathrm{S}^0$). Thus we call this tower the *W-Postnikov tower.* In this subsection we discuss the fibres of the morphisms in this tower.

2.3.2.1. Theorem. — *Let W and X be pointed connected homotopy types. Assuming that $\mathrm{P}_{\Sigma W}(X)$ is contractible, then the homotopy type $\mathrm{P}_{\Sigma^2 W}(X)$ is equivalent to a product of Eilenberg–MacLane spaces.*

Proof. — By Proposition 2.2.3.5 the morphism

$$\mathrm{P}_{\Sigma^2 W}\left(\Omega^\infty\Sigma^\infty X\right) \to \mathrm{P}_{\Sigma^2 W}\left(\mathrm{SP}^\infty(X)\right)$$

is an equivalence. Therefore, using Proposition 2.2.3.7, we see that $\mathrm{P}_{\Sigma^2 W}(X)$ is a retract of $\mathrm{P}_{\Sigma^2 W}\left(\mathrm{SP}^\infty(X)\right)$, which is a product of Eilenberg–MacLane spaces by Corollary 2.2.1.5. □

2.3.2.2. Theorem. — *Let W and X be pointed connected homotopy types. For all natural numbers $i \geq 1$, the fibre of the natural map $\mathrm{P}_{\Sigma^{i+1}W}(X) \to \mathrm{P}_{\Sigma^i(W)}(X)$ is equivalent to a product of Eilenberg–MacLane spaces.*

Proof. — It suffices to show the theorem for the case $i = 1$. Denote the fibre of the map $\mathrm{P}_{\Sigma^2 W}(X) \to \mathrm{P}_{\Sigma W}(X)$ by F . We know that $\mathrm{P}_{\Sigma W}(F) \simeq \mathrm{pt}$ and $\mathrm{P}_{\Sigma^2 W}(F) \simeq F$ by Theorem 2.2.2.5 and Proposition 2.1.3.1, respectively. By Theorem 2.3.2.1 the pointed homotopy type F is equivalent to a product of Eilenberg–MacLane spaces. ☐

2.3.2.3. Remark. — In the situation of the Theorem 2.3.2.2, we can not drop the suspension functor in the assumption in general. A counterexample is given by the homotopy type $W = \mathrm{BS}^3$. It is shown in [Zab87] that the connected component of $\mathcal{M}\mathrm{ap}_*(\mathrm{BS}^3, \mathrm{BS}^3)$ where the constant map sending BS^3 to the basepoint lies is contractible. And all other connected components of this mapping space have non-trivial homotopy groups, see [DM87, Theorem 1.1]. Thus we obtain the following equivalences

$$\mathcal{M}\mathrm{ap}_*\left(\Sigma(\mathrm{BS}^3), \mathrm{BS}^3\right) \simeq \mathcal{M}\mathrm{ap}_*\left(\mathrm{S}^1, \mathcal{M}\mathrm{ap}_*\left(\mathrm{BS}^3, \mathrm{BS}^3\right)\right) \simeq \mathrm{pt},$$

where the second equivalence holds because the image of S^1 lies in the connected component where the basepoint (the constant map) lies. Thus we have the equivalences $\mathrm{P}_{\mathrm{BS}^3}(\mathrm{BS}^3) \simeq \mathrm{pt}$ and $\mathrm{P}_{\Sigma \mathrm{BS}^3}(\mathrm{BS}^3) \simeq \mathrm{BS}^3$. However, we know that BS^3 is not a product of Eilenberg–MacLane space, since S^3 is not.

2.3.2.4. Question. — The counterexample above is the contraction of a homotopy type that is not finite. What is the fibre of the maps $\mathrm{P}_{\Sigma V}(X) \to \mathrm{P}_V(X)$ where V is a finite pointed homotopy type?

Let X be a pointed connected homotopy type. In §3.1.1 we will see that, under certain extra assumptions on the pointed homotopy type W, the fibre of the morphism $\mathrm{P}_{\Sigma^{i+1} W}(X) \to \mathrm{P}_{\Sigma^i W}(X)$ in the W-Postnikov tower of X is equivalent to an Eilenberg–MacLane space, for every $i \geq 1$ (see Theorem 3.1.1.5).

2.4. Arithmetic localisations

A connected homotopy type M is a *Moore space* if its non-trivial reduced singular homology groups are concentrated in one single degree, i.e. $\widetilde{\mathrm{H}}_n(M;\mathbb{Z}) \cong G$ for a fixed natural number $n \geq 1$ and $\widetilde{\mathrm{H}}_m(M;\mathbb{Z}) = 0$ for $m \neq n$. Contraction of a Moore space is very well-understood. We give an exposition of this topic in this section, summarising results from [Bou94; Bou96; Bou97]. Such contractions are also among the first examples of contractions of finite complexes, which will be discussed further in the next chapter.

We begin with some prerequisites and discuss the contraction of a single Moore space in §2.4.1. Then in §2.4.2 we summarise some results about the contraction of a wedge of Moore spaces. In the end of this section we present an example about the contraction of a wedge sum of Moore spaces associated with the singular homology groups of a pointed simply-connected homotopy type (see ¶2.4.2.6). Later in §3.1 we will see an applications of this example.

2.4.0.1. Notation. — We will use the following notations for Moore spaces. Let G be an abelian group.

(i) Let $\mathrm{M}(G,1)$ denote a Moore space modelled by a CW-complex which has only cells in dimension 0, 1 and 2 and $\widetilde{\mathrm{H}}_1(\mathrm{M}(G,1);\mathbb{Z}) \cong G$ and $\widetilde{\mathrm{H}}_i(\mathrm{M}(G,1);\mathbb{Z}) = 0$ for every $i \neq 1$. Define $M(0,1) \simeq \mathrm{pt}$.

(ii) Let $n \geq 2$ be a natural number. Let $\mathrm{M}(G,n)$ denote the *simply connected* Moore space with $\mathrm{H}_n(\mathrm{M}(G,n);\mathbb{Z}) \cong G$ and $\widetilde{\mathrm{H}}_i(\mathrm{M}(G,n);\mathbb{Z}) = 0$ for every $i \neq n$.

2.4.0.2. Convention. — In this section, let $\mathrm{Hom}(-,-)$ denote set of group homomorphisms, i.e. the morphism set of the category **Grp** of groups.

2.4.0.3. Proposition. — *Let G be an abelian group and let X be a pointed connected homotopy type.*

(i) *The homotopy type X is $\mathrm{M}(G,1)$-less if and only if the following joint conditions hold:*

  (a) $\mathrm{Hom}(\pi_1(\mathrm{M}(G,1)), \pi_1(X)) = \{\mathrm{pt}\}$;

  (b) $\mathrm{Hom}(G, \pi_i X) = \{\mathrm{pt}\}$ *for every natural number* $i \geq 2$;

  (c) $\mathrm{Ext}^1_{\mathbb{Z}}(G, \pi_i(X)) = 0$ *for every natural number* $i \geq 2$.

(ii) *Let $n \geq 2$ be a natural number. The homotopy type X is $\mathrm{M}(G,n)$-less if and only if the following joint conditions hold:*

  (e) $\mathrm{Hom}(G, \pi_i X) = \{\mathrm{pt}\}$ *for every natural number* $i \geq n$;

  (f) $\mathrm{Ext}^1(G, \pi_i(X)) = 0$ *for every natural number* $i \geq n+1$.

Proof. — The proposition follows from Lemma 2.4.0.5 below, which also provides a proof of [Bou94, Lemma 5.4]. □

2.4.0.4. Lemma. — *Let $k \geq n \geq 1$ be natural numbers. Let X be a pointed connected homotopy type. Any morphism $f\colon \mathrm{M}(G,n) \to \tau_{\leq k}X$ in $\mathcal{H}\mathrm{o}_*$ can be lifted to a morphism $\widetilde{f}\colon \mathrm{M}(G,n) \to X$ along the truncation $X \to \tau_{\leq k}X$, i.e. there exists a commutative diagram*

$$\begin{array}{ccc} & & X \\ & \overset{\widetilde{f}}{\nearrow} & \downarrow \\ \mathrm{M}(G,n) & \xrightarrow{f} & \tau_{\leq k}(X) \end{array}$$

of pointed homotopy types.

Proof. — The obstructions of finding a lift $\widetilde{f}$ of f lie in the cohomology groups

$$\mathrm{H}^i\left(\mathrm{M}(G,n); \pi_{i-1}(\tau_{>k}(X))\right)$$

with local coefficients for every natural number $1 \leq i \leq n+1$, which are all zero. See [DK, Theorem 7.37]. □

2.4.0.5. Lemma. — *Let G be an abelian group and let X be a pointed connected homotopy type.*

(i) *The set $[\mathrm{M}(G,1), X]$ of pointed homotopy class of maps consists of a single class if and only if $\mathrm{Hom}(\pi_1(\mathrm{M}(G,1)), \pi_1 X) = \{\mathrm{pt}\}$ and $\mathrm{Ext}^1(G, \pi_2(X)) = 0$;*

(ii) *Let $n \geq 2$ be a natural number. The set $[\mathrm{M}(G,m), X]$ consists of a single class if and only if $\mathrm{Hom}(G, \pi_n X) = \{\mathrm{pt}\}$ and $\mathrm{Ext}^1_{\mathbb{Z}}(G, \pi_{n+1}(X)) = 0$.*

Proof. — In this proof we abbreviate the homotopy type $\mathrm{M}(G,1)$ by M. Recall that $X \simeq \varprojlim_{k\geq 0} \tau_{\leq k}(X)$, see ¶1.3.0.2. Therefore, by Lemma 2.4.0.4, we have that $[M, X] = \{\mathrm{pt}\}$ exactly when $[M, \tau_{\leq j}X] = \{\mathrm{pt}\}$ for every $j \geq 1$. For $j = 1$, we have

$$[M, \tau_{\leq 1}(X)] = [\tau_{\leq 1}(M), \tau_{\leq 1}(X)] = [\mathrm{B}(\pi_1(M)), \mathrm{B}(\pi_1(X))].$$

So, $[M, \tau_{\leq 1}X] = \{\mathrm{pt}\}$ if and only if $\mathrm{Hom}(\pi_1(M), \pi_1(X)) = \{\mathrm{pt}\}$.

Assume that we have $[M, \tau_{\leq 1}X] = \{\mathrm{pt}\}$. Then any pointed map $f\colon M \to \tau_{\leq 2}(X)$ induces a pointed map $f'\colon M \to \mathrm{K}(\pi_2(X), 2)$, by the universal property of the fibre sequence $\mathrm{K}(\pi_2(X), 2) \to \tau_{\leq 2}(X) \to \tau_{\leq 1}(X)$. Therefore, $[M, \tau_{\leq 2}X] = \{\mathrm{pt}\}$ if and only if $\widetilde{\mathrm{H}}^2(M; \pi_2(X)) = 0$. By the Universal Coefficient Theorem $[M, \tau_{\leq 2}(X)] = \{\mathrm{pt}\}$ if and only if $\mathrm{Ext}^1(G, \pi_2(X)) = 0$. For $j \geq 3$, assume that $[M, \tau_{\leq j-1}X] = \{\mathrm{pt}\}$, we have $[M, \tau_{\leq j}(X)] = \widetilde{\mathrm{H}}^j(M; \pi_j(X)) = 0$. Using exactly the same procedure, we can prove (ii). □

***2.4.0.6.* Nullification of groups.** — The contraction of a Moore space relates closely with the theory of nullifications of groups. We will record necessary background on this topic and refer the interested reader to [Bou97, §7.1] and [Cas95, §3] for more details.

2.4.0.7. Definition. — Let G be a group. A group M is *G-null* if the set $\operatorname{Hom}(G, M)$ of group homomorphisms consists of the zero morphism.

2.4.0.8. Theorem. — *Let G be a group. There exists a functor $\mathrm{L}_G\colon \mathbf{Grp} \to \mathbf{Grp}$ together with natural transformations $\lambda_G\colon \mathrm{id} \to \mathrm{L}_G$ such that for every group H,*

(i) *the group $\mathrm{L}_G(H)$ is G-null, and*

(ii) *every group homomorphism $H \to M$, where M is a G-null, factors through $\mathrm{L}_G(H)$, i.e. there exists the following commutative diagram*

$$\begin{array}{ccc} H & \longrightarrow & M \\ \downarrow & \nearrow & \\ \mathrm{L}_G(H). & & \end{array}$$

of groups.

Proof. — See [Cas95, Theorem 3.1]. □

2.4.0.9. Definition. — In the situation of Theorem 2.4.0.8, the functor L_G together with the natural transformation $\mathrm{id} \to \mathrm{L}_G$ is called the *G-nullification*. Let H be a group. We call the group $\mathrm{L}_G(H)$ together with the group homomorphism $H \to \mathrm{L}_G(H)$ the *G-nullification of H*.

2.4.0.10. Proposition. — *For a pair (G, H) of groups, the G-nullification map $\lambda_G(H)\colon H \to \mathrm{L}_G(H)$ of H is a surjection.*

Proof. — See [Cas95, Theorem 3.2]. □

2.4.0.11. Notation. — By Proposition 2.4.0.10 we can consider $\mathrm{L}_G(H)$ as a quotient group of G. For a pair (G, H) of groups, define

$$H//G \coloneqq \mathrm{L}_G(H).$$

2.4.1. Contraction by a single Moore space. —

2.4.1.1. Notation. — Let G be an abelian group. Define the subset $\mathscr{P}_G$ of the set of prime numbers as follows: A prime number p is in $\mathscr{P}_G$ if the multiplication-by-p map $p\colon G \to G$ is an isomorphism. Define the ring R_G via

$$R_G \coloneqq \begin{cases} \mathbb{Z}_{(\mathscr{P}_G)}, & \text{if } G \text{ is torsion} \\ \oplus_{p \in \mathscr{P}_G} \mathbb{Z}/p\mathbb{Z}, & \text{otherwise.} \end{cases}$$

Let $\mathrm{H}R_G$ denote the Eilenberg–MacLane spectrum with $\pi_0(\mathrm{H}R_G) \cong R_G$.

2.4.1.2. Proposition. — *Let $n \geq 1$ be a natural number. For a pointed n-connected homotopy type Y, there exists a natural equivalence*

$$\mathrm{P}_{\mathrm{M}(G,n)}(Y) \xrightarrow{\sim} \mathrm{L}_{\mathrm{H}R_G}(Y).$$

Sketch. — We follow the proof idea of [Bou94, Theorem 5.2] and omit group theoretic technicalities. The Moore space $\mathrm{M}(G,n)$ is $(\mathrm{H}R_G)_\bullet$-acyclic by the definition of $\mathscr{P}_G$. By Proposition 2.1.2.10 we obtain a natural transformation

$$\mathrm{P}_{\mathrm{M}(G,n)} \to \mathrm{L}_{\mathrm{H}R_G}\,.$$

given by $(\mathrm{H}R_G)_\bullet$-localisation. In other words, for every $Z \in \mathcal{H}\mathrm{o}_*$, the evaluation $\mathrm{P}_{\mathrm{M}(G,n)}(Z) \to \mathrm{L}_{\mathrm{H}R_G}(Z)$ is an $(\mathrm{H}R_G)_\bullet$-equivalence. Thus, it suffices to show that $\mathrm{P}_{\mathrm{M}(G,n)}(Y)$ is $(\mathrm{H}R_G)_\bullet$-local. Since Y is n-connected, the homotopy type $\mathrm{P}_{\mathrm{M}(G,n)}(Y)$ is n-connected by Proposition 2.2.2.6. The homotopy groups of $\mathrm{P}_{\mathrm{M}(G,n)}(Y)$ satisfy the conditions from Proposition 2.4.0.3, which implies that $\mathrm{P}_{\mathrm{M}(G,n)}(Y)$ is $(\mathrm{H}R_G)_\bullet$-local by [Bou94, Lemma 5.5] and [Bou75, Theorem 5.5]. □

2.4.1.3. Remark. — Because of the property exhibits in Proposition 2.4.1.2, we call the contraction of a Moore space an *arithmetic localisation*.

2.4.1.4. Remark. — The hypothesis of being simply connected is important in Proposition 2.4.1.2. For example, we have $\mathrm{P}_{\mathrm{M}(\mathbb{Z}/p\mathbb{Z},1)}\,\mathrm{S}^1 \simeq \mathrm{S}^1$, because $\mathrm{Hom}(\mathbb{Z}/p\mathbb{Z},\mathbb{Z}) = 0$. However, S^1 is not $\mathrm{H}\mathbb{Z}[p^{-1}]$-local.

2.4.1.5. Theorem. — *Let G be an abelian group and let $n \geq 2$ be a natural number. For a pointed homotopy type $X \in \mathcal{H}\mathrm{o}_*$, the following statements hold:*

(i) *For every natural number $i \leq n-1$, we have an group isomorphism*

$$\pi_i\left(\mathrm{P}_{\mathrm{M}(G,n)}(X)\right) \cong \pi_i(X).$$

(ii) *There exists the following isomorphism of groups*

$$\pi_n\left(\mathrm{P}_{\mathrm{M}(G,n)}(X)\right) \cong \pi_n(X)//G.$$

(iii) *If G is torsion, then*

$$\pi_i\left(\mathrm{P}_{\mathrm{M}(G,n)}(X)\right) \cong \pi_i(X) \otimes \mathbb{Z}_{(\mathscr{P}_G)}$$

for every natural number $i \geq n+1$.

(iv) *If G is not torsion, then there is a splittable natural short exact sequence*

$$0 \to \prod_{p\in\mathscr{P}_G} \mathrm{Ext}^1(\mathbb{Z}/p^\infty,\pi_i X) \to \pi_i(\mathrm{P}_{\mathrm{M}(G,n)}(X)) \to \prod_{p\in\mathscr{P}_G} \mathrm{Hom}(\mathbb{Z}/p^\infty,\pi_{i-1} Y) \to 0$$

for every natural number $i \geq n+1$.

Sketch. — Statement (i) follows from Proposition 2.2.2.6. For proving the other statements, we can make the assumption that X is $(n-1)$-connected by Corollary 2.2.2.7.

To prove (ii), we show that the morphism $\pi_n(X) \to \pi_n\left(\mathrm{P}_{\mathrm{M}(G,n)}(X)\right)$, induced by the $\mathrm{M}(G,n)$-contraction $\lambda_{\mathrm{M}(G,n)}\colon X \to \mathrm{P}_{\mathrm{M}(G,n)}(X)$ of X, exhibits $\pi_n\left(\mathrm{P}_{\mathrm{M}(G,n)}(X)\right)$ as the G-nullification of $\pi_n(X)$.

Note that $\pi_n\left(\mathrm{P}_{\mathrm{M}(G,n)}(X)\right)$ is G-null by Proposition 2.4.0.3.(ii). For a G-null abelian group M let $f\colon \pi_n(X) \to M$ be a group homomorphism. It remains to show that f factors through $\pi_n(X) \to \pi_n\left(\mathrm{P}_{\mathrm{M}(G,n)}(X)\right)$. First, there exists a morphism $\overline{f}\colon X \to \tau_{\leq n}(X) \to \mathrm{K}(M,n)$ of pointed homotopy types such that the induced morphism $\pi_n(\overline{f}) \cong f$. Furthermore, $\overline{f}$ factors through $X \to \mathrm{P}_{\mathrm{M}(G,n)}(X)$, since $\mathrm{K}(M,n)$ is $\mathrm{M}(G,n)$-less by Proposition 2.4.0.3.(ii). Thus the map f factors through $\pi_n(X) \to \pi_n(\mathrm{P}_{\mathrm{M}(G,n)}(X))$.

Statements (iii) and (iv) is a consequence of Proposition 2.4.1.2 and the explicit description of the homotopy type of $\mathrm{L}_{\mathrm{H}R_G}(X)$ [Bou75, Proposition 4.3]. □

***2.4.1.6.* Contraction of $\mathrm{M}(G,1)$.** — Recall from Notation 2.4.0.1.(i) the construction the Moore space $\mathrm{M}(G,1)$. Let X be a pointed connected homotopy type. Because of the uncertainty with the fundamental group, the homotopy types $\mathrm{P}_{\mathrm{M}(G,1)}(X)$ is harder to determine completely in terms of that of X. However, there exists similar results as Theorem 2.4.1.5 for nilpotent homotopy types.

2.4.1.7. Theorem. — *Let Y be a nilpotent homotopy type. The following statements hold:*

(i) *We have an isomorphism $\pi_1(\mathrm{P}_{\mathrm{M}(G,1)}(Y)) \cong \pi_1(Y)//G$ of groups.*

(ii) *If G is torsion, then $\pi_i\left(\mathrm{P}_{\mathrm{M}(G,1)}(Y)\right) \cong \pi_i(Y) \otimes \mathbb{Z}_{(\mathscr{P}_G)}$, for every $i \geq 2$.*

(iii) *If G is not torsion, there exists a splittable natural short exact sequence*

$$0 \to \prod_{p\in\mathscr{P}_G} \mathrm{Ext}^1(\mathbb{Z}/p^\infty, \pi_i Y) \to \pi_i(\mathrm{P}_{\mathrm{M}(G,1)}(X)) \to \prod_{p\in\mathscr{P}_G} \mathrm{Hom}(\mathbb{Z}/p^\infty, \pi_{i-1} Y) \to 0$$

for every $i \geq 2$.

Sketch. — The idea of the proof is to reduce the computation to the simply connected $\mathrm{M}(G,1)$-less homotopy types and apply Proposition 2.4.1.2.

Consider the fibre sequence

$$\widetilde{Y} \to Y \to \mathrm{K}(\pi_1(Y)//G, 1)$$

where the second map induced by the group homomorphism $\pi_1(Y) \to \pi_1(Y)//G$. Note that we have $\pi_i(\widetilde{Y}) \cong \pi_i(Y)$ for every $i \geq 2$ and $\pi_1(\widetilde{Y})//G = 0$. Since $\mathrm{K}(\pi_1(Y)//G, 1)$ is $\mathrm{M}(G,1)$-less, applying $\mathrm{P}_{\mathrm{M}(G,1)}$ to the above fibre sequence gives a fibre sequence

$$\mathrm{P}_{\mathrm{M}(G,1)}(\widetilde{Y}) \to \mathrm{P}_{\mathrm{M}(G,1)}(Y) \to \mathrm{K}(\pi_1(Y)//G, 1),$$

by Theorem 2.2.2.5. Using the same proof as Theorem 2.4.1.5.(ii), we obtain an isomorphism $\pi_1(\mathrm{P}_{\mathrm{M}(G,1)}(\widetilde{Y})) \cong \pi_1(\widetilde{Y})//G$. For this we also use that $\pi_1(\widetilde{Y}) \subseteq \pi_1(Y)$ is nilpotent and [Bou97, Theorem 7.2]. Therefore, the homotopy type $\mathrm{P}_{\mathrm{M}(G,1)}(\widetilde{Y})$ is simply connected and $\pi_1(\mathrm{P}_{\mathrm{M}(G,1)}(Y)) \cong \pi_1(Y)//G$.

Since $\pi_i\left(\mathrm{P}_{\mathrm{M}(G,1)}(Y)\right) \cong \pi_i(\mathrm{P}_{\mathrm{M}(G,1)}(\widetilde{Y}))$ by the above fibre sequence, (ii) and (iii) follow from Proposition 2.4.1.2 and [Bou75, Proposition 4.3]. □

2.4.2. The Contraction of a wedge of Moore spaces. —

2.4.2.1. Notation. — Let $\underline{G} = (G_i)_{i\geq 1}$ be a sequence of abelian groups and let $n \geq 1$ be a natural number. Define

$$\mathrm{M}(\underline{G}) := \bigvee_{i=1}^{\infty} \mathrm{M}(G_i, i),$$
$$\mathrm{M}(\underline{G}, \underline{n}) := \mathrm{M}(G_1, 1) \vee \mathrm{M}(G_2, 2) \vee \cdots \vee \mathrm{M}(G_n, n).$$

2.4.2.2. Theorem. — *Let $\underline{G} = (G_i)_{i\geq 1}$ be a sequence of abelian groups and let G' be an abelian group. For natural numbers $n' > n \geq 1$, define the pointed homotopy type $W := \mathrm{M}(\underline{G}, \underline{n}) \vee \mathrm{M}(G', n')$ and $N := \mathrm{M}\left((\oplus_{i=1}^{n} G_i) \oplus G', n'\right)$. Then for every pointed homotopy type X, there exists a natural equivalence*

$$\mathrm{P}_W(X) \simeq \mathrm{P}_N\left(\mathrm{P}_{\mathrm{M}(\underline{G},\underline{n})}(X)\right).$$

Sketch. — We show that $\mathrm{P}_N\left(\mathrm{P}_{\mathrm{M}(\underline{G},\underline{n})}(X)\right)$ is the W-contraction of X; the theorem then follows from the uniqueness of W-contractions (see Proposition 2.1.2.6). Consider the following composition

$$\mathrm{L}_W \colon X \xrightarrow{\lambda_M} \mathrm{P}_{\mathrm{M}(\underline{G},\underline{n})}(X) \xrightarrow{\lambda_N} \mathrm{P}_N\left(\mathrm{P}_{\mathrm{M}(\underline{G},\underline{n})}(X)\right).$$

Claim. The morphism L_W is a W-equivalence.

By ¶2.3.1.6 the map $\lambda_M(X)$ is a W-equivalence. Thus it suffices to show that λ_N is a W-equivalence. Let $Y \in \mathcal{H}\mathrm{o}_*$ be W-less, i.e. Y is $\mathrm{M}(G', n')$-less and $\mathrm{M}(G_i, i)$-less for every $1 \leq i \leq n$. Using Proposition 2.4.0.3.ii), we see that Y is N-less. Since λ_N is an N-equivalence by definition, the induced map

$$(\lambda_N)^* \colon \mathrm{Map}_*(\mathrm{P}_N\left(\mathrm{P}_{\mathrm{M}(\underline{G},\underline{n})}(X)\right), Y) \to \mathrm{Map}_*(\mathrm{P}_{\mathrm{M}(\underline{G},\underline{n})}(X), Y)$$

is an equivalence. This shows that λ_N is a W-equivalence.

Claim. The homotopy type $\mathrm{P}_N\left(\mathrm{P}_{\mathrm{M}(\underline{G},\underline{n})}(X)\right)$ is W-less.

By Proposition 2.4.0.3.ii) we have $\langle N\rangle \geq \langle \mathrm{M}(G', n')\rangle$. Thus $\mathrm{P}_N\left(\mathrm{P}_{\mathrm{M}(\underline{G},\underline{n})}(X)\right)$ is $\mathrm{M}(G', n')$-less. Therefore, it suffices to show that it is also $\mathrm{M}(\underline{G}, \underline{n})$-less. Note that $\mathrm{P}_N\left(\mathrm{P}_{\mathrm{M}(\underline{G},\underline{n})}(X)\right)$ is N-less and $\pi_i\left(\mathrm{P}_N\left(\mathrm{P}_{\mathrm{M}(\underline{G},\underline{n})}(X)\right)\right) \cong \pi_i\left(\mathrm{P}_{\mathrm{M}(\underline{G},\underline{n})}(X)\right)$ for every $i < n'$, by Proposition 2.2.2.6. Thus, by Theorem 2.4.1.5 and Proposition 2.4.0.3, it suffices to show that for every natural number $k \geq n'$ the abelian group

$$H_k := \pi_{n'}(\mathrm{P}_N\left(\mathrm{P}_{\mathrm{M}(\underline{G},\underline{n})}(X)\right)) \cong \pi_{n'}\left(\mathrm{P}_{\mathrm{M}(\underline{G},\underline{n})}(X)\right) // \left((\oplus_{i=2}^{n} G_i) \oplus G'\right)$$

satisfies

$$\mathrm{Hom}(G_i, H_k) = 0 \text{ and } \mathrm{Ext}^1_{\mathbb{Z}}(G_i, H_k) = 0. \tag{2.4.2.1}$$

for every $1 \leq i \leq n$. Since $\mathrm{P}_{\mathrm{M}(\underline{G},\underline{n})}(X)$ is $\mathrm{M}(G_i, n)$-less for every $1 \leq i \leq n$, the homotopy group $\pi_k\left(\mathrm{P}_{\mathrm{M}(\underline{G},\underline{n})}(X)\right)$ satisfies

$$\mathrm{Hom}(G_i, \pi_{n'}\left(\mathrm{P}_{\mathrm{M}(\underline{G},\underline{n})}(X)\right)) = 0 \text{ and } \mathrm{Ext}^1_{\mathbb{Z}}(G_i, \pi_{n'}\left(\mathrm{P}_{\mathrm{M}(\underline{G},\underline{n})}(X)\right)) = 0$$

for every $k \geq n'$ and for every $1 \leq i \leq n$, by Proposition 2.4.0.3. Using [Bou94, Lemma 5.5] we see the quotient group H_k satisfies the conditions (2.4.2.1): If G_i is torsion, we apply [Bou96, Lemma 4.2]; Otherwise, we use [Bou94, Lemma 5.9]. See also [Bou94, Theorem 5.3]. □

2.4.2.3. Situation. — Let $\underline{G} = (G_i)_{i\geq 1}$ be a sequence of abelian groups, where the abelian group G_1 is trivial. Then the wedge $\mathrm{M}(\underline{G})$ of Moore spaces is simply-connected and is uniquely determined by $(G_i)_{i\geq 2}$ up to equivalence.

Let $n \geq 2$ be a natural number. Define the subset $\mathscr{P}_{(\underline{G},\underline{n})}$ of the set of prime numbers as follows: A prime number p is contained in $\mathscr{P}_{(\underline{G},\underline{n})}$ if for every natural number $2 \leq i \leq n$, the multiplication-by-p map $p\colon G_i \to G_i$ is an automorphism of G_i.

2.4.2.4. Theorem. — *Let X be a pointed homotopy type. In Situation 2.4.2.3 we can understand the homotopy groups of $\mathrm{P}_{\mathrm{M}(\underline{G},\underline{n})}(X)$ in terms of that of X. Let $n \geq 2$ be a natural number.*

(i) *If G_i is torsion for every $i \geq 2$, then for every $m \geq n+1$, there exists a natural isomorphism*

$$\pi_m\left(\mathrm{P}_{\mathrm{M}(\underline{G},\underline{n})}(X)\right) \cong \pi_m(X) \otimes \mathbb{Z}_{(\mathscr{P}_{(\underline{G},\underline{n})})}.$$

(ii) *If there exists a $2 \leq i \leq n$ such that G_i is not torsion, then for every $m \geq n+1$, there exists a splittable natural short exact sequence*

$$0 \to \prod_{p \in \mathscr{P}_{(\underline{G},\underline{n})}} \mathrm{Ext}^1(\mathbb{Z}/p^\infty, \pi_m(X)) \to \pi_m(\mathrm{P}_{\mathrm{M}(\underline{G},\underline{n})}(X)) \to \prod_{p \in \mathscr{P}_{(\underline{G},\underline{n})}} \mathrm{Hom}(\mathbb{Z}/p^\infty, \pi_{m-1}(X)) \to 0$$

(iii) *There exist natural isomorphisms*

$$\pi_1\left(\mathrm{P}_{\mathrm{M}(\underline{G})}(X)\right) \cong \pi_1(X) \text{ and}$$
$$\pi_n\left(\mathrm{P}_{\mathrm{M}(\underline{G})}(X)\right) \cong \pi_n(\mathrm{P}_{\mathrm{M}(\underline{G},\underline{n-1})}(X))//(\oplus_{i=2}^n G_i)$$

for every $n \geq 2$.

Proof. — (i) and (ii) follows from Theorem 2.4.1.5 and iterated applications of Theorem 2.4.2.2.

As for (iii), define $N := \mathrm{M}(\oplus_{i=1}^n G_i, n)$. By Theorem 2.4.2.2 and Proposition 2.2.2.6 there exist isomorphisms

$$\pi_n\left(\mathrm{P}_{\mathrm{M}(\underline{G})}(X)\right) \cong \pi_n\left(\mathrm{P}_{\mathrm{M}(\underline{G},\underline{n})}(X)\right) \cong \pi_n\left(\mathrm{P}_N\,\mathrm{P}_{\mathrm{M}(\underline{G},\underline{n-1})}(X)\right).$$

Therefore, (iii) follows from Theorem 2.4.1.5.ii). See also [Bou96, Theorems 4.1 and 4.3]. □

2.4.2.5. Corollary. — *Let Y be a pointed homotopy type. In Situation 2.4.2.3 the following statements hold:*

(i) *The homotopy type Y is $\mathrm{M}(\underline{G})$-less if and only if $\mathrm{K}(\pi_i(Y), i)$ is $\mathrm{M}(\underline{G})$-less for every $i \geq 2$;*

(ii) *The homotopy type Y is $\mathrm{M}(\underline{G})$-full if and only if Y is simply connected and $\mathrm{K}(\pi_i(Y), i)$ is $\mathrm{M}(\underline{G})$-full for every $i \geq 2$.*

Proof. — These are consequences of Proposition 2.4.0.3 and Theorem 2.4.2.4. See also [Bou96, Corollary 4.5]. □

2.4.2.6. **An example.** — We conclude this subsection with an example of the contraction of the wedge $\mathrm{M}(\underline{\mathrm{H}_\bullet(W)})$ of Moore spaces obtained from a pointed simply connected homotopy type W. In particular, we show that the contraction of W and the contraction of $\mathrm{M}(\underline{\mathrm{H}_\bullet(W)})$ coincide for a certain family of homotopy types, see Theorem 2.4.2.9. Later we will use this theorem to prove Theorem 3.1.2.15 in the next chapter.

2.4.2.7. Definition. — Let W be a simply connected pointed homotopy type. Define the sequence

$$\underline{\mathrm{H}_\bullet(W)} \coloneqq (\widetilde{\mathrm{H}}_i(W;\mathbb{Z}))_{i\geq 2}$$

of abelian groups associated with W.

2.4.2.8. Proposition. — *In the situation of Definition 2.4.2.7, the homotopy type W is $\mathrm{M}(\underline{\mathrm{H}_\bullet(W)})$-full.*

Proof. — In other words, we need to prove that $\langle W\rangle \leq \langle \mathrm{M}(\underline{\mathrm{H}_\bullet(W)})\rangle$.

Claim. Let G be an abelian group and let $n \geq 2$ be a natural number. If the Eilenberg–MacLane space $\mathrm{K}(G,n)$ is $\mathrm{M}(\underline{\mathrm{H}_\bullet(W)})$-less, then $\mathrm{K}(G,n)$ is W-less.

It is equivalent to show that $\widetilde{\mathrm{H}}^m(W;G) = 0$ for every $0 \leq m \leq n$, by Proposition 2.1.1.11. Consider the short exact sequence

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\widetilde{\mathrm{H}}_{m-1}(W;\mathbb{Z}), G) \to \widetilde{\mathrm{H}}^m(W;G) \to \mathrm{Hom}(\widetilde{\mathrm{H}}_m(W;\mathbb{Z}), G) \to 0.$$

obtained from the Universal Coefficient Theorem. Since $\mathrm{K}(G,n)$ is $\mathrm{M}(\underline{\mathrm{H}_\bullet(W)})$-less, it is $\mathrm{M}(\widetilde{\mathrm{H}}_i(W;\mathbb{Z})), i)$-less for every $i \geq 2$ (see ¶2.3.1.6). By Proposition 2.4.0.3 we have that $\mathrm{Ext}^1_{\mathbb{Z}}(\widetilde{\mathrm{H}}_{m-1}(W;\mathbb{Z}), G) = 0 = \mathrm{Hom}(\widetilde{\mathrm{H}}_m(W;\mathbb{Z}), G)$ for every $0 \leq m \leq n$. Thus we obtain $\widetilde{\mathrm{H}}^m(W;G) = 0$.

Claim. Let Y be a connected $\mathrm{M}(\underline{\mathrm{H}_\bullet(W)})$-less homotopy type. Then Y is W-less.

By Corollary 2.4.2.5 and the previous claim, the Eilenberg–MacLane space $\mathrm{K}(\pi_j Y, j)$ is W-less, for every natural number $j \geq 2$. Also, $\mathrm{K}(\pi_1 Y, 1)$ is W-less because W is simply connected. Since W-less homotopy types are closed under small limits (see Proposition 2.1.3.1), we obtain that Y is W-less by considering Y as the inverse limit of its Postnikov tower. □

2.4.2.9. Theorem. — *Let W be a simply connected homotopy type and let Y be a pointed connected homotopy type whose homotopy groups are non-trivial in finitely many degrees. Then canonical morphism*

$$\lambda_{\mathrm{M}(\underline{\mathrm{H}_\bullet(W)})}(\mathrm{P}_W(Y))\colon \mathrm{P}_W(Y) \to \mathrm{P}_{\mathrm{M}(\underline{\mathrm{H}_\bullet(W)})}(Y)$$

given by the natural transformation $\lambda_{\mathrm{M}(\underline{\mathrm{H}_\bullet(W)})}$ is an equivalence.

Proof. — We follow the proof of [Bou96, Theorem 4.6]. By Proposition 2.4.2.8, the homotopy type $\mathrm{P}_{\mathrm{M}(\underline{\mathrm{H}_\bullet(W)})}(Y)$ is W-less. Thus it suffices to show that $\lambda_{\mathrm{M}(\underline{\mathrm{H}_\bullet(W)})}(\mathrm{P}_W(Y))$ is a W-equivalence.

Denote the fibre of the map $\lambda_{\mathrm{M}(\underline{\mathrm{H}_\bullet(W)})}(\mathrm{P}_W(Y))$ by F. The homotopy type F is simply connected by Proposition 2.2.2.6, and it is $\mathrm{M}(\underline{\mathrm{H}_\bullet(W)})$-full by Theorem 2.2.2.5. Thus, by Corollary 2.4.2.5, we have $\langle \mathrm{M}(\underline{\mathrm{H}_\bullet(W)})\rangle \geq \langle \mathrm{K}(\pi_n F, n)\rangle$ for every natural number $n \geq 2$. So, by Proposition 2.3.1.14, we obtain

$$\langle W\rangle \geq \langle \mathrm{SP}^\infty W\rangle = \langle \mathrm{SP}^\infty(\mathrm{M}(\underline{\mathrm{H}_\bullet(W)}))\rangle \geq \langle \mathrm{SP}^\infty(\mathrm{K}(\pi_n F, n))\rangle = \langle \mathrm{K}(\pi_n F, n)\rangle.$$

Furthermore, by Proposition 2.2.2.8, the homotopy groups of F are non-trivial in finitely many degrees. Therefore, we have that F is W-full by applying Proposition 2.1.3.6 finitely many times. □

2.4.2.10. Remark. — Combining Theorem 2.4.1.7 and Theorem 2.4.2.2, one can completely determine the homotopy type of the $\mathrm{M}(\underline{G})$-contraction $\mathrm{P}_{\mathrm{M}(\underline{G})}(Y)$ of a nilpotent homotopy type Y, where $\mathrm{M}(\underline{G})$ is a wedge of Moore spaces, cf. [Bou97, Theorem 7.5]. As an application, one can generalise Theorem 2.4.2.9 to a lager family of homotopy types, the so-called *nilpotent generalised polyGEM*, see [Bou97, Theorem 8.8]. The interested reader shall check [Bou97, §§7–8] for more details and applications in this direction.

2.4.2.11. Proposition. — *Let W be a simply connected homotopy type. If there exists a natural number r such that $\langle \Sigma^r W\rangle = \langle \Sigma^r \mathrm{M}(\underline{\mathrm{H}_\bullet(W)})\rangle$, then $\langle W\rangle = \langle \mathrm{M}(\underline{\mathrm{H}_\bullet(W)})\rangle$.*

Proof. — We follow the proof of [Bou96, Theorem 4.7]. It suffices to show the implication under the assumption $\langle \Sigma W\rangle = \langle \Sigma \mathrm{M}(\underline{\mathrm{H}_\bullet(W)})\rangle$. Because $\langle W\rangle \leq \langle \mathrm{M}(\underline{\mathrm{H}_\bullet(W)})\rangle$ by Proposition 2.4.2.8, it remains to show that the canonical morphism

$$\lambda_{\mathrm{M}(\underline{\mathrm{H}_\bullet(W)})}(\mathrm{P}_W(X))\colon \mathrm{P}_W(X) \to \mathrm{P}_{\mathrm{M}(\underline{\mathrm{H}_\bullet(W)})}(X)$$

given by the natural transformation $\lambda_{\mathrm{M}(\underline{\mathrm{H}_\bullet(W)})}$ is a W-equivalence for every $X \in \mathcal{H}\mathrm{o}_*$; this would imply that $\lambda_{\mathrm{M}(\underline{\mathrm{H}_\bullet(W)})}(\mathrm{P}_W(X))$ is an equivalence since the homotopy type $\mathrm{P}_{\mathrm{M}(\underline{\mathrm{H}_\bullet(W)})}(X)$ is W-less.

Since we have $\langle W\rangle \geq \langle \Sigma W\rangle = \langle \Sigma \mathrm{M}(\underline{\mathrm{H}_\bullet(W)})\rangle$ by assumption, the homotopy type $\mathrm{P}_W(X)$ is $\Sigma \mathrm{M}(\underline{\mathrm{H}_\bullet(W)})$-less and thus

$$\mathrm{P}_W(X) \simeq \mathrm{P}_{\Sigma \mathrm{M}(\underline{\mathrm{H}_\bullet(W)})}(\mathrm{P}_W(X)).$$

So $\lambda_{\mathrm{M}(\underline{\mathrm{H}_\bullet(W)})}(\mathrm{P}_W(X))$ is equivalent to the following morphism

$$L\colon \mathrm{P}_{\Sigma\mathrm{M}(\underline{\mathrm{H}_\bullet(W)})}(\mathrm{P}_W(X)) \to \mathrm{P}_{\mathrm{M}(\underline{\mathrm{H}_\bullet(W)})}(\mathrm{P}_W(X)),$$

induced by the comparison $\langle \mathrm{M}(\underline{\mathrm{H}_\bullet(W)})\rangle \geq \langle \Sigma\mathrm{M}(\underline{\mathrm{H}_\bullet(W)})\rangle$.

Let F be the fibre of L. We have by Theorem 2.3.2.2 that F is equivalent to a product of Eilenberg–MacLane spaces, since $\mathrm{M}(\underline{\mathrm{H}_\bullet(W)})$ (a wedge of simply connected Moore spaces) is equivalent to a suspension of a homotopy type. By the same argument as in the proof of Theorem 2.4.2.9, the homotopy type F is simply connected and the Eilenberg–MacLane space $\mathrm{K}(\pi_n F, n)$ is W-full for every natural number $n \geq 2$. Therefore, the homotopy type F is W-full, by Propositions 2.1.3.1 and 2.1.3.5; an infinite product of Eilenberg–MacLane spaces is equivalent to a filtered colimit of all finite sub-products of those Eilenberg–MacLane spaces. By Proposition 2.1.3.6 the morphism L, equivalently $\lambda_{\mathrm{M}(\underline{\mathrm{H}_\bullet(W)})}(\mathrm{P}_W(X))$, is a W-equivalence. □

CHAPTER 3

The v_h-periodic localisation of homotopy types

3.1. The Unstable Class Invariance Theorem

Fix a prime number p. The (stable) Class Invariance Theorem (see Theorem 3.1.2.6) says that the Bousfield class of a finite p-local spectrum depends only on its type (see Definition 1.2.0.3). The goal of this section is to present a similar theorem (see Theorem 3.1.2.1) for the unstable Bousfield classes of finite p-local homotopy types, due to Bousfield, see §3.1.2. For this purpose, we begin in §3.1.1 by showing that the fibre of each morphism in the W-Postnikov tower for a p-local homotopy type W consists of a single p-local Eilenberg–MacLane space. In the last subsection §3.1.3 we discuss some fibration theorems in the p-local setting, which will be used extensively in later sections. The references for this expositional section are [Bou94; Bou96; DS95; Har18].

3.1.1. Layers of a W-Postnikov tower. —

3.1.1.1. Situation. — We fix a prime number p and work in the ∞-category $\mathcal{H}\mathrm{o}_*$ of pointed homotopy types. In particular, we consider a contraction functor as an endofunctor of $\mathcal{H}\mathrm{o}_*$.

3.1.1.2. Definition. — Recall the following definitions for abelian groups.

(i) An abelian group G is *p-local* if for every prime number $\ell \neq p$, the multiplication-by-ℓ map $\ell\colon G \to G$ is an automorphism of G.

(ii) An abelian group G is *p-primary* if the order of every element of G is p^n for some $n \in \mathbb{N}$.

3.1.1.3. Definition. — Let $n \geq 1$ be a natural number. A pointed homotopy type W satisfies the *n-supported p-torsion condition* if

(i) W is $(n-1)$-connected,

(ii) $\widetilde{\mathrm{H}}_\bullet(W;\mathbb{Z})$ is p-primary, and

(iii) $\mathrm{H}^n(W;\mathbb{Z}/p\mathbb{Z}) \neq 0$.

3.1.1.4. Situation. — Let $n \geq 1$ be a natural number, and let W be a pointed connected homotopy type satisfying the n-supported p-torsion condition.

3.1.1.5. Theorem **(Bousfield).** — *Let X be a pointed homotopy type and let $i \geq 1$ be a natural number. In Situation 3.1.1.4 there exists a fibre sequence*

$$\mathrm{K}(G_i, n+i) \to \mathrm{P}_{\Sigma^{i+1}W}\, X \to \mathrm{P}_{\Sigma^i W}\, X,$$

where G_i is a p-primary abelian group.

We will prove this theorem below.

3.1.1.6. Remark. — Bousfield proved a more general result, where he can consider several prime numbers at a time, see [Bou94, Theorem 7.2]. We will only present the p-torsion case, since we work later in the ∞-category $\mathcal{H}\mathrm{o}_{(p)}$ of p-local homotopy types. See also [DS95, Theorem A] for a more general statement.

3.1.1.7. Proposition. — *Let G be a p-primary abelian group. In Situation 3.1.1.4, for every natural number $\ell \geq n$, the Eilenberg–MacLane space $\mathrm{K}(G, \ell)$ is W-full.*

Proof. — Let Y be a pointed connected W-less homotopy type and let $\ell \geq n$ be a natural number. By Corollary 2.2.3.4 we have that $\mathcal{M}\mathrm{ap}_*(\mathrm{K}(\widetilde{\mathrm{H}}_\ell(W), \ell), Y)$ is contractible. Since $\mathrm{K}(\widetilde{\mathrm{H}}_\ell(W), \ell)$ is connected, it is W-less. By Proposition 2.1.3.6 the class of abelian groups G such that $\mathrm{K}(G, \ell)$ is W-less is closed under extensions, cokernels and directed colimits. Because the abelian group $\widetilde{\mathrm{H}}_\ell(W)$ is p-primary, we obtain that $\mathrm{K}(G, \ell)$ is W-less for every p-primary torsion abelian group G. See also [Bou94, Lemma 7.4]. □

Proof of Theorem 3.1.1.5. — We follow the proof ideas presented in [Har18, Lecture 3]. It suffices to show the $i = 1$ case. Denote the fibre of $\mathrm{P}_{\Sigma^2 W}\, X \to \mathrm{P}_{\Sigma W}\, X$ by F. The homotopy type F is n-connected by Proposition 2.2.2.6, and it is equivalent to a product of Eilenberg–MacLane spaces by Theorem 2.3.2.2. Write $F = \prod_{m \geq n+1} \mathrm{K}(G_m, m)$.

Claim. For every $m \geq n+1$, the Eilenberg–MacLane space $\mathrm{K}(G_m, m)$ is $\Sigma^2 W$-less and ΣW-full.

By construction F is $\Sigma^2 W$-less and ΣW-full. Since $\mathrm{K}(G_m, m)$ is a retract of F, it also enjoys these two properties.

Claim. For every $m \geq n+1$, the abelian group G_m is p-primary.

It suffices to show that $G_m \otimes \mathbb{Z}[1/p] = 0$ for every $m \geq n+1$. Consider the fibre sequence

$$\mathrm{K}(G_m \otimes \mathbb{Z}/p\mathbb{Z}, m-1) \to \mathrm{K}(G_m, m) \xrightarrow{\times p} \mathrm{K}(G_m, m).$$

The fibre $\mathrm{K}(G_m \otimes \mathbb{Z}/p\mathbb{Z}, m-1)$ is ΣW-full by Proposition 3.1.1.7. Thus the map $\times p$ is a ΣW-equivalence by Proposition 2.1.3.6. As ΣW-equivalences are preserved under passing to colimits (see Proposition 2.1.3.1), we obtain a ΣW-equivalence

$$f\colon \mathrm{K}(G_m, m) \to \mathrm{K}(G_m \otimes \mathbb{Z}[1/p], m);$$

recall that there is a group isomorphism $G_m \otimes \mathbb{Z}[1/p] \simeq \varinjlim(G_m \xrightarrow{\times p} G_m \xrightarrow{\times p} \cdots)$. Thus the Eilenberg–MacLane space $\mathrm{K}(G_m \otimes \mathbb{Z}[1/p], m)$ is ΣW-full by the first claim. Note that $\mathrm{K}(G_m \otimes \mathbb{Z}[1/p], m)$ is also ΣW-less: By the Brown Representability we have an isomorphism $[\Sigma^k W, \mathrm{K}(G_m \otimes \mathbb{Z}[1/p], m)] \cong \widetilde{\mathrm{H}}^{m-k}(W; G_m \otimes \mathbb{Z}[1/p]) = 0$, for every natural number k. So

$$\mathrm{K}(G_m \otimes \mathbb{Z}[1/p], m) \simeq \mathrm{pt},$$

that is, $G_m \otimes \mathbb{Z}[1/p] = 0$. From this we see that G_m is p-primary for every $m \geq n+1$.

Claim. For every $m \geq n+2$, the abelian group G_m is p-torsion free.

Let $T_{p,m}$ denote the p-power torsion subgroup of G_m, i.e.

$$T_{p,m} = \{x \in G \mid \exists n \in \mathbb{N}, \text{ such that } p^n(x) = 0\}$$

Consider the fibre sequence

$$\mathrm{K}(T_{p,m}, m) \to \mathrm{K}(G_m, m) \xrightarrow{q_m} \mathrm{K}(G_m/T_{p,m}, m)$$

induced by the quotient map $q_m\colon G_m \to G_m/T_{p,m}$. The fibre $\mathrm{K}(T_{p,m}, m)$ is $\Sigma^2 W$-full by Proposition 3.1.1.7. Thus the map q_m is a $\Sigma^2 W$-equivalence by Proposition 2.1.3.6. Moreover, $\mathrm{K}(G_m/T_{p,m}, m)$ is $\Sigma^2 W$-less, as one can check that the reduced singular cohomology groups of $\Sigma^2 W$ with $G_m/T_{p,m}$-coefficients vanish. This implies that the map q_m is an equivalence since both its source and target are $\Sigma^2 W$-less. Therefore, we obtain $G_m \cong G_m/T_{p,m}$, that is, G_m is p-torsion free.

The second and the third claims combined imply that

(i) G_{n+1} is a p-primary torsion abelian group,

(ii) $G_m = 0$ for every natural number $m \geq n+2$.

This concludes the proof of the theorem. □

3.1.1.8. Remark. — In the situation of Theorem 3.1.1.5 the group G_i is not p-torsion in general, cf. [Har18, Lecture 3, Theorem 2.12]. For example, let $W \simeq \mathrm{K}(\mathbb{Z}/p\mathbb{Z}, n)$ and $X \simeq \mathrm{K}(\mathbb{Z}/p^2\mathbb{Z}, n+1)$. We have $\mathrm{P}_{\Sigma W}(X) \simeq \mathrm{pt}$ and thus

$$\mathrm{K}(G_1, n+1) \simeq \mathrm{P}_{\Sigma^2 W}(X) \simeq K(\mathbb{Z}/p^2\mathbb{Z}, n+1).$$

There are some other variants of Theorem 3.1.1.5

3.1.1.9. Definition. — Let $n \geq 1$ be a natural number. A pointed homotopy type W satisfies the *n-supported p-local condition* if

(i) W is $(n-1)$-connected,

(ii) $\widetilde{\mathrm{H}}_\bullet(W; \mathbb{Z})$ is p-local, and

(iii) $\mathrm{H}^n\left(W; \mathbb{Z}_{(p)}\right) \neq 0$.

3.1.1.10. Remark. — In the situation of Theorem 3.1.1.5, if we replace the n-supported p-torsion condition by n-supported p-local condition, we obtain instead a fibre sequence

$$\mathrm{K}(G_i, n+i) \to \mathrm{P}_{\Sigma^{i+1}W}(X) \to \mathrm{P}_{\Sigma^i W}(X)$$

where G_i is a p-local abelian group. The proof works the same, where we use a "p-local version" of Proposition 3.1.1.7: If W satisfies the n-supported p-local condition, the Eilenberg–MacLane spaces $\mathrm{K}(G, \ell)$ is W-full for every $\ell \geq n$.

3.1.1.11. Remark. — In Bousfield's original definition of the n-supported p-torsion (respectively p-local) condition [Bou94, §7], the hypothesis (i) is replaced by the assumption that $\widetilde{\mathrm{H}}_i(W; \mathbb{Z}) = 0$ for all $i < n$. Both definitions imply that the pointed homotopy type ΣW is n-connected. We make this simplification in our definition since we are mostly concerned with contraction of ΣW in our later applications.

3.1.1.12. — A p-local homotopy type satisfies the n-supported p-local condition for some $n \in \mathbb{N}$. Let W be a p-torsion homotopy type, i.e. the homotopy groups of W are p-primary torsion groups in all degrees. Then W satisfies (i) and (ii) of Definition 3.1.1.3. However, W doesn't have to satisfy (iii). For these p-torsion homotopy types, we have a supplemental version of the Theorem 3.1.1.5, which is proved similarly.

3.1.1.13. Definition. — Let $n \geq 1$ be a natural number. A homotopy type $W \in \mathcal{H}\mathrm{o}$ satisfies the *n-supported divisible p-torsion condition* if

(i) W is $(n-1)$-connected,
(ii) $\widetilde{\mathrm{H}}_\bullet(W; \mathbb{Z})$ is p-primary, and
(iii) $\mathrm{H}_n(W; \mathbb{Z})$ is a non-trivial divisible abelian group.

3.1.1.14. Example. — Recall the Prüfer group $\mathbb{Z}/p^\infty$, see Example 2.3.1.11. For every natural number $n \geq 1$, the Eilenberg–MacLane space $\mathrm{K}(\mathbb{Z}/p^\infty, n)$ satisfies the n-supported divisible p-torsion condition. In particular, we have that $\mathrm{H}_n\left(\mathrm{K}(\mathbb{Z}/p^\infty, n); \mathbb{Z}/p\mathbb{Z}\right) = 0$ and $\mathrm{H}^n\left(\mathrm{K}(\mathbb{Z}/p^\infty, n); \mathbb{Z}/p\mathbb{Z}\right) = 0$.

***3.1.1.15. Theorem* (Bousfield).** — *Let W be a homotopy type satisfying the n-supported divisible p-torsion condition. For $X \in \mathcal{H}\mathrm{o}_*$ and $i \geq 1$, the fibre of the map $\mathrm{P}_{\Sigma^{i+1}W} X \to \mathrm{P}_{\Sigma^i W} X$ is equivalent to a product*

$$\mathrm{K}(G_i, n+i) \times \mathrm{K}(G_{i+1}, n+i+1),$$

where G_i is a divisible p-primary abelian group and G_{i+1} is a p-primary abelian group such that $\mathrm{Hom}_{\mathbf{AbGrp}}(\mathbb{Z}/p^\infty, G) = 0$.

Proof. — See [Bou94, Theorem 7.7]. ☐

3.1.2. Unstable Class Invariance Theorem. — The goal of this subsection is to prove the following theorem. Recall that we work with a fixed prime number p and recall the definition of the type of a p-local spectrum (see Definition 1.2.0.3).

3.1.2.1. Theorem **(Unstable Class Invariance Theorem).** — *Let V and V' be p-local finite homotopy types whose rational homology groups are trivial. The following statements are equivalent:*

(i) $\langle \Sigma V\rangle \leq \langle \Sigma V'\rangle$

(ii) $\mathrm{type}(\Sigma^\infty V) \geq \mathrm{type}(\Sigma^\infty V')$ *and* $\mathrm{conn}(\Sigma V) \geq \mathrm{conn}(\Sigma V')$.

Proof. — The proof idea is to relate the unstable Bousfield classes with the Bousfield classes of their suspension spectra (see ¶1.2.0.15). The theorem is a corollary of Theorem 3.1.2.7 and Theorem 3.1.2.6, which we prove in the following.

To apply Theorem 3.1.2.7, note that the hypothesis of the connectivities of the cohomology groups $\widetilde{\mathrm{H}}^\bullet(-;\mathbb{Z}/p\mathbb{Z})$ is satisfied from the finiteness assumption on V and V'. □

3.1.2.2. Definition. — Two homotopy types W and W' are *stably Bousfield equivalent* if there exist natural numbers j and k such that $\langle \Sigma^j W\rangle \leq \langle W'\rangle$ and $\langle \Sigma^k W'\rangle \leq \langle W\rangle$. This defines an equivalence relation of homotopy types.

3.1.2.3. Definition. — Let W and W' be homotopy types. Denote the *stable Bousfield equivalence class* of W by $\langle W\rangle_\Sigma$. We write $\langle W\rangle_\Sigma \leq \langle W'\rangle_\Sigma$ if there exists a natural number j such that $\langle \Sigma^j W\rangle \leq \langle W'\rangle$.

3.1.2.4. Remark. — Let W and W' be homotopy types. If there exist natural numbers h and i such that $\langle \Sigma^h W\rangle = \langle \Sigma^i W'\rangle$, then $\langle W\rangle_{\mathrm{st}} = \langle W'\rangle_{\mathrm{st}}$. However, the converse is not true in general. For example, the homotopy types $\mathrm{S}^1 \vee \mathrm{S}^2$ and $\mathrm{S}^1 \vee \mathrm{S}^3$ are stably Bousfield-equivalent, but non of their iterated suspensions are (unstably) Bousfield-equivalent.

3.1.2.5. Theorem **(Class Invariance Theorem).** — *Let F and F' be two p-local finite spectra. Then $\langle F\rangle \leq \langle F'\rangle$ if and only if* $\mathrm{type}(F) \geq \mathrm{type}(F')$.

Sketch. — This is a consequence of the Thick Subcategory Theorem, see Theorem 1.2.0.9. See also [HS98, Theorem 14]. □

3.1.2.6. Theorem. — *Let V and V' be pointed p-local finite homotopy types. The following statements are equivalent*

(i) $\langle V\rangle_\Sigma \leq \langle V'\rangle_\Sigma$.

(ii) $\mathrm{type}(\Sigma^\infty V) \geq \mathrm{type}(\Sigma^\infty V')$.

(iii) $\langle \Sigma^\infty V\rangle \leq \langle \Sigma^\infty V'\rangle$.

Proof. — The equivalence between (ii) and (iii) follows from Theorem 3.1.2.5. It suffices to show the equivalence between (i) and (ii). Recall from Definition 1.2.0.3 the p-local Morava K-theory spectrum $\mathrm{K}(h)$ of height h for $h \in \mathbb{N}$, which was used to define the type of a finite spectrum.

Assume (i). Let m be a natural number such that the Morava K-theory cohomology $\mathrm{K}(m)^\bullet(V')$ is trivial. Representing $\mathrm{K}(m)$ by an Ω-spectrum $(E_k)_{k\in\mathbb{Z}}$. Thus, for all $k \in \mathbb{Z}$, the pointed homotopy type E_k is V'-less. By our assumption the homotopy type E_k is also $\Sigma^j V$-less for some natural number j. This implies that $\mathrm{K}(m)^\bullet(V) = 0$. Since V and V' are finite homotopy types, the acyclicity of $\mathrm{K}(m)$-homology of V and V' follows from [AdaSH, Part III, Lemma 13.1], which is a universal coefficient theorem for generalised cohomology theories.

Assume (ii), we define a full ∞-subcategory $\mathcal{C}$ of the ∞-category $\mathcal{S}\mathrm{p}^{\mathrm{fin}}_{(p)}$ of p-local finite spectra as follows: A p-local spectrum F is object of $\mathcal{C}$ if there exist an integer z and a p-local finite complex V_F such that $\langle V_F\rangle_\Sigma \leq \langle V'\rangle_\Sigma$ and $F \simeq \Sigma^z\Sigma^\infty V_F$. By Proposition 2.3.1.7 the ∞-subcategory $\mathcal{C}$ is a thick ∞-subcategory of $\mathcal{S}\mathrm{p}^{\mathrm{fin}}_{(p)}$. By the Thick Subcategory Theorem (see Theorem 1.2.0.9), we have that $\Sigma^\infty V \in \mathcal{C}$, because $\Sigma^\infty V' \in \mathcal{C}$ and $\mathrm{type}(\Sigma^\infty V) \geq \mathrm{type}(\Sigma^\infty V')$. In particular, we conclude that $\langle V\rangle_\Sigma \leq \langle V'\rangle_\Sigma$. See also [Bou94, Theorem 9.14]. ☐

3.1.2.7. Theorem. — *Let W and W' be pointed homotopy types such that $\widetilde{\mathrm{H}}_\bullet(W;\mathbb{Z})$ and $\widetilde{\mathrm{H}}_\bullet(W';\mathbb{Z})$ are p-primary abelian groups. Then $\langle \Sigma W\rangle \leq \langle \Sigma W'\rangle$ if and only if the following combined conditions hold:*

(i) $\langle \Sigma W\rangle_\Sigma \leq \langle \Sigma W'\rangle_\Sigma$,

(ii) $\mathrm{conn}(\Sigma W) \geq \mathrm{conn}(\Sigma W')$ *(see Definition 1.3.0.4), and*

(iii) $\mathrm{conn}(\widetilde{\mathrm{H}}^\bullet(\Sigma W; \mathbb{Z}/p\mathbb{Z})) \geq \mathrm{conn}(\widetilde{\mathrm{H}}^\bullet(\Sigma W'; \mathbb{Z}/p\mathbb{Z}))$,

where the connectivity $\mathrm{conn}(\widetilde{\mathrm{H}}^\bullet(\Sigma W; \mathbb{Z}/p\mathbb{Z}))$ of the cohomology is the degree of the lowest non-trivial cohomology group minus one.

3.1.2.8. — In the above theorem the homotopy types W and W' satisfy either the n-supported p-torsion, or the n-supported divisible p-torsion condition. The proof of the theorem is at the end of this section, where we uses Theorem 3.1.2.10 and Theorem 3.1.2.12 below.

3.1.2.9. Remark. — Example 3.1.1.14 shows that the connectivity of the cohomology groups does not necessarily coincide with the connectivity of the homotopy type.

3.1.2.10. Theorem. — *Let $n \geq 1$ be a natural number, and let W be a pointed homotopy type satisfying the n-supported p-torsion condition. Then for every natural number $k \geq 1$, we have*

$$\langle \Sigma W\rangle = \langle \Sigma^k W\rangle \vee \langle \mathrm{K}(\mathbb{Z}/p\mathbb{Z}, n+1)\rangle.$$

Proof. — We present the proof of [Bou94, Theorem 9.10]. By Corollary 2.1.1.12 we have $\langle \Sigma W \rangle \geq \langle \Sigma^k W \rangle$ and by Proposition 3.1.1.7 we have $\langle \Sigma W \rangle \geq \langle \mathrm{K}(\mathbb{Z}/p\mathbb{Z}, n+1) \rangle$. Theses gives the following comparison

$$\langle \Sigma W \rangle \geq \langle \Sigma^k W \rangle \vee \langle \mathrm{K}(\mathbb{Z}/p\mathbb{Z}, n+1) \rangle.$$

For the other direction, let Y be a pointed connected homotopy type such that Y is $\Sigma^k W$-less and $\mathrm{K}(\mathbb{Z}/p\mathbb{Z}, n+1)$-less. Denote the fibre of the canonical map $\lambda_{\Sigma W}(Y)\colon Y \to \mathrm{P}_{\Sigma W} Y$ by F, given by the natural transformation $\lambda_{\Sigma W}$. To show that Y is ΣW-less, it is equivalent to show that F is contractible.

By Theorem 3.1.1.5 we know that $\pi_i(F)$ is p-primary for $n+1 \leq i \leq n+k-1$ and trivial otherwise. Since $\mathrm{P}_{\Sigma W} Y$ is ΣW-less, it is $\mathrm{K}(\mathbb{Z}/p\mathbb{Z}, i)$-less for all $i \geq n+1$, by Proposition 3.1.1.7. Thus F is $\mathrm{K}(\mathbb{Z}/p\mathbb{Z}, i)$-less for all $i \geq n+1$, by Proposition 2.1.3.1. Now we can show that the homotopy groups of F are all trivial using its Postnikov tower, which has finitely many non-trivial stages. Consider the fibre sequence

$$\mathrm{K}(\pi_{n+k-1}(F), n+k-1) \to F \to \tau_{\leq n+k-2} F.$$

By the universal property of the fibre any morphism $\mathrm{K}(\mathbb{Z}/p\mathbb{Z}, n+k-1) \to F$ factors through the fibre $\mathrm{K}(\pi_{n+k-1}(F), n+k-1)$. Since F is $\mathrm{K}(\mathbb{Z}/p\mathbb{Z}, n+k-1)$-less, every morphism $\mathrm{K}(\mathbb{Z}/p\mathbb{Z}, n+k-1) \to \mathrm{K}(\pi_{n+k-1}(F), n+k-1)$ is the null map, which implies that $\pi_{n+k-1}(F) = 0$. Repeating this procedure we conclude that F is weakly contractible. Therefore, $Y \simeq \mathrm{P}_{\Sigma W} Y$ and we obtain

$$\langle \Sigma W \rangle \leq \langle \Sigma^k W \rangle \vee \langle \mathrm{K}(\mathbb{Z}/p\mathbb{Z}, n+1) \rangle. \qquad \square$$

3.1.2.11. Remark. — Recall that we have the "p-local" or "multiple prime" version of the n-supported p-torsion condition, see Definition 3.1.1.3 and Remark 3.1.1.6. Theorem 3.1.2.10 has also corresponding generalisations, see [Bou94, Theorem 9.10].

3.1.2.12. Theorem. — *Let $n \geq 1$ be a natural number, and let W be a pointed homotopy type satisfying the n-supported divisible p-torsion condition. Then for every natural number $k \geq 1$, we have*

$$\langle \Sigma W \rangle = \langle \Sigma^k W \rangle \vee \langle \mathrm{K}(\mathbb{Z}/p^\infty, n+1) \rangle.$$

Proof. — The proof strategy is the same as that of Theorem 3.1.2.10, where we make use of Theorem 3.1.1.15. See also [Bou94, Theorem 9.11]. $\square$

Proof of Theorem 3.1.2.7. — We follow the proof of [Bou94, Theorem 9.12]. Assuming that $\langle \Sigma W \rangle \leq \langle \Sigma W' \rangle$, we have $\langle \Sigma W \rangle_\Sigma \leq \langle \Sigma W' \rangle_\Sigma$ by definition. Consider the $\Sigma W'$-contraction $\lambda_{\Sigma W'}(\Sigma W)\colon \Sigma W \to \mathrm{P}_{\Sigma W'}(\Sigma W)$ of ΣW. We have $\mathrm{P}_{\Sigma W'}(\Sigma W) \simeq \mathrm{pt}$ by assumption. Thus the map $\lambda_{\Sigma W'}(\Sigma W)$ induces isomorphisms on homotopy groups of degree less than or equal to $\mathrm{conn}(\Sigma W')$, by Proposition 2.2.2.6. So we have

$$\mathrm{conn}(\Sigma W) \geq \mathrm{conn}(\Sigma W').$$

Denote $d := \operatorname{conn}\left(\widetilde{\mathrm{H}}^{\bullet}(\Sigma W'; \mathbb{Z}/p\mathbb{Z})\right)$. The Eilenberg–MacLane space $\mathrm{K}(\mathbb{Z}/p\mathbb{Z}, m)$ is $\Sigma W'$-less for every natural number $m \leq d$, because of the Brown Representability $[\Sigma^{\bullet}\Sigma W', \mathrm{K}(\mathbb{Z}/p\mathbb{Z}, d)] \cong \widetilde{\mathrm{H}}^{d-\bullet}(\Sigma W'; \mathbb{Z}/p\mathbb{Z})$. By assumption $\mathrm{K}(\mathbb{Z}/p\mathbb{Z}, m)$ is also ΣW-less, for every $m \leq d$. Therefore, we obtain

$$\operatorname{conn}\left(\widetilde{\mathrm{H}}^{\bullet}(\Sigma W; \mathbb{Z}/p\mathbb{Z})\right) \geq \operatorname{conn}\left(\widetilde{\mathrm{H}}^{\bullet}(\Sigma W; \mathbb{Z}/p\mathbb{Z})\right).$$

Now assume that (i)-(iii) hold. The stable Bousfield class inequality means that $\langle \Sigma^k W \rangle \leq \langle \Sigma W' \rangle$ for some natural number $k \geq 1$. Denote the connectivities of ΣW and $\Sigma W'$ by c and c', respectively. Define the abelian groups

$$G = \begin{cases} \mathbb{Z}/p\mathbb{Z}, & \text{if } \widetilde{\mathrm{H}}^{n+1}(\Sigma W; \mathbb{Z}/p\mathbb{Z}) \neq 0 \\ \mathbb{Z}/\, p^{\infty}, & \text{otherwise} \end{cases}$$

and

$$G' = \begin{cases} \mathbb{Z}/p\mathbb{Z}, & \text{if } \widetilde{\mathrm{H}}^{n+1}(\Sigma W'; \mathbb{Z}/p\mathbb{Z}) \neq 0 \\ \mathbb{Z}/\, p^{\infty}, & \text{otherwise} \end{cases}$$

Then we have

$$\langle \Sigma W \rangle = \langle \Sigma^k W \rangle \vee \langle \mathrm{K}(G, c+1) \rangle \leq \langle \Sigma W \rangle \vee \langle \mathrm{K}(G', c'+1) \rangle = \langle \Sigma W' \rangle;$$

here middle comparison holds by (2.3.1.1) and (2.3.1.3) □

3.1.2.13. Definition. — A pointed homotopy type X is a *suspension homotopy type* if there exists a pointed homotopy type W such that $X \simeq \Sigma W$.

3.1.2.14. Remark. — Theorem 3.1.2.1 is about the comparison between the Bousfield classes of suspension homotopy types. However, statement (ii) of the theorem does not depends on the suspension. It is unclear whether there is a similar statement for all p-torsion finite complexes. Applying Proposition 2.4.2.11 we obtain a small extension of Theorem 3.1.2.1 to the Bousfield classes of simply connected finite complexes (that are not necessarily suspension homotopy types).

3.1.2.15. Theorem. — *Let V and W be pointed simply-connected p-local finite homotopy types. Assume that* $\operatorname{type}(\Sigma^\infty V), \operatorname{type}(\Sigma^\infty W) \in \{0, 1\}$. *Then $\langle V \rangle = \langle W \rangle$ if and only if the following combined conditions hold:*

(i) $\operatorname{conn}(V) = \operatorname{conn}(W)$, *and*

(ii) $\operatorname{type}(\Sigma^\infty V) = \operatorname{type}(\Sigma^\infty W)$.

Proof. — Assuming that $\langle V \rangle = \langle W \rangle$, we have $\operatorname{type}(\Sigma^\infty V) = \operatorname{type}(\Sigma^\infty W)$, because $\langle V \rangle_\Sigma = \langle W \rangle_\Sigma$. The W-contraction $\lambda_W(V)\colon V \to \mathrm{P}_W(V) \simeq \mathrm{pt}$ induces an isomorphism of homotopy groups in every degree $i < \operatorname{conn}(W)$. By the same reasons, the V-contraction $\lambda_V(W)\colon W \to \mathrm{P}_V(W) \simeq \mathrm{pt}$ induces an isomorphism of homotopy groups in every degree $i < \operatorname{conn}(V)$. These combined shows that $\operatorname{conn}(V) = \operatorname{conn}(W)$.

Assume that $\operatorname{type}(\Sigma^\infty V) = \operatorname{type}(\Sigma^\infty W) \in \{0, 1\}$ and $\operatorname{conn}(V) = \operatorname{conn}(W)$. Applying Theorem 3.1.2.1 we obtain that $\langle \mathrm{M}(\underline{\mathrm{H}_*(V)})\rangle = \langle \mathrm{M}(\underline{\mathrm{H}_*(W)})\rangle$, because they are suspension homotopy types. Furthermore, we have $\operatorname{type}(\Sigma^\infty V) = \operatorname{type}(\Sigma^\infty(\mathrm{M}(\underline{\mathrm{H}_*(V)})))$ by the following reasons:

(i) The suspension spectrum $\Sigma^\infty V$ is of type 0 if and only if its the $\mathrm{H}\mathbb{Z}$-homology of V is torsion free in some degrees. Since the wedge $\mathrm{M}(\underline{\mathrm{H}_*(V)})$ of Moore spaces encodes exactly the $\mathrm{H}\mathbb{Z}$-homology of V, its suspension spectrum is also of type 0.

(ii) If $\Sigma^\infty V$ is of type 1, the reduced homology groups $\widetilde{\mathrm{H}}^\bullet(V;\mathbb{Z})$ of V are p-primary in all degrees. Since V is p-locally finite, the Bousfield class of a wedge component of $\mathrm{M}(\underline{\mathrm{H}_*(V)})$ is the same as a mod p Moore space $\mathrm{M}(\mathbb{Z}/p\mathbb{Z}, n)$ for a natural number $n \geq 2$, by Example 2.3.1.10. From this, we see that the suspension spectrum of $\mathrm{M}(\underline{\mathrm{H}_*(V)})$ is of type 1.

By the same arguments we have $\operatorname{type}(\Sigma^\infty W) = \operatorname{type}(\Sigma^\infty(\mathrm{M}(\underline{\mathrm{H}_*(W)})))$. Using Theorem 3.1.2.1 again we obtain that

$$\langle \Sigma W\rangle = \langle \Sigma\mathrm{M}(\underline{\mathrm{H}_*(W)})\rangle \text{ and } \langle \Sigma V\rangle = \langle \Sigma\mathrm{M}\left(\underline{\mathrm{H}_*(V)}\right)\rangle.$$

Therefore, we have $\langle V\rangle = \langle W\rangle$ by Proposition 2.4.2.11. □

***3.1.2.16. Question* (Bousfield).** — Can we extend Theorem 3.1.2.15 to p-local finite homotopy type of other types? See also the end of [Bou96, Section 4].

3.1.3. Fibration theorems. — In this subsection we discuss the interaction of the contraction of an n-supported p-torsion (respectively p-local) homotopy type with fibre sequences. We work in the following situation.

3.1.3.1. Situation. — Fix a prime number p. Let $n \geq 1$ be a natural number, and let W be a pointed homotopy type satisfying the n-supported p-torsion (respectively p-local) condition. Consider the functor $\mathrm{P}_{\Sigma W}\colon \mathcal{H}\mathrm{o}_* \to \mathcal{H}\mathrm{o}_*$.

3.1.3.2. Theorem. — *Let $F \to X \xrightarrow{f} B$ be a fibre sequence in $\mathcal{H}\mathrm{o}_*$ where B is connected. In Situation 3.1.3.1 there exists an induced fibre sequence*

$$\mathrm{K}(G, n) \to \mathrm{P}_{\Sigma W}(F) \to \operatorname{fib}\left(\mathrm{P}_{\Sigma W}(X) \xrightarrow{f_*} \mathrm{P}_{\Sigma W}(B)\right),$$

where G is a p-primary (respectively p-local) abelian group.

Proof. — This is [Bou94, Theorem 8.1] and we give an outline of the proof. Using Theorem 2.2.2.3 we obtain a commutative diagram

$$\begin{array}{ccccc} F & \longrightarrow & X & \xrightarrow{f} & B \\ \downarrow & & \downarrow{\scriptstyle \overline{L}} & & \downarrow \\ \mathrm{P}_{\Sigma W}(F) & \longrightarrow & \overline{X} & \longrightarrow & \mathrm{P}_{\Sigma^2 W}(B) \end{array}$$

where the rows are fibre sequences, the map $\overline{L}$ is a ΣW-equivalence and the homotopy type $\overline{X}$ is $\Sigma^2 W$-less. Applying $\mathrm{P}_{\Sigma W}$ to the lower fibre sequence induces the commutative diagram

$$\begin{array}{ccccc}
E_1 & \longrightarrow & E_2 & \longrightarrow & E_3 \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{P}_{\Sigma W}(F) & \longrightarrow & \overline{X} \simeq \mathrm{P}_{\Sigma^2 W}(\overline{X}) & \longrightarrow & \mathrm{P}_{\Sigma^2 W}(B) \\
\downarrow & & \downarrow{\scriptstyle \lambda_{\Sigma W}(\overline{X})} & & \downarrow{\scriptstyle \lambda_{\Sigma W}(B)} \\
\mathrm{fib}\,(\mathrm{P}_{\Sigma W}(f)) & \longrightarrow & \mathrm{P}_{\Sigma W}(X) \simeq \mathrm{P}_{\Sigma W}(\overline{X}) & \xrightarrow{\mathrm{P}_{\Sigma W}(f)} & \mathrm{P}_{\Sigma W}(B)
\end{array}$$

where the rows and columns are fibre sequences. By Theorem 3.1.1.5 the fibres E_2 and E_3 are Eilenberg–MacLane spaces whose non-trivial homotopy groups are p-primary torsion (respectively p-local) abelian groups. One can calculate the homotopy type of E_1 from that of E_2 and E_3 using the long exact sequence of homotopy groups associated to upper horizontal fibre sequence. □

3.1.3.3. Remark. — The fibre of of the induced map

$$\mathrm{P}_{\Sigma W}(F) \to \mathrm{hofib}\,(\mathrm{P}_{\Sigma W}(X) \to \mathrm{P}_{\Sigma W}(B))$$

is called the *error term* of $\mathrm{P}_{\Sigma W}(f)$.

(i) Dror Farjoun–Smith gives a generalisation of Theorem 3.1.3.2 where they don't make the assumption that W is n-supported p-torsion (respectively p-local), see [DS95, Theorem D]. In this more general case, the error terms is a product of Eilenberg–MacLane spaces.

(ii) Later Bousfield proves more general versions of the above theorem where the ΣW-contraction is replaced by the localisation of $\mathcal{H}\mathrm{o}_*$ at g-equivalences for a fixed morphism g of pointed homotopy types, see [Bou97, Theorems 6.1 and 9.7]. However, in this general setting, the error term is also more complicated.

3.1.3.4. Theorem. — *Consider a pullback diagram*

$$\begin{array}{ccc}
U & \longrightarrow & V \\
\downarrow & \lrcorner & \downarrow \\
Y & \longrightarrow & Z
\end{array}$$

in $\mathcal{H}\mathrm{o}_*$. In Situation 3.1.3.1 define a pointed homotopy type F by the induced pullback diagram in $\mathcal{H}\mathrm{o}_*$ below

$$\begin{array}{ccc}
F & \longrightarrow & \mathrm{P}_{\Sigma W}(V) \\
\downarrow & \lrcorner & \downarrow \\
\mathrm{P}_{\Sigma W}(Y) & \longrightarrow & \mathrm{P}_{\Sigma W}(Z).
\end{array}$$

Then there exists an induced fibre sequence

$$\mathrm{K}(G,n) \to \mathrm{P}_{\Sigma W}(U) \to F$$

of pointed homotopy types, where G is a p-primary torsion (respectively p-local) abelian group.

Proof. — This is proven using Theorem 3.1.3.2 and Proposition 3.1.1.7. See also [Bou94, Theorem 8.2]. □

3.1.3.5. Theorem. — *Let V and Y be connected pointed homotopy types, where V is a finite complex. In Situation 3.1.3.1 the pointed mapping space $\mathcal{M}\mathrm{ap}_*\left(V, \mathrm{P}_{\Sigma W}(Y)\right)$ is ΣW-less. Furthermore, the fibre F_ϕ of the induced map*

$$\mathrm{P}_{\Sigma W}\left(\mathcal{M}\mathrm{ap}_*(V,Y)\right) \to \mathcal{M}\mathrm{ap}_*\left(V, \mathrm{P}_{\Sigma W}(Y)\right)$$

at a map $\phi \in \mathrm{Map}_*\left(V, \mathrm{P}_{\Sigma W}(Y)\right)$ satisfies the following properties:

(i) *In every degrees $i \geq n+1$ the homotopy group $\pi_i(F_\phi) = 0$*

(ii) *In every degree $j \geq 1$ the homotopy group $\pi_j(F_\phi)$ is a p-torsion (respectively p-local) and nilpotent.*

(iii) *The action of $\pi_1\left(\mathrm{P}_{\Sigma W}\,\mathcal{M}\mathrm{ap}_*(V,Y)\right)$ on $\pi_j(F_\phi)$ is nilpotent for $j \geq 1$.*

Proof. — Let $f\colon A \to B$ be a ΣW-equivalence in $\mathcal{H}\mathrm{o}_*$. Consider the following commutative diagram

$$\begin{array}{ccc} \mathcal{M}\mathrm{ap}_*\left(B, \mathcal{M}\mathrm{ap}_*\left(V, \mathrm{P}_{\Sigma W}(Y)\right)\right) & \xrightarrow{f^*} & \mathcal{M}\mathrm{ap}_*\left(A, \mathcal{M}\mathrm{ap}_*\left(V, \mathrm{P}_{\Sigma W}(Y)\right)\right) \\ \simeq\downarrow & & \downarrow\simeq \\ \mathcal{M}\mathrm{ap}_*\left(B \wedge V, \mathrm{P}_{\Sigma W}(Y)\right) & \xrightarrow[(f\wedge \mathrm{id})^*]{} & \mathcal{M}\mathrm{ap}_*\left(A \wedge V, \mathrm{P}_{\Sigma W}(Y)\right), \end{array}$$

where the vertical equivalences are given by the "tensor–Hom" adjunction. The lower horizontal arrow is also an equivalence, since $f \wedge \mathrm{id}_V$ is a ΣW-equivalence by Proposition 2.1.3.1 and $\mathrm{P}_{\Sigma W}(Y)$ is ΣW-less. Therefore, the morphism f^* is an equivalence, which implies that $\mathcal{M}\mathrm{ap}_*\left(V, \mathrm{P}_{\Sigma W}(Y)\right)$ is ΣW-less. For the rest of the proof, see [Bou94, Theorem 8.3]. □

3.1.3.6. Remark. — Recall that there is an equivalence

$$\mathrm{P}_W\left(\mathcal{M}\mathrm{ap}_*(\mathrm{S}^k, Y)\right) \simeq \mathcal{M}\mathrm{ap}_*(\mathrm{S}^k, \mathrm{P}_{\mathrm{S}^k \wedge W}(Y))$$

for every pointed connected homotopy type Y, see Corollary 2.2.1.3. In [DroCS, §3.A.2] Dror Farjoun posts the question whether $\mathrm{P}_W\left(\mathcal{M}\mathrm{ap}_*(V,Y)\right)$ is equivalent to a mapping space of the form $\mathrm{Map}_*(V, \widetilde{Y})$, for some suitable choices of pointed homotopy type Y and finite complex V.

3.2. v_h-periodic homotopy groups

From now on we fix a prime number p. In this section we recall the p-local v_h-periodic homotopy groups of a homotopy type and v_h-periodic equivalences of homotopy types. Everything is in this section is standard material, which one can also find for instance in [Bou01; Kuh08; Heu20b]. Recall the p-local Morava K-theory $\mathrm{K}(h)$ of height h, where $h \in \mathbb{N}$ (see §1.2).

3.2.0.1. Theorem. — *Let V be a p-local finite homotopy type. Then there exists a natural number h such that its suspension spectrum $\Sigma^\infty V$ is of type h (see Definition 1.2.0.3). Moreover, there exist natural numbers n and d_h such that the n-fold suspension $\Sigma^n V$ of V admits a map*

$$v_h \colon \Sigma^{d_h+n} V \to \Sigma^n V$$

which induces an isomorphism on $\mathrm{K}(h)$-homology and induces the zero map on $\mathrm{K}(m)$-homology for every natural numbers $m \neq h$.

Proof. — This is the "unstable version" of the Periodicity Theorem, and it follows from Theorem 1.2.0.6 and the Freundenthal Suspension Theorem, see [KocB, Corollary 3.2.3]. □

3.2.0.2. Convention. — In the situation of Theorem 3.2.0.1, we say V is of *type* h and call the map v_h a v_h *self-map* of V_h. Strictly speaking, the map v_h is not a self-map (a map from V_h to itself) of V_h. However, we follow this traditional abuse of notation, for example, cf. [Ada66; MRW77; HS98].

3.2.0.3. Situation. — Let $h \geq 1$ be a natural number, and let $V_h \in \mathcal{H}\mathrm{o}_{(p)}$ be a p-local finite complex of type h together with a v_h self-map $v_h \colon \Sigma^{d_{V_h}} V_h \to V_h$.

3.2.0.4. Definition. — Let X be a pointed connected homotopy type. In Situation 3.2.0.3, define a pointed homotopy type $\mathrm{T}_{V_h}(X)$ as the colimit of the following diagram in $\mathcal{H}\mathrm{o}_*$

$$\mathrm{Map}_*(V_h, X) \to \mathrm{Map}_*\left(\Sigma^{d_{V_h}} V_h, X\right) \to \cdots \to \mathrm{Map}_*\left(\Sigma^{k d_{V_h}} V_h, X\right) \to \cdots$$

induced by the v_h self-map of V_h.

3.2.0.5. Proposition. — *Let X be a pointed connected homotopy type. In Situation 3.2.0.3 there exist equivalences*

$$\mathrm{T}_{V_h}(X) \simeq \Omega^{d_{V_h}} \mathrm{T}_{V_h}(X) \simeq \mathrm{T}_{\Sigma^{d_{V_h}} V_h}(X)$$

of pointed homotopy types. In particular, $\mathrm{T}_{V_h}(X)$ is an infinite loop space.

Proof. — Since $\mathrm{T}_{V_h}(X)$ is formed by a filtered colimit in $\mathcal{H}\mathrm{o}_*$ and the d_{V_h}-dimensional sphere $\mathrm{S}^{d_{V_h}}$ is a compact object of $\mathcal{H}\mathrm{o}_*$, the d_{V_h}-fold loop space $\Omega^{d_{V_h}} \mathrm{T}_{V_h}(X)$ is equivalent to the colimit of the following diagram in $\mathcal{H}o_*$

$$\Omega^{d_{V_h}} \operatorname{Map}_* (V_h, X) \to \cdots \to \Omega^{d_{V_h}} \operatorname{Map}_* \left(\Sigma^{k d_{V_h}} V_h, X\right) \to \cdots,$$

because a compact object commutes with filtered colimits (see [HTT, §5.3.4]). We obtain the equivalence in the proposition by observing that

$$\Omega^{d_{V_h}} \operatorname{Map}_* \left(\Sigma^{k d_{V_h}} V_h, X\right) \simeq \operatorname{Map}_* \left(\Sigma^{(k+1) d_{V_h}} V_h, X\right). \qquad \square$$

3.2.0.6. Definition. — Let X be a pointed connected homotopy type. In Situation 3.2.0.3, define a spectrum $\Phi_{V_h}(X)$ modelled by an Ω-spectrum $(E_n)_{n\geq 0}$ where

(i) $E_{n+kd_{V_h}} \coloneqq \Omega^{d_{V_h}-n}(\mathrm{T}_{V_h}(X))$ for every natural number k, and
(ii) the structure maps are given by either the identity maps on iterated loop spaces of $\mathrm{T}_{V_h}(X)$ or the equivalence $\mathrm{T}_{V_h}(X) \simeq \Omega^{d_{V_h}} \mathrm{T}_{V_h}(X)$ from Proposition 3.2.0.5.

3.2.0.7. Proposition. — *Let X be a pointed connected homotopy type. In Situation 3.2.0.3, there exist equivalences*

(i) $\Omega^{\infty}\Phi_{V_h}(X) \simeq \mathrm{T}_{V_h}(X)$ *of pointed homotopy types, and*
(ii) $\Omega^{d_{V_h}} \Phi_{V_h}(X) \simeq \Phi_{V_h}(X)$ *of spectra.*

Proof. — This is by definition and by Proposition 3.2.0.5. $\square$

3.2.0.8. Definition. — Let X be a pointed connected homotopy type and let k be a natural number. In Situation 3.2.0.3, define the *v_h-periodic homotopy group $v_h^{-1}\pi_k(X;V_h)$ of X with coefficient V_h in degree k* as the stable homotopy group $\pi_k\left(\Phi_{V_h}(X)\right)$ of $\Phi_{V_h}(X)$ in degree k, or equivalently, as the homotopy group $\pi_k\left(\mathrm{T}_{V_h}(X)\right)$ of $\mathrm{T}_{V_h}(X)$ in degree k.

3.2.0.9. Remark. — In the situation of Definition 3.2.0.8 we consider the v_h-periodic homotopy groups of X as a graded abelian group, denoted by $v_h^{-1}\pi_\bullet(X;V_h)$. For a non-connected homotopy type, its v_h-periodic homotopy groups are defined separately on each of its connected components.

For $c \in \mathbb{N}$, recall that $\tau_{>c}X$ denotes the c-connected cover of X, see ¶1.3.0.5.

3.2.0.10. Proposition. — *Let X be a pointed connected homotopy type and let c be a natural number. In Situation 3.2.0.3 the natural map $\tau_{>c}X \to X$ induces a canonical isomorphism*

$$v_h^{-1}\pi_\bullet\left(X;V_h\right) \cong v_h^{-1}\pi_\bullet\left(\tau_{>c}X;V_h\right),$$

for every $c \geq 0$.

Proof. — By the $(\Sigma \dashv \Omega)$-adjunction there exists an equivalence

$$\mathcal{M}\mathrm{ap}_* \left(\Sigma^{t+kd_{V_n}} V_h, X\right) \simeq \mathcal{M}\mathrm{ap}_* \left(V_h, \Omega^{t+kd_{V_h}} X\right),$$

for every $t \geq 0$ and every $k \geq 0$. For a natural number $n > c$ there exists an equivalence $\Omega^n X \simeq \Omega^n(\tau_{>c} X)$. Therefore, if k is large enough so that $t + kd > c$, we have an equivalence $\mathcal{M}\mathrm{ap}_*\left(\Sigma^{t+kd} V_h, X\right) \simeq \mathcal{M}\mathrm{ap}_*\left(\Sigma^{t+kd} V_h, \tau_{>c} X\right)$. □

3.2.0.11. Proposition. — *Let $f\colon X \to Y$ be a morphism of pointed connected homotopy types. The following two statements are equivalent:*

(i) *The map f induces an isomorphism $v_h^{-1}\pi_\bullet(X; V_h) \xrightarrow{f_*} v_h^{-1}\pi_\bullet(X, ; V_h)$ for one p-local finite complex V_h of type h together with a v_h self-map.*

(ii) *The map f induces an isomorphism $v_h^{-1}\pi_\bullet(X; V_h) \xrightarrow{f_*} v_h^{-1}\pi_\bullet(X, ; V_h)$ for every p-local finite complex V_h of type h together with a v_h self-map.*

Proof. — (i) is a special case of (ii). We show that (i) implies (ii). Let V_h be the finite complex of type h in (i). First, note that the isomorphism f_* does not depend on the choice of v_h self-maps of V_h, since any two v_h self-maps become homotopic after suitable iterations, see [HS98, Corollary 3.7]. So, we just need to show the independence of f_* on the choice of V_h.

Define a full ∞-subcategory $\mathcal{C}$ of the ∞-category $\mathcal{S}\mathrm{p}^{\mathrm{fin}}_{(p)}$ of p-local finite spectra as follows: A p-local spectrum F is object of $\mathcal{C}$ if there exist an integer z and a p-local finite complex V_F of type h such that

(i) $F \simeq \Sigma^z \Sigma^\infty V_F$, and

(ii) for a natural number n there exists a v_h self-map of $\Sigma^n V_F$ inducing an isomorphism f_* on v_h-periodic homotopy groups with coefficient $\Sigma^n V_F$.

One can check that $\mathcal{C}$ is a thick ∞-subcategory of $\mathcal{S}\mathrm{p}^{\mathrm{fin}}_{(p)}$. Since $\mathcal{C}$ contains a type h finite spectra $\Sigma^\infty V_h$ by (i), we see that $\mathcal{C}$ is the full ∞-subcategory of $\mathcal{S}\mathrm{p}^{\mathrm{fin}}_{(p)}$ whose objects are finite spectra of type at least h, by the Thick Subcategory Theorem (see Theorem 1.2.0.9). □

3.2.0.12. Definition. — If a morphism $f\colon X \to Y$ of pointed connected homotopy types satisfies one of the equivalent conditions in Proposition 3.2.0.11, we say f is a *v_h-periodic equivalence*. If X and Y are nilpotent, we say by convention that f is a *v_0-periodic equivalence* if f is a rational homotopy equivalence, i.e. the induced map $f_*\colon \pi_\bullet(X) \otimes \mathbb{Q} \to \pi_\bullet(Y) \otimes \mathbb{Q}$ is an isomorphism (see ¶1.3.0.8).

3.2.0.13. Definition. — Let $f\colon X \to Y$ be a morphism of pointed homotopy types. We say f is a v_h-periodic equivalence if

(i) f induces an isomorphism on the sets of connected components, and

(ii) the restriction of f to each connected component of X, with an arbitrary chosen basepoint, is a v_h-periodic equivalence of pointed connected homotopy types.

3.2.0.14. Proposition. — *Let $h \geq 1$ be a natural number, and let $f\colon X \to Y$ be a v_h-periodic equivalence of pointed connected homotopy types. Then the v_h-periodic homotopy groups of the fibre F of f vanish.*

Proof. — Let V_h be a finite complex as in Situation 3.2.0.3. For every $k \in \mathbb{N}$ there is an induced fibre sequence

$$\mathcal{M}\mathrm{ap}_*\left(\Sigma^{kd_{V_h}} V_h, F\right) \to \mathcal{M}\mathrm{ap}_*\left(\Sigma^{kd_{V_h}} V_h, X\right) \to \mathcal{M}\mathrm{ap}_*\left(\Sigma^{kd_{V_h}} V_h, Y\right)$$

in of pointed homotopy types. Since filtered colimits commutes with finite limits, we obtain the following fibre sequence

$$\mathrm{T}_{V_h}(F) \to \mathrm{T}_{V_h}(X) \to \mathrm{T}_{V_h}(Y)$$

of pointed homotopy types. Recall the definition of v_h-periodic homotopy groups from Definition 3.2.0.8. By the hypothesis the second map in the above fibre sequence is an equivalence. Thus $\mathrm{T}_{V_h} F$ is contractible, which implies that the v_h-periodic homotopy groups of F are trivial. □

3.3. The contraction of a p-local finite complex

In this section we consider the contraction of the suspension of a p-local homotopy type which has the same stable Bousfield class as a p-local finite complex, see Situation 3.3.0.1. These kind of contractions relates closely with v_h-periodic equivalences of homotopy types, as we will present in §3.3.2. This section is expository, which also supplements [Bou94, §§10–13] with more details. We fix a prime number p and a natural number h throughout this section.

3.3.0.1. Situation. — Let W_{h+1} be a pointed p-local homotopy type satisfying the following hypotheses:

(i) There exists a finite complex V_{h+1} of type $h+1$ such that $\langle W_{h+1}\rangle_\Sigma = \langle V_{h+1}\rangle_\Sigma$. Denote $c^+(\Sigma W_{h+1}) \coloneqq \mathrm{conn}(\Sigma W_{h+1}) + 1$.

(ii) The homotopy type ΣW_{h+1} satisfies the $c^+(\Sigma W_{h+1})$-supported p-torsion condition (see Definition 3.1.1.3)

3.3.0.2. Remark. —

(i) The properties (i), (ii) in Situation 3.3.0.1 and the number $c^+(\Sigma W_{h+1})$ determine the Bousfield class of ΣW_{n+1}, by Theorem 3.1.2.7. In particular, there exist natural numbers j and k such that $\langle \Sigma^j V_{h+1}\rangle = \langle \Sigma^k W_{h+1}\rangle$.

(ii) Comparing with [Bou94, §10.1], we don't make the assumption that W_{h+1} is of minimal connectivity among the p-local homotopy types satisfying (i) and (ii), because we won't need it for the results concerning v_{h+1}-periodic homotopy theory in this chapter.

3.3.0.3. Proposition. — *In Situation 3.3.0.1,* $\mathrm{conn}(\Sigma W_{h+1}) \geq h+1$. *In other words,* $c^+(\Sigma W_{h+1}) \geq h+2$.

Proof. — By assumption $\langle W_{h+1}\rangle_\Sigma = \langle V_{h+1}\rangle_\Sigma$ and $\widetilde{\mathrm{K}(h)}^\bullet(\Sigma V_{h+1}) = 0$. Thus we have that $\widetilde{\mathrm{K}(h)}^\bullet(\Sigma W_{h+1}) = 0$. By Proposition 3.1.1.7 the Eilenberg–MacLane space $\mathrm{K}(\mathbb{Z}/p\mathbb{Z}, j)$ is $\widetilde{\mathrm{K}(h)}^\bullet$-acyclic for $j = \mathrm{conn}(\Sigma W_{h+1})$. The $\mathrm{K}(h)_\bullet$-acyclicity of $\mathrm{K}(\mathbb{Z}/p\mathbb{Z}, i)$, where $i \in \mathbb{N}$, is completely determined in [RW80]: It is $\mathrm{K}(h)_\bullet$-acyclic if and only if $i \geq h+1$ [RW80, Theorem 11.1]. Thus $\mathrm{conn}(\Sigma W_{h+1}) \geq h+1$. □

3.3.0.4. Example. — Here are some examples of p-local homotopy types satisfying the hypotheses of Situation 3.3.0.1.

(i) Every p-local finite complex of type $h+1$.

(ii) Let $n \geq 1$ be a natural number and let G be a finitely generated p-primary abelian group. The Moore space $\mathrm{M}(G, n)$ is an example of W_1.

(iii) For every natural number $m \geq h+1$, the homotopy type $V_{h+1} \vee \mathrm{K}(\mathbb{Z}/p\mathbb{Z}, m)$ is an example of W_{h+1}, for every finite complex V_{h+1} of type $h+1$, see Theorem 4.2.0.8. In particular, for $m = h+1$, the connectivity of $\Sigma(V_{h+1} \vee \mathrm{K}(\mathbb{Z}/p\mathbb{Z}, m))$ reaches the lower bound in Proposition 3.3.0.3.

3.3.0.5. Proposition. — *Let X be a pointed connected homotopy type whose homotopy groups are non-trivial in finitely many degrees. The ΣW_{h+1}-contraction map $\lambda_{\Sigma W_{h+1}}(X)\colon X \to \mathrm{P}_{\Sigma W_{h+1}}(X)$ of X induces isomorphisms*

$$\pi_i\left(\mathrm{P}_{\Sigma W_{h+1}}(X)\right) \cong \begin{cases} \pi_i Y, & \text{if } i < c^+(\Sigma W_{h+1}),\\ (\pi_i Y)//(\mathbb{Z}/p\mathbb{Z}), & \text{if } i = c^+(\Sigma W_{h+1}),\\ \pi_i Y \otimes \mathbb{Z}[1/p], & \text{otherwise.} \end{cases}$$

Proof. — We show that the $\mathrm{M}(\mathbb{Z}/p\mathbb{Z},\ c^+(\Sigma W_{h+1}))$-contraction of X is equivalent to $\mathrm{P}_{\Sigma W_{h+1}}(X)$. Then the isomorphisms follow from the calculation results in Theorem 2.4.1.5. Recall the definition of the wedge $\mathrm{M}(\underline{\mathrm{H}_\bullet(\Sigma W_{h+1})})$ of Moore spaces from Notation 2.4.2.1 and Definition 2.4.2.7. By Theorem 2.4.2.9 there exists a natural equivalence

$$\mathrm{P}_{\Sigma W_{h+1}}(X) \simeq \mathrm{P}_{\mathrm{M}(\underline{\mathrm{H}_\bullet(\Sigma W_{h+1})})}(X).$$

Since ΣW_{h+1} is $(c^+(\Sigma W_{h+1})-1)$-connected and its homology groups are p-primary, the Bousfield class $\langle \mathrm{M}(\underline{\mathrm{H}_\bullet(\Sigma W_{h+1})})\rangle$ is equal to

$$\langle \mathrm{M}(\mathbb{Z}/p\mathbb{Z}, c^+(\Sigma W_{h+1}))\rangle \vee \left(\bigvee_{i\in I} \langle \mathrm{M}(\mathbb{Z}/p\mathbb{Z}, c_i)\rangle\right),$$

where $c_i \geq c^+(\Sigma W_{h+1})$ for every $i \in I$, by Example 2.3.1.10. Moreover, for every $c \geq c^+(\Sigma W_{h+1})$, we have $\langle \mathrm{M}(\mathbb{Z}/p\mathbb{Z}, c^+(\Sigma W_{h+1}))\rangle \geq \langle \mathrm{M}(\mathbb{Z}/p\mathbb{Z}, c)\rangle$. Thus by Theorem 2.4.2.2 we obtain

$$\mathrm{P}_{\mathrm{M}(\underline{\mathrm{H}_\bullet(\Sigma W_{h+1})})}(X) \simeq \mathrm{P}_{\mathrm{M}(\mathbb{Z}/p\mathbb{Z},\ c^+(\Sigma W_{h+1}))}(X).$$

See also [Bou94, Proposition 10.5]. □

3.3.1. Properties of the ΣW_{h+1}-contraction. — In this section we list several consequences of the general theory of W-contraction applied to the ΣW_{h+1}-contraction for the pointed homotopy type W_{h+1} in Situation 3.3.0.1. Recall that V_{h+1} denotes a p-local finite complex of type $h+1$.

3.3.1.1. Proposition. — *let E be a spectrum with $\widetilde{E}^\bullet(V_{h+1}) = 0$ and let X be a pointed connected homotopy type. Then the ΣW_{h+1}-contraction $\lambda_{\Sigma W_{h+1}}(X)\colon X \to \mathrm{P}_{\Sigma W_{h+1}}(X)$ is an $E_\bullet$-equivalence and an $E^\bullet$-equivalence (the latter notation means E-cohomology equivalence).*

Proof. — By the hypothesis $\langle W_{h+1}\rangle_\Sigma = \langle V_{h+1}\rangle_\Sigma$ there exist natural numbers j and k such that $\langle W\rangle \geq \langle \Sigma^j V\rangle$ and $\langle V\rangle \geq \langle \Sigma^k W\rangle$. Because V_{h+1} is finite, we have that $\widetilde{E}_\bullet(V_{h+1}) = 0$. Since $\Sigma^k W$ is a V-full, we see that W is $E_\bullet$-acyclic by Proposition 2.1.2.10. By loc. cit. the ΣW_{h+1}-equivalence $X \to \mathrm{P}_{\Sigma W_{h+1}} X$ is an $E_\bullet$-equivalence.

Consider $E = (E_i)_{i\geq 0}$ as an Ω-spectrum. The condition $\widetilde{E}^\bullet(V_{n+1}) = 0$ means that the pointed homotopy type E_i is V_{h+1}-less for each $i \geq 0$. Thus the homotopy type E_i is also $\Sigma^k W_{n+1}$-less for every $i \geq 0$. So $\widetilde{E}^\bullet\left(\Sigma^k W_{h+1}\right) = 0$, which also implies that $\widetilde{E}^\bullet(\Sigma W_{h+1}) = 0$. In other words, E_i is ΣW_{h+1}-less for each $i \geq 0$. Therefore, the ΣW_{h+1}-equivalence $X \to \mathrm{P}_{\Sigma_{h+1}W}\, X$ is a $E^\bullet$-equivalence. See also [Bou94, Proposition 12.1] □

3.3.1.2. Example. — Recall we work with a fixed prime number p. Some examples of spectra that satisfy the hypothesis of Proposition 3.3.1.1 are the Eilenberg–MacLane spectrum $\mathrm{H}\mathbb{Z}[1/p]$, the BP-module spectrum $v_h^{-1}\mathrm{BP}$, the Johnson–Wilson spectrum $\mathrm{E}(h)$, the Morava K-theory spectrum $\mathrm{K}(h)$ of height h, the telescopic spectrum $\mathrm{T}(h)$ of height h and the spectrum $\mathrm{S}(h)$, see §1.2 and ¶1.2.0.13 for an introduction of some of those p-local spectra.

3.3.1.3. Proposition. — *Let $F \to X \to B$ be a fibre sequence in $\mathcal{H}\mathrm{o}_*$ where B is connected. Denote the fibre of the induced map $\mathrm{P}_{\Sigma W_{h+1}}(X) \to \mathrm{P}_{\Sigma W_{h+1}}(B)$ by F'. There exists an induced fibre sequence*

$$\mathrm{K}(G, c^+(\Sigma W_{h+1}) - 1) \to \mathrm{P}_{\Sigma W_{h+1}}(F) \to F'$$

where G is a p-primary torsion abelian group.

Proof. — Note that W_{h+1} is $(c^+(\Sigma W_{h+1}) - 1)$-supported p-torsion. The proposition is an application of Theorem 3.1.3.2. □

3.3.1.4. Proposition. — *The ΣW_{h+1}-contraction functor $\mathrm{P}_{\Sigma W_{h+1}}$ commutes with taking $c^+(\Sigma W_{h+1})$-connected covers, i.e. for every pointed connected homotopy type X, there exists a natural equivalence*

$$\tau_{>c^+(\Sigma W_{h+1})}\left(\mathrm{P}_{\Sigma W_{h+1}}(X)\right) \simeq \mathrm{P}_{\Sigma W_{h+1}}\left(\tau_{>c^+(\Sigma W_{h+1})}(X)\right).$$

Proof. — We apply Proposition 3.3.1.3 to the fibre sequence

$$\tau_{>c^+(\Sigma W_{h+1})}(X) \to X \to \tau_{\leq c^+(\Sigma W_{h+1})}(X).$$

Denote the fibre of the induced map $\mathrm{P}_{\Sigma W_{h+1}}(X) \to \mathrm{P}_{\Sigma W_{h+1}}\left(\tau_{\leq c^+(\Sigma W_{h+1})}(X)\right)$ by F'. Applying the computation results in Proposition 3.3.0.5 to $\mathrm{P}_{\Sigma W_{h+1}}\left(\tau_{\leq c^+(\Sigma W_{h+1})}(X)\right)$, we obtain

$$\pi_i(F') \cong \pi_i\left(\mathrm{P}_{\Sigma W_{h+1}}(X)\right)$$

for every $i \geq c^+(\Sigma W_{h+1}) + 1$. So the map $\mathrm{P}_{\Sigma W_{h+1}}\left(\tau_{>c^+(\Sigma W_{h+1})}(X)\right) \to \mathrm{P}_{\Sigma W_{h+1}}(X)$ induces an isomorphism of homotopy groups in all degree $i \geq c^+(\Sigma W_{h+1}) + 1$, by Proposition 3.3.1.3. It remains to show that $\mathrm{P}_{\Sigma W_{h+1}}\left(\tau_{>c^+(\Sigma W_{h+1})}(X)\right)$ is $c^+(\Sigma W_{h+1})$-connected, which holds by Proposition 2.2.2.6. See also [Bou94, Proposition 13.1]. □

3.3.1.5. Remark. — Recall that the ΣW_{h+1}-contraction functor also commutes with taking $(c^+(\Sigma W_{h+1}) - 1)$-connected cover, see Corollary 2.2.2.7.

3.3.1.6. Remark. — Let $\mathcal{H}\mathrm{o}_*^{>c^+(\Sigma W_{h+1})}$ denote the full ∞-subcategory of $\mathcal{H}\mathrm{o}_*$ whose objects are $c^+(\Sigma W_{h+1})$-connected pointed homotopy types. The functor $\tau_{>c^+(\Sigma W_{h+1})}$ of taking $c^+(\Sigma W_{h+1})$-connected covers is right adjoint to the fully faithful inclusion $\mathcal{H}\mathrm{o}_*^{>c^+(\Sigma W_{h+1})} \hookrightarrow \mathcal{H}\mathrm{o}_*^{>0}$.

3.3.1.7. Theorem. — *The functor*

$$\tau_{>c^+(\Sigma W_{h+1})} \circ \mathrm{P}_{\Sigma W_{h+1}} \colon \mathcal{H}\mathrm{o}_* \to \mathcal{H}\mathrm{o}_*^{>c^+(\Sigma W_{h+1})}$$

preserves finite limits.

Proof. — By [HTT, Corollary 4.4.2.5] it is equivalent to show that the functor $\tau_{>c^+(\Sigma W_{h+1})} \circ \mathrm{P}_{\Sigma W_{h+1}}$ preserves the terminal objects and pullbacks. Since every contractible object is ΣW_{h+1}-less, the functor preserves terminal objects. Using Theorem 3.1.3.4 and Remark 3.3.1.6 wee see that $\tau_{>c^+(\Sigma W_{h+1})} \circ \mathrm{P}_{\Sigma W_{h+1}}$ preserves pullback diagrams. □

3.3.2. ΣW_{h+1}-equivalences and v_h-periodic equivalences. — We continue to work in Situation 3.3.0.1. In this subsection we discuss the relationship between ΣW_{h+1}-equivalences and v_n-periodic equivalences.

3.3.2.1. Proposition. — *Let n be a natural number such that $1 \le n \le h$, and let V_n be a p-local finite complex of type n together with a v_n self-map. For every pointed connected homotopy type X, the pointed homotopy type $\mathrm{T}_{V_n}(X)$ is ΣW_{h+1}-less.*

Proof. — Recall from Situation 3.3.0.1 that $\langle \Sigma^j W_{h+1} \rangle \le \langle \Sigma V_{h+1} \rangle$ for some natural number $j \ge 1$. It suffices to show that $\mathrm{T}_{V_n}(X)$ is ΣV_{h+1}-less. Indeed, assuming this, we have that $\mathrm{T}_{V_n}(X)$ is $\Sigma^{j+i} W_{h+1}$-less for all $i \ge 0$, i.e.

$$\mathcal{M}\mathrm{ap}_* \left(\Sigma^{j+i} W_{h+1}, \mathrm{T}_{V_n}(X)\right) \simeq \mathrm{pt}.$$

By the $\Sigma \dashv \Omega$-adjunction of $\mathcal{H}\mathrm{o}_*$, we obtain an equivalence

$$\mathcal{M}\mathrm{ap}_* \left(\Sigma W_{h+1}, \Omega^{j+i-1} \mathrm{T}_{V_n}(X)\right) \simeq \mathrm{pt}.$$

Choose i such that $j + i - 1 = kd_{V_n}$ for some natural number i. Since the homotopy type $\mathrm{T}_{V_n}(X)$ is d_{V_n}-periodic with respect to taking the loop functor (see Proposition 3.2.0.5), we obtain

$$\mathcal{M}\mathrm{ap}_* \left(\Sigma W_{h+1}, \mathrm{T}_{V_n}(X)\right) \simeq \mathcal{M}\mathrm{ap}_* \left(\Sigma W_{h+1}, \Omega^{kd_{V_n}} \mathrm{T}_{V_n}(X)\right) \simeq \mathrm{pt},$$

that is, $\mathrm{T}_{V_n}(X)$ is ΣW_{h+1}-less.

Therefore, we would like to show that $\mathcal{M}\mathrm{ap}_* (\Sigma V_{h+1}, \mathrm{T}_{V_n}(X))$ is contractible. Since $\mathrm{T}_{V_n}(X)$ is constructed via a filtered colimit and ΣV_{h+1} is finite, the mapping space $\mathcal{M}\mathrm{ap}_*(\Sigma V_{h+1}, \mathrm{T}_{V_n}(X))$ is equivalent to the colimit of the filtered diagram

$$\mathrm{Map}_*(\Sigma V_{h+1} \wedge V_n, \mathrm{T}_{V_n}(X)) \to \mathrm{Map}_*\left(\Sigma V_{h+1} \wedge \Sigma^{d_{V_n}} V_n, \mathrm{T}_{V_n}(X)\right) \to \cdots \qquad (3.3.2.1)$$

The induced self-map

$$\mathrm{id}_{\Sigma V_{h+1}} \wedge v_n \colon \Sigma^{d_{V_n}}(\Sigma V_{h+1} \wedge V_n) \to \Sigma V_{h+1} \wedge V_n$$

induces the zero map on $\mathrm{K}(j)$-homology for all $j \geq 0$, because V_{h+1} and V_n are of different type. Thus, by the Nilpotence Theorem (see Theorem 1.2.0.2), the map $\mathrm{id}_{\Sigma V_{h+1}} \wedge v_n$ is nilpotent. This implies that almost all terms in the diagram (3.3.2.1) are contractible. So its colimit is contractible. See also [Bou94, Lemma 11.4]. □

3.3.2.2. Theorem. — *Let X be a pointed connected homotopy type. For every natural number $1 \leq n \leq h$, the canonical map $\lambda_{\Sigma W_{h+1}}(X)\colon X \to \mathrm{P}_{\Sigma W_{h+1}}(X)$ given by ΣW_{h+1}-contraction is a v_n-periodic equivalence. If X is simply connected, then $\lambda_{\Sigma W_{h+1}}(X)$ induces an isomorphism*

$$\pi_\bullet(X) \otimes \mathbb{Z}[1/p] \xrightarrow{\simeq} \pi_\bullet(\mathrm{P}_{\Sigma W_{h+1}}(X)) \otimes \mathbb{Z}[1/p]$$

of graded abelian groups.

Proof. — Let n be a natural number such that $1 \leq n \leq h$, and let V_n be a p-local finite complex of type n together with a v_n self-map. We begin by showing the following claim.

Claim. The canonical morphism $\lambda_{\Sigma V_{h+1}}(X)\colon X \to \mathrm{P}_{\Sigma V_{h+1}}(X)$ given by ΣV_{h+1}-contraction is a v_n-periodic equivalence.

The map $X \to \mathrm{P}_{\Sigma V_{h+1}}(X)$ and the v_n self-map of V_n induce a commutative diagram

$$\begin{array}{ccccc}
\mathrm{P}_{\Sigma V_{h+1}}(\mathrm{Map}_*(V_n, X)) & \longrightarrow & \mathrm{P}_{\Sigma V_{h+1}}\left(\mathrm{Map}_*\left(\Sigma^{d_{V_n}} V_n, X\right)\right) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Map}_*\left(V_n, \mathrm{P}_{\Sigma V_{h+1}}(X)\right) & \longrightarrow & \mathrm{Map}_*\left(\Sigma^{d_{V_n}} V_n, \mathrm{P}_{\Sigma V_{h+1}}(X)\right) & \longrightarrow & \cdots
\end{array}$$

in $\mathcal{H}\mathrm{o}_*$, where each vertical map is given by Theorem 3.1.3.5. The colimit of the upper row is equivalent to $\mathrm{P}_{\Sigma V_{h+1}}(\mathrm{T}_{V_n}(X))$: Indeed, since ΣV_{h+1} is equivalent to a finite complex (see Definition 1.3.0.10), the functor $\mathrm{P}_{\Sigma V_{h+1}}$ preserves filtered colimits in $\mathcal{H}\mathrm{o}_*$, see [Bou94, Lemma 11.6]. By Proposition 3.3.2.1 the homotopy type $\mathrm{P}_{\Sigma V_{h+1}}(\mathrm{T}_{V_n}(X))$ is equivalent to $\mathrm{T}_{V_n}(X)$. By Theorem 3.1.3.5 each vertical map induces an isomorphism on homotopy groups in all degree $i \geq \mathrm{conn}(\Sigma W_{h+1}) + 2$. Thus, the induced map

$$\mathrm{T}_{V_n}X \to \mathrm{T}_{V_n}(\mathrm{P}_{\Sigma W_{h+1}}(X)) \qquad (3.3.2.2)$$

of the colimits of the rows in the above diagram also induces an isomorphism on homotopy groups in all degree $i \geq \mathrm{conn}(\Sigma W_{h+1}) + 2$. This actually shows that the map (3.3.2.2) is an equivalence, since $\mathrm{T}_{V_n}(X)$ and $\mathrm{T}_{V_n}(\mathrm{P}_{\Sigma V_{h+1}}(X))$ are periodic (see Proposition 3.2.0.5). Since the v_n-homotopy groups of X are isomorphic to the homotopy groups of $\mathrm{T}_{V_n}X$, the canonical morphism $\lambda_{\Sigma V_{h+1}}(X)$ induces isomorphisms of v_n-periodic homotopy groups for every $1 \leq n \leq h-1$.

Recall the hypothesis $\langle W_{h+1}\rangle_\Sigma = \langle V_{h+1}\rangle_\Sigma$ from Situation 3.3.0.1. Using the fact that $\mathrm{T}_{V_n}(\mathrm{P}_{\Sigma V_{h+1}}(X))$ and $\mathrm{T}_{V_n}(\mathrm{P}_{\Sigma W_{h+1}}(X))$ are periodic, there exists an natural equivalence $\mathrm{T}_{V_n}(\mathrm{P}_{\Sigma V_{h+1}}(X)) \xrightarrow{\sim} \mathrm{T}_{V_n}(\mathrm{P}_{\Sigma W_{h+1}}(X))$, similarly as we did in the proof of Proposition 3.3.2.1. Together with the claim, this shows that the natural map $\lambda_{\Sigma W_{h+1}}(X)$ is a v_n-periodic equivalence for every $1 \leq n \leq h$.

By Proposition 2.1.2.10 the map $\lambda_{\Sigma W_{h+1}}(X)$ is a $\mathrm{H}\mathbb{Z}[1/p]$-homology equivalence. The last part of the theorem follows since X and thus $\mathrm{P}_{\Sigma W_{h+1}}(X)$ are simply connected. See also [Bou94, Theorem 11.5]. □

3.3.2.3. Corollary. — *Every ΣW_{h+1}-equivalence in $\mathcal{H}\mathrm{o}_*$ is a v_n-periodic equivalence for each $1 \leq n \leq h$. A ΣW_{h+1}-equivalence of simply connected p-local homotopy types is also a v_0-periodic equivalence, i.e. a rational homotopy equivalence.* □

3.3.2.4. Proposition. — *Let X be a pointed connected homotopy type. For every natural number $n \geq h+1$, the v_n-periodic homotopy groups of $\mathrm{P}_{\Sigma W_{h+1}}(X)$ vanish.*

Proof. — Let V_n be a finite complex of type n together with a v_n-self map. We have

(i) $\langle V_n\rangle_\Sigma \leq \langle W_{h+1}\rangle_\Sigma = \langle V_{h+1}\rangle_\Sigma$

(ii) $\mathrm{conn}\left(\Sigma^k V_n\right) \geq \mathrm{conn}(\Sigma W_{h+1})$ for sufficiently large $k \in \mathbb{N}$.

By Theorem 3.1.2.7 we have $\langle \Sigma^{k-1}V\rangle \leq \langle \Sigma W_{h+1}\rangle$, i.e. a ΣW_{h+1}-less homotopy type is $\Sigma^{k-1}V$-less. So the pointed mapping space $\mathcal{M}\mathrm{ap}_*\left(\Sigma^{k-1}V_n, \mathrm{P}_{\Sigma W_{h+1}}(X)\right)$ is contractible. Thus $\mathrm{T}_{V_n}(\mathrm{P}_{\Sigma W_{h+1}}(X))$ is also contractible, that is, the v_n-periodic homotopy groups of X vanish. □

3.3.2.5. Theorem. — *Let $f\colon X \to Y$ be a morphism of pointed connected homotopy types. The following statements are equivalent:*

(i) *The induced map $f_*\colon \pi_i\left(\mathrm{P}_{\Sigma W_{h+1}}(X)\right) \to \pi_i\left(\mathrm{P}_{\Sigma W_{h+1}}(X)\right)$ is an isomorphism for sufficiently large i.*

(ii) *The map f is a v_n-periodic equivalence for all $1 \leq n \leq h$ and f induces an isomorphism $f_*\colon \pi_j(X) \otimes \mathbb{Z}[1/p] \to \pi_j(Y) \otimes \mathbb{Z}[1/p]$ for sufficiently large $j > 1$.*

Proof. — We follow the proof idea of [Bou94, Theorem 11.10]. By Proposition 3.3.1.3 it suffices to show that for a pointed connected homotopy type F, the following statements are equivalent:

(i) The homotopy groups $\pi_i\left(\mathrm{P}_{\Sigma W_{h+1}}(F)\right)$ vanish for sufficiently large i.

(ii) We have that $\pi_j(F) \otimes \mathbb{Z}[1/p] = 0$ for sufficiently large $j > 1$, and the v_n-periodic homotopy groups of F vanish for every $1 \leq n \leq h$.

Assume (i). By Proposition 3.2.0.10 and Theorem 3.3.2.2 the v_n-periodic homotopy groups of F vanish for all $0 \leq n \leq h$. Choose a Moore space $\mathrm{M}(\mathbb{Z}/p\mathbb{Z}, c)$ such that $\langle \mathrm{M}(\mathbb{Z}/p\mathbb{Z}, c)\rangle \geq \langle \Sigma W_{h+1}\rangle$. Then

$$\mathrm{P}_{\mathrm{M}(\mathbb{Z}/p\mathbb{Z},c)}(F) \simeq \mathrm{P}_{\mathrm{M}(\mathbb{Z}/p\mathbb{Z},c)}\left(\mathrm{P}_{\Sigma W_{h+1}}(F)\right).$$

For sufficiently large j, we have that

$$\pi_j(F) \otimes \mathbb{Z}[1/p] \cong \pi_j\left(\mathrm{P}_{\mathrm{M}(\mathbb{Z}/p\mathbb{Z},c)}(F)\right) = 0$$

by Theorem 2.4.1.5.

Assume (ii). First we show the following claim.

Claim. There exists a (sufficiently large) natural number $k \geq 1$ such that $\mathrm{P}_{\Sigma W_{h+1}}(F)$ is $\Sigma\mathrm{M}(\mathbb{Z}/p\mathbb{Z}, k)$-less.

Let V_h be a finite complex of type h together with a v_h self-map $\Sigma^{d_{V_h}} V_h \to V_h$. The cofibre C of the v_h self-map is a finite complex of type at least $h+1$. Choose a natural number $k_h \geq 1$ such that the connectivity of $\Sigma^{k_h} C_h$ is no less than the connectivity of ΣW_h. Then $\langle \Sigma^{k_h} C\rangle \leq \langle \Sigma W_{h+1}\rangle$ by Theorem 3.1.2.7. Consider the induced fibre sequence

$$\begin{gathered}\mathcal{M}\mathrm{ap}_*(\Sigma^{k_h} C_h, \mathrm{P}_{\Sigma W_{h+1}}(F)) \\ \downarrow \\ \mathcal{M}\mathrm{ap}_*(\Sigma^{k_h} V_h, \mathrm{P}_{\Sigma W_{h+1}}(F)) \\ \downarrow \\ \mathcal{M}\mathrm{ap}_*(\Sigma^{k_h + d_{V_h}} V_h, \mathrm{P}_{\Sigma W_{h+1}}(F)).\end{gathered}$$

Then the lower arrow in the above fibre sequence is an equivalence. In particular,

$$\pi_\bullet\left(\mathcal{M}\mathrm{ap}_*(\Sigma^{k_h} V_h, \mathrm{P}_{\Sigma W_{h+1}}(F))\right) \cong v_h^{-1}\pi_\bullet\left(\mathrm{P}_{\Sigma W_{h+1}}(F)); \Sigma^{k_h} V_h\right) = 0.$$

In other words, the homotopy type $\mathrm{P}_{\Sigma W_{h+1}}(F)$ is $\Sigma^{k_h} V_h$-less. Iterating this procedure and using the assumption that the v_n-periodic homotopy groups of F vanish for *all* natural number $1 \leq n \leq h$, we can find a (sufficiently large) $k \geq 1$ such that $\mathrm{P}_{\Sigma W_{h+1}}(F)$ is $\Sigma\mathrm{M}(\mathbb{Z}/p\mathbb{Z}, k)$-less.

By this claim, we have $\mathrm{P}_{\Sigma W_{h+1}}(F) \simeq \mathrm{P}_{\Sigma\mathrm{M}(\mathbb{Z}/p\mathbb{Z},k)}(F)$. Therefore for sufficiently large j we have by Theorem 2.4.1.5

$$\pi_j\left(\mathrm{P}_{\Sigma W_{h+1}}(F)\right) \cong \pi_j\left(\mathrm{P}_{\Sigma\mathrm{M}(\mathbb{Z}/p\mathbb{Z},k)}(F)\right) \cong \pi_j(F) \otimes \mathbb{Z}[1/p] = 0. \qquad \square$$

3.3.2.6. Theorem. — *Let $f\colon X \to Y$ be a morphism of $c^+(\Sigma W_{h+1})$-connected pointed homotopy types. The following statements are equivalent:*

(i) *The map f is a ΣW_{h+1}-equivalence,*

(ii) *The map f is a v_n-periodic equivalences for every $1 \leq n \leq h$ and f induces an isomorphism*

$$\pi_\bullet(X) \otimes \mathbb{Z}[1/p] \xrightarrow{f_*} \pi_\bullet(Y) \otimes \mathbb{Z}[1/p]. \tag{3.3.2.3}$$

Proof. — By Theorem 3.3.2.2 (i) implies (ii). Assuming (ii), let F' be the fibre of the induced map $\mathrm{P}_{\Sigma W_{h+1}}(X) \to \mathrm{P}_{\Sigma W_{h+1}}(Y)$. Thus F' is ΣW_{h+1}-less by Proposition 2.1.3.1 and the v_n-homotopy groups of F' vanish for every $1 \leq n \leq h$, by Proposition 3.2.0.14.

Let F be the fibre of f. Then F is p-torsion by (3.3.2.3) and the v_n-homotopy groups of F' vanish for every $1 \leq n \leq h$, by assumption and by Proposition 3.2.0.14. Thus $\pi_j(\mathrm{P}_{\Sigma W_{h+1}}(F)) = 0$ for sufficiently large j by Theorem 3.3.2.5.

Therefore by Proposition 3.3.1.3 the fibre F' is also p-torsion and the homotopy groups of F vanish in sufficiently large degrees. Using Proposition 3.3.0.5 to calculate the homotopy groups of F' we obtain that $\pi_i(F) \cong \pi_i(\mathrm{P}_{\Sigma W_{h+1}}(F)) = 0$ for every natural number $i \geq c^+(\Sigma W_{h+1})$. By the assumption that X and Y are $c^+(\Sigma W_{h+1})$-connected, the other homotopy groups of F' also vanish. In other word F' is contractible. Therefore, the induced map $\mathrm{P}_{\Sigma W_{h+1}}(X) \to \mathrm{P}_{\Sigma W_{h+1}}(Y)$ is an equivalence, that is, f is a ΣW_{h+1}-equivalence. See also [Bou94, Theorem 13.3]. ☐

3.3.2.7. Corollary. — *Let X be a pointed homotopy type. Then $\mathrm{P}_{\Sigma W_{h+1}}(X) \simeq \mathrm{pt}$ if and only if the following combined conditions hold:*

(i) *The homotopy type X is $(c^+(\Sigma W_{h+1}) - 1)$-connected.*

(ii) *The homotopy group of X in each degree is a p-primary torsion abelian group.*

(iii) *The v_n-periodic homotopy groups of X vanish for every $1 \leq n \leq h$.*

Proof. — Recall that $c^+(\Sigma W_{h+1}) = \mathrm{conn}(\Sigma W_{h+1})+1$. Assume that $\mathrm{P}_{\Sigma W_{h+1}}(X) \simeq \mathrm{pt}$. Then (i) follows from Proposition 2.2.2.6, and (ii) and (iii) follow from Theorem 3.3.2.2 and Proposition 3.3.0.3.

For the other direction of implication, we make use of the commutative diagram below

$$\begin{array}{ccccc}
\tau_{>c^+(\Sigma W_{h+1})}(X) & \longrightarrow & X & \longrightarrow & \tau_{\leq c^+(\Sigma W_{h+1})}(X) \\
\big\downarrow \lambda_> & & \big\downarrow L & & \big\downarrow \lambda_\leq \\
\mathrm{P}_{\Sigma W_{h+1}}\left(\tau_{>c^+(\Sigma W_{h+1})}(X)\right) & \longrightarrow & \overline{X} & \longrightarrow & \mathrm{P}_{\Sigma^2 W_{h+1}}\left(\tau_{\leq c^+(\Sigma W_{h+1})}(X)\right),
\end{array}$$

obtained by Theorem 2.2.2.3; the rows are fibres sequences in $\mathcal{H}\mathrm{o}_*$, the pointed homotopy type $\overline{X}$ is $\Sigma^2 W_{h+1}$-less and L is a ΣW_{h+1}-equivalence. Conditions (ii) and (iii) together with Theorem 3.3.2.6 imply that $\mathrm{P}_{\Sigma W_{h+1}}\left(\tau_{>c^+(\Sigma W_{h+1})}(X)\right)$ is contractible. Thus we have $\overline{X} \simeq \mathrm{P}_{\Sigma^2 W_{h+1}}\left(\tau_{\leq c^+(\Sigma W_{h+1})}(X)\right)$ in the above diagram. Therefore, we obtain equivalences

$$\mathrm{P}_{\Sigma W_{h+1}}(X) \xrightarrow{\sim} \mathrm{P}_{\Sigma W_{h+1}}\left(\overline{X}\right) \xrightarrow{\sim} \mathrm{P}_{\Sigma W_{h+1}}\left(\tau_{\leq c^+(\Sigma W_{h+1})}(X)\right).$$

Using the calculation in Proposition 3.3.0.5 and (i) we see that $\mathrm{P}_{\Sigma W_{h+1}}(X) \simeq \mathrm{pt}$. See also [Bou94, Corollary 13.4]. ☐

3.4. v_h-periodic-localisations of homotopy types

Consider a sequence $(W_{h+1})_{h\in\mathbb{N}}$ of p-local homotopy types, where each W_{h+1} satisfies the hypotheses of Situation 3.3.0.1 and $\operatorname{conn}(W_m) \geq \operatorname{conn}(W_l)$ for every pair (m, l) of natural numbers with $m \geq l$. Then $\langle \Sigma W_m \rangle < \langle \Sigma W_l \rangle$ for $m \geq l$ by Theorem 3.1.2.7. Thus we obtain a tower

$$\cdots \to \mathrm{P}_{\Sigma W_{h+1}} \to \mathrm{P}_{\Sigma W_h} \to \cdots \to \mathrm{P}_{\Sigma W_2} \to \mathrm{P}_{\Sigma W_1} \tag{3.4.0.1}$$

of functors. This is known as the *unstable chromatic tower*, because of Theorem 3.3.2.2 and Proposition 3.3.2.4, cf. [Bou94, §11.9]. The main content of this section is the construction of the localisation $\mathcal{H}\mathrm{o}_{v_h}$ of the ∞-category $\mathcal{H}\mathrm{o}_*$ of pointed connected homotopy types at the set of v_h-periodic equivalences, using the natural transformations in the unstable chromatic tower, following [Heu21] and [Har18]. In §3.4.1 we discuss briefly the relationship between the stable and unstable periodic equivalences and recall the Bousfield–Kuhn functor, following [Bou01; Kuh08]. We work in the following situation.

3.4.0.1. Situation. — We fix a prime number p and a natural number $h \geq 1$. Let W_{h+1} and W_h be pointed p-local homotopy types satisfying Situation 3.3.0.1.(i) and (ii). Furthermore, we assume that $\operatorname{conn}(W_{h+1}) \geq \operatorname{conn}(W_h)$. In particular, we have that $c^+(\Sigma W_{h+1}) \geq c^+(\Sigma W_h)$. By Corollary 2.3.1.5 there exists a natural transformation $\mathrm{P}_{\Sigma W_{h+1}} \to \mathrm{P}_{\Sigma W_h}$, given by ΣW_h-contraction.

3.4.0.2. Construction. — Consider the natural transformation $\mathrm{P}_{\Sigma W_{h+1}} \to \mathrm{P}_{\Sigma W_h}$ as a morphism in ∞-category $\mathfrak{Fun}(\mathcal{H}\mathrm{o}_*, \mathcal{H}\mathrm{o}_*)$ of functors. Denote its fibre by $\mathrm{F}_{\Sigma W_{h+1}, \Sigma W_h}$. In particular, for every pointed connected homotopy type X, there exists a fibre sequence

$$\mathrm{F}_{\Sigma W_{h+1}, \Sigma W_h}(X) \to \mathrm{P}_{\Sigma W_{h+1}}(X) \to \mathrm{P}_{\Sigma W_h}(X)$$

of pointed connected homotopy types, since limits of functors are computed pointwise.

3.4.0.3. Corollary. — *Let X be a pointed homotopy type. Then the homotopy type $\mathrm{F}_{\Sigma W_{h+1}, \Sigma W_h}(X)$ is ΣW_{h+1}-less and ΣW_h-full.*

Proof. — This is by Proposition 2.1.3.1 and Theorem 2.2.2.5, respectively. ☐

3.4.0.4. Proposition. — *Let X be a pointed connected homotopy type. Then the homotopy type $\mathrm{F}_{\Sigma W_{h+1}, \Sigma W_h}(X)$ is also connected. Furthermore:*

(i) *For every natural number $n \neq h$ and $n \geq 1$, the v_n-periodic homotopy groups of the homotopy type $\mathrm{F}_{\Sigma W_{h+1}, \Sigma W_h}(X)$ vanishes.*

(ii) *The v_h-periodic homotopy groups of the homotopy type $\mathrm{F}_{\Sigma W_{h+1}, \Sigma W_h}(X)$ are isomorphic to those of X.*

Proof. — This is a consequence of Proposition 3.2.0.14 and Theorem 3.3.2.2. ☐

3.4.0.5. Proposition. — *Let $f\colon X \to Y$ be a morphism of pointed connected homotopy types. The following statements are equivalent:*

(i) *The map f is a v_h-periodic equivalence.*

(ii) *The induced map $(\tau_{>c^+(\Sigma W_{h+1})} \circ \mathrm{F}_{\Sigma W_{h+1},\Sigma W_h})(f)$ is a v_h-periodic equivalence.*

(iii) *The induced map $(\tau_{>c^+(\Sigma W_{h+1})} \circ \mathrm{F}_{\Sigma W_{h+1},\Sigma W_h})(f)$ is an equivalence in $\mathcal{H}\mathrm{o}_*$.*

Proof. — It suffices to prove the proposition for connected homotopy types. For the general case, we can consider each connected component separately. We show the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). Consider the commutative diagram

$$\begin{array}{ccccc} \mathrm{F}_{\Sigma W_{h+1},\Sigma W_h}(X) & \xrightarrow{i_X} & \mathrm{P}_{\Sigma W_{h+1}}(X) & \longrightarrow & \mathrm{P}_{\Sigma W_h}(X) \\ \downarrow{\scriptstyle \widetilde{f}} & & \downarrow{\scriptstyle f_{h+1}} & & \downarrow{\scriptstyle f_h} \\ \mathrm{F}_{\Sigma W_{h+1},\Sigma W_h}(Y) & \xrightarrow{i_Y} & \mathrm{P}_{\Sigma W_{h+1}}(Y) & \longrightarrow & \mathrm{P}_{\Sigma W_h}(Y) \end{array}$$

of pointed homotopy types, where the rows are fibre sequences and the vertical maps are induced by f.

Assume (i). Since f is a v_h-periodic equivalence, the middle map f_{h+1} is also a v_n-periodic equivalence, by Theorem 3.3.2.2. Recall that the v_h-periodic homotopy groups of $\mathrm{P}_{\Sigma W_h}(X)$ and $\mathrm{P}_{\Sigma W_h}(Y)$ are both trivial, see Proposition 3.3.2.4. Therefore, the maps i_X and i_Y are also v_h-periodic equivalences, by the proof of Proposition 3.2.0.14. Thus $\widetilde{f}$ is a v_h-periodic equivalence. Using Proposition 3.2.0.10 we conclude that the map $(\tau_{>c^+(\Sigma W_{h+1})} \circ \mathrm{F}_{\Sigma W_{h+1},\Sigma W_h})(f)$ is a v_h-periodic equivalence.

Assume (ii). By Proposition 3.4.0.4 the map $(\tau_{>c^+(\Sigma W_{h+1})} \circ \mathrm{F}_{\Sigma W_{h+1},\Sigma W_h})(f)$ is a v_n-periodic equivalence for $1 \leq n \leq h$; for $i \geq h$ the v_i-periodic homotopy groups of the source and the target are trivial.

Claim. The pointed homotopy types

$$(\tau_{>c^+(\Sigma W_{h+1})} \circ \mathrm{F}_{\Sigma W_{h+1},\Sigma W_h})(X) \text{ and } (\tau_{>c^+(\Sigma W_{h+1})} \circ \mathrm{F}_{\Sigma W_{h+1},\Sigma W_h})(Y)$$

are ΣW_h-full. In particular, their homotopy groups are p-primary torsion abelian groups by Corollary 3.3.2.7.

Since $\mathrm{F}_{\Sigma W_{h+1},\Sigma W_h}(X)$ is ΣW_h-full, its homotopy groups are p-primary by Corollary 3.3.2.7. In particular, its $c^+(\Sigma W_{h+1})$-connected cover is a p-torsion homotopy type. The claim then follows from Corollary 3.4.0.3 and Corollary 3.3.2.7. The same proof works for Y in place of X.

Claim. The map $(\tau_{>c^+(\Sigma W_{h+1})} \circ \mathrm{F}_{\Sigma W_{h+1},\Sigma W_h})(f)$ is a ΣW_{h+1}-equivalence.

This is follows from the previous claim and the fact that the map is a v_n-periodic equivalence for $1 \leq n \leq h$, by applying Theorem 3.3.2.6.

Claim. The pointed homotopy types

$$(\tau_{>c^+(\Sigma W_{h+1})} \circ \mathrm{F}_{\Sigma W_{h+1},\Sigma W_h})(X) \text{ and } (\tau_{>c^+(\Sigma W_{h+1})} \circ \mathrm{F}_{\Sigma W_{h+1},\Sigma W_h})(Y)$$

are ΣW_{h+1}-less.

We show it for X and the same proof works for Y. We have

$$\begin{aligned}&(\mathrm{P}_{\Sigma W_{h+1}} \circ \tau_{>c^+(\Sigma W_{h+1})} \circ \mathrm{F}_{\Sigma W_{h+1}, \Sigma W_h})(X)\\ \simeq&(\tau_{>c^+(\Sigma W_{h+1})} \circ \mathrm{P}_{\Sigma W_{h+1}} \circ \mathrm{F}_{\Sigma W_{h+1}, \Sigma W_h})(X)\\ \simeq&(\tau_{>c^+(\Sigma W_{h+1})} \circ \mathrm{F}_{\Sigma W_{h+1}, \Sigma W_h})(X)\end{aligned}$$

by Proposition 3.3.1.4 and Corollary 3.4.0.3.

The second and the third claim combined implies (iii).

Assuming (iii), the first statement follows Propositions 3.2.0.10 and 3.4.0.4. See [Heu21, Theorem 3.7]. □

3.4.0.6. Definition. — We define the functor

$$\begin{aligned}\mathrm{L}_{v_h} \colon \mathcal{H}\mathrm{o}_* &\to \mathcal{H}\mathrm{o}_*^{>c^+(\Sigma W_{h+1})}\\ X &\mapsto \left(\tau_{>c^+(\Sigma W_{h+1})} \circ \mathrm{F}_{\Sigma W_{h+1}, \Sigma W_h}\right)(X).\end{aligned}$$

Let $\mathcal{H}\mathrm{o}_{v_h}$ denote the full ∞-subcategory of $\mathcal{H}\mathrm{o}_*$ whose objects are pointed connected homotopy types that are equivalent to $\left(\tau_{>c^+(\Sigma W_{h+1})} \circ \mathrm{F}_{\Sigma W_{h+1}, \Sigma W_h}\right)(X)$ for some pointed homotopy type X.

3.4.0.7. Theorem. — *The functor L_{v_h} exhibits $\mathcal{H}\mathrm{o}_{v_h}$ as a localisation of the ∞-category $\mathcal{H}\mathrm{o}_*$ at v_h-periodic equivalences. In particular, for every ∞-category $\mathcal{C}$, composing with L_{v_h} induces an equivalence*

$$\mathcal{F}\mathrm{un}\left(\mathcal{H}\mathrm{o}_{v_h}, \mathcal{C}\right) \xrightarrow{\sim} \mathcal{F}\mathrm{un}^{v_h}\left(\mathcal{H}\mathrm{o}_*, \mathcal{C}\right)$$

of ∞-category of functors where $\mathcal{F}\mathrm{un}^{v_h}$ denotes the ∞-category of functors that send v_h-periodic equivalences in $\mathcal{H}\mathrm{o}_*$ to equivalences in $\mathcal{C}$.

Proof. — We follow the proof of [Heu21, Theorem 2.2], which goes through without the p-local assumption. First, note that the composition

$$\mathcal{H}\mathrm{o}_{v_h} \overset{i}{\hookrightarrow} \mathcal{H}\mathrm{o}_* \xrightarrow{\mathrm{L}_{v_h}} \mathcal{H}\mathrm{o}_{v_h}$$

is equivalent to the identity by the proof of Proposition 3.4.0.5, that is, the homotopy type $\left(\tau_{>c^+(\Sigma W_{h+1})} \circ \mathrm{F}_{\Sigma W_{h+1}, \Sigma W_h}\right)(X)$ is ΣW_{h+1}-less and ΣW_h-full for every pointed homotopy type X. Thus, the induced composition

$$\mathcal{F}\mathrm{un}(\mathcal{H}\mathrm{o}_{v_h}, \mathcal{C}) \xrightarrow{(\mathrm{L}_{v_h})^*} \mathcal{F}\mathrm{un}\left(\mathcal{H}\mathrm{o}_*, \mathcal{C}\right) \xrightarrow{i^*} \mathcal{F}\mathrm{un}(\mathcal{H}\mathrm{o}_{v_h}, \mathcal{C})$$

is equivalent to the identity functor on $\mathcal{F}\mathrm{un}\left(\mathcal{H}\mathrm{o}_{v_h}, \mathcal{C}\right)$. Furthermore, the image of $(\mathrm{L}_{v_h})^*$ is contained in $\mathcal{F}\mathrm{un}^{v_h}\left(\mathcal{H}\mathrm{o}_*, \mathcal{C}\right)$ by construction. Now we show that the composition

$$\mathcal{F}\mathrm{un}\left(\mathcal{H}\mathrm{o}_*, \mathcal{C}\right) \xrightarrow{i^*} \mathcal{F}\mathrm{un}(\mathcal{H}\mathrm{o}_{v_h}, \mathcal{C}) \xrightarrow{(\mathrm{L}_{v_h})^*} \mathcal{F}\mathrm{un}\left(\mathcal{H}\mathrm{o}_*, \mathcal{C}\right)$$

restricts to the identity functor on the full ∞-subcategory $\mathcal{F}\mathrm{un}^{v_h}\left(\mathcal{H}\mathrm{o}_*, \mathcal{C}\right)$. For every $F \in \mathcal{F}\mathrm{un}^{v_h}\left(\mathcal{H}\mathrm{o}_*, \mathcal{C}\right)$, we need show that $F(X) \simeq F\left(\mathrm{L}_{v_h}(X)\right)$. It suffices to show that there are v_h-periodic equivalences connecting X and $\mathrm{L}_{v_h}(X)$. The following

$$\begin{array}{ccc} X & \longleftarrow & \tau_{>c^+(\Sigma W_{h+1})}(X) \\ & & \downarrow \\ & & \tau_{>c^+(\Sigma W_{h+1})}\left(\mathrm{P}_{\Sigma W_{h+1}}(X)\right) \longleftarrow \tau_{>c^+(\Sigma W_{h+1})}\left(\mathrm{F}_{\Sigma W_{h+1},\Sigma W_h}(X)\right) \simeq \mathrm{L}_{v_h}(X) \end{array}$$

provides such a zig-zag of v_h-periodic equivalences. □

3.4.0.8. Theorem. — *Let V_h be a pointed p-local $(c^+(\Sigma W_{h+1})-1)$-connected finite complex of type h. The ∞-category $\mathcal{H}\mathrm{o}_{v_h}$ is compactly generated and $\mathrm{P}_{\Sigma W_{h+1}}(\Sigma V_h)$ is a compact generator.*

Proof. — See [Heu21, Proposition 3.14] for a detailed proof. □

3.4.0.9. Situation. — Recall the p-local homotopy type W_{h+1} from Situation 3.4.0.1. Let $1 \leq k < h$ be a natural number, and let W_k be a pointed p-local homotopy type satisfying the hypotheses of Situation 3.3.0.1 and $c^+(\Sigma W_k) \leq c^+(\Sigma W_h)$. Thus we obtain a natural transformation $\mathrm{P}_{\Sigma W_{h+1}} \to \mathrm{P}_{\Sigma W_k}$ of endo-functors of $\mathcal{H}\mathrm{o}_*$.

3.4.0.10. Construction. — Let $\mathrm{F}_{\Sigma W_{h+1},\Sigma W_k}$ denote the fibre of the natural transformation $\mathrm{P}_{\Sigma W_{h+1}} \to \mathrm{P}_{\Sigma W_k}$, i.e. there exists a fibre sequence

$$\mathrm{F}_{\Sigma W_{h+1},\Sigma W_k} \to \mathrm{P}_{\Sigma W_{h+1}} \to \mathrm{P}_{\Sigma W_k}$$

in the ∞-category $\mathcal{F}\mathrm{un}\left(\mathcal{H}\mathrm{o}_*, \mathcal{H}\mathrm{o}_*\right)$ of functors. Define the functor

$$\begin{aligned} \mathrm{L}_{v_{[k,h]}} \colon \mathcal{H}\mathrm{o}_* &\to \mathcal{H}\mathrm{o}_*^{>c^+(\Sigma W_{h+1})} \\ X &\mapsto \left(\tau_{>c^+(\Sigma W_{h+1})} \circ \mathrm{F}_{\Sigma W_{h+1},\Sigma W_k}\right)(X). \end{aligned}$$

Let $\mathcal{H}\mathrm{o}_{v_{[k,h]}}$ denote the full ∞-subcategory of $\mathcal{H}\mathrm{o}_*$ whose objects are connected pointed homotopy types that are equivalent to $\left(\tau_{>c^+(\Sigma W_{h+1})} \circ \mathrm{F}_{\Sigma W_{h+1},\Sigma W_k}\right)(X)$ for some $X \in \mathcal{H}\mathrm{o}_*$.

3.4.0.11. Theorem. — *The functor $\mathrm{L}_{v_{[k,h]}}$ exhibits $\mathcal{H}\mathrm{o}_{v_{[k,h]}}$ as the localisation of the ∞-category $\mathcal{H}\mathrm{o}_*$ at v_n-periodic equivalences for all $k \leq n \leq h$.*

Proof. — The proof is the same as that of Theorem 3.4.0.7. □

3.4.1. Stable and unstable periodic localisations. — In this subsection we discuss briefly the relationship between unstable and stable periodic localisations and recall the Bousfield–Kuhn functor, following [Bou01]. We continue to work with a fixed prime number p and recall that $\mathcal{S}\mathrm{p}_{(p)}$ denotes the ∞-category of p-local spectra.

***3.4.1.1.* Finite chromatic localisation of spectra.** — For every natural number h, recall the finite localisation

$$\mathrm{L}_h^{\mathrm{f}} \colon \mathcal{S}\mathrm{p}_{(p)} \to \mathcal{L}_h^{\mathrm{f}}\left(\mathcal{S}\mathrm{p}_{(p)}\right) \simeq \mathcal{S}\mathrm{p}_{(p)} \big/ \mathcal{S}\mathrm{p}_{\geq h+1}$$

of height h from §1.2. This localisation functor can alternatively be constructed in an analogous way to the contraction functor of a homotopy type. Let F_{h+1} be a (p-local) finite spectrum of type $h+1$. A p-local spectrum E is F_{h+1}-*null* if the *mapping spectrum* $\mathbb{M}\mathrm{ap}(F_{h+1}, E)$ is equivalent to the zero spectrum, cf. ¶2.2.1.4. A morphism $E_1 \to E_2$ of p-local spectra is a F_{h+1}-*equivalence* if the induced map $\mathbb{M}\mathrm{ap}(E_2, E) \to \mathbb{M}\mathrm{ap}(E_1, E)$ of spectra is an equivalence for every F_{h+1}-null p-local spectrum E. By the Thick Subcategory Theorem (see Theorem 1.2.0.9) a morphism of p-local spectra is a F_{h+1}-equivalence if and only if it is a F'_{h+1}-equivalence for any p-local finite spectrum F'_{h+1} of type $h+1$.

The functor $\mathrm{L}^{\mathrm{f}}_h$ exhibits $\mathcal{L}^{\mathrm{f}}_h\left(\mathcal{S}\mathrm{p}_{(p)}\right)$ as the localisation of the ∞-category $\mathcal{S}\mathrm{p}_{(p)}$ at the set of F_{h+1}-equivalences of p-local spectra, see [Bou01, §2, §3.2]. Let F_h be a p-local type n spectrum together with a $v_h$self-map. For every $E \in \mathcal{S}\mathrm{p}_{(p)}$, one can define analogues the v_h-*periodic homotopy groups* $v_h^{-1}\pi_\bullet(E; F_h)$ of E and define v_h-*periodic equivalences* of p-local spectra: In Definitions 3.2.0.8 and 3.2.0.12 replace the finite complex V_h by the finite spectrum F_h. In a similar proof as Theorem 3.3.2.6, one can show that a morphism of p-local spectra is a F_{h+1}-equivalence if and only if it is a v_n-periodic equivalence for all $0 \leq n \leq h$.

By the (stable) Class Invariance Theorem, we have a fibre sequence

$$\mathrm{M}^{\mathrm{f}}_h \to \mathrm{L}^{\mathrm{f}}_h \to \mathrm{L}^{\mathrm{f}}_{h-1}$$

of functors for $h \geq 1$. We say a p-local spectrum E is v_h-*periodic* if it is in the essential image of $\mathrm{M}^{\mathrm{f}}_h$, i.e. $E \simeq \mathrm{L}^{\mathrm{f}}_h(E)$ and the v_i-periodic homotopy groups of E vanishes for $i \neq h$. Let $\mathcal{M}^{\mathrm{f}}_h(\mathcal{S}\mathrm{p}_{(p)})$ denotes the full ∞-subcategory of $\mathcal{S}\mathrm{p}_{(p)}$ whose objects are v_h-periodic spectra. A morphism in $\mathcal{M}^{\mathrm{f}}_h(\mathcal{S}\mathrm{p}_{(p)})$ is an equivalence if and only if the underlying morphism of spectra is a v_h-periodic equivalence.

Recall from Definition 3.2.0.4 that the v_h-periodic homotopy groups of a homotopy type are the homotopy groups of the infinite loop space $\mathrm{T}_{V_h}(X)$ defined as the colimit

$$\mathcal{M}\mathrm{ap}_*\left(V_h, X\right) \to \mathcal{M}\mathrm{ap}_*\left(\Sigma^{d_{V_h}} V_h, X\right) \to \cdots \to \mathcal{M}\mathrm{ap}_*\left(\Sigma^{k d_{V_h}} V_h, X\right) \to \cdots$$

of mapping spaces. Replace V_h by F_h, and X by E and the mapping spaces by mapping spectrum, we see that the above colimit is equivalent to $\mathrm{T}(h) \otimes_{\mathcal{S}\mathrm{p}} E$ where $\mathrm{T}(h)$ is the telescope spectrum defined using the Spanier–Whitehead dual of F_{h+1}, see ¶1.2.0.13. Thus, a morphism of p-local spectra is a v_h-periodic equivalence if and only if it is a $\mathrm{T}(h)$-homology equivalences of spectra. Moreover, we have an equivalence

$$\mathcal{M}^{\mathrm{f}}_h(\mathcal{S}\mathrm{p}_{(p)}) \xrightarrow{\sim} \mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$$

of ∞-categories, see [Bou01, Theorem 3.3].

3.4.1.2. Situation. — Let W_{h+1} be a p-local homotopy type satisfying the conditions in Situation 3.3.0.1, where $h \geq 1$. Recall the number $c^+(\Sigma W_{h+1}) = \mathrm{conn}(\Sigma W_{h+1}) + 1$.

3.4.1.3. Proposition. — *Let V_h be a p-local finite complex of type h together with a v_h self-map. For every p-local spectrum E, there exists an isomorphism*

$$v_h^{-1}\pi_\bullet(\Omega^\infty E; V_h) \cong v_h^{-1}\pi_\bullet(E; \Sigma^\infty V_h).$$

Proof. — The proposition follows from the definition of v_h-periodic homotopy groups and the $\Sigma^\infty \dashv \Omega^\infty$-adjunction. See [Bou01, Proposition 5.1]. □

3.4.1.4. Proposition. — *For a p-local spectrum E, there are equivalences*

$$\tau_{>c^+(\Sigma W_{h+1})}\left(\mathrm{P}_{\Sigma W_{h+1}}(\Omega^\infty E)\right) \simeq \tau_{>c^+(\Sigma W_{h+1})}\left(\Omega^\infty \mathrm{L}_h^{\mathrm{f}}(E)\right)$$
$$\mathrm{L}_{v_h}(\Omega^\infty E) \simeq \tau_{>c^+(\Sigma W_{h+1})}\left(\Omega^\infty \mathrm{M}_h^{\mathrm{f}}(E)\right).$$

Proof. — Since the homotopy types involved are $c^+(\Sigma W_{h+1})$-connected and p-local, it suffices to check that the induced maps on v_i-periodic homotopy groups are isomorphisms for every $0 \leq i \leq h$, by Theorem 3.3.2.6 and Proposition 3.4.0.5. The proposition then follows by applying Proposition 3.4.0.5. □

***3.4.1.5. Theorem* (Bousfield–Kuhn functor).** — *Let $h \geq 1$ be a natural number. There exists a functor $\widetilde{\Phi}_h \colon \mathcal{H}\mathrm{o}_* \to \mathcal{S}\mathrm{p}$ satisfying the following properties:*

(i) *For every $X \in \mathcal{H}\mathrm{o}_*$, the spectrum $\widetilde{\Phi}_h(X)$ is* $\mathrm{T}(h)_\bullet$*-local.*

(ii) *For every p-local finite complex V_h of type h, there exist isomorphisms*

$$v_h^{-1}\pi_\bullet(X; V_h) \cong \pi_\bullet \operatorname{Map}(\Sigma^\infty V_h, \widetilde{\Phi}_h(X)) \cong v_h^{-1}\pi_\bullet(\widetilde{\Phi}_h(X); \Sigma^\infty V_h),$$

which are natural in X.

(iii) *For every spectrum W, there exists a natural equivalence $\widetilde{\Phi}_h(\Omega^\infty E) \simeq \mathrm{L}_{\mathrm{T}(h)}(E)$.*

(iv) *The functor $\widetilde{\Phi}_h$ send a v_h-periodic equivalence of pointed homotopy types to an equivalence of spectra.*

Proof. — See [Bou01, Theorem 5.3]. We refer the reader to [Bou01] and [Kuh08] a more detailed study of the Bousfield–Kuhn functor. □

3.4.1.6. Theorem. — *Let $h \geq 1$ be a natural number. The the induced functor $\widetilde{\Phi}_h \colon \mathcal{H}\mathrm{o}_* \to \mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$ factors through $\mathcal{H}\mathrm{o}_{v_h}$, i.e. there exists a functor*

$$\Phi_h \colon \mathcal{H}\mathrm{o}_{v_h} \to \mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$$

such that $\widetilde{\Phi}_h \simeq \Phi_h \circ \mathrm{L}_{v_h}$. Moreover, the functor Φ_h admits a left adjoint Θ_h.

Proof. — The factorisation follows from Theorem 3.4.1.5.(iv). See [Bou01, Theorem 5.4 and Corollary 5.6] for the existence of the left adjoint. The functor Φ_h is also known under the name the *Bousfield–Kuhn functor.* □

CHAPTER 4

Homological and homotopical periodic localisations

4.1. Properties of homological localisations of homotopy types

In this section we include several useful properties of homological localisations of homotopy types for later use. Fix a prime number p. We prove that the localisations of homotopy types with respect to certain p-local homology theories preserves suitable highly connectedness: For example, the $\mathrm{K}(h)_\bullet$- and $\mathrm{T}(h)_\bullet$-homology localisations preserve $d \leq h$-connectedness. This is original work, see Theorem 4.1.0.14.

4.1.0.1. Definition. — Let p be a prime number. The *mod-p Moore spectrum* $\mathbb{S}/p$ is a spectrum whose $\mathrm{H}\mathbb{Z}$-homology is concentrated in degree 0 and is isomorphic to the abelian group $\mathbb{Z}/p\mathbb{Z}$. One can construct $\mathbb{S}/p$ by the following cofibre sequence

$$\mathbb{S} \xrightarrow{\times p} \mathbb{S} \to \mathbb{S}/p$$

where the first map is the degree p map of the sphere spectrum $\mathbb{S}$.

Let E be a spectrum. For later applications, we record without proofs several useful properties about the relationships between $E_\bullet$-equivalences of spectra and rational homology equivalences or mod-p homology equivalences of spectra. Recall that $\otimes$ denote the smash product of spectra.

4.1.0.2. Notation. — Let E be a spectrum.

(i) For a prime number p, denote the spectrum $E \otimes \mathbb{S}/p$ by E/p.

(ii) Define the subset $\mathscr{P}(E_\bullet)$ of the set of prime numbers as follows: A prime number p is in $\mathscr{P}(E_\bullet)$ if the multiplication-by-p map is *not* an isomorphism of the coefficient group $E_\bullet \cong \pi_\bullet^{\mathrm{st}}(E)$ of E.

(iii) Recall that $\mathbb{Z}_{(\mathscr{P}(E_\bullet))}$ denotes the localisation of $\mathbb{Z}$ where every prime number q not contained in $\mathscr{P}(E_\bullet)$ is inverted. Define the abelian group

$$G(E_\bullet) := \begin{cases} \oplus_{p \in \mathscr{P}(E_\bullet)} \mathbb{Z}/p\mathbb{Z}, & \text{if } E \otimes \mathrm{H}\mathbb{Q} = 0 \\ \mathbb{Z}_{(\mathscr{P}(E_\bullet))}, & \text{otherwise.} \end{cases}$$

***4.1.0.3. Lemma* ([Bou82, Lemma 3.3]).** — *Let E be a spectrum and let $f\colon X \to Y$ be a morphism of pointed homotopy types. If $E \otimes \mathrm{H}\mathbb{Q} \not\simeq 0$, then the following are equivalent.*

(i) *The map f is an $E_\bullet$-equivalence.*

(ii) *The map f is a $\mathrm{H}\mathbb{Q}_\bullet$-equivalence, and is an $(E/p)_\bullet$-equivalence for each prime number $p \in \mathscr{P}(E_\bullet)$.*

***4.1.0.4. Lemma* ([Bou82, Lemma 3.4]).** — *Let E be a spectrum and let $f\colon X \to Y$ be a morphism of pointed homotopy types. If $E \otimes \mathrm{H}\mathbb{Q} = 0$, then the following are equivalent:*

(i) *The map f is an $E_\bullet$-equivalence.*

(ii) *The map f is an $(E/p)_\bullet$-equivalence for each $p \in \mathscr{P}(E_\bullet)$.*

***4.1.0.5. Proposition* ([Bou82, Proposition 2.1]).** — *Let $f\colon X \to Y$ be a morphism of pointed homotopy types and let E be a spectrum. Fix a prime number p, and assume that $\widetilde{E}_\bullet\left(\mathrm{K}(\mathbb{Z}/p\mathbb{Z}, n)\right) \neq 0$ for a natural number $n \geq 1$.*

If f is an $E_\bullet$-equivalence, then f induces an isomorphism

$$f_*\colon\ \mathrm{H}_n\left(X; \mathbb{Z}/p\mathbb{Z}\right) \xrightarrow{\simeq} \mathrm{H}_n\left(Y; \mathbb{Z}/p\mathbb{Z}\right).$$

***4.1.0.6. Theorem* ([Bou82, Theorem 3.1]).** — *Let E be a spectrum. Abbreviate the abelian group $G(E_\bullet)$ by G. The following statements are equivalent:*

(i) *A morphism of pointed homotopy types is an $E_\bullet$-equivalence if and only if it is a $\mathrm{H}G$-homology equivalence.*

(ii) *For every natural number $n \geq 1$ and for every $p \in \mathscr{P}(E_\bullet)$, we have that $\widetilde{E}_\bullet\left(\mathrm{K}(\mathbb{Z}/p\mathbb{Z}, n)\right) \neq 0$.*

4.1.0.7. Remark. — Let E be a spectrum. The reader can find a detailed explanation of types of $E_\bullet$-acyclicity of Eilenberg–MacLane spaces in [Bou82].

4.1.0.8. Situation. — We fix a prime number p for the rest of this section.

4.1.0.9. Definition. — Let E be a spectrum. Define the *(mod p) transitional dimension*[(1)] $\mathrm{tran}_p E$ of E to be the largest natural number n such that $\widetilde{E}_\bullet\left(\mathrm{K}(\mathbb{Z}/p\mathbb{Z}, n)\right) \neq 0$. Define $\mathrm{tran}_p E = \infty$ if $\widetilde{E}_\bullet\left(\mathrm{K}(\mathbb{Z}/p\mathbb{Z}, n)\right)$ is non-trivial for every $n \in \mathbb{N}$.

4.1.0.10. Theorem. — *Let E be a spectrum.*

(i) *If $\mathrm{tran}_p(E) = 0$, then $E/p \simeq 0$.*

(ii) *If $\mathrm{tran}_p(E) = \infty$, then $(E/p)_\bullet$-equivalences of homotopy types are the same as $(\mathrm{H}\mathbb{Z}/p\mathbb{Z})_\bullet$-equivalences.*

(1) We take the definition from [Bou97, §10], which is slightly different from the one in [Bou82, 8.1]

Proof. — (i) is by [Bou82, Proposition 2.2], which proves that the multiplication-by-p endomorphism of the coefficient group $E_\bullet$ is an isomorphism of under the assumption $\mathrm{tran}_p(E) = 0$. (ii) is by Theorem 4.1.0.6. □

4.1.0.11. Example. — Let $h \geq 1$ be a natural number. The Morava K-theory $\mathrm{K}(h)$, the Johnson–Wilson spectrum $\mathrm{E}(h)$, the telescope spectrum $\mathrm{T}(h)$ and $\mathrm{S}(h)$ all have transitional dimension h, as shown in [RW80] and [CSY22], respectively.

***4.1.0.12. Theorem* ([Bou82, Theorems 8.2]).** — *Let E be a p-local spectrum and let $f\colon X \to Y$ be an $E_\bullet$-equivalence of pointed homotopy types.*

If $E \otimes \mathrm{H}\mathbb{Q} = 0$, then f induces an isomorphism $f_*\colon \mathrm{H}_i(X; \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\simeq} \mathrm{H}_i(Y; \mathbb{Z}/p\mathbb{Z})$ for every $i \leq \mathrm{tran}_p E$.

If $E \otimes \mathrm{H}\mathbb{Q} \neq 0$, then f induces an isomorphism $f_*\colon \mathrm{H}_i\left(X; \mathbb{Z}_{(p)}\right) \xrightarrow{\simeq} \mathrm{H}_i\left(Y; \mathbb{Z}_{(p)}\right)$ for every $i \leq \mathrm{tran}_p E$.

***4.1.0.13. Theorem* ([Bou82, Theorems 8.3]).** — *Let E be a p-local spectrum and let X be a pointed homotopy type.*

If $E \otimes \mathrm{H}\mathbb{Q} = 0$, then the map $\mathrm{L}_{\mathrm{H}\mathbb{Z}/p\mathbb{Z}}(X) \to \mathrm{L}_E(X)$ given by $E_\bullet$-localisation induces an isomorphism of homotopy groups in degrees less than $\mathrm{tran}_p(E)$ and induces an epimorphism of homotopy groups in degree $\mathrm{tran}_p E$.

If $E \otimes \mathrm{H}\mathbb{Q} \neq 0$, then the map $\mathrm{L}_{\mathrm{H}\mathbb{Z}_{(p)}}(X) \to \mathrm{L}_E(Y)$ given by $E_\bullet$-localisation induces an isomorphism of homotopy groups in degrees less than $\mathrm{tran}_p(E)$ and induces an epimorphism of homotopy groups in degree $\mathrm{tran}_p E$.

4.1.0.14. Theorem. — *Let E be a p-local spectrum such that $0 < \mathrm{tran}_p E < \infty$. For a d-connected pointed homotopy type X with $1 \leq d \leq \mathrm{tran}_p E$, the localisation $\mathrm{L}_E(X)$ is also d-connected.*

Proof. — Denote

$$R = \begin{cases} \mathbb{Z}/p\mathbb{Z}, & \text{if } E \otimes \mathrm{H}\mathbb{Q} = 0 \\ \mathbb{Z}_{(p)}, & \text{otherwise.} \end{cases}$$

By Theorem 4.1.0.13 the induced map $\pi_k(\mathrm{L}_{\mathrm{H}R}(X)) \to \pi_k(\mathrm{L}_E(X))$ is an isomorphism for every $k \leq \mathrm{tran}_p E - 1$ and is an epimorphism for $k = \mathrm{tran}_p E$. Furthermore, we know that $\pi_i(\mathrm{L}_{\mathrm{H}R}(X)) = 0$ for $i \leq d$, e.g. by [Bou75, Proposition 4.3].

Therefore, we have that $\pi_i(X) \to \pi_i(\mathrm{L}_E(X))$ is an isomorphism for $i < d$ and is an epimorphism for $i = d$. In other words, the localisation functor L_E preserves d-connectedness for $d \leq \mathrm{tran}_p E$. □

4.1.0.15. Remark. — Let $h \in \mathbb{N}$. As a consequence, the chromatic localisation functors $\mathrm{L}_{\mathrm{K}(h)}$, $\mathrm{L}_{\mathrm{E}(h)}$, $\mathrm{L}_{\mathrm{T}(h)}$ and $\mathrm{L}_{\mathrm{S}(h)}$ preserve d-connectedness for $d \leq h$. Thus, the above theorem generalises the result in [Tai98] where it is shown that the $(\mathrm{KU}/p)_\bullet$-localisation of homotopy types preserves 1-connectedness.

4.1.1. Homology localisations and fibre sequences. — In this subsection we recall several useful interactions of homology localisation of homotopy types with fibre sequences.

4.1.1.1. Theorem. — *Consider the following commutative diagram*

$$\begin{array}{ccccc} F & \longrightarrow & X & \longrightarrow & B \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ F' & \longrightarrow & X' & \longrightarrow & B' \end{array} \tag{4.1.1.1}$$

in $\mathcal{H}o_*$ where the rows are fibre sequences and B and B' are connected. Assume that the maps f and Ωh induce equivalences $f_*\colon \mathcal{M}\mathrm{ap}(F',Y) \xrightarrow{\sim} \mathcal{M}\mathrm{ap}(F,Y)$ and $(\Omega h)_*\colon \mathcal{M}\mathrm{ap}(\Omega B',Y) \xrightarrow{\sim} \mathcal{M}\mathrm{ap}(\Omega B,Y)$ for a homotopy type Y. Then the map g also induces an equivalence $g_*\colon \mathcal{M}\mathrm{ap}(X',Y) \xrightarrow{\sim} \mathcal{M}\mathrm{ap}(X,Y)$.

Proof. — We show that the map $f \times \Omega h$ induces an equivalence

$$\mathcal{M}\mathrm{ap}(F' \times \Omega B', Y) \xrightarrow{\sim} \mathcal{M}\mathrm{ap}(F \times \Omega B, Y).$$

Then the theorem follows from ¶1.3.0.6.(ii). We have

$$\begin{aligned} \mathcal{M}\mathrm{ap}(F' \times \Omega B', Y) &\simeq \mathcal{M}\mathrm{ap}(F', \mathcal{M}\mathrm{ap}(\Omega B', Y)) \\ &\simeq \mathcal{M}\mathrm{ap}(F', \mathcal{M}\mathrm{ap}(\Omega B, Y)) \\ &\simeq \mathcal{M}\mathrm{ap}(F' \times \Omega B, Y) \\ &\simeq \mathcal{M}\mathrm{ap}(\Omega B, \mathcal{M}\mathrm{ap}(F', Y)) \\ &\simeq \mathcal{M}\mathrm{ap}(\Omega B, \mathcal{M}\mathrm{ap}(F, Y)) \\ &\simeq \mathcal{M}\mathrm{ap}(F \times \Omega B, Y) \end{aligned}$$

See also [Bou94, Theorem 4.6]. □

4.1.1.2. Corollary. — *Let E be a spectrum. In the commutative diagram (4.1.1.1), if f and Ωh are $E_\bullet$-equivalences, then g is also an $E_\bullet$-equivalence.*

Proof. — In Theorem 4.1.1.1 we can replace the unpointed mapping spaces $\mathcal{M}\mathrm{ap}$ by pointed mapping spaces $\mathcal{M}\mathrm{ap}_*$ if Y is pointed and connected. As a corollary, if f and Ωh induces an isomorphism on E-cohomology, then so does g. The statement for E-homology follows by applying a universal coefficient theorem for generalised homology theories, see [Kai71; Yos75]. See also [Bou94, Theorem 4.6]. □

4.1.1.3. Corollary. — *Let E be a spectrum and let $f\colon X \to Y$ be a morphism of pointed connected homotopy types. If the induced map Ωf is an $E_\bullet$-equivalence, then the map f is an $E_\bullet$-equivalence.* □

4.1.1.4. Remark. — The converse of the above corollary does not always hold, see also Theorem 4.4.0.2.

4.2. Telescopic homological and homotopical localisations

Fix a prime number p for this section. Recall the p-local telescope spectrum $\mathrm{T}(h)$ of height h and the spectrum $\mathrm{S}(h)$ from ¶1.2.0.13. In this expositional section we discuss the relationship between $\mathrm{T}(h)$-homology equivalences and v_h-periodic equivalences of homotopy types. Here we update several results in [Bou94, §13] using the recent work [CSY22]: Starting from Theorem 4.2.0.8 the reader can compare our formulations of the statements with Bousfield's original ones.

4.2.0.1. Situation. — Let V_h be a p-local finite complex of type h together with a v_h self-map $\Sigma^{d_{V_h}} V_h \to V_h$, where $h \geq 1$. Modelling V_h by a finite CW-complex (see Remark 1.3.0.11), let $\dim(V_h)$ denote the dimension of the top cells of V_{n+1}.

4.2.0.2. Proposition. — *Let $\varphi\colon X \to Y$ be a morphism of pointed homotopy types. In Situation 4.2.0.1, assuming that there exists a natural number $k \geq \dim(V_h)$ such that $\Omega^k X \to \Omega^k Y$ is a $\mathrm{T}(h)_\bullet$-equivalence, then φ induces an isomorphism*

$$\varphi_*\colon v_h^{-1}\pi_\bullet\,(X; V_h) \to v_h^{-1}\pi_\bullet\,(Y; V_h)$$

of v_h-periodic homotopy groups.

Proof. — We follow the proof idea of [Bou94, Theorem 12.5]. Since v_h-periodic homotopy groups are independent of taking higher connected covers, we can assume that X and Y are k-connected. By Corollary 4.1.1.3 the map $\Omega^j\varphi$ is a $\mathrm{T}(h)_\bullet$-equivalence for every $0 \leq j \leq k$. Thus, the induced map $\mathrm{Map}_*\,(V_h, X) \to \mathrm{Map}_*\,(V_h, Y)$ is a $\mathrm{T}(h)_\bullet$-equivalence, because V_h is equivalent to a colimit of spheres of dimension at most k.

Recall the homotopy type $\mathrm{T}_{V_h}(X)$ from Definition 3.2.0.4, which is $\mathrm{T}(h)_\bullet$-local by Proposition 3.3.2.1, Proposition 3.3.1.1 and Example 3.3.1.2. Thus we obtain the following commutative diagram

$$\begin{array}{ccc} \mathrm{Map}_*\,(V_h, X) & \longrightarrow & \mathrm{T}_{V_h} X \\ {\scriptstyle\varphi_*}\downarrow & \overset{\lambda}{\nearrow} & \downarrow{\scriptstyle\varphi_*} \\ \mathrm{Map}_*\,(V_h, Y) & \longrightarrow & \mathrm{T}_{V_h} Y, \end{array} \tag{4.2.0.1}$$

where the vertical maps are induced by φ, the horizontal maps are the canonical maps obtained from the colimit construction of $\mathrm{T}_{V_h}(X)$ and $\mathrm{T}_{V_h}(Y)$, and the lift λ exists because $\mathrm{T}_{V_h}(X)$ and $\mathrm{T}_{V_h}(Y)$ are $\mathrm{T}(h)_\bullet$-local and φ_* is a $\mathrm{T}(h)_\bullet$-equivalence. Furthermore, the map $v_h\colon \Sigma^{d_{V_h}} V_h \to V_h$ induces the commutative diagram below

$$\begin{array}{ccc} \mathrm{Map}_*\,(V_h, Y) & \xrightarrow{\ \lambda\ } & \mathrm{T}_{V_h}(X) \\ \downarrow & & \downarrow{\scriptstyle\simeq} \\ \Omega^{d_{V_h}}\mathrm{Map}_*\,(V_h, Y) & \xrightarrow[\Omega^{d_{V_h}}\lambda]{} & \Omega^{d_{V_h}}(\mathrm{T}_{V_h}(X))\,. \end{array}$$

Thus the diagram (4.2.0.1) induces the following commutative diagram on v_h-periodic homotopy groups

$$\begin{array}{ccc} v_h^{-1}\pi_\bullet(X;V_h) & \xrightarrow{\mathrm{id}} & v_h^{-1}\pi_\bullet(X;V_h) \\ {\scriptstyle\varphi_*}\downarrow & \overset{\lambda_*}{\nearrow} & \downarrow{\scriptstyle\varphi_*} \\ v_h^{-1}\pi_\bullet(Y;V_h) & \xrightarrow[\mathrm{id}]{} & v_h^{-1}\pi_\bullet(Y;V_h)\,. \end{array}$$

As shown in the above diagram, the map λ_* is a two-sided inverse of φ_*. Therefore, the induced map φ_* of v_h-periodic homotopy groups is an isomorphism. □

4.2.0.3. Remark. — The proposition is not very practical for applications, since it depends on the choice of the CW-structures of the finite complex V_h: Given a $\mathrm{T}(h)_\bullet$-equivalence of k-fold loop spaces, it is not trivial to find a type h finite complex of dimension at most k and equipped with a v_h-self map. The corollary below is a bit more convenient.

4.2.0.4. Corollary. — *Let $h \geq 1$ be a natural number. If φ is a $\mathrm{T}(h)_\bullet$-equivalence of pointed connected homotopy types, then there exists a natural number k such that $\Sigma^k\varphi$ is a v_h-periodic equivalence.*

Proof. — It is shown in the proof of [Tho93, Lemma 2.3] that the functor $\Omega^m\Sigma^m$ preserves generalised homology equivalences of pointed connected homotopy types, for all natural numbers. Let V_h a finite complex as in Situation 4.2.0.1. Thus, for every natural number $k \geq \dim(V_h)$, the morphism $\Sigma^k\varphi$ induces an isomorphism of v_h-periodic homotopy groups by Proposition 4.2.0.2, because $\Omega^k\Sigma^k(\varphi)$ is a $\mathrm{T}(h)_\bullet$-equivalence. See also [Bou94, Corollary 12.6]. □

4.2.0.5. Corollary. — *Let $h \geq 1$ be a natural number. If a map $\psi\colon X \to Y$ of pointed connected homotopy types is an $\mathrm{S}(h)_\bullet$-equivalence, then there exists a natural number k such that $\Sigma^k\psi$ is a v_n-periodic equivalences for all $0 \leq n \leq h$.*

Proof. — Recall that an $\mathrm{S}(h)_\bullet$-equivalence is a $\mathrm{T}(n)_\bullet$-equivalences for every $0 \leq n \leq h$, where $\mathrm{T}(0) = \mathrm{H}\mathbb{Q}$, see ¶1.2.0.15. Thus the map ψ and any iterated suspensions of ψ are rational homology equivalences. Then iterated suspensions of ψ is a rational homotopy equivalence, since the source and target become simply connected.

For each $1 \leq n \leq h$, choose a finite complex V_n as in Situation 4.2.0.1. Define

$$k \coloneqq \max\{1, \dim(V_1), \ldots, \dim(V_n), \ldots, \dim(V_h)\}.$$

By Corollary 4.2.0.4 the map ψ induces an isomorphism

$$v_n^{-1}\pi_\bullet\left(\Sigma^k X; V_n\right) \to v_n^{-1}\pi_\bullet\left(\Sigma^k Y; V_n\right),$$

for all $1 \leq n \leq h$. See also [Bou94, Theorem 13.5]. □

Recall the stable Bousfield class $\langle - \rangle_\Sigma$ of homotopy types from Definition 3.1.2.3.

4.2.0.6. Proposition. — *Let X be a pointed homotopy type, and let V_{h+1} be a finite complex of type $h+1$, where $h \in \mathbb{N}$. Then $\langle X\rangle_\Sigma \leq \langle V_{h+1}\rangle_\Sigma$ if and only if X is $\mathrm{S}(h)_\bullet$-acyclic and $\widetilde{\mathrm{H}}_\bullet(X;\mathbb{Z})$ is p-local.*

Proof. — Assume we have $\langle X\rangle_\Sigma \leq \langle V_{h+1}\rangle_\Sigma$. Thus, $\langle \Sigma^j X\rangle \leq \langle V_{h+1}\rangle$ for some $j \in \mathbb{N}$. By Proposition 2.1.2.10 the homotopy type X is $\mathrm{S}(h)_\bullet$-acyclic, since $\Sigma^j X$ is V_{h+1}-full and V_h is $\mathrm{S}(h)_\bullet$-acyclic. The same arguments show that $\widetilde{\mathrm{H}}_*(X;\mathbb{Z}/q\mathbb{Z}) = 0$ for any prime number $q \neq p$, because $\widetilde{\mathrm{H}}_*(V_h;\mathbb{Z}/q\mathbb{Z}) = 0$. This shows that $\widetilde{\mathrm{H}}_*(X;\mathbb{Z})$ is p-local.

As for the other direction of implication, we can apply Corollary 4.2.0.5 to show that $\mathrm{P}_{\Sigma V_{h+1}}(\Sigma^k X) \simeq \mathrm{pt}$, for sufficiently large k. Indeed, for $k \geq \mathrm{conn}(\Sigma V_{h+1}) + 1$, recall that $\mathrm{P}_{\Sigma V_{h+1}}(\Sigma^k X) \simeq \mathrm{pt}$ if and only if the v_n-periodic homotopy groups of $\Sigma^k X$ vanish for all $1 \leq n \leq h+1$ and $\pi_\bullet(\Sigma^k X) \otimes \mathbb{Z}[1/p] = 0$, see Corollary 3.3.2.7. The latter condition is guaranteed by the assumptions and Corollary 4.2.0.5. Thus we obtain $\langle \Sigma^k X\rangle \leq \langle \Sigma V_{h+1}\rangle$. Therefore, we have $\langle X\rangle_\Sigma = \langle V_{h+1}\rangle_\Sigma$. See also [Bou94, Proposition 13.6]. □

4.2.0.7. Remark. — In Corollary 4.2.0.4 and Corollary 4.2.0.5, the choice of k can be very large, depending on our choice of the finite complexes. However, recall that v_h-periodic equivalences of homotopy types should not depend on the choices of the "coefficients" V_h, cf. Proposition 3.2.0.11. In the rest of this section we present stronger versions of these two corollaries which are in particular independent of the choice of the finite complex V_h. The idea is to use the relationship between v_h-periodic equivalences and ΣW_h-equivalences for a specific p-local homotopy type W_h which satisfies the conditions of Situation 3.3.0.1. Recall that ΣW_h is of connectivity at least $h+1$, see Proposition 3.3.0.3.

4.2.0.8. Theorem. — *Let h be a natural number. There exists a pointed p-local homotopy type $W_{h+1,0}$ such that*

(i) *it satisfies the hypotheses in Situation 3.3.0.1, and*
(ii) *its suspension $\Sigma W_{h+1,0}$ has connectivity $\mathrm{conn}(\Sigma W_{h+1,0}) = h+1$.*

As a result, we have $c^+(\Sigma W_{h+1,0}) = h+2$.

Proof. — In [CSY22, Theorem 5.3.5] Carmeli–Schlank–Yanovski show that the Eilenberg–MacLane space $\mathrm{K}(\mathbb{Z}/p\mathbb{Z}, m)$ is $\mathrm{T}(h)_\bullet$ acyclic if and only if $m \geq h+1$, which verifies [Bou94, Conjecture 12.4]. The theorem follows by applying this result to [Bou94, Proposition 13.7, §§13.8–13.9]. More concretely, we can choose $W_{h+1,0}$ to be of the form

$$V_{h+1} \vee \mathrm{K}(\mathbb{Z}/p\mathbb{Z}, h+1)$$

for any p-local finite complex V_{h+1} of type $h+1$. To verify (i), we see that $\mathrm{K}(\mathbb{Z}/p\mathbb{Z}, h+1)$ is $\mathrm{S}(h)_\bullet$-acyclic and its reduced integral homology is p-local. So by Proposition 4.2.0.6

we have $\langle \mathrm{K}(\mathbb{Z}/p\mathbb{Z}, h+1)\rangle_\Sigma \leq \langle V_{h+1}\rangle_\Sigma$. Therefore, we have

$$\langle V_{h+1}\rangle_\Sigma \leq \langle V_{h+1} \vee \mathrm{K}(\mathbb{Z}/p\mathbb{Z}, h+1)\rangle_\Sigma = \langle V_{h+1}\rangle_\Sigma \vee \langle \mathrm{K}(\mathbb{Z}/p\mathbb{Z}, h+1)\rangle_\Sigma \leq \langle V_{h+1}\rangle_\Sigma.$$

In other words, $\langle W_{h+1}\rangle_\Sigma = \langle V_{h+1}\rangle_\Sigma$. □

4.2.0.9. Construction. — We can use the construction in the proof of Theorem 4.2.0.8 to build a homotopy type $W_{h+1,i}$ satisfying Situation 3.3.0.1 and $\mathrm{conn}(W_{h+1,i}) = h+i$ for any $i \in \mathbb{N}$. For example, we let

$$W_{h+1,i} \coloneqq V_{h+1} \vee \mathrm{K}(\mathbb{Z}/p\mathbb{Z}, h+i)$$

where V_{h+1} is a p-local finite complex of type $h+1$.

4.2.0.10. Theorem. — *Let h be a natural number and let X be an $\mathrm{S}(h)_\bullet$-acyclic pointed homotopy type. Then for every $k \geq 1$ and for every $0 \leq n \leq h$, the v_n-periodic homotopy groups of the p-localisation $\left(\Sigma^k X\right)_{(p)}$ vanish.*

Proof. — We present the proof of [Bou94, Theorem 13.10]. We show that the homotopy type $\mathrm{P}_{\Sigma W_{h+1,0}}\left((\Sigma^k X)_{(p)}\right)$ is contractible. Then the theorem follows from Corollary 3.3.2.7. Since $\widetilde{\mathrm{S}(h)_\bullet}(X) = 0$, we have that $\widetilde{\mathrm{H}}_*(X;\mathbb{Q}) = 0$ and $\widetilde{\mathrm{H}}_*(X;\mathbb{Z}_{(p)})$ is p-primary. Let $\widetilde{\mathrm{H}}_m(X;\mathbb{Z}_{(p)})$ be the first non-trivial singular homotology group of X with $\mathbb{Z}_{(p)}$-coefficient. Set

$$G \coloneqq \begin{cases} \mathbb{Z}/p^\infty, & \text{if } \widetilde{\mathrm{H}}_m(X;\mathbb{Z}_{(p)}) \text{ is divisible by } p,\\ \mathbb{Z}/p\mathbb{Z}, & \text{otherwise.}\end{cases}$$

Then $\Omega^\infty\left(\Sigma^\infty X \otimes_{\mathcal{S}\mathrm{p}} \mathrm{H}\mathbb{Z}_{(p)}\right)$ and $\mathrm{K}(G,m)$ are $\mathrm{S}(h)_\bullet$-acyclic by Proposition 2.2.3.3 and Theorem 4.1.1.1, respectively.

Hence, we have $m \geq h+1$ by [CSY22, Theorem 5.3.5], in particular $m \geq 2$. Let $k \geq 1$ be a natural number. The p-localisation $X_{(p)}$ of X satisfies

$$\widetilde{\mathrm{H}}_*(X_{(p)};\mathbb{Z}) \cong \widetilde{\mathrm{H}}_*(X;\mathbb{Z}_{(p)}) \text{ and } (\Sigma^k X)_{(p)} \simeq \Sigma^k(X_{(p)}).$$

So $(\Sigma^k X)_{(p)}$ is $(m+k)$-supported and p-torsion or p-divisible. Thus we can apply Theorem 3.1.2.10 to obtain the equality of unstable Bousfield class

$$\langle (\Sigma^k X)_{(p)}\rangle = \langle (\Sigma^{j+k} X)_{(p)}\rangle \vee \langle \mathrm{K}(G, m+k)\rangle,$$

for any natural number j. Furthermore, we know $\mathrm{P}_{\Sigma W_{h+1,0}}(\Sigma^{j+k}X)_{(p)}) = 0$ for sufficiently large j by Corollary 4.2.0.5, $\mathrm{P}_{\Sigma W_{h+1,0}}(\mathrm{K}(G,m+k)) \simeq \mathrm{pt}$ by Proposition 3.3.0.5, and the fact that $m+k \geq h+2$. Therefore, we obtain $\langle (\Sigma^k X)_{(p)}\rangle \leq \langle \Sigma W_{h+1,0}\rangle$, that is $\mathrm{P}_{\Sigma W_{h+1,0}}\left((\Sigma^k X)_{(p)}\right) \simeq \mathrm{pt}$. □

4.2.0.11. Corollary. — *Let $h \in \mathbb{N}$, and let $f\colon X \to Y$ be an $\mathrm{S}(h)_\bullet$-equivalence of pointed homotopy types. Then for every $k \geq 2$ and every $0 \leq n \leq h$, the induced map $(\Sigma^k X)_{(p)} \to (\Sigma^k Y)_{(p)}$ induces isomorphisms on v_n-periodic homotopy groups.*

Proof. — The cofibre $\mathrm{cofib}(f)$ of f is $\mathrm{S}(h)_\bullet$-acyclic. For every natural number $k \geq 2$, we have $\mathrm{P}_{\Sigma W_{h+1,0}}\left((\Sigma^{k-1}\mathrm{cofib}(\psi))_{(p)}\right) \simeq \mathrm{pt}$, by Theorem 4.2.0.10. Consider the cofibre sequence

$$(\Sigma^{k-1}\mathrm{cofib}(f))_{(p)} \to (\Sigma^k X)_{(p)} \xrightarrow{\Sigma^k f} (\Sigma^k Y)_{(p)}$$

of pointed connected homotopy types. Let Z be a pointed connected $\Sigma W_{h+1,0}$-less homotopy type. The above cofibre sequence induces a fibre sequence

$$\mathcal{M}\mathrm{ap}_*\left((\Sigma^k Y)_{(p)}, Z\right) \to \mathcal{M}\mathrm{ap}_*\left((\Sigma^k Y)_{(p)}, Z\right) \to \mathcal{M}\mathrm{ap}_*\left((\Sigma^{k-1}\mathrm{cofib}(f))_{(p)}, Z\right),$$

where $\mathcal{M}\mathrm{ap}_*\left((\Sigma^k Y)_{(p)}, Z\right)$ is contractible. Thus the map $\Sigma^k f$ is a $\Sigma W_{h+1,0}$-equivalence. By Corollary 3.3.2.3 it is a v_n-periodic equivalence for every $0 \leq n \leq h$. See [Bou94, Corollary 13.11]. □

4.2.0.12. Corollary. — *Let h and $k \geq 2$ be natural numbers. If a pointed homotopy type Y is $\Sigma W_{h+1,0}$-less and its homotopy groups $\pi_i Y$ is p-local for all $i \geq k+1$, then $\Omega^k Y$ is $\mathrm{S}(h)_\bullet$-local.*

Proof. — This is a consequence of applying the $\Sigma \dashv \Omega$-adjunction to Corollary 4.2.0.11. See [Bou94, Corollary 13.12]. □

4.2.0.13. Theorem. — *Let $k \geq 1$ and h be natural numbers. For a k-connected pointed p-local homotopy type X with $\widetilde{\mathrm{S}(h)}_\bullet(\Omega^k X) = 0$, the v_n-periodic homotopy groups of X vanish for every $0 \leq n \leq h$.*

Proof. — We have the following comparisons of Bousfield classes

$$\langle X \rangle \leq \langle \Sigma^k \Omega^k (X) \rangle = \langle (\Sigma^k \Omega^k X)_{(p)} \rangle \leq \langle \Sigma W_{h+1,0} \rangle$$

of homotopy types. The first inequality is due to Corollary 2.3.1.13, the second equality holds because X is p-local and the last inequality is by Theorem 4.2.0.10. Then the v_n-periodic homotopy groups of X vanish for every $0 \leq n \leq h$, by Corollary 3.3.2.7. See [Bou94, Theorem 13.13]. □

4.2.0.14. Corollary. — *Let h and $k \geq 2$ be natural numbers. Let $f\colon X \to Y$ be a morphism of pointed k-connected homotopy types such that $\Omega^k f$ is a $\mathrm{S}(h)_\bullet$-equivalence and $f_*\colon \pi_\bullet(X) \otimes \mathbb{Z}[1/p] \to \pi_\bullet(Y) \otimes \mathbb{Z}[1/p]$ is an isomorphism, then f is a v_n-periodic equivalences for every $0 \leq n \leq h$.*

Proof. — By assumption we see that the fibre $\mathrm{fib}(f)$ is p-local and $(k-1)$-connected. Furthermore, the homotopy type $\Omega^{k-1}(\mathrm{fib}(f))$ is $\mathrm{S}(h)_\bullet$-acyclic by Corollary 4.1.1.2. Thus, the v_n-periodic homotopy groups of $\mathrm{fib}(f)$ vanish for every $0 \leq n \leq h$, by Theorem 4.2.0.13. Therefore, f is v_n-periodic equivalences for every $0 \leq n \leq h$. See [Bou94, Corollary 13.14]. □

4.3. Virtual homology equivalences

In this section we recall the basics of the theory of virtual homology equivalences of pointed connected homotopy types, following [Bou97].

4.3.0.1. Definition. —
(i) A spectrum E has *h-elevated acyclicity* if $\mathrm{K}(\mathbb{Z}/p\mathbb{Z}, h+1)$ is $E_\bullet$-acyclic for each prime number p.
(ii) A spectrum E has *elevated acyclicity* if it has h-elevated acyclicity for some natural number h.

4.3.0.2. Remark. — By Corollary 4.1.1.3, if $\mathrm{K}(\mathbb{Z}/p\mathbb{Z}, h+1)$ is $E_\bullet$-acyclic, then $\mathrm{K}(\mathbb{Z}/p\mathbb{Z}, m)$ is $E_\bullet$-acyclic for all $m \geq h+1$. Note that if E is p-local, the notion of h-elevated acyclicity coincides with the notion of mod-p transitional dimension, cf. Definition 4.1.0.9.

4.3.0.3. Example. — Let p be a prime number. The p-local Morava K-theory spectrum $\mathrm{K}(h)$, the telescope spectrum $\mathrm{T}(h)$, the Johnson–Wilson spectrum $\mathrm{E}(h)$ and the spectrum $\mathrm{S}(h)$ have h-elevated acyclicity for $h \in \mathbb{N}$, see ¶1.2.0.15 and [CSY22, Theorem 5.3.5].

4.3.0.4. Proposition. — *Let p be a prime number. A p-local spectrum E has elevated acyclicity if E-homology equivalences of homotopy types is* not *the same as $\mathrm{H}(\mathbb{Z}/p\mathbb{Z})$-homology equivalences or $\mathrm{H}\mathbb{Z}_{(p)}$-homology equivalences of homotopy types.*

Proof. — This follows from Theorem 4.1.0.6. □

4.3.0.5. Theorem. — *Let E be a spectrum of h-elevated acyclicity.*
(i) *Let $F \to X \to B$ be a fibre sequence of pointed connected homotopy types. Consider the $E_\bullet$-localisation applied to the fibre sequence $\Omega F \to \Omega X \to \Omega B$. Then there exist a double loop space $\Omega^2 D$, whose homotopy groups in degrees above $h+1$ are trivial, fitting into the fibre sequence*

$$\Omega^2 D \to \mathrm{L}_E(\Omega F) \to \mathrm{fib}\left(\mathrm{L}_E(\Omega X) \to \mathrm{L}_E(\Omega B)\right).$$

(ii) *Let Y be a connected H-space, considered as an object in the homotopy category $\mathrm{ho}(\mathcal{H}\mathrm{o}_*)$ of the ∞-category of pointed homotopy types. Then the natural map $\mathrm{L}_E(\Omega Y) \to \Omega(\mathrm{L}_E(Y))$ induces a fibre sequence*

$$F \to \mathrm{Bar}\left(\mathrm{L}_E(\Omega Y)\right) \to \mathrm{L}_E(Y)$$

where the fibre F is a loop space whose homotopy groups in degree above $h+2$ are trivial.

(iii) *Let $H_f \to H_t \to H_b$ be a fibre sequence in* $\mathrm{ho}(\mathcal{H}\mathrm{o}_*)$ *of pointed connected H-space. Then there exists a fibre sequence*

$$F \to \mathrm{L}_E(H_f) \to \mathrm{fib}\,(\mathrm{L}_E(H_t) \to \mathrm{L}_E(H_b))$$

where the homotopy groups of the fibre F in degrees above $h+1$ are trivial.

Proof. — See [Bou97, Theorem 11.2]. If we make further assumptions on the fundamental groups of the homotopy types involved in the statement, e.g. simply connected, one can show that the non-trivial homotopy groups of the error terms concentrates in three single degrees, see [Bou97, Theorems 10.8, 10.9, and 10.10]. □

4.3.0.6. Situation. — Let h be a natural number. In the rest of this section, let E be a spectrum of h-elevated acyclicity.

4.3.0.7. Definition. —

(i) A morphism $f\colon X \to Y$ of pointed homotopy types is a *virtual $E_\bullet$-equivalence* if the induced map $\mathrm{L}_E(f)\colon \mathrm{L}_E(\Omega X) \to \mathrm{L}_E(\Omega Y)$ is an equivalence after taking sufficiently highly connected covers. In this case, we say X and Y are *virtually $E_\bullet$-equivalent.*

(ii) If a pointed homotopy type X is virtually $E_\bullet$-equivalent to a point, we say X is *virtually $E_\bullet$-acyclic*

4.3.0.8. Remark. — In the situation of Definition 4.3.0.7, the morphism f is a virtual $E_\bullet$-equivalence if the induced map $\pi_i\,(\mathrm{L}_E(\Omega X)) \to \pi_i\,(\mathrm{L}_E(\Omega Y))$ of homotopy groups, with every choice of basepoints, is an isomorphism for all sufficiently large natural number i.

4.3.0.9. Proposition. — *Consider the commutative diagram*

$$\begin{array}{ccccc} F & \longrightarrow & X & \longrightarrow & B \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle h} \\ F' & \longrightarrow & X' & \longrightarrow & B' \end{array}$$

of pointed homotopy types where the rows are fibre sequences. If any two of the maps in $\{f, g, h\}$ are virtual $E_\bullet$-equivalences, then so is remaining one.

Proof. — A map $\psi\colon Y \to Z$ of pointed homotopy types is a virtual $E_\bullet$-equivalence if and only if $\tau_{\geq 1}(\psi)$ is a virtual $E_\bullet$-equivalence, because $\Omega(\tau_{\geq 1}(Y)) \simeq \Omega Y$ for every pointed connected homotopy type Y. Thus, we can assume that all homotopy types in the commutative diagrams are connected. Consider induced commutative diagram

$$\begin{array}{ccccc} \Omega F & \longrightarrow & \Omega X & \longrightarrow & \Omega B \\ \big\downarrow{\scriptstyle \Omega f} & & \big\downarrow{\scriptstyle \Omega g} & & \big\downarrow{\scriptstyle \Omega h} \\ \Omega F' & \longrightarrow & \Omega X' & \longrightarrow & \Omega B'. \end{array}$$

Applying $E_\bullet$-localisation, we obtain by Theorem 4.3.0.5.(i) the commutative diagram below

$$\begin{array}{ccccc} M & \longrightarrow & \mathrm{L}_E\,(\Omega X) & \longrightarrow & \mathrm{L}_E\,(\Omega B) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ M' & \longrightarrow & \mathrm{L}_E\,(\Omega X') & \longrightarrow & \mathrm{L}_E\,(\Omega B')\,, \end{array}$$

where the rows are fibre sequences, $\pi_i(M) \cong \pi_i(\mathrm{L}_E\,(\Omega F))$, and $\pi_i(M') \cong \pi_i(\mathrm{L}_E\,(\Omega F'))$ for every $i \geq h+2$. The theorem now follows by applying the Five Lemma to the induced maps of long exact sequence of homotopy groups of fibre sequences. See also [Bou97, Theorem 11.4]. □

4.3.0.10. Corollary. — *Let $F \to X \to B$ be a fibre sequence of homotopy types.*

(i) *The homotopy type F is virtually $E_\bullet$-acyclic if and only if the map $X \to B$ is a virtual $E_\bullet$-equivalence.*

(ii) *The homotopy type B is virtually $E_\bullet$-acyclic if and only if the map $F \to X$ is a virtual $E_\bullet$-equivalence.*

(iii) *The homotopy type X is virtually $E_\bullet$-acyclic if and only if the induced morphism $\Omega B \to F$ is a virtual $E_\bullet$-equivalence.*

Proof. — (i) and (ii) follows from applying Proposition 4.3.0.9 to the following two maps of fibre sequences respectively

$$\begin{array}{ccccc} F & \longrightarrow & X & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{pt} & \longrightarrow & B & \longrightarrow & B, \end{array} \qquad \begin{array}{ccccc} F & \longrightarrow & X & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & X & \longrightarrow & \mathrm{pt}. \end{array}$$

(iii) follows by applying (ii) to the fibre sequence $\Omega B \to F \to X$. □

4.3.0.11. Corollary. — *Let $f\colon X \to Y$ be a morphism of pointed homotopy types. Then the following statements are equivalent.*

(i) *The map f is a virtual $E_\bullet$-equivalence.*

(ii) *The induced map $\Omega^i f$ is a virtual $E_\bullet$-equivalence for all $i \in \mathbb{N}$.*

(iii) *The induced map $\Omega^i f$ is a virtual $E_\bullet$-equivalence for some $i \in \mathbb{N}$*

Proof. — We apply Proposition 4.3.0.9 to the following commutative diagram

$$\begin{array}{ccccccccccccccc} \cdots & \longrightarrow & \Omega^{i+1}X & \longrightarrow & \mathrm{pt} & \longrightarrow & \Omega^i X & \longrightarrow & \cdots & \longrightarrow & \Omega X & \longrightarrow & \mathrm{pt} & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \Omega^i f} & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f} \\ \cdots & \longrightarrow & \Omega^{i+1}Y & \longrightarrow & \mathrm{pt} & \longrightarrow & \Omega^i Y & \longrightarrow & \cdots & \longrightarrow & \Omega Y & \longrightarrow & \mathrm{pt} & \longrightarrow & Y. \end{array}$$

See [Bou97, Corollary 11.5]. □

4.3.0.12. Corollary. — *Let $f\colon X \to Y$ be a morphism of pointed homotopy types. Then the following statements are equivalent:*

(i) *The map f is a virtual $E_\bullet$-equivalence.*

(ii) *The induced map $\tau_{>i}(f)\colon \tau_{>i}(X) \to \tau_{>i}(Y)$ is a virtual $E_\bullet$-equivalence for every natural number i.*

(iii) *The induced map $\tau_{>i}(f)\colon \tau_{>i}(X) \to \tau_{>i}(Y)$ is a virtual $E_\bullet$-equivalence for some natural number i.*

Proof. — Consider the following commutative diagram

$$\begin{array}{ccccc} \tau_{>i}(X) & \longrightarrow & X & \longrightarrow & \tau_{\leq i}(X) \\ \downarrow{\scriptstyle f_{>i}} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_{\leq i}} \\ \tau_{>i}(Y) & \longrightarrow & Y & \longrightarrow & \tau_{\leq i}(Y) \end{array}$$

in $\mathcal{H}\mathrm{o}_*$. By Corollary 4.3.0.12 the morphism $f_{\leq i}$ is a virtual $E_\bullet$-equivalence. Now the corollary follows from Proposition 4.3.0.9. See also [Bou97, Corollary 11.6]. □

4.3.0.13. Theorem. — *Assume that $\pi_\bullet E$ is torsion. Let G be an abelian group. Then the Eilenberg–MacLane space $\mathrm{K}(G, n)$ is $E_\bullet$-acyclic for $n \geq h+2$.*

Proof. — Bousfield calculates the $(E/p)_\bullet$-localisation of every Eilenberg–MacLane spaces, see [Bou82, §6]. Our situation corresponds to [Bou82, §§ 6.3–6.4]. □

4.3.0.14. Proposition. — *Assume that $\pi_\bullet E$ is torsion. Let $f\colon X \to Y$ be a morphism of pointed homotopy types. If f is a virtual $E_\bullet$-equivalence, then the induced map $\tau_{>i}(f)\colon \tau_{>i}(X) \to \tau_{>i}(Y)$ is an $E_\bullet$-equivalence for all $i \geq h+2$.*

Proof. — The canonical maps $\tau_{>h+3}(X) \to \tau_{>h+2}(X)$ and $\tau_{>h+3}(Y) \to \tau_{>h+2}(Y)$ are $E_\bullet$-equivalences, because their fibres are $E_\bullet$-acyclic Eilenberg–MacLane spaces by Theorem 4.3.0.13. Thus it suffices to show that $\tau_{>i}(f)$ is an $E_\bullet$-equivalence for every $i \geq h+3$. By Corollary 4.3.0.12, the map $\tau_{>i}(f)$ is a virtual $E_\bullet$-equivalence. Denote the fibre of $\tau_{>i}(f)$ by F_i, which is virtually $E_\bullet$-acyclic by Corollary 4.3.0.10. We show that F_i is $E_\bullet$-acyclic.

The homotopy type $\mathrm{L}_E(\Omega F_i)$ is an $E_\bullet$-local with non-trivial homotopy groups in finitely many degrees, by Theorem 4.3.0.5.(i). Moreover, the homotopy type $\mathrm{L}_E(\Omega F_i)$ is $\mathrm{K}(\mathbb{Z}/p\mathbb{Z}, j)$-less for each $j \geq h+1$ and for each prime number p, since the latter are $E_\bullet$-acyclic. By Theorem 2.4.2.9 and Corollary 2.4.2.5 the Eilenberg–MacLane space $\mathrm{K}(\pi_i(F), i)$ is $\mathrm{K}(\mathbb{Z}/p\mathbb{Z}, j)$-less for every $i \geq 2$ and every $j \geq h+1$ and for every primer number p. Thus the homotopy groups of $\mathrm{L}_E(\Omega F_i)$ in degree above $h+1$ is trivial, since $\pi_\bullet(E)$ is torsion, see also [Bou97, Theorem 10.7]. This implies that the localisation map $\Omega F_i \to \mathrm{L}_E(\Omega F_i)$ is null, because ΩF_i is $(h+1)$-connected. So ΩF_i is $E_\bullet$-acyclic and thus F_i is $E_\bullet$-acyclic by Corollary 4.1.1.2. Therefore, the map $\tau_{>i}(f)$ is an $E_\bullet$-equivalence for all $i \geq h+2$. See [Bou97, Theorem 11.7]. □

4.3.0.15. Theorem. — *Assume that $\pi_\bullet E$ is torsion. Let $f\colon X \to Y$ be a morphism of pointed homotopy types. Then the following statements are equivalent:*

(i) *The map f is a virtual $E_\bullet$-equivalence.*

(ii) *The map $\tau_{>i}(\Omega^k f)\colon \tau_{>i}(\Omega^k X) \to \tau_{>i}(\Omega^k Y)$ is an $E_\bullet$-equivalence for all $k \geq 1$ and all $i \geq h+2$.*

(iii) *The map $\tau_{>i}(\Omega^k f)$ is an $E_\bullet$-equivalence for some $k \geq 1$ and some $i \geq h+2$.*

Proof. — This theorem is the combination of Corollary 4.3.0.11, Corollary 4.3.0.12 and Proposition 4.3.0.14. □

4.3.0.16. Remark. — In the situation of Theorem 4.3.0.15, a map f satisfying (ii) is also known as f being a *durable $E_\bullet$-equivalence*, introduced in [Bou94, Section 13].

4.3.0.17. Situation. — From now on we fix a prime number p.

4.3.0.18. Definition. — Let X be a pointed nilpotent homotopy type. The *p-torsion component* of X is a p-torsion nilpotent homotopy type $\mathrm{tors}_p X$ together with a map $\eta\colon \mathrm{tors}_p X \to X$ such that η induces an isomorphism of their mod-p singular homology groups. See [Bou94, §14.1].

4.3.0.19. Example. — Let X be a pointed nilpotent homotopy type. A model of $\mathrm{tors}_p X$ is given by the fibre of the localisation $X \to \mathrm{L}_{\mathrm{H}\mathbb{Z}[1/p]}(X)$. Recall that for an n-connected pointed homotopy type X, a model of the localisation $\mathrm{L}_{\mathrm{H}\mathbb{Z}[1/p]}(X)$ is given by $\mathrm{M}(\mathbb{Z}/p\mathbb{Z}, n)$-contraction, see Proposition 2.4.1.2.

4.3.0.20. Proposition. — *For a pointed 2-connected homotopy type X, the natural map $\eta\colon \mathrm{tors}_p X \to X$ is a $\mathrm{T}(h)_\bullet$-equivalence and a v_h-periodic equivalence for all $h \geq 1$.*

Proof. — This proposition is a more general version of [Bou94, Lemma 14.2]. By Proposition 2.4.1.2 there exists a natural equivalence $\mathrm{L}_{\mathrm{H}\mathbb{Z}[1/p]}(X) \simeq \mathrm{P}_{\mathrm{M}(\mathbb{Z}/p\mathbb{Z},2)}\, X$. Note that the homotopy type $\mathrm{M}(\mathbb{Z}/p\mathbb{Z}, 2) \simeq \Sigma\mathrm{M}(\mathbb{Z}/p\mathbb{Z}, 1)$ satisfies Situation 3.3.0.1. Therefore, the v_h-periodic homotopy groups of $\mathrm{L}_{\mathrm{H}\mathbb{Z}[1/p]}(X)$ vanish for all $h \geq 1$ by Proposition 3.3.2.4. This implies that η induces an isomorphism on v_h-periodic homotopy groups for all $h \geq 1$, as we showed in the proof of Proposition 3.2.0.14. Furthermore, the map η induces an isomorphism of $\mathrm{T}(h)$-homology for all $h \geq 1$ because η is a $(\mathrm{H}\mathbb{Z}/p\mathbb{Z})_\bullet$-equivalence. □

4.3.0.21. Theorem. — *Let $f\colon X \to Y$ be a map of pointed homotopy types and let $h \geq 1$ be a natural number. If f is a v_n-periodic equivalence for every $1 \leq n \leq h$, then f is a virtual $\mathrm{T}(h)_\bullet$-equivalence.*

Proof. — By assumption $\tau_{>j}(\Omega f)$ is a v_n-equivalence for every $1 \leq n \leq h$ and all $j \geq 0$. Let ΣW_{h+1} be a pointed p-local homotopy type satisfying the hypotheses of Situation 3.3.0.1. Then for sufficiently large j, the induced map $\mathrm{tors}_p\left(\tau_{>j}(\Omega f)\right)$ on the p-torsion component is a ΣW_{h+1}-equivalence, by Theorem 3.3.2.6 and Proposition 4.3.0.20. Using Proposition 2.1.2.10 and Proposition 4.3.0.20 we can choose large enough j such that $\tau_{>j}(\Omega f)$ is a $\mathrm{T}(h)_\bullet$-equivalence. Therefore, the morphism f is a virtual $\mathrm{T}(h)_\bullet$-equivalence by Theorem 4.3.0.15. See also [Bou97, Theorem 11.13]. □

4.3.0.22. Theorem. — *A morphism $f\colon X \to Y$ of pointed homotopy types is a virtual $\mathrm{K}(1)_\bullet$-equivalence if and only if f is a v_1-periodic equivalence.*

Proof. — One direction follows from Theorem 4.3.0.21. For the other direction, assume that f is a virtual $\mathrm{K}(1)_\bullet$-equivalence. Thus $\tau_{>3}\left(\Omega^2(f)\right)$ is a $\mathrm{K}(1)_\bullet$-equivalence, by Theorem 4.3.0.15. Note that $\tau_{>3}\left(\Omega^2(f)\right) = \Omega^2(\tau_{>5}(f))$. It suffices to show that $\tau_{>5}(f)$ is a v_1-periodic equivalence.

Denote the fibre of $\tau_{>5}(f)$ by F. Applying Corollary 4.1.1.2 to the commutative diagram in $\mathcal{H}\mathrm{o}_*$ below

$$\begin{array}{ccccc} \Omega^2(\tau_{>5}(Y)) & \longrightarrow & \Omega F & \longrightarrow & \Omega(\tau_{>5}(X)) \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow{\scriptstyle \Omega(\tau_{>5}(f))} \\ \Omega^2(\tau_{>5}(Y)) & \longrightarrow & \mathrm{pt} & \longrightarrow & \Omega(\tau_{>5}(Y)) \end{array}$$

where the horizontal rows are fibre sequences with connected bases, we obtain that ΩF is $\mathrm{K}(1)_\bullet$-acyclic. Thus, its p-torsion component $\mathrm{tors}_p(\Omega F)$ is $\mathrm{K}(1)_\bullet$-acyclic. Moreover, the rational homology of the loop space $\Omega(\mathrm{tors}_p(F)) \simeq \mathrm{tors}_p(\Omega F)$ is trivial. Hence, $\Omega(\mathrm{tors}_p(F))$ is $\mathrm{S}(1)_\bullet$-acyclic; recall that $\langle \mathrm{S}(1)\rangle = \langle \mathrm{H}\mathbb{Q}\rangle \vee \langle \mathrm{K}(1)\rangle$. Then, by Theorem 4.2.0.13, the v_1-periodic homotopy groups of $\Omega(\mathrm{tors}_p(F))$ vanish. By Proposition 4.3.0.20 the v_1-periodic homotopy groups of F also vanish. Therefore, the map $\tau_{>5}(f)$ and thus f induces isomorphisms of v_1-periodic homotopy groups, by Proposition 3.2.0.14. See also [Bou97, Theorem 11.11] and [Bou94, §14]. □

***4.3.0.23. Conjecture* (Bousfield).** — *Let $h \geq 1$ be a natural number. A virtual $\mathrm{K}(h)_\bullet$-equivalence of pointed homotopy types is a v_h-periodic equivalence.*

***4.3.0.24. Theorem* ([Bou01, Theorem 6.5].)** — *For every natural number $h \geq 1$, the above conjecture is equivalent to the (stable) telescope conjecture.*

4.3.0.25. Remark. — Burklund–Hahn–Levy–Schlank announced recently at a conference [BHLS23] that the telescope conjecture, which suggests that $\langle \mathrm{T}(h)\rangle = \langle \mathrm{K}(h)\rangle$, is wrong.

4.4. Chromatic localisations of H-spaces

Let p be a fixed prime number. In this section we prove that an $\mathrm{S}(h)_\bullet$-equivalence of connected H-spaces is a v_n-periodic equivalence for all $0 \leq n \leq h$, improving Corollary 4.2.0.11 for H-spaces, see Theorem 4.4.0.4. Based on this and the results in [Bou99a] we conjecture a relationship between the $\mathrm{T}(h)_\bullet$-equivalences of homotopy types and v_h-periodic equivalences of loop spaces, see Conjecture 4.4.0.8.

We begin this section with some nice properties of homological localisations of H-spaces. From Theorem 4.4.0.4 on the content of this section is original.

4.4.0.1. Theorem. — *Let E be a spectrum of elevated acyclicity. If a morphism of connected H-space is an $E_\bullet$-equivalence, then it is a virtual $E_\bullet$-equivalence.*

Proof. — This follows from Theorem 4.3.0.5.(iii). See [Bou97, Theorem 11.3]. □

4.4.0.2. Theorem. — *Let $h \geq 1$ be a natural number. Let E be a p-local spectrum whose mod-p transitional dimension $\mathrm{tran}_p E$ is h (see Definition 4.1.0.9). If X is a $(h+1)$-connected $E_\bullet$-acyclic H-space and $\pi_{h+2}(X)$ is torsion, then ΩX is $E_\bullet$-acyclic.*

Proof. — See [Bou96, Theorem 7.4]. Compare this with Corollary 4.1.1.3. □

4.4.0.3. Corollary. — *Let f be a morphism of connected* H*-space such that f is a $\mathrm{T}(1)_\bullet$-equivalence. Then f is a v_1-periodic equivalence.*

Proof. — This is a consequence of Theorem 4.3.0.22 and Theorem 4.4.0.1. □

4.4.0.4. Theorem. — *Let $f\colon X \to Y$ be a morphism of connected* H*-space. If f is a $\mathrm{S}(h)_\bullet$-equivalence and f induces an isomorphism*

$$f_*\colon \pi_\bullet(X) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] \xrightarrow{\simeq} \pi_\bullet(Y) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p],$$

then f a v_n-equivalence for every $0 \leq n \leq h$.

Proof. — Since f is a $\mathrm{S}(h)_\bullet$-equivalence, it is a $\mathrm{T}(n)_\bullet$-equivalence for every $0 \leq n \leq h$.

By Theorem 4.4.0.1 the map f is a virtual $\mathrm{T}(n)_\bullet$-equivalences for every $0 \leq n \leq h$. By Proposition 4.3.0.14 the induced morphism $\tau_{\geq h+2}\left(\Omega^2(f)\right)$ is a $\mathrm{T}(n)_\bullet$-equivalences for every $1 \leq n \leq h$. Recall that we have $\tau_{\geq h+2}\left(\Omega^2(f)\right) \simeq \Omega^2(\tau_{\geq h+4}(f))$.

By assumption f is a rational homology equivalence. Since f is a morphism of connected H-spaces, it is also a rational homotopy equivalence by [BK, Chapter V, Proposition 3.2]; connected H-spaces are nilpotent. Then the induced map $\Omega^2(\tau_{\geq h+4}(f))$ is a rational homotopy equivalence of simply connected spaces, thus it is a rational homology equivalence.

In all the map $\Omega^2(\tau_{\geq h+4}(f))$ is a $\mathrm{S}(h)_\bullet$-equivalence. Therefore, by Corollary 4.2.0.14 the map f is a v_n-periodic equivalence for every $0 \leq n \leq h$ □

4.4.0.5. Remark. — In the situation of Theorem 4.4.0.4, if we assume in addition that X and Y are p-local, then the induced map $f_*\colon \pi_\bullet(X)\otimes_{\mathbb{Z}}\mathbb{Z}[1/p] \to \pi_\bullet(Y)\otimes\mathbb{Z}[1/p]$ is an isomorphism.

4.4.0.6. Situation. — Let h be a natural number and let W_{h+1} be a pointed homotopy type satisfying Situation 3.3.0.1. Recall from Proposition 3.3.0.3 the notation $c^+(\Sigma W_{h+1}) = \mathrm{conn}(\Sigma W_{h+1}) + 1 \geq h+2$.

4.4.0.7. Theorem. — *In Situation 4.4.0.6 let X be a p-local H-space such that it is $c^+(\Sigma W_{h+1})$-connected and ΣW_{h+1}-less. Then X is $\mathrm{S}(h)_\bullet$-local in the ∞-category $\mathcal{H}\mathrm{o}_*^{>c^+(\Sigma W_{h+1})}$ of $c^+(\Sigma W_{h+1})$-connected homotopy types.*

Proof. — Denote the localisation map $X \to \mathrm{L}_{\mathrm{S}(h)}(X)$ by λ. We prove the theorem in the following steps

Claim. The induced map

$$\tau_{>c^+(\Sigma W_{h+1})}(\lambda)\colon X \simeq \tau_{>c^+(\Sigma W_{h+1})}(X) \to \tau_{>c^+(\Sigma W_{h+1})}\left(\mathrm{L}_{\mathrm{S}(h)}(X)\right)$$

is a ΣW_{h+1}-equivalence.

By Theorem 4.4.0.4 $\lambda\colon X \to \mathrm{L}_{\mathrm{S}(h)}\, X$ is v_n-equivalence for $0 \leq n \leq h$. The claim then follows from Theorem 3.3.2.6.

Claim. The morphism $\tau_{>c^+(\Sigma W_{h+1})}(\lambda)$ exhibits the pointed homotopy type

$$\tau_{>c^+(\Sigma W_{h+1})}\left(\mathrm{L}_{\mathrm{S}(h)}(X)\right)$$

as the ΣW_{h+1}-contraction of X.

Indeed, the homotopy type $\tau_{>c^+(\Sigma W_{h+1})}\left(\mathrm{L}_{\mathrm{S}(h)}(X)\right)$ is ΣW_{h+1}-less by Proposition 3.3.1.4, since $\mathrm{L}_{\mathrm{S}(h)}(X)$ is ΣW_{h+1}-less by Proposition 2.1.2.10. The statement then follows from the previous claim.

In particular, we have

$$X \simeq \tau_{>c^+(\Sigma W_{h+1})}\left(\mathrm{L}_{\mathrm{S}(h)}(X)\right),$$

since X is ΣW_{h+1}-less.

To finish the proof, let Y be a $c^+(\Sigma W_{h+1})$-connected $\mathrm{S}(h)_\bullet$-acyclic homotopy type. The fibre sequence

$$\tau_{>c^+(\Sigma W_{h+1})}\left(\mathrm{L}_{\mathrm{S}(h)}(X)\right) \to \mathrm{L}_{\mathrm{S}(h)}(X) \to \tau_{\leq c^+(\Sigma W_{h+1})}\left(\mathrm{L}_{\mathrm{S}(h)}(X)\right)$$

induces the following fibre sequence

$$\begin{gathered}\mathrm{Map}_*\left(Y, \tau_{>c^+(\Sigma W_{h+1})}\left(\mathrm{L}_{\mathrm{S}(h)}(X)\right)\right)\\ \downarrow\\ \mathrm{Map}_*\left(Y, \mathrm{L}_{\mathrm{S}(h)}(X)\right)\\ \downarrow\\ \mathrm{Map}_*\left(Y, \tau_{\leq c^+(\Sigma W_{h+1})}\left(\mathrm{L}_{\mathrm{S}(h)}(X)\right)\right)\end{gathered}$$

on the pointed mapping spaces. The second and the third pointed mapping spaces are contractible, using that Y is $\mathrm{S}(h)_\bullet$-acyclic and $c^+(\Sigma W_{h+1})$-connected respectively. Thus the mapping space $\mathrm{Map}_*\left(Y, \tau_{>c^+(\Sigma W_{h+1})}\left(\mathrm{L}_{\mathrm{S}(h)}(X)\right)\right)$ is contractible. Therefore, we obtain that the homotopy type $X \simeq \tau_{>c^+(\Sigma W_{h+1})}\left(\mathrm{L}_{\mathrm{S}(h)}(X)\right)$ is $\mathrm{S}(h)_\bullet$-local in the ∞-category $\mathcal{H}\mathrm{o}_*^{>c^+(\Sigma W_{h+1})}$. □

4.4.0.8. Conjecture. — *Let $f\colon X \to Y$ be a morphism of pointed simply connected homotopy types such that the homotopy groups of X and Y are p-primary torsion abelian groups. Fix a natural number $h \geq 1$. The following statements are equivalent*

(i) *The map Ωf is a $\mathrm{T}(h)_\bullet$-equivalence.*

(ii) *The map Ωf is a $\mathrm{S}(h)_\bullet$-equivalence.*

(iii) *The map f is a v_n-periodic equivalence for every $0 \leq n \leq h$ and f induces an isomorphism $\pi_k(f)$ on homotopy groups of degree $k \leq h+1$.*

4.4.0.9. **Explanation.** — (ii) implies (i) by comparison of the Bousfield classes of spectra. We conjecture that (i) implies (ii) because of the fact that $\mathrm{K}(m)_*$-equivalence of homotopy types implies $\mathrm{K}(m-1)$-equivalence of homotopy types for every $m \geq 2$, see [Bou99a]. Assuming (ii), using Theorem 4.4.0.4 we obtain that Ωf and thus f is a v_h-periodic equivalence for $0 \leq n \leq h$.

Part II

Spectral Lie Algebras in Monochromatic Layers

CHAPTER 5

Introduction on ∞-operads

5.1. Operads, categories of operators and symmetric sequences

In this section we recall the classical (1-categorical) theory of operads, in order to motivate the theory of ∞-operads in §5.2. For this purpose we present three equivalent ways to think about operads with values in a symmetric monoidal category: We begin by recalling the classical definition of operads using operations and structure maps, then discuss in §5.1.1 the category of operators associated with an operad and close with a brief exposition about monads and symmetric sequences (§5.1.2). The latter two approaches will be generalised to define ∞-operads in the next section. Examples of operads in this section lead to examples of ∞-operads, which are important for later applications.

This section is expository and the main references are [FreHO; HM22; HA].

5.1.0.1. Notation. — We write a (symmetric) monoidal category as

$$\mathbf{V} = (\underline{\mathbf{V}}, \otimes, \mathbb{1}_{\mathbf{V}}),$$

where $\underline{\mathbf{V}}$ is the underlying category, $\otimes \colon \underline{\mathbf{V}} \times \underline{\mathbf{V}} \to \underline{\mathbf{V}}$ is the (symmetric) monoidal product, and $\mathbb{1}_{\mathbf{V}}$ is the unit of the (symmetric) monoidal structure.

5.1.0.2. Definition. — Let $\mathbf{V}$ be a symmetric monoidal category. An *operad* $\mathbf{O}$ with values in $\mathbf{V}$ consists of

(i) a set $\mathrm{Col}(\mathbf{O})$ of colours, and
(ii) an object $\mathbf{O}((c_i)_{i=1}^r; c) \in \mathbf{V}$, for every pair $((c_i)_{i=1}^r, c)$ of a colour $c \in \mathrm{Col}(\mathbf{O})$ and an r-tuple $(c_i)_{i=1}^r \in \mathrm{Col}(\mathbf{O})^r$ of colours, for every $r \in \mathbb{N}$,

together with the following structure maps:

(iii) A unit map $1_c \colon \mathbb{1}_{\mathbf{V}} \to \mathbf{O}(c, c)$, for every colour $c \in \mathrm{Col}(\mathbf{O})$.
(iv) A morphism $\sigma^* \colon \mathbf{O}(c_1, c_2, \dots, c_r; c) \to \mathbf{O}(c_{\sigma(1)}, c_{\sigma(2)}, \cdots, c_{\sigma(r)}, c)$ in $\mathbf{V}$, for every element σ of the symmetric group $\mathfrak{S}_r$, and every pair $((c_i)_{i=1}^r, c)$ of an r-tuple $(c_i)_{i=1}^r$ of colours and a colour c, for every $r \in \mathbb{N}$.

(v) A morphism

$$\mathbf{O}((c_i)_{i=1}^r; c) \otimes \left(\bigotimes_{i=1}^r \mathbf{O}\left((c_{i,j})_{j=1}^{m_i}; c_i\right) \right) \to \mathbf{O}\left((c_{1,j_1})_{j_1=1}^{m_1}, \dots, (c_{r,j_r})_{j_r=1}^{m_r}; c\right)$$

in $\mathbf{V}$, called a *composition map*, for every r-tuple $(c_i)_{i=1}^r$ of colours, every r-tuple $((c_{i,j})_{j=1}^{m_i})_{i=1}^r$ of finite sequences $(c_{i,j})_{j=1}^{m_i}$ of colours and every colour c, for every $r \in \mathbb{N}$,

satisfying a number of axioms (which we omit writing them down here) requiring that

(vi) the composition maps are associative and unital,

(vii) for every $r \in \mathbb{N}$ and every colour c, the set $\{\sigma^* \mid \sigma \in \mathfrak{S}_r\}$ of morphisms induces a right $\mathfrak{S}_r$-action on the set $\{\mathbf{O}((c_i)_{i=1}^r; c) \mid (c_i)_{i=1}^r \in \mathrm{Col}(\mathbf{O})^r\}$ of objects, and

(viii) the right symmetric group actions are compatible with the composition maps.

We refer the reader to [HM22, Definition 1.1] for a detailed and rigorous presentation of these axioms.

5.1.0.3. Definition. — In the situation of Definition 5.1.0.2, a *one-coloured operad* is an operad $\mathbf{O}$ with values in $\mathbf{V}$ whose set of colours contains only one element.

5.1.0.4. Remark. — In the situation of Definition 5.1.0.2, one should think of the object $\mathbf{O}(c_1, c_2, \dots, c_r; c)$ as describing an operation having r-number of inputs of "types" $c_1, c_2, \cdots c_r$ respectively and has one output of "type" c. We call the object $\mathbf{O}(c_1, c_2, \dots, c_r; c)$ an *operation of* $\mathbf{O}$ *of arity* r. See [FreHO, §1.1.5] for a graphical representation and a detailed explanation of the definition of an operad from this point of view.

Assume that $\mathbf{O}$ is an one-coloured operad with $\mathrm{Col}(\mathbf{O}) = \{c\}$. For each $r \in \mathbb{N}$ and for every r-tuple $(c)_{i=1}^r$ of colours, we denote

$$\mathbf{O}(r) \coloneqq \mathbf{O}((c_i)_{i=1}^r; c).$$

We call $\mathbf{O}(r)$ the *operations of* $\mathbf{O}$ *of arity* r. In particular, the object $\mathbf{O}(r)$ admits a right symmetric group $\mathfrak{S}_r$ action by definition, for every $r \in \mathbb{N}$.

5.1.0.5. Definition. — Let $\mathbf{V}$ be a symmetric monoidal category. For two operads $\mathbf{O}$ and $\mathbf{P}$ with values in $\mathbf{V}$, an *operad map* $\mathbf{O} \to \mathbf{P}$ consists of

(i) a morphism $f\colon \mathrm{Col}(\mathbf{O}) \to \mathrm{Col}(\mathbf{P})$ of sets, and

(ii) a morphism

$$f((c_i)_{i=1}^r; c))\colon \mathbf{O}(c_1, c_2, \dots, c_r; c) \to \mathbf{P}(f(c_1), f(c_2), \dots, f(c_r); f(c))$$

in $\mathbf{V}$, for every operation of $\mathbf{O}$ of arity r, for every $r \in \mathbb{N}$,

such that they are compatible with the structure maps of $\mathbf{O}$ and $\mathbf{P}$. See [YauCO, Definition 11.2.12] for the commutative diagrams describing those compatibility.

5.1.0.6. Definition. — In the situation of Definition 5.1.0.5 the operad map $\mathbf{O} \to \mathbf{P}$ is an *inclusion* of $\mathbf{O}$ in $\mathbf{P}$ if

(i) f is an injective morphism of sets, and
(ii) the morphism $f((c_i)_{i=1}^r; c))$ is an isomorphism in $\mathbf{V}$, for every operation of $\mathbf{O}$ of arity r, for every $r \in \mathbb{N}$.

In this case, we say $\mathbf{O}$ is a *suboperad* of $\mathbf{P}$.

5.1.0.7. Example. — In the situation of Definition 5.1.0.2, let $\mathbf{C}$ be a symmetric monoidal category enriched over $\mathbf{V}$. For objects Y and Z of $\mathbf{C}$, let $\mathrm{Map}_{\mathbf{C}}(Y, Z)$ denote the object of $\mathbf{V}$ underlying the set of of morphisms from Y to Z in $\mathbf{C}$. Let S be a set of objects of $\mathbf{C}$. We can define the *mapping operad* $\mathbf{Map}(\mathbf{C}, S)$ with values in $\mathbf{V}$ as follows:

(i) The set $\mathrm{Col}\left(\mathbf{Map}(\mathbf{C}, S)\right)$ of colours is the set S.
(ii) For every object $X \in S$, define the operation of arity 0 as

$$\mathbf{Map}(\mathcal{C}, S)(0, X) := \mathrm{Map}_{\mathbf{C}}(\mathbb{1}_{\mathbf{C}}, X).$$

(iii) For every natural number $r \geq 1$, an operation of arity r is defined as

$$\mathbf{Map}(\mathbf{C}, S)(X_1, X_2, \dots, X_r; X) := \mathrm{Map}_{\mathbf{C}}(X_1 \otimes X_2 \otimes \cdots \otimes X_r, X),$$

for $X \in S$ and $X_i \in S$ for every $1 \leq i \leq r$.

The unit maps are given by taking the identity morphism, symmetric groups act by permuting the indices of the colours and composition maps are induced by compositions of morphisms and tensor products of morphisms. If $S = \{X\}$, we denote the operad $\mathbf{Map}(\mathbf{C}, \{X\})$ by $\mathbf{End}(X)$, called the *endomorphism operad* of X.

5.1.0.8. Definition. — Let $\mathbf{C} = (\underline{\mathbf{C}}, \otimes_{\mathbf{C}}, \mathbb{1}_{\mathbf{C}})$ be a symmetric monoidal category enriched over a closed symmetric monoidal category $\mathbf{V} = (\underline{\mathbf{V}}, \otimes_{\mathbf{V}}, \mathbb{1}_{\mathbf{V}})$. We say $\mathbf{C}$ is *copowered* over $\mathbf{V}$ if the following two conditions hold:

(i) The category $\mathbf{C}$ is *tensored* over $\mathbf{V}$: For every object V of $\mathbf{V}$ there exists a functor $V \otimes - \colon \mathbf{C} \to \mathbf{C}$ such that, for every pair (V, V') of objects of $\mathbf{V}$ and every pair (C, C') of objects of $\mathbf{C}$, there exist isomorphisms $V' \otimes (V \otimes C) \cong (V' \otimes_{\mathbf{V}} V) \otimes C$ and $(V \otimes C) \otimes_{\mathbf{C}} C' \cong V \otimes (C \otimes_{\mathbf{C}} C')$.
(ii) The functor $V \otimes -$ defined in (i) satisfies the following property: For every object $V \in \mathbf{V}$ and every pair (C, C') of objects of $\mathbf{C}$, there exists a natural isomorphism $\mathrm{Map}_{\mathbf{C}}\left(V \otimes C, C'\right) \cong \mathrm{Map}_{\mathbf{V}}\left(V, \mathrm{Map}_{\mathbf{C}}(C, C')\right)$ in $\mathbf{V}$.

5.1.0.9. Definition. — Let $\mathbf{C}$ be a symmetric monoidal category enriched over a closed symmetric monoidal category $\mathbf{V}$. Let $\mathbf{O}$ be an operad with values in $\mathbf{V}$. An $\mathbf{O}$-*algebra* in $\mathbf{C}$ is a set $S_{\mathbf{O}} = \{X_i\}_{i \in \mathrm{Col}(\mathbf{O})}$ of objects of $\mathbf{C}$ together with a map $\mathbf{O} \to \mathbf{Map}(\mathbf{C}, S_{\mathbf{O}})$ of operads with values in $\mathbf{V}$.

5.1.0.10. Example. — In the situation of Definition 5.1.0.9, let $\mathbf{O}$ be a one-coloured operad with values in $\mathbf{V}$. Then an $\mathbf{O}$-algebra in $\mathbf{C}$ is an object $X \in \mathbf{C}$ together with a structure map (as a morphism in $\mathbf{V}$)

$$\mathbf{O}(r) \to \mathrm{Map}_{\mathbf{C}}\left(X^{\otimes r}, X\right)$$

for every $r \geq 0$, compatible with the structure maps of $\mathbf{O}$ and $\mathbf{End}(X)$.

Assume that $\mathbf{C}$ is copowered over $\mathbf{V}$. Then an $\mathbf{O}$-algebra in $\mathbf{C}$ is equivalently an object $X \in \mathbf{C}$ together with a structure map (as s morphism in $\mathbf{C}$)

$$a_r\colon\ \mathbf{O}(r) \otimes X^{\otimes r} \to X. \tag{5.1.0.1}$$

for every $r \geq 0$ with suitable compatibilities, by Definition 5.1.0.8.(ii). The morphisms (5.1.0.1) resemble the structure maps of a left module over an algebra. Thus one can think of an $\mathbf{O}$-algebra as a left module over the operad $\mathbf{O}$. Later we will make this point of view precise by considering an operad as an associative algebra in a suitable category and an algebra over the operad as a left module over this associative algebra, see Theorem 5.1.2.10.(iv).

5.1.0.11. Notation. — Let $(\mathbf{Set}, \times, \{\mathrm{pt}\})$ denote the symmetric monoidal category of sets with the cartesian monoidal structure.

5.1.0.12. Example. — The *trivial operad* $\mathbf{Triv}$ is a one-coloured operad with values in $(\mathbf{Set}, \times, \{\mathrm{pt}\})$ where $\mathbf{Triv}(1) = \{\mathrm{pt}\}$ and $\mathbf{Triv}(r) = \emptyset$ for all $r \neq 1$. The structure maps of $\mathbf{Triv}$ are obvious.

Every object X in a symmetric monoidal category $\mathbf{C}$ admits a unique trivial algebra structure: The structure maps are given by the *trivial multiplications* $\emptyset \to \mathrm{Hom}_{\mathbf{C}}(X^{\otimes r}, X)$ for every $r \geq 2$ and $r = 0$.

5.1.0.13. Example. — The *unital operad* $\mathbf{E}_0$ is a one-coloured operad with values in $(\mathbf{Set}, \times, \{\mathrm{pt}\})$ where $\mathbf{E}_0(0) = \mathbf{E}_1(1) = \{\mathrm{pt}\}$ and $\mathbf{E}_0(r) = \emptyset$ for all $r \geq 2$. The structure maps of $\mathbf{E}_0$ are again rather obvious.

An $\mathbf{E}_0$-algebra in a symmetric monoidal category $\mathbf{C}$ is an object $X \in \mathbf{C}$ together with a morphism $\mathbb{1}_{\mathbf{C}} \to X$.

5.1.0.14. Example. — The *associative operad* $\mathbf{Ass}$ is a one-coloured operad with values in $(\mathbf{Set}, \times, \{\mathrm{pt}\})$ where $\mathbf{Ass}(0) = \{\mathrm{pt}\}$ and $\mathbf{Ass}(r) = \mathfrak{S}_r$ for every $r \geq 1$. The *right* symmetric group action is given by group multiplication. The composition maps

$$\mathfrak{S}_r \times \mathfrak{S}_{b_1} \times \cdots \times \mathfrak{S}_{b_r} \to \mathfrak{S}_{b_1+\cdots+b_r}$$

are given by the so-called *block permutations* as we explain now. Given a tuple $(\sigma_0, \sigma_1, \cdots, \sigma_r)$ with $\sigma_0 \in \mathfrak{S}_r$ and $\sigma_i \in \mathfrak{S}_{b_i}$ for $i \in \{1, 2, \ldots, r\}$, we will define a permutation $\sigma \in \mathfrak{S}_{b_1+\cdots+b_r}$ of a set S of $b_1 + \cdots + b_r$ elements. We can consider a decomposition $S = \sqcup_{i=1}^r S_i$ of S where $S_i = \{a_{i,1}, a_{i,2}, \ldots, a_{i,b_i}\}$ is a set of b_i elements.

Define the bijection

$$\sigma\colon S \to S$$
$$a_{i,j} \mapsto a_{\sigma_0(i),\sigma_i(j)}.$$

Informally speaking, one decompose the set S into r-number of "blocks" of cardinality b_i, for $1 \leq i \leq r$, respectively. The element $\sigma_0 \in \mathfrak{S}_r$ permutes the blocks and the element $\sigma_i \in \mathfrak{S}_{b_i}$ permutes the elements in the block with with cardinality b_i.

An **Ass**-algebra in a symmetric monoidal category **C** is a $\otimes$-*monoid* in $\mathcal{C}$, i.e. it is an object $X \in \mathbf{C}$ together with a unit morphism $\mathbb{1}_{\mathbf{C}} \to X$ and a multiplication morphism $X \otimes X \to X$ such that the multiplication is unital and associative.(1) See [HM22, Example 1.14 ii)] for more details. We also say that X is an *associative algebra (object)* in **C**.

5.1.0.15. Example. — The *left module operad* **LM** is an operad with values in $(\mathbf{Set}, \times, \{\mathrm{pt}\})$, defined as follows:

(i) $\mathrm{Col}(\mathbf{LM}) \coloneqq \{a, m\}$.

(ii) For every ordered pair $1 \leq j \leq r$ of natural numbers, let $\mathfrak{S}_r\binom{j}{r}$ denote the subgroup of $\mathfrak{S}_r$ whose elements send j to r. Then define

$$\mathbf{LM}((c_i)_{i=1}^r; c) \coloneqq \begin{cases} \mathfrak{S}_r, & \text{if } c = c_i = a \text{ for all } 1 \leq i \leq r, \\ \mathfrak{S}_r\binom{j}{r}, & \text{if } c = c_j = m \text{ for exactly one } j \text{ and} \\ & c_i = a \text{ for } i \neq j, \\ \emptyset, & \text{otherwise.} \end{cases}$$

The structure maps of the associative operads give the structure maps of **LM**; take the restriction of the structure maps of **Ass** to the subgroups $\mathfrak{S}_r\binom{j}{r}$ for every $r \geq 1$ and every $r \geq j \geq 1$ if necessary. There exists a canonical inclusion $\mathbf{Ass} \hookrightarrow \mathbf{LM}$ of operads sending the single colour of **Ass** to a.

An algebra over the operad **LM** in a symmetric monoidal category **C** is a pair (A, M) of objects of **C** such that A is an associative algebra object in **C** and M is a left module over A.

5.1.0.16. Example. — Let **C** be a symmetric monoidal category. The *commutative operad* $\mathbf{Com}_{\mathbf{C}}$ with $\mathbf{Com}_{\mathbf{C}}(r) = \mathbb{1}_{\mathbf{C}}$ for every $r \in \mathbb{N}$ is a one-coloured operad with values in **C**. The structure maps are obvious.

A $\mathbf{Com}_{\mathbf{C}}$-algebra in **C** is a *commutative* $\otimes$-*monoid*, i.e. an object $X \in \mathbf{C}$ together with a multiplication $X \otimes X \to X$ and a unit morphism $\mathbb{1}_{\mathbf{C}} \to X$ such that the multiplication is unital, associative and commutative.

(1)A *monoid* in a category **C** with finite products and terminal objet pt is an object X together with a multiplication $X \times X \to X$ and a unit morphism $\mathrm{pt} \to X$ such that the multiplication is unital and associative, see [MacCWM, §III.6].

5.1.0.17. Definition. — A *Lie algebra* over $\mathbb{Z}$ is a $\mathbb{Z}$-module $\mathfrak{g}$ together with a $\mathbb{Z}$-bilinear operation $[-,-]\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$, called the *Lie bracket*, such that

(i) for every pair (x,y) of elements of $\mathfrak{g}$ the *antisymmetry* relation

$$[x,y] = -[x,y]$$

holds, and

(ii) for every tuple (x,y,z) of elements of $\mathfrak{g}$ the *Jacobi identity*

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$$

condition holds.

5.1.0.18. Definition. — Let S be a finite set. Denote the cardinality of S by $\sharp S$.

(i) Let $\mathrm{Lie}\langle S\rangle$ denote the free Lie algebra generated by the elements of S. A *Lie monomial* of S is an elements of $\mathrm{Lie}\langle S\rangle$ given by iterated bracketing of elements of S. For the set $\underline{r} = \{1,2,\ldots,r\}$, denote the free Lie algebra $\mathrm{Lie}\langle\underline{r}\rangle$ by $\mathrm{Lie}\langle r\rangle$.

(ii) Let $\mathrm{Lie}(S)$ denote the submodule of $\mathrm{Lie}\langle S\rangle$ generated by Lie monomials which contain each element of S exactly once. The symmetric group $\mathfrak{S}_{\sharp S}$ acts on $\mathrm{Lie}(S)$ on the right by permutation. For $S=\underline{r}$, denote $\mathrm{Lie}(S)$ by $\mathrm{Lie}(r)$, which is also known as the *degree r Lie representation of the permutation group* $\mathfrak{S}_r$, see [ReuFLA, Section 8.2].

5.1.0.19. Example. — The Lie operad **Lie** is a one-coloured operad with values in the symmetric monoidal category $(\mathbf{Mod}_{\mathbb{Z}},\otimes_{\mathbb{Z}},\mathbb{Z})$ of $\mathbb{Z}$-modules. The operations $\mathbf{Lie}(r)$ of arity r are given by $\mathrm{Lie}(r)$ for every $r\geq 1$ and $\mathbf{Lie}(0)=0$, the trivial abelian group. The unit map $\mathbb{Z}\to\mathbf{Lie}(1)\cong\mathbb{Z}$ is the identity morphism and the action by the permutation groups is described in Definition 5.1.0.18.(ii). The composition map is given by “inserting Lie monomials”, see [AB21, Definition 4.10] for more details. An algebra over the Lie operad **Lie** is a Lie algebra (see Definition 5.1.0.17).

5.1.0.20. Definition. — A *topological operad* is an operad with values in the symmetric monoidal category $(\mathbf{CGH},\times,\mathrm{pt})$ of (compactly generated weakly Hausdorff) topological spaces.

5.1.0.21. Example. — Fix a natural number n. For every $r\in\mathbb{N}$, recall the set $\underline{r}=\{1,2,\ldots,r\}$ and $\underline{0}\coloneqq\emptyset$. Denote the open unit disc in $\mathbb{R}^n$ with the Euclidean norm by $\mathring{\mathrm{D}}^n$. Consider the embedding space $\mathrm{Emb}(\mathring{\mathrm{D}}^n\times\underline{r},\mathring{\mathrm{D}}^n)$ (with the compact-open topology) of labelled open discs, for every $r\in\mathbb{N}$. Define the subspace $\mathbf{D}_n(r)$ of the embedding space $\mathrm{Emb}(\mathring{\mathrm{D}}^n\times\underline{r},\mathring{\mathrm{D}}^n)$ as follows: An embedding f is a point in $\mathbf{D}_n(r)$ if the restriction f_j of f to the j-th disc $\mathring{\mathrm{D}}^n\times\{j\}$ is an affine map of the form

$$\mathring{\mathrm{D}}^n\to\mathring{\mathrm{D}}^n,\ (x_i)_{1\leq i\leq n}\mapsto(ax_i+b_i)_{1\leq j\leq n}$$

for some $a\in\mathbb{R}$ and $b_i\in\mathbb{R}$.

The sequence $(\mathbf{D}_n(r))_{r\in\mathbb{N}}$ admits the structure of a one-coloured topological operad, called the *little n-disc operad* $\mathbf{D}_n$. The structure maps are given by composition of embeddings:

$$\mathbf{D}_n(r)\times\mathbf{D}_n(b_1)\times\cdots\times\mathbf{D}_n(b_r)\to\mathbf{D}_n(b_1+\cdots b_r)$$
$$(f_j)_{j=1}^r\times(g_{j_1})_{j_1=1}^{b_1}\times\cdots\times(g_{j_r})_{j_r=1}^{b_r}\mapsto(f_m\circ((g_{j_m})_{j_m=1}^{b_m}))_{m=1}^r.$$

We refer the reader to [FreHO, §4.1] for pictorial presentations of the structure maps.

An $\mathbf{E}_n$*-operad* is a topological operad $\mathbf{E}_n$ admitting a map $f\colon\ \mathbf{E}_n\to\mathbf{D}_n$ of operads which is arity-wise a weak homotopy equivalence. In other words, the map f induces a weak homotopy equivalence $\mathbf{E}_n(r)\xrightarrow{\sim}\mathbf{D}_n(r)$, for all $r\in\mathbb{N}$. A topological operad $\mathbf{O}$ is a *model* for the $\mathbf{E}_n$-operad if there exists a map $\mathbf{O}\to\mathbf{E}_n$ of operads which is arity-wise a weak homotopy equivalence. Some other well-known models for the $\mathbf{E}_n$-operad are the little n-cubes operad [HM22, Example 1.8] and the Fulton–MacPherson operad [HM22, §§1.8–1.9]. We refer the reader to [BM23b] for an exposition about the E_n-operad and to [CS22] for an application of various models for the $\mathbf{E}_n$-operad.

The set $\{\mathrm{D}^n\hookrightarrow\mathrm{D}^{n+1}, x\mapsto(x,0)\mid n\in\mathbb{N}\}$ induces a sequence

$$\mathbf{D}_0\hookrightarrow\mathbf{D}_1\hookrightarrow\cdots\hookrightarrow\mathbf{D}_n\hookrightarrow\mathbf{D}_{n+1}\hookrightarrow\cdots \tag{5.1.0.2}$$

of inclusions of topological operads. We form the following colimit

$$\mathbf{D}_\infty(r):=\varinjlim_{n\geq 0}\mathbf{D}_n(r)$$

in **CGH**. The sequence $(\mathbf{D}_\infty(r))_{r\in\mathbb{N}}$ admits the structure of a one-coloured topological operad, denoted by $\mathbf{D}_\infty$.

An $\mathbf{E}_\infty$*-operad* is a topological operad $\mathbf{E}_\infty$ such that

(i) the symmetric group action on the operations of each arity is free, and

(ii) $\mathbf{E}_\infty$ admits a map $f\colon\ \mathbf{E}_\infty\to\mathbf{D}_\infty$ of operads which is arity-wise a weak homotopy equivalence.

Similarly, we say $\mathbf{D}_\infty$ is a model for the $\mathbf{E}_\infty$-operad. In particular, we obtain a sequence

$$\mathbf{E}_0\hookrightarrow\mathbf{E}_1\hookrightarrow\cdots\hookrightarrow\mathbf{E}_n\hookrightarrow\mathbf{E}_{n+1}\hookrightarrow\cdots\hookrightarrow\mathbf{E}_\infty$$

of inclusions of topological operads.

5.1.0.22. Remark. —

(i) By definition the operad $\mathbf{D}_0$ is isomorphic, as an operad with values in $(\mathbf{Set},\times,\mathrm{pt})$, to the operad $\mathbf{E}_0$ (see Example 5.1.0.13).

(ii) Taking arity-wise the set π_0 of connected components of $\mathbf{E}_n$, we obtain an operad $\pi_0(\mathbf{E}_n)$ with values in $(\mathbf{Set},\times,\mathrm{pt})$. In particular, we have isomorphisms

$$\pi_0(\mathbf{E}_1)\cong\mathbf{Ass}\ \text{ and }\ \pi_0(\mathbf{E}_n)\cong\mathbf{Com}$$

of operads for every $n\geq 2$.

(iii) There exists an equivalence $\mathbf{E}_1 \to \pi_0(\mathbf{E}_1) \cong \mathbf{Ass}$ of topological operads, i.e. an operad map which is an arity-wise weak homotopy equivalence.

(iv) For every pair (r, n) of natural numbers, the topological space $\mathbf{E}_n(r)$ is weakly homotopy equivalent to the ordered configuration space $\mathrm{Conf}_r(\mathbb{R}^n)$ of r-points in $\mathbb{R}^n$.

(v) There exists an equivalence $\mathbf{E}_\infty \to \pi_0(\mathbf{E}_\infty) \cong \mathbf{Com}$ of topological operads, becuase, for every $r \geq 0$, the connectivity of the topological space $\mathbf{E}_n(r)$ increases with n.

See [FreHO, §§4.1–4.2] for more details.

5.1.0.23. Example. — Let $\mathbf{C}$ be a topologically enriched symmetric monoidal category. By Remark 5.1.0.22 one shall think about an $\mathbf{E}_1$-algebra in $\mathbf{C}$ as an object X of $\mathcal{C}$ together with a multiplication morphism $X \otimes X \to X$ which is unital and associative up to homotopy; in particular X becomes an $\mathbf{Ass}$-algebra in the homotopy category of $\mathbf{C}$. Similarly, an $\mathbf{E}_\infty$-algebra in $\mathbf{C}$ is an object X equipped with a multiplication $X \otimes X \to X$ which is unital, associative and commutative up to homotopy. For every $1 \leq m < \infty$, the Dunn Additivity Theorem [Dun88] shows that an $\mathbf{E}_n$-algebra structure on an object $X \in \mathbf{C}$ is the same as n-many $\mathbf{E}_1$-algebra structures on X which satisfy certain compatibility conditions.

The theory of $\mathbf{E}_n$-algebras is motivated by the following example: The n-fold loop space $\Omega^n X$ of a pointed topological space X is an $\mathbf{E}_n$-algebra in $\mathbf{CGH}$. Moreover, any group-like $\mathbf{E}_n$-algebra in $\mathbf{CGH}$ is weakly homotopy equivalent to an n-fold loop space,(2) known as the Recognition Principle. For more details about those iterated loop spaces, see [MayGIL]. We also refer the interested reader to [Law20a] for applications of $\mathbf{E}_n$-algebras in spectra.

5.1.0.24. Example. — Let $n \in \mathbb{N}$. Define an operad $\mathrm{H}_\bullet(\mathbf{E}_n; \mathbb{Z})$ by setting

$$\mathrm{H}_\bullet(\mathbf{E}_n; \mathbb{Z})(r) \coloneqq \mathrm{H}_\bullet(\mathbf{E}_n(r); \mathbb{Z}), \text{ for } r \in \mathbb{N}.$$

We consider $\mathrm{H}_\bullet(\mathbf{E}_n; \mathbb{Z})$ as an operad with values in the symmetric monoidal category $(\mathbf{Ch}_\mathbb{Z}, \otimes_\mathbb{Z}, \mathbb{Z})$ of chain complexes over $\mathbb{Z}$; in each arity $r \in \mathbb{N}$ the operation $\mathrm{H}_\bullet(\mathbf{E}_n; \mathbb{Z})(r)$ is a chain complex with zero differentials.

The structure maps $\mathrm{H}_\bullet(\mathbf{E}_n; \mathbb{Z})$ are the maps on singular homology groups induced by the structure maps of the $\mathbf{E}_n$-operad; the homology functor $\mathrm{H}_\bullet(-; \mathbb{Z})$ is lax monoidal, see [FreHO, Proposition 4.2.11] for more details.

We abbreviate the commutative operad with values in $(\mathbf{Ch}_\mathbb{Z}, \otimes_\mathbb{Z}, \mathbb{Z})$ by $\mathbf{Com}_\mathbb{Z}$. The operad $\mathrm{H}_\bullet(\mathbf{E}_n; \mathbb{Z})$ relates to $\mathbf{Com}_\mathbb{Z}$ via the following theorem. Recall the inclusion $i_n^{n+1} \colon \mathbf{E}_n \hookrightarrow \mathbf{E}_{n+1}$ of operads, see (5.2.5.1).

(2) A topological space is *group-like* if its set of connected components with the operation of taking disjoint union is an abelian group.

5.1.0.25. Theorem. — *Let $n \geq 1$ be a natural number.*

(i) *There exists a map $\pi_c\colon \mathrm{H}_\bullet(\mathbf{E}_n;\mathbb{Z}) \to \mathbf{Com}_\mathbb{Z}$ of operads.*

(ii) *For every $m \geq 2$, there exists a map*

$$\iota_c\colon \mathbf{Com}_\mathbb{Z} \to \mathrm{H}_\bullet(\mathbf{E}_m;\mathbb{Z})$$

of operads. Moreover, we have $\pi_c \circ \iota_c = \mathrm{id}$.

(iii) *There exists the following commutative diagram*

$$\begin{array}{ccc} \mathrm{H}_\bullet(\mathbf{E}_n;\mathbb{Z}) & \xrightarrow{\pi_c} & \mathbf{Com}_\mathbb{Z} \\ {\scriptstyle (i_n^{n+1})_*}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathrm{H}_\bullet(\mathbf{E}_{n+1};\mathbb{Z}) & \xleftarrow[\iota_c]{} & \mathbf{Com}_\mathbb{Z}\,. \end{array}$$

Proof. — Recall the little n-disks operad $\mathbf{D}_n$ as a model for the $\mathbf{E}_n$-operad.

(i) The map π_c is induced by the unique map $\pi\colon \mathbf{D}_n(r) \to \mathrm{pt}$ of topological spaces for every $r \in \mathbb{N}$

(ii) The map ι_c is induced by an inclusion $\iota\colon \mathrm{pt} \hookrightarrow \mathbf{D}_m(r)$ for every $r \in \mathbb{N}$. This is a well-defined map of operads because $\mathbf{D}_m(r)$ is connected for every $m \geq 2$. In particular, $\pi_c \circ \iota_c$ is induced by the composition $\pi \circ \iota$.

(iii) The commutativity of the diagram follows from the observation that in each arity $r \in \mathbb{N}$ the map $i_n^{n+1}(r)\colon \mathbf{E}_n(r) \to \mathbf{E}_{n+1}(r)$ is homotopic to the following null-homotopic map

$$\begin{aligned} (\mathrm{Conf}_r(\mathbb{R}^n), (e_1, \ldots, e_n)) &\hookrightarrow \big(\mathrm{Conf}_r(\mathbb{R}^{n+1}), ((e_1, 0), \ldots, (e_n, 0))\big) \\ (x_1, \ldots, x_n) &\mapsto ((x_1, 0), \ldots, (x_n, 0)) \end{aligned} \tag{5.1.0.3}$$

of pointed topological spaces. We construct an explicit null-homotopy of the map (5.1.0.3). Fix an $r \in \mathbb{N}$. Define the null-homotopy

$$h_a\colon \mathbb{R}^n \times [0,1] \to \mathbb{R}^{n+1},\ (x,t) \mapsto (t^a x, 0),$$

for every natural number $1 \leq a \leq r$, of the inclusion $\mathbb{R}^n \to \mathbb{R}^{n+1},\ x \mapsto (x, 0)$. The following map

$$\begin{aligned} (\mathrm{Conf}_r(\mathbb{R}^n), (e_1, \ldots, e_n)) \times [0,1] &\to \big(\mathrm{Conf}_r(\mathbb{R}^{n+1}), ((e_1, 0), \ldots, (e_n, 0))\big) \\ ((x_i)_{1\leq a\leq r}, t) &\mapsto (h_a(x_a, t))_{1\leq a\leq r} \end{aligned}$$

is a null-homotopy of the map (5.1.0.3). See also [Fre11, Proposition 0.3.5.a]. □

5.1.0.26. Remark. — This theorem is actually part of a structural theorem about the operad $\mathrm{H}_\bullet(\mathbf{E}_m;\mathbb{Z})$ with $m \geq 2$: There exists an isomorphism

$$\mathbf{Gerst}_m \to \mathrm{H}_*(\mathbf{E}_m;\mathbb{Z})$$

of operads where $\mathbf{Gerst}_m$ denotes the *m-Gerstenhaber operad*. We refer the reader to [FreHO, §4.2] for a detailed exposition on this topic.

5.1.1. The category of operators associated to an operad. — To generalise the notion of operads to an ∞-category $\mathcal{C}$, we might start with a coloured sequence of objects in $\mathcal{C}$ together with structure maps, similar as in Definition 5.1.0.2. However, it is then not straightforward how we could document the unitality, associativity and equivariance of the structure maps up to coherent homotopy in an efficient way.

In this subsection we give a brief introduction to the theory of categories of operators and explain why it is equivalent to the theory of operads. Generalising this approach, one can define ∞-operads with values in the ∞-category $\mathcal{H}o$ of homotopy types, see §5.2.1.

5.1.1.1. Definition. — Let $\mathbf{O}$ be an operad with values in $(\mathbf{Set}, \times, \{\mathrm{pt}\})$. The *category* $\mathbf{O}^{\otimes}$ *of operators* associated to $\mathbf{O}$ consists of the following data:

(i) An object of $\mathbf{O}^{\otimes}$ is a finite sequence of colours of $\mathbf{O}$.

(ii) Let $C = (c_i)_{i=1}^m$ and $D = (d_j)_{j=1}^{\ell}$ be two objects of $\mathbf{O}^{\otimes}$. A morphism from C to D consists of a pair of

  (a) a morphism $\alpha\colon \{\mathrm{pt}, 1, 2, \ldots, m\} \to \{\mathrm{pt}, 1, 2, \ldots, \ell\}$ of pointed sets, and

  (b) a ℓ-tuple $(\phi_1, \phi_2, \ldots, \phi_\ell) \in \mathbf{O}\left((c_i)_{i\in\alpha^{-1}(1)}; d_1\right) \times \cdots \times \mathbf{O}\left((c_i)_{i\in\alpha^{-1}(\ell)}; d_\ell\right)$ of operations where $(c_i)_{i\in\alpha^{-1}(j)}$ denotes the subsequence of C where the indices of the elements are the preimages of j under α, for $j = 1, 2, \ldots, \ell$.

(iii) The composition of two morphisms in $\mathbf{O}^{\otimes}$ is given pairwise by the composition of the morphisms of pointed sets and the composition maps of operations of $\mathbf{O}$.

5.1.1.2. Definition. — Define a category $\mathbf{Fin}_*$ as follows: An object of $\mathbf{Fin}_*$ is a finite pointed set

$$\langle n\rangle := \{\mathrm{pt}, 1, 2, \ldots, n\}$$

for a natural number n, where pt denotes the basepoint. We set $\langle 0\rangle := \{\mathrm{pt}\}$. For a pair (n, m) of natural numbers, a morphism $\langle n\rangle \to \langle m\rangle$ is a morphism of pointed sets.

5.1.1.3. Remark. — The inclusion functor of $\mathbf{Fin}_*$ to the category of finite pointed sets is an equivalence of categories. Because of this we call $\mathbf{Fin}_*$ the *category of finite pointed sets.*

5.1.1.4. Definition. — A morphism $i\colon \langle m\rangle \to \langle n\rangle$ in $\mathbf{Fin}_*$ is *inert* if for every element $\mathrm{pt} \neq k \in \langle n\rangle$ the preimage $i^{-1}(k)$ of k contains exactly one element.

5.1.1.5. Definition. — For $1 \leq i \leq n$, let $\rho_i\colon \langle n\rangle \to \langle 1\rangle$ be the inert morphism in $\mathbf{Fin}_*$ sending i to 1 and everything else to the basepoint.

5.1.1.6. Definition. — Let $p\colon \mathbf{C} \to \mathbf{Fin}_*$ be a functor and let $n \in \mathbb{N}$. Define the subcategory $\mathbf{C}_{\langle n\rangle}$ of $\mathbf{C}$ as follows: An object $X \in \mathbf{C}$ is in $\mathbf{C}_{\langle n\rangle}$ if $p(X) = \langle n\rangle$. Given two objects X and Y in $\mathbf{C}_{\langle n\rangle}$, a morphism $f\colon X \to Y$ in $\mathbf{C}$ is in $\mathbf{C}_{\langle n\rangle}$ if $p(f) = \mathrm{id}_{\langle n\rangle}$.

5.1.1.7. Definition. — Let $p\colon \mathbf{C} \to \mathbf{Fin}_*$ be a functor. A morphism $f\colon X \to Y$ in $\mathbf{C}$ is *p-cocartesian* if the following condition is satisfied: For every tuple (Z, g, α), where Z is an object of $\mathbf{C}$ and $g\colon X \to Z$ is a morphism in $\mathbf{C}$ and $\alpha\colon p(Y) \to p(Z)$ is a morphism in $\mathbf{Fin}_*$, satisfying $p(g) = \alpha \circ p(f)$, there exists a unique morphism $h\colon Y \to Z$ such that $p(h) = \alpha$.

5.1.1.8. Proposition. — *In the situation of Definition 5.1.1.1 the category $\mathbf{O}^{\otimes}$ is equipped with a functor $p\colon \mathbf{O}^{\otimes} \to \mathbf{Fin}_*$ satisfying the following properties:*

(i) *For every object $(c_i)_{i=1}^m$ in $\mathbf{O}^{\otimes}$ and every inert morphism $i\colon \langle m\rangle \to \langle \ell\rangle$ of pointed finite sets, there exists a p-*cocartesian lift *$\bar{i}\colon (c_j)_{j=1}^m \to (d_k)_{k=1}^{\ell}$ of i, i.e. the morphism $\bar{i}$ is p-cocartesian and $p(\bar{i}) = i$.*

(ii) *For every natural number $m \geq 1$ and every $1 \leq n \leq m$, the inert morphism ρ_i induces a functor $R_{m,n}\colon \mathbf{O}^{\otimes}_{\langle m\rangle} \to \mathbf{O}^{\otimes}_{\langle 1\rangle}$ by taking p-cocartesian lifts: It assigns an object $(c_j)_{j=1}^m$ in $\mathbf{O}^{\otimes}_{\langle m\rangle}$ the object c_n in $\mathbf{O}^{\otimes}_{\langle 1\rangle}$.*

(iii) *For every natural number $m \geq 1$, the sequence $(R_{m,i})_{i=1}^m$ of functors in (ii) induces the equivalence of categories below*

$$R_m\colon \mathbf{O}^{\otimes}_{\langle m\rangle} \xrightarrow{\sim} \left(\mathbf{O}^{\otimes}_{\langle 1\rangle}\right)^{\times m}.$$

Sketch. — For $r \in \mathbb{N}$, the functor p assigns to a an r-tuple of colours the pointed set $\langle r\rangle$, and p maps a morphism $(\alpha, \phi_1, \phi_2, \ldots, \phi_l)$ to its component α.

(i) We define $d_k \coloneqq c_{i^{-1}(k)}$ for $1 \leq k \leq \ell$; the preimage $i^{-1}(k)$ is well-defined because i is inert. Recall the unit map 1_c for a colour c in $\mathbf{O}$ (Definition 5.1.0.2). Define $\bar{i} \coloneqq (i, (1_{c_{i^{-1}(k)}})_{1\leq k\leq \ell})$. The uniqueness follows from the unitality of the structure maps of operads. We leave it for the reader to check that $\bar{i}$ is p-cocartesian.

(ii) It follows from the construction in (i).

(iii) The functor R_m sends an object $(c_1, c_2, \ldots, c_m)$ to $c_1 \times c_2 \times \cdots \times c_m$, which is essentially surjective. The morphism set from $(c_j)_{j=1}^m$ to $(d_j)_{j=1}^m$ is $\prod_{j=1}^m \mathbf{O}(c_j; d_j)$, which implies that R_m is fully faithful. □

5.1.1.9. Notation. — Let $p\colon \mathbf{C} \to \mathbf{Fin}_*$ be a functor. For a tuple (α, X, Y) of a morphism $\alpha\colon \langle m\rangle \to \langle \ell\rangle$ of pointed finite sets, an object X of $\mathbf{C}_{\langle m\rangle}$ and an object Y of $\mathbf{C}_{\langle \ell\rangle}$, let $\mathrm{Hom}^{\alpha}_{\mathbf{C}}(X, Y)$ denote the subset of $\mathrm{Hom}_{\mathbf{C}}(X, Y)$ whose elements are morphisms f such that $p(f) = \alpha$.

5.1.1.10. Proposition. — *Let $p\colon \mathbf{C} \to \mathbf{Fin}_*$ be a functor satisfying Proposition 5.1.1.8.(i), (iii) and the following condition: For every tuple (α, X, Y) as in Notation 5.1.1.9 and every sequence $(\overline{\rho_i}\colon Y \to Y_i)_{i=1}^{\ell}$ of p-cocartesian lifts $\overline{\rho_i}$ of ρ_i, there exists an isomorphism $\mathrm{Hom}^{\alpha}_{\mathbf{C}}(X, Y) \cong \prod_{i=1}^{\ell} \mathrm{Hom}^{\rho_i \circ \alpha}_{\mathbf{C}}(X, Y_i)$ induced by composition of morphisms. Then there exists an operad $\mathbf{O}_{\mathbf{C}}$ with values in $(\mathbf{Set}, \times, \{\mathrm{pt}\}$ whose associated category $\mathbf{O}^{\otimes}_{\mathbf{C}}$ of operators is equivalent to $\mathbf{C}$.*

Sketch. — We construct the set of colours, the sets of operations and the structure maps of the operad $\mathbf{O_C}$ and omit the verification of the axioms/compatibilities.

The set of colours of $\mathbf{O_C}$ is the set of objects of $\mathbf{C}_{\langle 1\rangle}$. Then a finite sequence $(c_i)_{i=1}^r$ of colours of $\mathbf{O_C}$ corresponds to an object $(c_1, c_2, \cdots, c_r) \in \mathbf{C}_{\langle r\rangle}$ under the equivalence R_r in Proposition 5.1.1.8.(iii). For each $r \geq 0$, denote by $f_r \colon \langle r\rangle \to \langle 1\rangle$ the unique morphism in $\mathbf{Fin}_*$ with $f^{-1}(\mathrm{pt}) = \{\mathrm{pt}\}$. The set $\mathbf{O_C}((c_i)_{i=1}^r; c)$ of operations is defined as the set $\mathrm{Hom}_{\mathbf{C}}^{f_r}((c_1, c_2, \ldots, c_r), c)$ morphisms in $\mathbf{C}$.

The unit morphism $\{\mathrm{pt}\} \to \mathbf{O_C}(c; c)$ is given by choosing the identity morphism of $c \in \mathbf{C}_{\langle 1\rangle}$. Now let us consider the construction of the composition map

$$\mathrm{Hom}_{\mathbf{C}}^{f_r}((c_i)_{i=1}^r, c) \times \left(\prod_{i=1}^r \mathrm{Hom}_{\mathbf{C}}^{f_{m_i}}(d_i, c_i)\right) \to \mathrm{Hom}_{\mathbf{C}}^{f_m}(d, c)\,, \tag{5.1.1.1}$$

where $d_i \coloneqq (c_{i,j})_{j=1}^{m_i} \in \mathbf{C}_{\langle m_i\rangle}$, and $m \coloneqq \sum_{i=1}^r m_i$ and $d \coloneqq (d_i)_{i=1}^r \in \mathbf{C}_{\langle m\rangle}$. Consider the pointed set $\langle m\rangle$ as the wedge sum $\vee_{i=1}^r \langle m_i\rangle$, and define the inert morphism $v_i \colon \langle m\rangle \to \langle m_i\rangle$ which sends the wedge component $\langle m_i\rangle$ identically to $\langle m_i\rangle$ and everything else to the basepoint, for every $1 \leq i \leq r$. Then there exists a p-cocartesian lift $\overline{v_i} \colon d \to d_i$ of the morphism v_i, by Proposition 5.1.1.8.(iii). Precomposing with $\overline{v_i}$ induces a morphism

$$\begin{gathered}\mathrm{Hom}_{\mathbf{C}}^{f_r}((c_i)_{i=1}^r, c) \times \left(\textstyle\prod_{i=1}^r \mathrm{Hom}_{\mathbf{C}}^{f_{m_i}}(d_i, c_i)\right) \\ \downarrow \\ \mathrm{Hom}_{\mathbf{C}}^{f_r}((c_i)_{i=1}^r, c) \times \left(\textstyle\prod_{i=1}^r \mathrm{Hom}_{\mathbf{C}}^{f_{m_i} \circ v_i}(d, c_i)\right).\end{gathered}$$

Define the map $\alpha \colon \langle m\rangle \to \langle r\rangle$ sending every non-basepoint element in the wedge component $\langle m_i\rangle$ to i, for every $1 \leq i \leq r$. The condition (iii) implies that

$$\mathrm{Hom}_{\mathbf{C}}^{\alpha}(d, (c_i)_{i=1}^r) \cong \prod_{i=1}^r \mathrm{Hom}_{\mathbf{C}}^{f_{m_i} \circ v_i}(d, c_i)\,.$$

Thus, composition

$$\mathrm{Hom}_{\mathbf{C}}^{f_r}((c_i)_{i=1}^r, c) \times \mathrm{Hom}_{\mathbf{C}}^{\alpha}(d, (c_i)_{i=1}^r) \to \mathrm{Hom}_{\mathbf{C}}^{f_m}(d, c)$$

in $\mathbf{C}$ gives the resulting composition map (5.1.1.1) of $\mathbf{O_C}$. Every element $\sigma \in \mathfrak{S}_r$ gives an automorphism σ_* of $\langle r\rangle$. The symmetric group action on $\mathbf{O_C}(c_1, c_2, \ldots, c_r; c)$ is induced by the p-cocartesian lift of σ_*. □

5.1.1.11. Remark. — A category satisfying the hypotheses of Proposition 5.1.1.10 is called a *category of operators*. With Propositions 5.1.1.8 and 5.1.1.10 we present the idea that there exists a one-to-one correspondence between operads with values in $(\mathbf{Set}, \times, \{\mathrm{pt}\})$ and categories of operators. One can also generalise this to give a correspondence between operads with values in a nice symmetric monoidal category and enriched categories of operators, using similar proofs.

The notion of algebras over an operad (see Definition 5.1.0.9) can also be translated, under the above correspondence, to the setting of categories of operators. Let $\mathbf{O}$ and $\mathbf{P}$ be two operads with values in $(\mathbf{Set}, \times, \{\mathrm{pt}\})$. The data of an operad map (see Definition 5.1.0.5) $\mathbf{O} \to \mathbf{P}$ is the same as a functor $F\colon \mathbf{O}^{\otimes} \to \mathbf{P}^{\otimes}$ of their associated categories of operators such that the diagram

$$\begin{array}{ccc} \mathbf{O}^{\otimes} & \xrightarrow{F} & \mathbf{P}^{\otimes} \\ & \searrow \quad \swarrow & \\ & \mathbf{Fin}_* & \end{array}$$

of categories commutes and F preserves morphisms that are are p-cocartesian and are lifts of inert morphisms of pointed finite sets.

5.1.1.12. Remark. — In §5.2 we will generalise the notion of categories of operators to ∞-categorical setting to define ∞-*operads*, i.e. ∞-categorial version of operads with values in topological spaces. In later chapters we will need ∞-categorical generalisations of operads with values in other symmetric monoidal categories, e.g. the category of spectra. For this purpose, we introduce another viewpoint on operads using the theory of monads in this last part of the section.

5.1.2. Monads and symmetric sequences. — The aim of this short subsection is to point out the relationship among operads, monads and symmetric sequences, in order to motivate the construction of ∞-operads in §5.2.4.

5.1.2.1. Definition. — Let $\mathbf{M}$ be a monoidal category. A $\otimes$-*monoid* in $\mathbf{M}$ is an object $A \in \mathbf{M}$ together with morphisms

$$\mu\colon A \otimes A \to A \text{ and } \iota\colon \mathbb{1}_{\mathbf{M}} \to A$$

such that μ is associative and unital, expressed by the commutativity of the following commutative diagrams:

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mu \circ \mathrm{id}_A} & A \circ A \\ {\scriptstyle \mathrm{id} \circ \mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ A \otimes A & \xrightarrow[\mu]{} & A \end{array} \qquad \begin{array}{ccccc} A \otimes A & \xleftarrow{\mathrm{id} \otimes \iota} & A & \xrightarrow{\iota \otimes \mathrm{id}} & A \circ A \\ & {\scriptstyle \mu}\searrow & \downarrow{\scriptstyle \mathrm{id}} & \swarrow{\scriptstyle \mu} & \\ & & A. & & \end{array}$$

5.1.2.2. Situation. — Let $\mathbf{C}$ be a symmetric monoidal category. The functor category $\mathbf{Fun}(\mathbf{C}, \mathbf{C})$ equipped with the composition $\circ$ of functors and the identity natural transformation becomes a monoidal category.

5.1.2.3. Definition. — In Situation 5.1.2.2 a *monad* on $\mathbf{C}$ is a $\circ$-monoid in the monoidal category $(\mathbf{Fun}(\mathbf{C}, \mathbf{C}), \circ, \mathrm{id}_{\mathbf{C}})$.

5.1.2.4. Definition. — Let $\mathbf{C}$ be a symmetric monoidal category and let T be a monad on $\mathbf{C}$. A *left module over T* in $\mathbf{C}$ is an object $M \in \mathbf{C}$ together with a morphism $a\colon T(M) \to M$ in $\mathbf{C}$ such that the following diagrams commute:

$$\begin{array}{ccc} (T \circ T)(M) & \xrightarrow{\mu(M)} & F(M) \\ {\scriptstyle T(a)}\downarrow & & \downarrow{\scriptstyle a} \\ T(M) & \xrightarrow[a]{} & M, \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{\mathrm{id}} & M \\ & {\scriptstyle \iota(M)}\searrow \quad \nearrow{\scriptstyle a} & \\ & T(M). & \end{array}$$

5.1.2.5. Notation. —

(i) Let $\mathbf{Fin}$ denote the full subcategory of the category $\mathbf{Set}$ of sets whose objects are

$$\underline{n} = \{1, 2, \dots, n\} \text{ for } n \in \mathbb{N},$$

where $\underline{0} := \emptyset$. The category $\mathbf{Fin}$ is equivalent to the category of finite sets.

(ii) Let $\mathbf{Fin}^{\cong}$ denote the maximal subgroupoid of $\mathbf{Fin}$. In other words, $\mathbf{Fin}^{\cong}$ has the same set of objects as $\mathbf{Fin}$ and morphisms in $\mathbf{Fin}^{\cong}$ are bijections of finite sets.

5.1.2.6. Definition. — Let $\mathbf{V}$ be a symmetric monoidal category. The *category* $\mathbf{SymSeq}(\mathbf{V})$ *of symmetric sequences in* $\mathbf{V}$ is the functor category $\mathbf{Fun}(\mathbf{Fin}^{\cong}, V)$. An object of $\mathbf{SymSeq}(\mathbf{V})$ is called a *symmetric sequence* in $\mathbf{V}$.

5.1.2.7. Remark. — A symmetric sequence in $\mathbf{V}$ is a sequence $M = (M(r))_{r \in \mathbb{N}}$ of objects of $\mathbf{V}$ where each $M(r)$ is equipped with a symmetric group $\mathfrak{S}_r$ action.

5.1.2.8. Example. — Let $\mathbf{V}$ be a symmetric monoidal category admitting an initial object.

(i) Every object $X \in \mathbf{V}$ defines a symmetric sequence $X^{\mathfrak{S}}$ where $X^{\mathfrak{S}}(1) := X$ and $X^{\mathfrak{S}}(r)$ is the initial object of $\mathbf{V}$ for every $r \neq 1$.

(ii) Let $\mathbf{O}$ be a one-coloured operad with values in $\mathbf{V}$. The sequence

$$M_{\mathbf{O}} := (\mathbf{O}(r))_{r \in \mathbb{N}}$$

of operations (see Remark 5.1.0.4) is a symmetric sequence in $\mathbf{V}$.

5.1.2.9. Construction. — Let $\mathbf{C}$ be a cocomplete symmetric monoidal category copowered over (see Definition 5.1.0.8) a closed symmetric monoidal category $\mathbf{V}$. Every symmetric sequence $M = (M(r))_{r \geq 0}$ in $\mathbf{V}$ induces a functor

$$F_M\colon \mathbf{C} \to \mathbf{C},\ X \mapsto \coprod_{r \geq 0} \left(M(r) \otimes X^{\otimes r}\right)_{\mathfrak{S}_r}.$$

***5.1.2.10. Theorem* (Kelly).** — *In the situation of Construction 5.1.2.9, we obtain the following statements:*

(i) *There exists a functor* $\circledcirc\colon \mathbf{SymSeq}(\mathbf{V}) \times \mathbf{SymSeq}(\mathbf{V}) \to \mathbf{SymSeq}(\mathbf{V})$, *called the* composition product, *such that* $(\mathbf{SymSeq}(\mathbf{V}), \circledcirc, \mathbb{1}^{\mathfrak{S}}_{\mathbf{V}})$ *is a monoidal category.*

(ii) *There exists a one-to-one correspondence between one-coloured operads with values in* $\mathbf{V}$ *and* $\circledcirc$*-monoids in* $\mathbf{SymSeq}(\mathbf{V})$*, given by assigning to a one-coloured operad its underlying sequence of operations (see Example 5.1.2.8.(ii)).*

(iii) *Construction 5.1.2.9 leads to a monoidal functor*

$$F_{(-)}\colon \mathbf{SymSeq}(\mathbf{V}) \to \mathbf{Fun}(\mathbf{C}, \mathbf{C})$$

of categories of $\otimes$*-monoids.*

(iv) *Under the correspondence in (ii), an algebra in* $\mathbf{C}$ *over an operad with values in* $\mathbf{V}$ *is a left module over the associated monad* $F_{M_{\mathbf{O}}}$.

Proof. — See [Kel05, §3] for (i) and [Kel05, §4] for (ii), (iii) and (iv). □

5.1.2.11. Remark. — In this section we omit the detailed construction of the composition product and refer the reader to [Chi12, §2] for an alternative exposition of the above theorems and to [Tri] for a construction of the composition product that is different than the standard approach as in [Kel05; Chi05]. We will give a more detailed presentation about monads, symmetric sequences and composition products in the ∞-categorical setting, see §§5.2.3 and 5.2.4.

5.1.2.12. Remark. — One can also relate an operad with "coloured" symmetric sequences, see [BM07, Appendix 7]. For some textbook references, see also [HM22, §1.4] or [YauCO, Definition 9.3.2].

5.2. A quick course on ∞-operads

We give an introduction on the theory of ∞-operads in this section, including necessary prerequisites that we need for later chapters. First we recall Lurie's theory of ∞-operads with values in the ∞-category $\mathcal{H}o$ of homotopy types (§5.2.1). Using this theory we can define, among others, (symmetric) monoidal ∞-categories, and associative algebras and left modules in a monoidal ∞-category (see §5.2.2). We also review briefly the theory of monads and monadic adjunctions for later applications (see §5.2.3). The main reference is [HA].

The above prepares us well to introduce the theory of ∞-operads with values in an arbitrary presentable symmetric monoidal ∞-category, where we use the theory of symmetric sequences and monads (see §5.2.4). This point of view of ∞-operads is known to experts, but not much has been written about it in full details. Our exposition on this topic supplements [Bra17, §4.1.2]. To conclude we prove that the notions of algebras over an ∞-operad with values in $\mathcal{H}o$, obtained from §5.2.1 and §5.2.4 respectively, are equivalent (§5.2.5). To the best of our knowledge, this is not available in the literature.

5.2.1. ∞-operads as ∞-categories of operators. — In this subsection we recall the basic notions of Lurie's theory of ∞-operads, following [HA, §2]. This theory generalises the notion of topological operad to ∞-categorical settings.

5.2.1.1. Notation. — Let $\mathcal{F}\mathrm{in}_*$ denote the ∞-category $\mathrm{N}(\mathbf{Fin}_*)$ of the nerve of $\mathbf{Fin}_*$. Note that morphisms in $\mathcal{F}\mathrm{in}_*$ are represented by morphisms of $\mathbf{Fin}_*$. We therefore use the same definitions and the properties of the morphisms in $\mathcal{F}\mathrm{in}_*$ as for those in $\mathbf{Fin}_*$. For example, a morphism in $\mathcal{F}\mathrm{in}_*$ is *inert* if it is represented by an inert morphism (see Definition 5.1.1.4) in $\mathbf{Fin}_*$.

5.2.1.2. Definition. — Let $p\colon \mathcal{C} \to \mathcal{F}\mathrm{in}_*$ be a functor of ∞-categories.

(i) Let n be a natural number. Define the ∞-subcategory $\mathcal{C}_{\langle n\rangle}$ of $\mathcal{C}$ via the following pullback diagram of ∞-categories:

$$\begin{array}{ccc} \mathcal{C}_{\langle n\rangle} & \longrightarrow & \mathcal{C} \\ \big\downarrow & \lrcorner & \big\downarrow p \\ \Delta^0 & \xrightarrow{\langle n\rangle} & \mathcal{F}\mathrm{in}_* . \end{array}$$

(ii) Let (α, X, Y) be a tupe of a morphism $\alpha\colon \langle m\rangle \to \langle \ell\rangle$ in $\mathcal{F}\mathrm{in}_*$, an object X of $\mathcal{C}_{\langle m\rangle}$ and an object Y of $\mathcal{C}_{\langle \ell\rangle}$. Define the ∞-subgroupoid $\mathcal{M}\mathrm{ap}^{\alpha}_{\mathcal{C}}(X,Y)$ as the union of connected components of $\mathrm{Map}_{\mathcal{C}}(X,Y)$ where a morphism $f\colon X \to Y$ is in $\mathcal{M}\mathrm{ap}^{\alpha}_{\mathcal{C}}(X,Y)$ if $p(f) \simeq \alpha$.

(iii) We refer the reader to [HTT, Definition 2.4.1.1] for the definition of *p-cocartesian* morphisms, which is the ∞-categorical generalisation of Definition 5.1.1.7.

5.2.1.3. Definition. — An *∞-operad* is an ∞-category $\mathcal{O}^{\otimes}$ together with a functor $p\colon \mathcal{O}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ of ∞-categories satisfying the following conditions:

(i) For every inert morphism i in $\mathcal{F}\mathrm{in}_*$, there exists a p-cocartesian morphism $\bar{i}$ in $\mathcal{O}^{\otimes}$ such that $p(\bar{i}) \simeq i$.

(ii) Let (α, C, D) be a tuple as in Definition 5.2.1.2.(ii), and let $(\overline{\rho_i}\colon D \to D_i)_{i=1}^{\ell}$ be a sequence of p-cocartesian lifts $\bar{i}$ of ρ_i (see Definition 5.1.1.5). Then there exists an equivalence

$$\mathcal{M}\mathrm{ap}^{\alpha}_{\mathcal{O}^{\otimes}}(C, D) \simeq \prod_{i=1}^{l} \mathcal{M}\mathrm{ap}^{\rho_i \circ \alpha}_{\mathcal{O}^{\otimes}}(C, D_i)$$

of ∞-groupoids, induced by composition of morphisms.

(iii) Let $m \geq 1$. For every m-tuple of objects $(C_1, C_2, \ldots, C_m)$ of $\mathcal{O}^{\otimes}_{\langle 1\rangle}$, there exists an object X of $\mathcal{O}^{\otimes}_{\langle m\rangle}$ and a p-cocartesian lift $\overline{\rho_i}\colon C \to C_i$ of ρ_i for every $1 \leq i \leq n$.

We call $\mathcal{O}^{\otimes}_{\langle 1\rangle}$ the ∞-groupoid of *colours* of $\mathcal{O}^{\otimes}$ and p the *structure map* of $\mathcal{O}^{\otimes}$. An object of $\mathcal{O}^{\otimes}_{\langle 1\rangle}$ is called a *colour* of $\mathcal{O}^{\otimes}$.

5.2.1.4. Remark. — This definition is a natural generalisation of that of ordinary categories of operators, cf. Remark 5.1.1.11. In particular, for every $m \geq 1$ we obtain from Definition 5.2.1.3.(ii) and (iii) an equivalence

$$\mathcal{O}^{\otimes}_{\langle m\rangle} \simeq \left(\mathcal{O}^{\otimes}_{\langle 1\rangle}\right)^{m} \tag{5.2.1.1}$$

of ∞-categories; (iii) provides the construction of an essentially surjective functor and (ii) gives the fully faithfulness of this functor.

5.2.1.5. Definition. — A *one-coloured ∞-operad* is an ∞-operad $\mathcal{O}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ together with an essentially surjective functor $\Delta^0 \to \mathcal{O}^{\otimes}_{\langle 1\rangle}$. In writing we usually omit the latter functor from the notation.

5.2.1.6. Example. — Let $\mathbf{O}$ be an operad with values in $(\mathbf{Set}, \times, \{\mathrm{pt}\})$. Recall the associated category $p\colon \mathbf{O}^{\otimes} \to \mathbf{Fin}_*$ of operators of $\mathbf{O}$. By taking the nerves, the induced functor $\mathrm{N}(p)\colon \mathrm{N}(\mathbf{O}^{\otimes}) \to \mathrm{N}(\mathbf{Fin}_*) = \mathcal{F}\mathrm{in}_*$ exhibits $\mathrm{N}(\mathbf{O}^{\otimes})$ as an ∞-operad.

5.2.1.7. Example. — Applying the construction of Example 5.2.1.6 to some of the examples of operads in §5.1, we obtain the following ∞-operads:

(i) The *trivial ∞-operad* $\mathcal{T}\mathrm{riv}^{\otimes} := \mathrm{N}\left(\mathbf{Triv}^{\otimes}\right)$. The category $\mathbf{Triv}^{\otimes}$ is the subcategory of $\mathbf{Fin}_*$ of finite pointed sets with inert morphisms. The structure map $p\colon \mathcal{T}\mathrm{riv}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ is induced by the canonical inclusion $\mathbf{Triv}^{\otimes} \hookrightarrow \mathbf{Fin}_*$.

(ii) The *unital ∞-operad* $\mathcal{E}^{\otimes}_0 := \mathrm{N}\left(\mathbf{E}^{\otimes}_0\right)$. The category $\mathbf{E}^{\otimes}_0$ has the same set of objects as $\mathbf{Fin}_*$. A morphism in $\mathbf{E}^{\otimes}_0$ is a morphism f of finite pointed sets such that $f^{-1}(i)$ contains at most one element for every element $i \neq \mathrm{pt}$ in the target. The structure map $p\colon \mathcal{E}^{\otimes}_0 \to \mathcal{F}\mathrm{in}_*$ is induced by the canonical inclusion $\mathbf{E}^{\otimes}_0 \hookrightarrow \mathbf{Fin}_*$.

(iii) The *associative ∞-operad* $\mathcal{A}ss^{\otimes} := \mathrm{N}\left(\mathbf{Ass}^{\otimes}\right)$. See [HA, Remark 4.1.1.4] for a detailed description of the ∞-category $\mathcal{A}ss^{\otimes}$.

(iv) The *commutative ∞-operad* $\mathcal{C}om^{\otimes} := \mathrm{N}\left(\mathbf{Com}^{\otimes}_{\mathbf{Set}}\right)$. Under the isomorphism $\mathbf{Com}^{\otimes}_{\mathbf{Set}} \cong \mathbf{Fin}_*$ of ordinary categories the structure map of $\mathcal{C}om^{\otimes}$ is the identity functor of $\mathbf{Fin}_*$.

(v) The *left module ∞-operad* $\mathcal{LM}^{\otimes} := \mathrm{N}\left(\mathbf{LM}^{\otimes}\right)$. The inclusion $\mathbf{Ass} \to \mathbf{LM}$ of operads induces a fully faithful functor $\mathcal{A}ss^{\otimes} \to \mathcal{LM}^{\otimes}$ of ∞-categories, compatible with their structure maps.

5.2.1.8. Definition. — A *simplicial operad* is an operad with values in the symmetric monoidal category $(\mathbf{sSet}, \times, \mathrm{pt})$ of simplicial sets with the cartesian symmetric monoidal structure.

5.2.1.9. Remark. — Let $\mathbf{O}$ be a simplicial operad. Then its category $\mathbf{O}^{\otimes}$ of operators is a simplicially enriched category. The structure map $p\colon \mathbf{O}^{\otimes} \to \mathbf{Fin}_*$ induces a morphism $\mathrm{N}(\pi)\colon \mathcal{N}(\mathbf{O}^{\otimes}) \to \mathcal{N}(\mathbf{Fin}_*)$ of simplicial sets, where $\mathcal{N}$ denotes the simplicial nerve functor (see [HTT, Definition 1.1.5.5]). It is shown in [HTT, Proposition 1.1.5.10] that $\mathcal{N}(\mathbf{O}^{\otimes})$ is an ∞-category if $\mathbf{O}^{\otimes}$ is a fibrant simplicially enriched category, i.e. if all mapping simplicial sets of $\mathbf{O}^{\otimes}$ are Kan complexes.(3)

5.2.1.10. Definition. — Let $\mathbf{O}$ be a simplicial operad. We say $\mathbf{O}$ is a *fibrant simplicial operad* if $\mathbf{O}((c_i)_{i=1}^r; c)$ is a fibrant simplicial set (Kan complex), for every pair $((c_i)_{i=1}^r, c)$ of a finite sequence $(c_i)_{i=1}^r$ of colours and a colour c of $\mathbf{O}$.

5.2.1.11. Proposition. — *Let $\mathbf{O}$ be a fibrant simplicial operad. The simplicial nerve $\mathcal{N}(\mathbf{O}^{\otimes})$ of the category $p\colon \mathbf{O}^{\otimes} \to \mathbf{Fin}_*$ of operators together with the induced functor $\mathcal{N}(p)$ is an ∞-operad.*

Proof. — See [HA, Proposition 2.1.1.27]. □

5.2.1.12. Example. — There is a Quillen equivalence(4)

$$|-|\colon \mathbf{sSet} \rightleftarrows \mathbf{CGH} : \mathrm{Sing},$$

where $|-|$ denotes the geometric realisation of a simplicial set and Sing denotes the singular simplicial complex functor, see [QuiHA; Hir19]. In particular, the functor Sing preserves fibrant objects.

Let $\mathbf{O}$ be a topological operad (Definition 5.1.0.20). Thus, we obtain a fibrant simplicial operad $\mathrm{Sing}\,(\mathbf{O})$ by applying the functor Sing to $\mathbf{O}(c_1, c_2, \ldots, c_r; c)$, for every pair $((c_i)_{i=1}^r, c)$ of a finite sequence $(c_i)_{i=1}^r$ of colours and a colour c of $\mathbf{O}$. Using Proposition 5.2.1.11 we obtain an ∞-operad $\mathcal{N}(\mathrm{Sing}\,(\mathbf{O})^{\otimes})$.

(3) Here we consider the Quillen model structure on $\mathbf{sSet}$, see [QuiHA]

(4) The categories on both sides are equipped with the Quillen model structures, see [QuiHA]

5.2.1.13. Example. — Recall the topological operad $\mathbf{E}_n$ from Example 5.1.0.21. The $\mathcal{E}_n$ *∞-operad* is defined by applying the construction in Example 5.2.1.12 to $\mathbf{E}_n$.

5.2.1.14. Definition. — Let $p\colon \mathcal{O}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ be an ∞-operad. We say a morphism f in $\mathcal{O}^{\otimes}$ is *inert* if $p(f)$ is inert and f is p-cocartesian.

5.2.1.15. Definition. — Let $\mathcal{O}^{\otimes}$ and $\mathcal{P}^{\otimes}$ be two ∞-operads. An *∞-operad map* from $\mathcal{O}^{\otimes}$ to $\mathcal{P}^{\otimes}$ is a morphism $f\colon \mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ of underlying simplicial sets such that

(i) the following diagram

$$\begin{array}{ccc} \mathcal{O}^{\otimes} & \xrightarrow{\quad f \quad} & \mathcal{P}^{\otimes} \\ & {\scriptstyle p_{\mathcal{O}}}\searrow \quad \swarrow{\scriptstyle p_{\mathcal{P}}} & \\ & \mathcal{F}\mathrm{in}_* & \end{array}$$

of simplicial sets commutes, and

(ii) the map f preserves inert morphisms.

5.2.1.16. Definition. — Let $p\colon \mathcal{O}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ be an ∞-operad and let $q\colon \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a cocartesian fibration of ∞-categories. We say q exhibits $\mathcal{C}^{\otimes}$ as an *$\mathcal{O}$-monoidal ∞-category* if the composition $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ exhibits $\mathcal{C}^{\otimes}$ as an ∞-operad. For an object $X \in \mathcal{O}^{\otimes}$, let $\mathcal{C}^{\otimes}_X$ denote the ∞-category $\mathcal{C}^{\otimes} \times_{\mathcal{O}^{\otimes}} \Delta^0$ of fibres over X.

5.2.1.17. Proposition. — *Let $p\colon \mathcal{O}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ be an ∞-operad and let $q\colon \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a cocartesian fibration of ∞-categories. Recall the inert morphism ρ_i from Definition 5.1.1.5. Then $q\colon \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ is an $\mathcal{O}$-monoidal ∞-category if and only if every sequence $(\overline{\rho_i}\colon C \to C_i)_{i=1}^m$ of p-cocartesian lifts $\overline{\rho_i}$ of ρ_i induces an equivalence*

$$\mathcal{C}^{\otimes}_C \simeq \prod_{i=1}^{m} \mathcal{C}^{\otimes}_{C_i} \tag{5.2.1.2}$$

of ∞-categories, for every $m \geq 1$.

Proof. — See [HA, Proposition 2.1.2.12]. □

5.2.1.18. Definition. — Let $p\colon \mathcal{O}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ be an ∞-operad and let $q\colon \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be an $\mathcal{O}$-monoidal ∞-category. The *underlying $\mathcal{O}$-monoidal ∞-category* $\mathcal{C}$ of $\mathcal{C}^{\otimes}$ is defined as the fibre product $\mathcal{C}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{O}^{\otimes}_{\langle 1 \rangle}$.

By abuse of notation, we conflate $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ and $\mathcal{C}$ whenever convenient.

5.2.1.19. Definition. — Let $p\colon \mathcal{O}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ be an ∞-operad. For $\mathcal{O}$-monoidal ∞-categories $q_{\mathcal{C}}\colon \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ and $q_{\mathcal{D}}\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes}$, an *$\mathcal{O}$-monoidal functor* from $\mathcal{C}$ to $\mathcal{D}$ is an ∞-operad map from $\mathcal{C}^{\otimes}$ to $\mathcal{D}^{\otimes}$ which carries $q_{\mathcal{C}}$-cocartesian morphisms to $q_{\mathcal{D}}$-cocartesian morphisms. Let $\mathcal{F}\mathrm{un}^{\otimes}_{\mathcal{O}}(\mathcal{C}, \mathcal{D})$ denote the ∞-category of $\mathcal{O}$-monoidal functors from $\mathcal{C}$ to $\mathcal{D}$.

5.2.1.20. Definition. — Let $\mathcal{D}$ be an ∞-category admits finite products and let $p\colon \mathcal{O}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ be an ∞-operad. An $\mathcal{O}$*-monoid* in $\mathcal{D}$ is a functor $F\colon \mathcal{O}^{\otimes} \to \mathcal{D}$ of ∞-categories such that every sequence $(\overline{\rho_i}\colon C \to C_i)_{i=1}^{m}$ of p-cocartesian lifts $\overline{\rho_i}$ of ρ_i induces an equivalence

$$F(C) \xrightarrow{\sim} \prod_{i=1}^{m} F(C_i) \tag{5.2.1.3}$$

in the ∞-category $\mathcal{D}$, for every $m \geq 1$.

5.2.1.21. Remark. — Definition 5.2.1.20 is a generalisation of Segal's condition [Seg74] for commutative topological monoid to an arbitrary ∞-operad. In the situation of Definition 5.2.1.16, a cocartesian fibration $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ is equivalent to a functor $\mathcal{O}^{\otimes} \to \mathcal{C}\mathrm{at}_{\infty}$ of ∞-categories, by straightening [HTT, Theorem 3.2.0.1]. Under this correspondence, an $\mathcal{O}^{\otimes}$-monoidal ∞-category is an $\mathcal{O}$-monoid in the ∞-category $\mathcal{C}\mathrm{at}_{\infty}$ of ∞-categories, by Proposition 5.2.1.17.

For every natural number r, recall the unique morphism $f_r\colon \langle r\rangle \to \langle 1\rangle$ of finite pointed sets such that $f_r^{-1}(\mathrm{pt}) = \mathrm{pt}$.

5.2.1.22. Definition. — Let $p\colon \mathcal{O}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ be an ∞-operad. For $r \in \mathbb{N}$, an *r-ary operation* $f_r(\mathcal{O})$ consists of the following data:

(i) A colour C and a sequence $(C_i)_{1\leq i\leq r}$ of colours of $\mathcal{O}^{\otimes}$ (see Definition 5.2.1.3).

(ii) An object $C_{\underline{r}}$ of $\mathcal{O}^{\otimes}_{\langle r\rangle}$ corresponding to the sequence $(C_i)_{1\leq i\leq r}$ under the equivalence (5.2.1.1).

(iii) A morphism $f_r(\mathcal{O})\colon C_{\underline{r}} \to C$ such that $p\left(f_r(\mathcal{O})\right) \simeq f_r$.

5.2.1.23. Example. — In the situation of Definition 5.2.1.20, an $\mathcal{O}$-monoid in $\mathcal{D}$ consists of the following data (informally speaking):

(i) For each colour $C_i \in \mathcal{O}^{\otimes}_{\langle 1\rangle}$, an object $X_{C_i} \coloneqq F(C_i)$ in $\mathcal{D}$.

(ii) For each r-ary operation $f_r(\mathcal{O})$, an "$\mathcal{O}$-multiplication"

$$(f_r)_*\colon X_{C_1} \times X_{C_2} \times \cdots X_{C_r} \simeq F(C_{\underline{r}}) \to F(C) = X_C,$$

induced by the morphism f_r and the equivalence (5.2.1.3), for every $r \in \mathbb{N}$.

(iii) Suitable structures and compatibilities among the $\mathcal{O}$-multiplication maps up to coherent homotopy, which are described by the evaluations of F at morphisms of $\mathcal{O}^{\otimes}$.

For example, let $\mathcal{O}$ be the commutative ∞-operad $\mathcal{C}\mathrm{om}$. Then a $\mathcal{C}\mathrm{om}$-monoid is an object $X = F(\langle 1\rangle)$ of $\mathcal{D}$ together with a multiplication

$$X \times X \xrightarrow{\sim} F(\langle 2\rangle) \xrightarrow{f_*} F(\langle 1\rangle) = X$$

induced by the morphism $f\colon \langle 2\rangle \to \langle 1\rangle$, and a unit map $F(\{\mathrm{pt}\}) \to X$, where the multiplication is unital and commutative up to coherent homotopy. In other word, it is a (∞-categorical) commutative monoidal in $\mathcal{D}$.

5.2.1.24. Definition. — Let $f\colon \mathcal{P}^\otimes \to \mathcal{O}^\otimes$ be an ∞-operad map and let $q\colon \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ be an $\mathcal{O}$-monoidal ∞-category. A *$\mathcal{P}$-algebra in $\mathcal{C}$* is a map $\alpha\colon \mathcal{P}^\otimes \to \mathcal{C}^\otimes$ of ∞-operads such that $q \circ \alpha \simeq f$.

The *∞-category $\mathcal{A}\mathrm{lg}_{\mathcal{P}/\mathcal{O}}(\mathcal{C})$ of $\mathcal{P}$-algebras in $\mathcal{C}$* is defined as the full ∞-subcategory of the ∞-category $\mathcal{F}\mathrm{un}_{/\mathcal{O}^\otimes}(\mathcal{P}^\otimes, \mathcal{C}^\otimes)$ of functors over $\mathcal{O}^\otimes$ whose objects are $\mathcal{P}$-algebras in $\mathcal{C}$.

5.2.1.25. Notation. — In the situation of Definition 5.2.1.24, we write $\mathcal{A}\mathrm{lg}_{/\mathcal{O}}(\mathcal{C})$ instead of $\mathcal{A}\mathrm{lg}_{\mathcal{O}/\mathcal{O}}(\mathcal{C})$ if f is the identity map of $\mathcal{O}^\otimes$. If $\mathcal{O}^\otimes$ is $\mathcal{C}\mathrm{om}^\otimes \simeq \mathcal{F}\mathrm{in}_*$ and f becomes the structure map of the ∞-operad $\mathcal{P}^\otimes$, we simplify $\mathcal{A}\mathrm{lg}_{\mathcal{P}/\mathcal{O}}(\mathcal{C})$ by $\mathcal{A}\mathrm{lg}_{\mathcal{P}}(\mathcal{C})$.

5.2.1.26. Remark. — Using the notation of Definition 5.2.1.24, one can define a more general notion of algebra over an ∞-operad by assuming only that q is a categorical fibration of the underlying simplicial sets of the ∞-operads,[(5)] see [HA, Remark 2.1.1.13, Definition 2.1.3.1]. In this thesis, we work only with the definition of algebras over an ∞-operad as Definition 5.2.1.24. Also note that our notation $\mathcal{A}\mathrm{lg}_{\mathcal{P}/\mathcal{O}}(\mathcal{C})$ differs in certain situations from Lurie's notation, cf. [HA, Definition 2.1.2.7].

5.2.1.27. Proposition. — *In the situation of Definition 5.2.1.24, the fibre product $(\mathcal{C} \times_{\mathcal{O}} \mathcal{P})^\otimes := \mathcal{C}^\otimes \times_{\mathcal{O}^\otimes} \mathcal{P}^\otimes$ together with the canonical map*

$$(\mathcal{C} \times_{\mathcal{O}} \mathcal{P})^\otimes \to \mathcal{P}^\otimes$$

is a $\mathcal{P}$-monoidal ∞-category whose underlying $\mathcal{P}$-monoidal ∞-category is

$$\mathcal{C} \times_{\mathcal{O}} \mathcal{P} \simeq \mathcal{C} \times_{\mathcal{O}^\otimes_{\langle 1\rangle}} \mathcal{P}^\otimes_{\langle 1\rangle}.$$

Furthermore, there exists the following equivalence of ∞-categories:

$$\mathcal{A}\mathrm{lg}_{/\mathcal{P}}(\mathcal{C} \times_{\mathcal{O}} \mathcal{P}) \simeq \mathcal{A}\mathrm{lg}_{\mathcal{P}/\mathcal{O}}(\mathcal{C}).$$

Proof. — The proposition follows from the fact that cocartesian morphisms are stable under base change, see [HTT, Proposition 2.4.2.3]. □

5.2.1.28. Remark. — Let $\mathcal{O}^\otimes \to \mathcal{F}\mathrm{in}_*$ be an ∞-operad and let $q\colon \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ be an $\mathcal{O}$-monoidal ∞-category. Recall that q is equivalently to an $\mathcal{O}$-monoid in the ∞-category $\mathcal{C}\mathrm{at}_\infty$ of small ∞-categories (see Remark 5.2.1.21). Thus the $\mathcal{O}$-multiplications on q, as described in Example 5.2.1.23, are functors

$$\otimes_{\mathcal{C}}\colon \mathcal{C}_{C_1} \times \mathcal{C}_{C_1} \times \cdots \mathcal{C}_{C_r} \to \mathcal{C}_C,$$

for every pair $((C_i)_{1\le i\le r}, C)$ of an r-tuple (C_i) of colours of $\mathcal{O}^\otimes$ and a colour C of $\mathcal{O}$.

(5) A categorical fibration is a fibration in the Joyal model structure for simplicial sets [HTT, Theorem 2.2.5.1].

Recall the notations about operations of $\mathcal{O}^{\otimes}$ from Definition 5.2.1.22. An $\mathcal{O}$-algebra in $\mathcal{C}$ is a ∞-operad map $A\colon \mathcal{O}^{\otimes} \to \mathcal{C}^{\otimes}$ together with a commutative diagram

$$\begin{array}{ccc} \mathcal{O}^{\otimes} & \xrightarrow{\quad A \quad} & \mathcal{C}^{\otimes} \\ & {\scriptstyle \mathrm{id}} \searrow \quad \swarrow {\scriptstyle q} & \\ & \mathcal{O}^{\otimes} & \end{array}$$

of ∞-categories, which we can informally consider as the following data:

(i) For every colour $C \in \mathcal{O}^{\otimes}_{\langle 1 \rangle}$, an object $X_{\mathcal{C}} := A(C)$ in $\mathcal{C}_C$.

(ii) For every r-ary operation $f_r(\mathcal{O})\colon C_{\underline{r}} \to C$, a morphism

$$m_r\colon X_{C_1} \otimes_{\mathcal{C}} X_{C_2} \otimes_{\mathcal{C}} \cdots \otimes_{\mathcal{C}} X_{C_r} \to X_C,$$

obtained as follows: Setting $X_{\underline{r}} = A(C_{\underline{r}})$, there are two morphisms

$$\overline{f_r(\mathcal{O})}\colon X_{\underline{r}} \to X_{C_1} \otimes_{\mathcal{C}} X_{C_2} \otimes_{\mathcal{C}} \cdots X_{C_r} \text{ and } A(f_r(\mathcal{O}))\colon X_{\underline{r}} \to X_C$$

lifting $f_r(\mathcal{O})$, where $\overline{f_r(\mathcal{O})}$ is the q-cocartesian lift. Thus the universal property of q-cocartesian morphisms induces the morphism m_r.

(iii) Structures and compatibility of the "multiplications" m_r for $r \geq 0$ up to coherent homotopy, obtained from operations of $\mathcal{O}^{\otimes}$ and the universal property of the cocartesian fibration q, see also [HTT, Remark 2.4.1.4].

Therefore, we regard Definition 5.2.1.24 is an infinity categorical generalisation of the classical definition of algebras over an operad, cf. Definition 5.1.0.9.

5.2.1.29. Proposition. — *Let $\mathcal{O}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ be an ∞-operad and let $q\colon \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be an $\mathcal{O}$-monoidal ∞-category. Assume that there exists a map $f\colon \mathcal{T}\mathrm{riv}^{\otimes} \to \mathcal{O}^{\otimes}$ of ∞-operads. Then the evaluation of f at $\langle 1 \rangle$ induces a trivial Kan fibration*

$$\mathcal{A}\mathrm{lg}_{\mathcal{T}\mathrm{riv}\,/\mathcal{O}}(\mathcal{C}) \xrightarrow{\sim} \mathcal{C}^{\otimes}_{f(\langle 1 \rangle)}.$$

Proof. — This is a consequence of [HA, Example 2.1.3.5, Remark 2.1.3.6]. □

5.2.1.30. Definition. — Recall the commutative ∞-operad $\mathcal{C}\mathrm{om}^{\otimes}$ from Example 5.2.1.7.(iv).

(i) A *symmetric monoidal ∞-category* is a $\mathcal{C}\mathrm{om}$-monoidal ∞-category $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$. By Example 5.2.1.23, the underlying ∞-category $\mathcal{C}$ is equipped with a *symmetric monoidal product* $\otimes_{\mathcal{C}}\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a *symmetric monoidal unit* $\mathbb{1}_{\mathcal{C}} \in \mathcal{C}$ where $\otimes_{\mathcal{C}}$ is associative, commutative and unital up to coherent homotopy.

(ii) A *symmetric monoidal functor* is a $\mathcal{C}\mathrm{om}$-monoidal functor between symmetric monoidal ∞-categories.

(iii) A *lax symmetric monoidal functor* between symmetric monoidal ∞-categories is an ∞-operad map between the underlying ∞-operads of the source and target symmetric monoidal ∞-categories.

5.2.1.31. Example. —
(i) The ∞-category of $\mathcal{H}o$ of homotopy types can be endowed with the cartesian symmetric monoidal structure: the symmetric monoidal product is the cartesian product of homotopy types and the symmetric monoidal unit is the point pt.
(ii) The ∞-category $\mathcal{S}p$ of spectra admits a unique symmetric monoidal structure where the symmetric monoidal product $\otimes$ is the *smash product of spectra* and the unit of the symmetric monoidal structure is the sphere spectrum. Furthermore, the smash product preserves small colimits in each variable, see [HA, Corollary 4.8.2.19].

5.2.1.32. Proposition. — *let $F\colon \mathcal{C} \to \mathcal{D}$ be a lax symmetric monoidal functor between symmetric monoidal ∞-categories. Let $p\colon \mathcal{O}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ be an ∞-operad. Then there exists an induced functor*

$$F_*\colon \mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C}) \to \mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{D})$$

of ∞-categories of $\mathcal{O}$-algebras.

Proof. — This follows from Definitions 5.2.1.24 and 5.2.1.30. □

5.2.1.33. Proposition. — *Let $\mathcal{C}$ be a symmetric monoidal ∞-category. There exists an equivalence $\mathcal{A}\mathrm{lg}_{\mathcal{E}_0}(\mathcal{C}) \simeq \mathcal{C}_{\mathbb{1}_{\mathcal{C}}/}$ of ∞-categories.*

Proof. — See [HA, Proposition 2.1.3.9]. □

5.2.1.34. Example. — A *commutative algebra object* in a symmetric monoidal ∞-category $\mathcal{C}$ is an object of $\mathcal{A}\mathrm{lg}_{\mathcal{C}\mathrm{om}}(\mathcal{C})$. By Remark 5.2.1.28 it is an object $X \in \mathcal{C}$ together with a unit map $\mathbb{1}_{\mathcal{C}} \to X$ and a multiplication $\mu\colon X \otimes_{\mathcal{C}} X \to X$ such that μ is unital, associative and commutative up to coherent homotopy.

5.2.1.35. Example. — Let $p\colon \mathcal{O}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ be an ∞-operad. For a natural number $m \geq 1$ recall the set $\underline{m} = \{1, 2, \ldots, m\}$. Let $\kappa\colon \underline{m} \to \mathcal{O}^{\otimes}_{\langle 1\rangle} \hookrightarrow \mathcal{O}^{\otimes}$ be a diagram in $\mathcal{O}^{\otimes}$. For every $1 \leq i \leq m$ denote $C_i := \kappa(i)$. Let $C_{\underline{m}}$ be an object of $\mathcal{O}^{\otimes}_{\langle m\rangle}$ corresponding to the sequence $(C_1, C_2, \ldots, C_m)$ of colours under the equivalence (5.2.1.1). By Definition 5.2.1.3.(ii) the p-limit $\overline{\kappa}\colon \underline{m}^{\triangleleft} \to \mathcal{O}^{\otimes}$ of κ exists and is given by $\overline{\kappa}(\triangleleft) \simeq C_{\underline{m}}$.

Let $q\colon \mathcal{C}^{\otimes} \to \mathcal{C}\mathrm{om}^{\otimes}$ be a symmetric monoidal ∞-category and $F\colon \mathcal{O}^{\otimes} \to \mathcal{C}^{\otimes}$ be an $\mathcal{O}$-algebra in $\mathcal{C}$. By the definition of maps of ∞-operads we see that $F(C_{\underline{m}})$ is a q-limit of $F \circ \kappa$. In particular, if the symmetric monoidal structure on $\mathcal{C}$ (modelled by q) is cartesian, i.e. $\otimes_{\mathcal{C}}$ is equivalent to the cartesian product, then $F(C_{\underline{m}})$ is equivalent to the product of $F(C_i)$'s, that is, F is an $\mathcal{O}$-monoid, cf. Definition 5.2.1.20 and see [HA, Proposition 2.4.2.5].

5.2.2. Associative algebras and left modules. —

5.2.2.1. Definition. — A *monoidal ∞-category* is an $\mathcal{A}$ss-monoidal category. Similarly as in Definition 5.2.1.30 we can define *monoidal functors* ($\mathcal{A}$ss-monoidal functors) and *lax monoidal functors.*

5.2.2.2. Corollary. — *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a lax monoidal functor between monoidal ∞-categories. Then there exists an induced functor*

$$F_*\colon \mathcal{A}\mathrm{lg}_{/\mathcal{A}\mathrm{ss}}(\mathcal{C}) \to \mathcal{A}\mathrm{lg}_{/\mathcal{A}\mathrm{ss}}(\mathcal{D})$$

of ∞-categories of associative algebras. □

5.2.2.3. Remark. — By Definition 5.2.1.16 and Example 5.2.1.23, the underlying ∞-category $\mathcal{C}$ is equipped with a unit $\mathbb{1}_{\mathcal{C}}$ object and a monoidal tensor product functor $\otimes_{\mathcal{C}}\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which is unital and associative up to coherent homotopy.

5.2.2.4. Definition. — Let $\mathcal{C}$ be a monoidal ∞-category. The ∞-category of associative algebras in $\mathcal{C}$ is defined as the ∞-category $\mathcal{A}\mathrm{lg}_{/\mathcal{A}\mathrm{ss}}(\mathcal{C})$.

5.2.2.5. Remark. — By similar analysis as in Remark 5.2.1.28, one can consider an associative algebra in $\mathcal{C}$, i.e. an object of $\mathcal{A}\mathrm{lg}_{/\mathcal{A}\mathrm{ss}}(\mathcal{C})$, as an object X of $\mathcal{C}$ together with a unit map $\mathbb{1}_{\mathcal{C}} \to X$ and a multiplication $X \otimes_{\mathcal{C}} X \to X$ which is associative and unital up to coherent homotopy.

5.2.2.6. Example. — Let $\mathcal{C}$ be an ∞-category. The ∞-category $\mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{C})$ of functors is the underlying monoidal ∞-category of a monoidal ∞-category $\mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{C})^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes}$. The monoidal tensor product is given by the composition of functors and the monoidal unit is the identity functor. See [HA, Example 4.1.8.7, §4.7].

A *monad* of $\mathcal{C}$ is an associative algebra in the monoidal ∞-category $\mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{C})$, cf. Definition 5.1.2.3.

5.2.2.7. Situation. — Recall the left module ∞-operad $\mathcal{LM}^{\otimes}$ (see Example 5.1.0.15) and the map $\mathcal{A}\mathrm{ss}^{\otimes} \hookrightarrow \mathcal{LM}^{\otimes}$ of ∞-operads (see Example 5.2.1.7). Let $\mathcal{C}^{\otimes} \to \mathcal{LM}^{\otimes}$ be a $\mathcal{LM}$-monoidal ∞-category. Denote

$$\mathcal{C}_a^{\otimes} := \mathcal{C}^{\otimes} \otimes_{\mathcal{LM}^{\otimes}} \mathcal{A}\mathrm{ss}^{\otimes} .$$

Then $\mathcal{C}_a^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes}$ is a monoidal ∞-category whose underlying monoidal ∞-category is denoted by $\mathcal{C}_a$.

By Example 5.2.1.23 we can consider the underlying $\mathcal{LM}$-monoidal ∞-category of $\mathcal{C}^{\otimes}$ as a pair $(\mathcal{C}_a, \mathcal{C}_m)$ of ∞-categories together with a functor $\mathcal{C}_a \times \mathcal{C}_m \to \mathcal{C}_m$ *exhibiting* $\mathcal{C}_m$ *as a left module over* $\mathcal{C}_a$, see also [HA, Remark 4.2.1.2].

5.2.2.8. Proposition. — *In Situation 5.2.2.7, the map* $\mathcal{A}\mathrm{ss}^{\otimes} \to \mathcal{LM}^{\otimes}$ *of* ∞*-operad induces a functor*

$$\mathrm{forg}_m \colon \mathcal{A}\mathrm{lg}_{/\,\mathcal{LM}}(\mathcal{C}) \to \mathcal{A}\mathrm{lg}_{\mathcal{A}\mathrm{ss}\,/\,\mathcal{LM}}(\mathcal{C}) \simeq \mathcal{A}\mathrm{lg}_{/\,\mathcal{A}\mathrm{ss}}(\mathcal{C}_a),$$

which is a cartesian fibration.

Proof. — The first arrow is induced by the composition of functors and the second equivalence is given by Proposition 5.2.1.27. It is proven in [HA, Corollary 4.2.3.2] that forg_m is cartesian. □

5.2.2.9. Remark. — An object of $\mathcal{A}\mathrm{lg}_{/\,\mathcal{LM}}(\mathcal{C})$ is a pair $(A, M) \in \mathcal{C}_a \times \mathcal{C}_m$ together with morphisms

$$A \otimes A \to A \text{ and } A \otimes M \to M$$

in $\mathcal{C}$ satisfying certain compatibility conditions, see Remark 5.2.1.28. In particular, A is an associative algebra object of $\mathcal{C}_a$ and these data exhibits M as a left module over A. The functor forg_m shall be considered as the forgetful functor sending (A, M) to A.

5.2.2.10. Definition. — In Situation 5.2.2.7, let $A \in \mathcal{A}\mathrm{lg}_{\mathcal{A}\mathrm{ss}}(\mathcal{C}_a)$ be an associative algebra of $\mathcal{C}$.

(i) The ∞*-category* $\mathcal{L}\mathrm{Mod}_A(\mathcal{C})$ *of left* A*-modules* is defined as the fibre product $\mathcal{A}\mathrm{lg}_{/\,\mathcal{LM}}(\mathcal{C}) \times_{\mathcal{A}\mathrm{lg}_{/\,\mathcal{A}\mathrm{ss}}(\mathcal{C}_a)} \Delta^0$ where the functor $\Delta^0 \to \mathcal{A}\mathrm{lg}_{\mathcal{A}\mathrm{ss}\,/\,\mathcal{LM}}(\mathcal{C})$ sends the vertex to A.

(ii) There exists a forgetful functor

$$\mathrm{forg}_A \colon \mathcal{L}\mathrm{Mod}_A(\mathcal{C}) \to \mathcal{C}_m$$

defined as the composition

$$\mathcal{L}\mathrm{Mod}_A(\mathcal{C}) \xrightarrow{(a)} \mathcal{A}\mathrm{lg}_{/\,\mathcal{LM}}(\mathcal{C}) \xrightarrow{(b)} \mathcal{F}\mathrm{un}(\mathcal{LM}^{\otimes}_{\langle 1 \rangle}, \mathcal{C}) \xrightarrow{\mathrm{ev}_m} \mathcal{C}_m,$$

where (a) is the canonical functor in the pullback diagram defining $\mathcal{L}\mathrm{Mod}_A(\mathcal{C})$ and (b) is given by restricting to the ∞-subcategory $\mathcal{LM}^{\otimes}_{\langle 1 \rangle}$ of $\mathcal{LM}^{\otimes}$.

5.2.2.11. Proposition. — *In Situation 5.2.2.7 a morphism* $f \colon A \to B \in \mathcal{A}\mathrm{lg}_{/\,\mathcal{A}\mathrm{ss}}(\mathcal{C}_a)$ *of associative algebras of* $\mathcal{C}_a$ *induces a commutative diagram*

$$\begin{array}{ccc} \mathcal{L}\mathrm{Mod}_B(\mathcal{C}) & \xrightarrow{f^*} & \mathcal{L}\mathrm{Mod}_A(\mathcal{C}) \\ & {\scriptstyle \mathrm{forg}_B} \searrow \quad \swarrow {\scriptstyle \mathrm{forg}_A} & \\ & \mathcal{C}_m & \end{array}$$

of ∞*-categories.*

Proof. — Recall the cartesian fibration forg_m from Proposition 5.2.2.8. By straightening it corresponds to a functor

$$\mathcal{A}\mathrm{lg}_{/\mathcal{A}\mathrm{ss}}(\mathcal{C}_a)^{\mathrm{op}} \to \mathcal{CAT}_\infty$$

sending an associative algebra A to the ∞-category $\mathcal{LM}\mathrm{od}_A(\mathcal{C})$. Furthermore, a morphism $g \in \mathcal{A}\mathrm{lg}_{/\mathcal{LM}}(\mathcal{C})$ is a forg_m-cartesian morphism if and only if the induced map $\mathrm{ev}_m(g)$ is an equivalence.

The functors f^* and forg_B and forg_A are defined as the forg_m-cartesian lifts of the morphisms f and $\mathbb{1}_{\mathcal{C}_a} \to B$ and $\mathbb{1}_{\mathcal{C}_a} \to A$, respectively, where we consider the monoidal unit $\mathbb{1}_{\mathcal{C}_a}$ an associative algebra with trivial multiplication. □

5.2.2.12. Example. — Let $\mathcal{C}$ be an ∞-category. There exists a $\mathcal{LM}$-monoidal ∞-category $\mathcal{M}(\mathcal{C})^\otimes \to \mathcal{LM}^\otimes$ such that

$$\mathcal{M}(\mathcal{C})_a \simeq \mathcal{F}\mathrm{un}(\mathcal{C},\mathcal{C}) \text{ and } \mathcal{M}(\mathcal{C})_m \simeq \mathcal{C}.$$

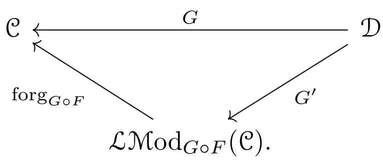
Thus an object of the ∞-category $\mathcal{A}\mathrm{lg}_{/\mathcal{LM}}(\mathcal{M}(\mathcal{C}))$ is a pair (T, M) of a monad T of $\mathcal{C}$ and a left module M over the monad T. See [HA, §4.7] for more details about ∞-categorical monads and left modules over a monad.

5.2.3. Monads and monadic adjunctions. — We need the following statements about monadic adjunctions for our later applications.

5.2.3.1. Proposition. — *Let $F\colon \mathcal{C} \rightleftarrows \mathcal{D} : G$ be an adjunction of ∞-categories. The following statement hold:*

(i) *The composition $G \circ F$ admits the structure of a monad on $\mathcal{C}$: The unit map is the adjunction-unit $\mathrm{id}_\mathcal{C} \to G \circ F$ and the multiplication $(G \circ F) \circ (G \circ F) \to G \circ F$ is induced by the adjunction-counit.*

(ii) *The functor G admits the following factorisation*



Proof. — See [HA, Proposition 4.7.3.3]. □

5.2.3.2. Definition. — In the situation of Proposition 5.2.3.1, the adjunction $F \dashv G$ is *monadic* if the induced morphism G' is an equivalence of ∞-categories.

***5.2.3.3. Theorem* (Barr–Beck, Lurie).** — *Let $F\colon \mathcal{C} \rightleftarrows \mathcal{D} : G$ be an adjunction of ∞-categories. Then the following statements are equivalent:*

(i) *The adjunction (F, G) is a monadic adjunction.*

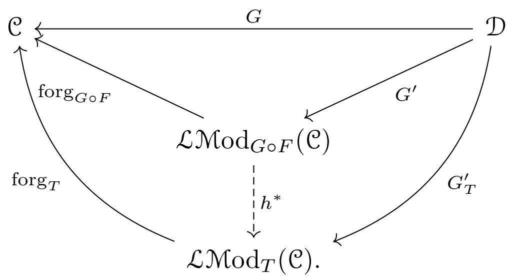
(ii) *The right adjoint G satisfies the following properties:*

(a) *It is conservative, i.e. a morphism f in $\mathcal{D}$ is an equivalence if and only if $G(f)$ is an equivalence.*
(b) *The colimit of every G-split simplicial object (see [HA, Definition 4.7.2.2]) in $\mathcal{D}$ exists and is preserved by G.*

5.2.3.4. Example. — In the situation of Example 5.2.2.12, let T be a monad on $\mathcal{C}$. The forgetful functor $\mathrm{forg}_T\colon \mathcal{LM}\mathrm{od}_T(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint free_T, see [HA, Corollary 4.2.4.8]. Furthermore, this adjunction $\mathrm{free}_T \dashv \mathrm{forg}_T$ is monadic by the Barr–Beck theorem, see [HA, Lemma 3.2.2.6, Example 4.7.2.5]. In particular, there exists an equivalence $T \simeq \mathrm{forg}_T \circ \mathrm{free}_T$ of monads.

5.2.3.5. Proposition. — *Let $F\colon \mathcal{C} \rightleftarrows \mathcal{D} : G$ be an adjunction of ∞-categories. Let $T\colon \mathcal{C} \to \mathcal{C}$ be a monad on $\mathcal{C}$ such that G admits a factorisation $G \simeq \mathrm{forg}_T \circ G'_T$ where $G'_T\colon \mathcal{D} \to \mathcal{LM}\mathrm{od}_T(\mathcal{C})$.*

Then there exists a morphism $h\colon T \to G \circ F$ of monads, uniquely up to contractible choice, inducing the following commutative diagram of ∞-categories:



Proof. — This follows from the universal property of $G \circ F$, being the endomorphism object of G, see [HA, p.658, §4.7.1, Lemma 4.7.3.1]. It suffices to show that the factorisation G'_T induces a natural transformation $T \circ G \to G$. Note that we can write $T \circ G$ as the following composition

$$\mathcal{D} \xrightarrow{G'_T} \mathcal{LM}\mathrm{od}_T(\mathcal{C}) \xrightarrow{\mathrm{forg}_T} \mathcal{C} \xrightarrow{\mathrm{free}_T} \mathcal{LM}\mathrm{od}_T(\mathcal{C}) \xrightarrow{\mathrm{forg}_T} \mathcal{C}.$$

Thus the morphism h is induced by the counit $\mathrm{free}_T \circ \mathrm{forg}_T \to \mathrm{id}$ of the adjunction. □

5.2.4. ∞-operads in a presentable symmetric monoidal ∞-category. — The notion of ∞-operad generalises the classical notion of simplicial or topological operads to ∞-categorical settings; the collection of r-ary operations of an ∞-operad (Definition 5.2.1.22) with a fixed sequence of colours forms an ∞-groupoid, unique up to homotopy, see [HA, Notation 2.1.1.16]. For our later applications, we need the notion of ∞-categorical operad with values in some other ∞-categories, e.g. the ∞-category $\mathcal{S}\mathrm{p}$ of spectra. The goal of this section is to introduce a model for one-coloured ∞-operads with values in an arbitrary presentable symmetric monoidal ∞-categories $\mathcal{C}$ using symmetric sequences and monads on $\mathcal{C}$, generalising the correspondence between operads and monads in 1-categorical setting Theorem 5.1.2.10.

5.2.4.1. Definition. — Let $\mathcal{O}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ be an ∞-operad and let $q\colon \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be an $\mathcal{O}$-monoidal ∞-category. For a set $\mathscr{K}$ of simplicial sets, we say that $\mathcal{C}$ is *compatible with $\mathscr{K}$-indexed colimits* if for every $K \in \mathscr{K}$

(i) K-indexed colimits exists in $\mathcal{C}^{\otimes}_{\langle m\rangle}$ for every $m \geq 0$, and
(ii) the ($\mathcal{O}$-monoidal) tensor product $\otimes_{\mathcal{C}}$ of $\mathcal{C}$ preserves K-indexed colimits in each variable, cf. [HA, Definition 3.1.1.18].

5.2.4.2. Definition. — An ∞-category $\mathcal{C}^{\otimes}$ together with a functor $q\colon \mathcal{C}^{\otimes} \to \mathcal{C}\mathrm{om}^{\otimes}$ is a *presentable symmetric monoidal ∞-category* if

(i) $\mathcal{C}^{\otimes}$ with q is a symmetric monoidal ∞-category,
(ii) $\mathcal{C}^{\otimes}$ with q is compatible with small colimits, and
(iii) the underlying $\mathcal{C}\mathrm{om}$-monoidal ∞-category is a presentable ∞-category.

We say $\mathcal{C}$ is the *underlying presentable symmetric monoidal ∞-category* of $\mathcal{C}^{\otimes}$. By abuse of notation we conflate $\mathcal{C}^{\otimes} \to \mathcal{C}\mathrm{om}^{\otimes}$ and $\mathcal{C}$ whenever convenient.

5.2.4.3. Remark. — Endow the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ of presentable ∞-categories and cocontinuous functors (see Definition 1.1.2.14) with the symmetric monoidal structure constructed in [HA, Proposition 4.8.1.15]. Then a presentable symmetric monoidal ∞-category is a commutative algebra object in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$.

Let $\mathcal{C}$ and $\mathcal{D}$ be presentable symmetric monoidal ∞-categories. A morphism from $\mathcal{C}$ to $\mathcal{D}$ in $\mathcal{A}\mathrm{lg}_{\mathcal{C}\mathrm{om}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ is a cocontinuous symmetric monoidal functor. Let

$$\mathcal{F}\mathrm{un}^{\otimes}_{\mathcal{P}\mathrm{r}^{\mathrm{L}}}(\mathcal{C}, \mathcal{D}) \coloneqq \mathcal{M}\mathrm{or}_{\mathcal{A}\mathrm{lg}_{\mathcal{C}\mathrm{om}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})}(\mathcal{C}, \mathcal{D})$$

denote the ∞-category of cocontinuous symmetric monoidal functors from $\mathcal{C}$ to $\mathcal{D}$.

5.2.4.4. Situation. — In the rest of this section we work with a presentable symmetric monoidal ∞-category $\mathcal{C}$.

5.2.4.5. Definition. — Recall the category **Fin** of finite sets and its maximal subgroupoid $\mathbf{Fin}^{\cong}$ from Notation 5.1.2.5. Define the ∞-category

$$\mathcal{F}\mathrm{in} \coloneqq \mathrm{N}\,(\mathbf{Fin})$$

of finite sets and the ∞-category

$$\mathcal{F}\mathrm{in}^{\simeq} \coloneqq \mathrm{N}\left(\mathbf{Fin}^{\cong}\right)$$

of finite sets and bijections. Note that $\mathcal{F}\mathrm{in}^{\simeq}$ is the maximal ∞-groupoid of $\mathcal{F}\mathrm{in}$.

5.2.4.6. Definition. — Define the ∞-category $\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C})$ of *symmetric sequences in* $\mathcal{C}$ as the ∞-category $\mathcal{F}\mathrm{un}(\mathcal{F}\mathrm{in}^{\simeq}, \mathcal{C})$ of functors. For $F \in \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C})$, let $F(r)$ denote the evaluation of F on the object $\underline{r} = \{1, 2, \ldots, r\}$ for $r \in \mathbb{N}$, where $\underline{0} \coloneqq \emptyset$.

5.2.4.7. Example. — Let X be an object of $\mathcal{C}$.

(i) Define the symmetric sequence $X^{\mathfrak{S}}$ in $\mathcal{C}$ as $X^{\mathfrak{S}}(1) := X$ and $X^{\mathfrak{S}}(r)$ is the initial object of $\mathcal{C}$ for every $r \neq 1$.

For $X = \mathbb{1}_{\mathcal{C}}$ we call $\mathbb{1}_{\mathcal{C}}^{\mathfrak{S}}$ the *unit symmetric sequence* in $\mathcal{C}$.

(ii) Define the symmetric sequence $\underline{X}$ in $\mathcal{C}$ as $\underline{X}(0) := X$ and $\underline{X}(r)$ is the initial object of $\mathcal{C}$ for every $r \neq 0$.

5.2.4.8. Construction. — Here we construct a monoidal structure on the ∞-category $\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C})$ of symmetric sequences where the monoidal tensor product

$$\circledcirc\colon \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}) \times \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}) \to \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}),$$

is called the *composition product* and the unit of the monoidal structure is the unit symmetric sequence $\mathbb{1}_{\mathcal{C}}^{\mathfrak{S}}$. We learnt the construction from [Bra17, §4.1.2]. As the name suggests, we consider this as the ∞-categorical generalisation of the ordinary-categorical composition product (see Theorem 5.1.2.10), see also (5.2.4.3). Briefly speaking, this monoidal structure is induced by the composition of endofunctors of the ∞-category $\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C})$.

First let us review some ∞-categories:

(i) The ∞-category $\mathcal{F}\mathrm{in}^{\simeq}$ admits a symmetric monoidal structure where the symmetric monoidal product is given by disjoint unions of sets and the symmetric monoidal unit is the empty set. Endowed with this symmetric monoidal structure, the ∞-category $\mathcal{F}\mathrm{in}^{\simeq}$ is the free symmetric monoidal ∞-category generated by the one-point set. For a proof, see [Hei18, §2.2.1, Proposition 6.82].

(ii) The ∞-category $\mathcal{H}\mathrm{o}$ of homotopy types is the free presentable ∞-category generated by a point under small colimits, see [HTT, Theorem 5.1.5.6].

(iii) The ∞-category $\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{H}\mathrm{o})$ of symmetric sequences in homotopy types admits a symmetric monoidal structure by Day convolution [HA, §2.2.6]. The symmetric monoidal unit is the symmetric sequence $\underline{\mathrm{pt}}$ (see Example 5.2.4.7). With this symmetric monoidal structure $\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{H}\mathrm{o})$ is the free presentable symmetric monoidal ∞-category generated by the unit symmetric sequence $\mathbb{1}_{\mathcal{H}\mathrm{o}}^{\mathfrak{S}}$, by the universal property of the Day convolution, see [Hei18, §6.1.2].

The Day convolution endows the ∞-category $\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C})$ with the structure of a presentable *symmetric* monoidal ∞-category, where the symmetric monoidal unit is $\underline{\mathbb{1}}_{\mathcal{C}}$. Evaluation on the generator induces the following commutative diagram

$$\begin{array}{ccc}
\mathcal{F}\mathrm{un}_{\mathcal{P}\mathrm{r}^{\mathrm{L}}}(\mathcal{H}\mathrm{o}, \mathcal{C}) & \xrightarrow[\mathrm{ev}_{\mathrm{pt}}]{\sim} & \mathcal{C} \\
\downarrow & & \downarrow \underline{(-)} \\
\mathcal{F}\mathrm{un}_{\mathcal{P}\mathrm{r}^{\mathrm{L}}}(\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{H}\mathrm{o}), \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C})) & \xrightarrow{\mathrm{ev}_{\underline{\mathrm{pt}}}} & \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C})
\end{array}$$

where the left vertical arrow is given by compositions.

Using the functor $\underline{(-)}$ we can consider $\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C})$ as an object in the slice $(\infty,2)$-category $\mathcal{A}\mathrm{lg}_{\mathcal{C}\mathrm{om}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})_{\mathcal{C}/}$ of presentable symmetric monoidal ∞-categories under $\mathcal{C}$. Let

$$\mathcal{F}\mathrm{un}^{\otimes}_{\mathcal{P}\mathrm{r}^{\mathrm{L}},\mathcal{C}/}(-,-)$$

denote $(\infty,1)$-category of morphisms in $\mathcal{A}\mathrm{lg}_{\mathcal{C}\mathrm{om}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})_{\mathcal{C}/}$. The ∞-category $\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C})$ is the free presentable symmetric monoidal ∞-category under $\mathcal{C}$ generated by the unit symmetric sequence $\mathbb{1}^{\mathfrak{S}}_{\mathcal{C}}$, by the equivalences

$$\mathcal{F}\mathrm{un}^{\otimes}_{\mathcal{P}\mathrm{r}^{\mathrm{L}},\mathcal{C}/}(\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}),\mathcal{D}) \overset{(a)}{\simeq} \mathcal{F}\mathrm{un}^{\otimes}_{\mathcal{P}\mathrm{r}^{\mathrm{L}}}(\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{H}\mathrm{o}),\mathcal{D}) \simeq \mathcal{D} \tag{5.2.4.1}$$

for every $\mathcal{D} \in \mathcal{A}\mathrm{lg}_{\mathcal{C}\mathrm{om}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})_{\mathcal{C}/}$. The equivalence (a) is obtained from the adjunction [HA, Remark 4.3.3.3]

$$-\otimes_{\mathcal{P}\mathrm{r}^{\mathrm{L}}}\mathcal{C}\colon\ \mathcal{A}\mathrm{lg}_{\mathcal{C}\mathrm{om}}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \rightleftarrows \mathcal{A}\mathrm{lg}_{\mathcal{C}\mathrm{om}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})_{\mathcal{C}/} : \mathrm{forg}_{\mathcal{C}}$$

and the following equivalences

$$\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{H}\mathrm{o}) \otimes_{\mathcal{P}\mathrm{r}^{\mathrm{L}}} \mathcal{C} \simeq \mathcal{F}\mathrm{un}^{R}(\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{H}\mathrm{o}),\mathcal{C}) \simeq \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}).$$

of ∞-categories, where $\mathcal{F}\mathrm{un}^{R}(-,-)$ denotes the ∞-category of functors admitting left adjoints. The above two equivalences hold by [HA, Proposition 4.8.1.16, Lemma 4.8.1.17].

Taking $\mathcal{D} = \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C})$ the equivalence (5.2.4.1) becomes the equivalence

$$\mathrm{ev}\colon\ \mathcal{F}\mathrm{un}^{\otimes}_{\mathcal{P}\mathrm{r}^{\mathrm{L}}{}_{\mathcal{C}/}}(\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}),\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C})) \xrightarrow{\sim} \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}), \tag{5.2.4.2}$$

given by evaluating at $\mathbb{1}^{\mathfrak{S}}_{\mathcal{C}}$. Via this equivalence, the monoidal structure on the ∞-category $\mathcal{F}\mathrm{un}^{\otimes}_{\mathcal{P}\mathrm{r}^{\mathrm{L}}{}_{\mathcal{C}/}}(\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}),\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}))$ (with compositions of endofunctors as the monoidal product) induces a monoidal structure on the ∞-category $\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C})$. The monoidal unit is the unit symmetric sequence $\mathbb{1}^{\mathfrak{S}}_{\mathcal{C}}$. Note that we need to make a choice of the monoidal structures, since compositions of endofunctors is not commutative.

Let $(F = \mathrm{ev}(\overline{F}), G = \mathrm{ev}(\overline{G}))$ be a pair of two symmetric sequences in $\mathcal{C}$. The *composition product* $F \circledcirc G$ of F with G is defined as

$$F \circledcirc G \coloneqq \mathrm{ev}(\overline{G} \circ \overline{F}).$$

Furthermore, we have the evaluation

$$(F \circledcirc G)(r) \simeq \coprod_{n\geq 0}\left(\coprod_{\underline{r}=\sqcup_{i=1}^{n} S_i} F(n) \otimes_{\mathfrak{S}_n} (\otimes_{i=1}^{n} G(S_i))\right) \tag{5.2.4.3}$$

for every $r \in \mathbb{N}$. We postpone the calculation of (5.2.4.3) to Appendix A.1.

5.2.4.9. Convention. — Let $q_{\circledcirc}\colon\ \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C})^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes}$ denote the (just-defined) monoidal ∞-category whose underlying monoidal ∞-category is $\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C})$ and the monoidal tensor product is the composition product (5.2.4.3).

5.2.4.10. Proposition. — *We can generalise results of Theorem 5.1.2.10 to ∞-categorical settings:*

(i) *The composition product induces a monoidal functor*

$$\begin{aligned} \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}) &\to \mathcal{F}\mathrm{un}(\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}), \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C})) \\ F &\mapsto (G \mapsto F \circledcirc G)\,, \end{aligned}$$

with respect to the composition product and the composition of endofunctors of the ∞-category $\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C})$.

(ii) *Recall the symmetric sequence* $\underline{X}$ *(see Example 5.2.4.7) for an object* $X \in \mathcal{C}$*. We have the following equivalences in* $\mathcal{C}$*:*

$$(F \circledcirc \underline{X})(r) \simeq \begin{cases} \coprod_{r\geq 0} (F(n) \otimes X^{\otimes r})_{\mathfrak{S}_r}\,, & \textit{for } r = 0, \\ \textit{the initial object of } \mathcal{C}, & \textit{otherwise.} \end{cases}$$

(iii) *Consider* $\mathcal{C}$ *as an ∞-subcategory of* $\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C})$ *via the functor* $\underline{(-)}$*. Then there exists an induced monoidal functor*

$$\begin{aligned} \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}) &\to \mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{C}) \\ F &\mapsto (\underline{X} \mapsto F \circledcirc \underline{X})\,, \end{aligned}$$

with respect to the composition product and the composition of endofunctors of the ∞-category $\mathcal{C}$

Proof. — (i) The inclusion $\mathcal{P}\mathrm{r}^{\mathrm{L}} \hookrightarrow \mathcal{CAT}_\infty$ is lax symmetric monoidal, by [HA, Propositions 4.8.1.4 and 4.8.1.15]. Thus we can consider $\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C})$ as a monoidal ∞-category in the cartesian monoidal ∞-category $\mathcal{CAT}_\infty$ (see [HA, Remark 4.8.1.5, p.185]).[6] Therefore, the composition product induces a morphism

$$\begin{aligned} \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}) &\to \mathcal{F}\mathrm{un}(\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}), \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C})) \\ F &\mapsto (G \mapsto F \circledcirc G)\,, \end{aligned}$$

in $\mathcal{CAT}_\infty$ by the (Tensor ⊣ Hom)-adjunction, since the symmetric monoidal structure on $\mathcal{CAT}_\infty$ is closed [HA, Remark 4.8.1.6]. The monoidality of the above functor (morphism) follows from the associativity of the composition product.

(ii) This follows by computations with the formula (5.2.4.3).

(iii) It is a corollary of (i) and (ii). □

5.2.4.11. Definition. — A *(one-coloured) ∞-operad with values in* $\mathcal{C}$ is an associative algebra object in the monoidal ∞-category $q_\circledcirc \colon \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C})^\otimes \to \mathcal{A}\mathrm{ss}^\otimes$. Define the ∞-category $\mathcal{O}\mathrm{pd}(\mathcal{C})$ of ∞-operads with values in $\mathcal{C}$ as

$$\mathcal{O}\mathrm{pd}(\mathcal{C}) \coloneqq \mathcal{A}\mathrm{lg}_{/\mathcal{A}\mathrm{ss}}(\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}))$$

[6] A monoidal ∞-category is cartesian if the monoidal product is given by the cartesian product.

5.2.4.12. Corollary. — *The monoidal functor from Proposition 5.2.4.10.(iii) induces a functor*

$$\mathcal{O}\mathrm{pd}(\mathcal{C}) \simeq \mathcal{A}\mathrm{lg}_{\mathcal{A}\mathrm{ss}}(\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C})) \to \mathcal{A}\mathrm{lg}_{\mathcal{A}\mathrm{ss}}(\mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{C}))$$

assigning to an ∞-operad $\mathcal{O}$ *with values in* $\mathcal{C}$ *its* associated monad $\mathrm{T}_{\mathcal{O}}$. *Moreover, we obtain an equivalence*

$$\mathrm{T}_{\mathcal{O}}(X) \simeq \coprod_{r \geq 0} \left(\mathcal{O}(r) \otimes X^{\otimes r}\right)_{\mathfrak{S}_r}$$

for every object $X \in \mathcal{C}$. □

5.2.4.13. Definition. — Let $\mathcal{O}$ be an ∞-operad with values in $\mathcal{C}$. An $\mathcal{O}$*-algebra in* $\mathcal{C}$ is a left module in $\mathcal{C}$ over the associated monad $\mathrm{T}_{\mathcal{O}}$ (see Example 5.2.2.12). We set the following notations:

(i) Let $\mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C})$ denote the ∞-category $\mathcal{L}\mathcal{M}\mathrm{od}_{\mathrm{T}_{\mathcal{O}}}(\mathcal{C})$ of $\mathcal{O}$-algebras in $\mathcal{C}$.

(ii) Whenever we denote $\mathcal{L}\mathcal{M}\mathrm{od}_{\mathrm{T}_{\mathcal{O}}}(\mathcal{C})$ by $\mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C})$, we abbreviate the forgetful functor $\mathrm{forg}_{\mathrm{T}_{\mathcal{O}}} \colon \mathcal{L}\mathcal{M}\mathrm{od}_{\mathrm{T}_{\mathcal{O}}}(\mathcal{C}) \to \mathcal{C}$ (see Definition 5.2.2.10.(ii)) by $\mathrm{forg}_{\mathcal{O}}$.

5.2.4.14. Example. — Consider the following elementary example. The unit symmetric sequence $\mathbb{1}_{\mathcal{C}}^{\mathfrak{S}}$ is an ∞-operad with values in $\mathcal{C}$, since it is the image of the identity functor under the equivalence (5.2.4.2). Moreover, every object of $\mathcal{C}$ is canonically an algebra over $\mathbb{1}_{\mathcal{C}}^{\mathfrak{S}}$ via the identity morphism of X. Thus we say that $\mathbb{1}_{\mathcal{C}}^{\mathfrak{S}}$ is the *trivial ∞-operad* with values in $\mathcal{C}$, denoted also by $\mathcal{T}\mathrm{riv}_{\mathcal{C}}$.

5.2.4.15. Proposition. — *A morphism* $f \colon \mathcal{O} \to \mathcal{P}$ *of ∞-operads with values in* $\mathcal{C}$ *induces a forgetful functor* $f^* \colon \mathcal{A}\mathrm{lg}_{\mathcal{P}}(\mathcal{C}) \to \mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C})$ *such that* $\mathrm{forg}_{\mathcal{O}} \circ f^* \simeq \mathrm{forg}_{\mathcal{P}}$. *This is illustrated by the following commutative diagram of ∞-categories:*

$$\begin{array}{ccc} \mathcal{A}\mathrm{lg}_{\mathcal{P}}(\mathcal{C}) & \xrightarrow{\quad f^* \quad} & \mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C}) \\ & {\scriptstyle \mathrm{forg}_{\mathcal{P}}} \searrow \quad \swarrow {\scriptstyle \mathrm{forg}_{\mathcal{O}}} & \\ & \mathcal{C}. & \end{array}$$

Proof. — This is an example of Proposition 5.2.2.11. In particular, the forgetful functor $\mathrm{forg}_{\mathcal{O}}$ and $\mathrm{forg}_{\mathcal{P}}$ are induced by the canonical morphisms $\mathcal{T}\mathrm{riv}_{\mathcal{C}} \to \mathcal{O}$ and $\mathcal{T}\mathrm{riv}_{\mathcal{C}} \to \mathcal{P}$ in $\mathcal{O}\mathrm{pd}(\mathcal{C})$ respectively, see Example 5.2.4.14. □

5.2.4.16. Proposition. — *Let* $f \colon \mathcal{O} \to \mathcal{P}$ *be a morphism of ∞-operads with values in* $\mathcal{C}$. *There exists an adjunction*

$$f_! \colon \mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C}) \rightleftarrows \mathcal{A}\mathrm{lg}_{\mathcal{P}}(\mathcal{C}) : f^*.$$

Proof. — It is shown in [HA, Corollary 4.2.3.3] that f^* preserves small limits. Since $\mathcal{C}$ is presentable, the ∞-category $\mathcal{L}\mathcal{M}\mathrm{od}_F(\mathcal{C})$ is presentable for every monad on $\mathcal{C}$ and f^* is accessible by [HA, Corollary 4.2.3.7]. Thus, the existence of the adjunction follows from the Adjoint Functor Theorem [HTT, Corollary 5.5.2.9]. □

5.2.4.17. Corollary. — *The forgetful functor* $\mathrm{forg}_{\mathcal{O}}$ *admits a left adjoint*

$$\mathrm{free}_{\mathcal{O}} \colon \mathcal{C} \to \mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C}),$$

and the monad $\mathrm{forg}_{\mathcal{O}} \circ \mathrm{free}_{\mathcal{O}}$ *is equivalent to the monad* $\mathrm{T}_{\mathcal{O}}$*, i.e. the adjunction is monadic.*

Proof. — The existence of the adjunction follows from Proposition 5.2.4.16. The monadicity of the adjunction is explained in Example 5.2.3.4. □

5.2.4.18. Proposition. — *In the situation of Proposition 5.2.4.16, for every* $\mathcal{O}$*-algebra* $X \in \mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C})$*, the* $\mathcal{P}$*-algebra* $f_!(X)$ *is equivalent to the geometric realisation of the following simplicial object*

$$\mathrm{Bar}(\mathrm{free}_{\mathcal{P}}, \mathrm{T}_{\mathcal{O}}, \mathrm{forg}_{\mathcal{O}}(X)) \coloneqq \left(\cdots (\mathrm{free}_{\mathcal{P}}\, \mathrm{T}^2_{\mathcal{O}})(\underline{X}) \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} (\mathrm{free}_{\mathcal{P}}\, \mathrm{T}_{\mathcal{O}})(\underline{X}) \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow} \mathrm{free}_{\mathcal{P}}(\underline{X}) \right),$$

where $\underline{X}$ *in the formula is an abbreviation for* $\mathrm{forg}_{\mathcal{O}}(X)$ *(meaning the underlying object, not to be confused with the symmetric sequence).*

Proof. — Let X be an $\mathcal{O}$-algebra. Then X is equivalent, as an $\mathcal{O}$-algebra, to the geometric realisation of the following simplicial object

$$\mathrm{Bar}(\mathrm{free}_{\mathcal{O}}, \mathrm{T}_{\mathcal{O}}, \mathrm{forg}_{\mathcal{O}}(X)) \coloneqq \left(\cdots (\mathrm{free}_{\mathcal{O}}\, \mathrm{T}^2_{\mathcal{O}})(\underline{X}) \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} (\mathrm{free}_{\mathcal{O}}\, \mathrm{T}_{\mathcal{O}})(\underline{X}) \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow} \mathrm{free}_{\mathcal{O}}(\underline{X}) \right)$$

in $\mathcal{A}\mathrm{lg}_{\mathcal{O}}(X)$. Indeed, the simplicial object $\mathrm{Bar}(\mathrm{free}_{\mathcal{O}}, \mathrm{T}_{\mathcal{O}}, \mathrm{forg}_{\mathcal{O}}(X))$ admits an augmentation to X given by the counit of the adjunction $\mathrm{free}_{\mathcal{O}} \dashv \mathrm{forg}_{\mathcal{O}}$, which induces the following morphism in $\mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C})$.

$$c \colon |\mathrm{Bar}(\mathrm{free}_{\mathcal{O}}, \mathrm{T}_{\mathcal{O}}, \mathrm{forg}_{\mathcal{O}}(X))| \to X,$$

where $|-|$ denotes the geometric realisation. Applying $\mathrm{forg}_{\mathcal{O}}$ to c induces an equivalence between the geometric realisations of the induced simplicial object $\mathrm{forg}_{\mathcal{O}}(\mathrm{Bar}(\mathrm{free}_{\mathcal{O}}, \mathrm{T}_{\mathcal{O}}, \mathrm{forg}_{\mathcal{O}}(X)))$ and $\mathrm{forg}_{\mathcal{O}}(X)$. Since the functor $\mathrm{forg}_{\mathcal{O}}$ is conservative (see Theorem 5.2.3.3), the morphism c is an equivalence in $\mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C})$ at the first place.

Applying the cocontinuous functor $f_!$, we obtain

$$\begin{aligned} f_! X &\simeq f_! \left(\varinjlim \mathrm{Bar}(\mathrm{free}_{\mathcal{O}}, \mathrm{T}_{\mathcal{O}}, \underline{X}) \right) \\ &\simeq \varinjlim \left(\cdots f_!((\mathrm{free}_{\mathcal{O}}\, \mathrm{T}^2_{\mathcal{O}})(\underline{X})) \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} f_!((\mathrm{free}_{\mathcal{O}}\, \mathrm{T}_{\mathcal{O}})(\underline{X})) \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow} f_!(\mathrm{free}_{\mathcal{O}}(\underline{X})) \right) \\ &\simeq \varinjlim \left(\cdots (\mathrm{free}_{\mathcal{P}}\, \mathrm{T}^2_{\mathcal{O}})(\underline{X}) \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} (\mathrm{free}_{\mathcal{P}}\, \mathrm{T}_{\mathcal{O}})(\underline{X}) \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow} \mathrm{free}_{\mathcal{P}}(\underline{X}) \right), \end{aligned}$$

where the last equivalence holds by the equivalence $f_! \circ \mathrm{free}_{\mathcal{O}}) \simeq \mathrm{free}_{\mathcal{P}}$ (the functor $f_!$ preserves free algebras) which is a consequence of Proposition 5.2.4.15 and Corollary 5.2.4.17. □

5.2.4.19. Definition. — An *augmentation* of an ∞-operad $\mathcal{O}$ with values in $\mathcal{C}$ is a morphism $\epsilon\colon \mathcal{O} \to \mathbb{1}^{\mathfrak{S}}_{\mathcal{C}}$ of ∞-operads. An *augmented ∞-operad* is an ∞-operad together with an augmentation.

5.2.4.20. Example. — Let $\epsilon\colon \mathcal{O} \to \mathbb{1}^{\mathfrak{S}}_{\mathcal{C}}$ be an augmented ∞-operad with values in $\mathcal{C}$. The augmentation ϵ induces an adjunction

$$\mathrm{indec}_{\mathcal{O}}\colon \mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C}) \rightleftarrows \mathcal{C} : \mathrm{triv}_{\mathcal{O}}\,.$$

Every object $X \in \mathcal{C}$ is canonically an algebra over the unit symmetric sequence (see Example 5.2.4.14). The induced $\mathcal{O}$-algebra structure on X is given by $\mathrm{T}_{\mathcal{O}}(X) \to \mathrm{T}_{\mathbb{1}^{\mathfrak{S}}_{\mathcal{V}}}(X) \to X$. Thus the right adjoint is denoted as the trivial $\mathcal{O}$-algebra functor $\mathrm{triv}_{\mathcal{O}}$. By adjunction we have an equivalence

$$\mathrm{Map}_{\mathcal{C}}\left(\mathrm{indec}_{\mathcal{O}}(Y), X\right) \simeq \mathrm{Map}_{\mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C})}\left(Y, \mathrm{triv}_{\mathcal{O}}(X)\right).$$

Informally, for a morphism $Y \to \mathrm{triv}_{\mathcal{O}}(X)$ of $\mathcal{O}$-algebras, any decomposable elements in Y, i.e. elements that are "$\mathcal{O}$-multiplications" of other elements must be sent to zero, since the $\mathcal{O}$-algebra multiplication on $\mathrm{triv}_{\mathcal{O}}(X)$ is trivial. Therefore, we call the left adjoint $\mathrm{indec}_{\mathcal{O}}$ the "indecomposables".

5.2.4.21. Example. — Let $\epsilon\colon \mathcal{O} \to \mathbb{1}^{\mathfrak{S}}_{\mathcal{C}}$ be an augmented ∞-operad with values in $\mathcal{C}$. We have the following composition of adjunctions:

$$\mathcal{C} \underset{\mathrm{forg}}{\overset{\mathrm{free}}{\rightleftarrows}} \mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C}) \underset{\mathrm{triv}}{\overset{\mathrm{indec}}{\rightleftarrows}} \mathcal{C}$$

from Corollary 5.2.4.17 and Example 5.2.4.20, where we omitted the subscript $\mathcal{O}$ from the notations. In particular, the compositions $\mathrm{indec}\circ\mathrm{free}$ and $\mathrm{forg}\circ\mathrm{triv}$ are both equivalent to the identity functor of $\mathcal{C}$, since $\epsilon \circ i = \mathrm{id}_{\mathbb{1}^{\mathfrak{S}}_{\mathcal{C}}}$.

5.2.5. Comparison of models of ∞-operads with values in $\mathcal{H}$o. — We have given two definitions of ∞-operads with values in $\mathcal{H}$o, one using the ∞-category of operators (see §5.2.1) and the other using symmetric sequences and monads (see §5.2.4). In this subsection we show that given an ∞-operad $\mathcal{O}^{\otimes}$ (Definition 5.2.1.3) there exists an associated ∞-operad $\mathcal{O}$ (Definition 5.2.4.11) with values in $\mathcal{H}$o such that the ∞-category of $\mathcal{O}^{\otimes}$-algebras in $\mathcal{C}$ (Definition 5.2.1.24) are equivalent to the ∞-category of $\mathcal{O}$-algebras in $\mathcal{C}$ (Definition 5.2.4.13), for a presentable symmetric monoidal ∞-category $\mathcal{C}$.

5.2.5.1. Proposition. — *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a symmetric monoidal functor in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. Then F induces a monoidal functor*

$$\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}) \to \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{D}),$$

which assigns to a symmetric sequence $(M(r))_{r\geq 0}$ in $\mathcal{C}$ the symmetric sequence $(F(M(r)))_{r\geq 0}$ in $\mathcal{D}$.

Proof. — Post-composing with F induces a symmetric monoidal functor

$$F_*\colon \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}) \to \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{D})$$

in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ with respect to the Day convolution symmetric products. We show that this functor is also monoidal with respect to composition products from Construction 5.2.4.8, defined using (5.2.4.2).

The functor F induces an adjunction

$$- \otimes_{\mathcal{C}} \mathcal{D}\colon \mathcal{A}\mathrm{lg}_{\mathcal{C}\mathrm{om}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})_{\mathcal{C}/} \rightleftarrows \mathcal{A}\mathrm{lg}_{\mathcal{C}\mathrm{om}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})_{\mathcal{D}/} : \mathrm{forg}^{\mathcal{C}}_{\mathcal{D}}$$

where we have

$$\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{D}) \simeq \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}) \otimes_{\mathcal{C}} \mathcal{D} \in \mathcal{A}\mathrm{lg}_{\mathcal{C}\mathrm{om}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})_{\mathcal{D}/}.$$

Thus we can define a functor

$$F'\colon \mathcal{F}\mathrm{un}^{\otimes}_{\mathcal{P}\mathrm{r}^{\mathrm{L}},\mathcal{C}/}(\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}), \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C})) \to \mathcal{F}\mathrm{un}^{\otimes}_{\mathcal{P}\mathrm{r}^{\mathrm{L}},\mathcal{D}/}(\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{D}), \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{D}))$$

via the composition depicted below

$$\begin{array}{ccc}
\mathcal{F}\mathrm{un}^{\otimes}_{\mathcal{P}\mathrm{r}^{\mathrm{L}},\mathcal{C}/}(\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}), \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C})) & \xrightarrow{F_*\circ -} & \\
\downarrow F' & & \mathcal{F}\mathrm{un}^{\otimes}_{\mathcal{P}\mathrm{r}^{\mathrm{L}},\mathcal{C}/}(\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}), \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{D})) \\
\mathcal{F}\mathrm{un}^{\otimes}_{\mathcal{P}\mathrm{r}^{\mathrm{L}},\mathcal{D}/}(\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{D}), \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{D})), & \xrightarrow{\simeq} &
\end{array}$$

where the equivalence is given by the adjunction. This gives us the following commutative diagram

$$\begin{array}{ccc}
\mathcal{F}\mathrm{un}^{\otimes}_{\mathcal{P}\mathrm{r}^{\mathrm{L}},\mathcal{C}/}(\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}), \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C})) & \xrightarrow{F'} & \mathcal{F}\mathrm{un}^{\otimes}_{\mathcal{P}\mathrm{r}^{\mathrm{L}},\mathcal{D}/}(\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{D}), \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{D})) \\
\mathrm{ev}_{\mathbb{1}^{\mathfrak{S}}_{\mathcal{C}}} \downarrow \simeq & & \simeq \downarrow \mathrm{ev}_{\mathbb{1}^{\mathfrak{S}}_{\mathcal{D}}} \\
\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}) & \xrightarrow{F_*} & \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{D})
\end{array}$$

which shows that F_* is monoidal with respect to the composition product on the source and target ∞-category, since the upper horizontal arrow preserves composition of functors by functoriality. □

5.2.5.2. Remark. — Since the induced monoidal functor $\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}) \to \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{D})$ in Proposition 5.2.5.1 is given by level-wise applying the functor F on objects, we abuse the notation F for the monoidal functor $\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}) \to \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{D})$. In particular, we obtain an induced functor

$$F\colon \mathcal{O}\mathrm{pd}(\mathcal{C}) \to \mathcal{O}\mathrm{pd}(\mathcal{D}).$$

5.2.5.3. Situation. — Let $p\colon \mathcal{O}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ together with an essentially surjective morphism $\Delta^0 \to \mathcal{O}^{\otimes}_{\langle 1\rangle}$ be a one-coloured ∞-operad. Let c denote the image of Δ^0 in $\mathcal{O}^{\otimes}_{\langle 1\rangle}$, which is considered as the colour of $\mathcal{O}^{\otimes}$. For every $r \in \mathbb{N}$, recall

(i) the equivalence (see Remark 5.2.1.4)

$$R_r\colon \mathcal{O}^{\otimes}_{\langle r\rangle} \xrightarrow{\sim} \left(\mathcal{O}^{\otimes}_{\langle 1\rangle}\right)^{\times r}$$

of ∞-categories, induced by the sequence $(\rho_i)_{i\geq 1}$ of morphisms of pointed finite sets (see Definition 5.1.1.5), and

(ii) the morphism $f_r\colon \langle r\rangle \to \langle 1\rangle$ of pointed finite sets satisfying $f_r^{-1}(\mathrm{pt}) = \mathrm{pt}$.

For every $r \in \mathbb{N}$, fix an inverse Q_r of R_r and define the ∞-groupoid

$$\mathcal{O}(r) \coloneqq \mathcal{M}\mathrm{ap}^{f_r}_{\mathcal{O}^{\otimes}}\left(Q_r(c^{\times r}), c\right)$$

of morphisms lifting f_r (see Definition 5.2.1.2). Note that $\mathcal{O}(r)$ admits a $\mathfrak{S}_r$-action induced by the permutation group action on $\langle r\rangle$. We call

$$\mathcal{O} \coloneqq \left(\mathcal{O}(r)\right)_{r\geq 0} \in \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{H}\mathrm{o})$$

the *underlying symmetric sequence* of the ∞-operad $\mathcal{O}^{\otimes}$.

5.2.5.4. Situation. — Let $\mathcal{C}$ be a presentable symmetric monoidal ∞-category. Then there exists a symmetric monoidal functor $F\colon \mathcal{H}\mathrm{o} \to \mathcal{C}$ in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$, unique up to contractible choice, since $\mathcal{H}\mathrm{o}$ is symmetric monoidal and is the free presentable ∞-category generated by a point.

5.2.5.5. Theorem. — *In Situations 5.2.5.3 and 5.2.5.4, we obtain the following statements:*

(i) *The symmetric sequence $\mathcal{O}$ admits the structure of an associative algebra object in the monoidal ∞-category $q_{\circledcirc}\colon \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{H}\mathrm{o})^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes}$.*

(ii) *Consider $\mathcal{O}$ as an ∞-operad with values in $\mathcal{H}\mathrm{o}$ by (i). There exists an equivalence*

$$\mathcal{A}\mathrm{lg}_{\mathcal{O}/\,\mathcal{C}\mathrm{om}}(\mathcal{C}) \simeq \mathcal{L}\mathcal{M}\mathrm{od}_{\mathrm{T}_{F(\mathcal{O})}}(\mathcal{C})$$

of ∞-categories, cf. Definitions 5.2.1.24 and 5.2.4.13.

Proof. — The idea is to show that the free-forgetful adjunction for $\mathcal{A}\mathrm{lg}_{\mathcal{O}/\,\mathcal{F}\mathrm{in}_*}(\mathcal{C})$ is monadic and the associated monad is equivalent to $\mathrm{T}_{F(\mathcal{O})}$. For the detailed proof see the proof of Theorem A.2.0.3. □

5.2.5.6. Definition. — Using the notations from Situations 5.2.5.3 and 5.2.5.4, we define the ∞-category $\mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C})$ of $\mathcal{O}$-algebras in $\mathcal{C}$ as the ∞-category $\mathcal{L}\mathcal{M}\mathrm{od}_{T_{F(\mathcal{O})}}(\mathcal{C})$. By the previous proposition this should not cause confusion with the notation introduced in Notation 5.2.1.25

5.2.5.7. Convention. — Consider an ∞-operad $\mathcal{O}$ with values in $\mathcal{H}\mathrm{o}$ and an ∞-operad $\mathcal{P}$ with values in $\mathcal{C}$. Recall that there exists a unique symmetric monoidal functor $F\colon \mathcal{H}\mathrm{o} \to \mathcal{C}$ in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. A morphism $\mathcal{O} \to \mathcal{P}$ of ∞-operads denotes a morphism $F(\mathcal{O}) \to \mathcal{P}$ of ∞-operads with values in $\mathcal{C}$.

5.2.5.8. Remark. — Note that we denote an ∞-operad from Definition 5.2.1.3 by $\mathcal{O}^{\otimes}$ and an ∞-operad with values in $\mathcal{H}\mathrm{o}$ via symmetric sequences by $\mathcal{O}$, because the reader should think of $\mathcal{O}^{\otimes}$ the ∞-category of operators associated to $\mathcal{O}$, analogous to the ordinary categorical situation, cf. Definition 5.1.1.1.

5.2.5.9. Convention. — We have already omited the adjective "one-coloured" when we consider ∞-operads as symmetric sequences. If it is clear from the context in which ∞-category $\mathcal{C}$ we are working, we will also abbreviate "∞-operads with values in $\mathcal{C}$" by "∞-operads".

5.2.5.10. Remark. — Let $\mathcal{V}$ be a symmetric monoidal ∞-category compatible with colimits indexed by ∞-groupoids (see Definition 5.2.4.1). In [Hau22] Haugseng gives a model of (not necessarily one-coloured) ∞-operads with values in $\mathcal{V}$ using ("coloured") symmetric sequences. To obtain the composition product monoidal structure on the ∞-category of symmetric sequences, he generalises another construction of the ordinary-categorical composition product by Dwyer and Hess [DH14] to the ∞-categorical settings. A key ingredient in doing the generalisation is the construction of a monoidal structure on a certain "double ∞-category", using the theory of spans [Bar17], see [Hau22, Definition 4.2.5], and relates algebras in this double ∞-category with ∞-operads with values in $\mathcal{V}$, see [Hau22, Corollary 4.2.8].(7) If $\mathcal{V}$ is in addition presentable, then the formula for the composition product of two symmetric sequences by the above construction coincides with our formula (5.2.4.3), see [Hau22, Remark 4.1.18].

Haugseng shows that his model of ∞-operads with values in $\mathcal{H}\mathrm{o}$ is equivalent to Lurie's model of ∞-operads (Definition 5.2.1.3), by comparing both to Barwick's model of ∞-operads [Bar18], see [Hau22, Corollaries 4.1.12, 4.2.8]. Restricting to one-coloured ∞-operads, it is expected that these models are also equivalent to our model of ∞-operad with values in $\mathcal{H}\mathrm{o}$ (defined using symmetric sequences in §5.2.4). Let $\mathcal{O}^{\otimes}$ be an ∞-operad (Lurie's model), denote the equivalent ∞-operad with values in $\mathcal{H}\mathrm{o}$ using Haugseng's construction by $\mathcal{O}_{\mathrm{Hau}}$. In [Hau19] Haugseng defines the notion of algebras over $\mathcal{O}_{\mathrm{Hau}}$ in $\mathcal{V}$ and shows that it is equivalent to Lurie's notion of algebras over $\mathcal{O}^{\otimes}$ in $\mathcal{V}$. Thus, by Theorem 5.2.5.5, it is also equivalent to our notion of algebras over the ∞-operad $\mathcal{O}$ (see Definition 5.2.4.13), if $\mathcal{V}$ is presentable.

Haugseng's approach to ∞-operads with values in $\mathcal{V}$ is certainly more general. However, our model in the special case where $\mathcal{V}$ is presentable requires much fewer prerequisites and suffices for the purpose of the thesis.

(7) For the theory of double ∞-categories, see [Hau21].

5.2.5.11. Example. — Define the symmetric sequence $\mathcal{C}om_{\mathcal{C}}$ with values in $\mathcal{C}$ by $\mathcal{C}om_{\mathcal{C}}(n) \coloneqq \mathbb{1}_{\mathcal{C}}$ for $n \geq 0$. Then, by Proposition 5.2.5.1, $\mathcal{C}om_{\mathcal{C}}$ is an ∞-operad with values in $\mathcal{C}$, called the commutative ∞-operad: The underlying symmetric sequence $\mathcal{C}om$ of the commutative ∞-operad $\mathcal{C}om^{\otimes}$ is of the form $\mathcal{C}om(r) \simeq \mathrm{pt}$ for every $r \in \mathbb{N}$. There exists a unique symmetric monoidal functor $F\colon \mathcal{H}o \to \mathcal{C}$ in $\mathcal{P}r^{\mathrm{L}}$ and it sends the symmetric monoidal unit pt of $\mathcal{H}o$ (with the cartesian monoidal structure) to the symmetric monoidal unit $\mathbb{1}_{\mathcal{C}}$ of $\mathcal{C}$. Thus, by Theorem 5.2.5.5, the ∞-category $\mathcal{A}lg_{\mathcal{C}om_{\mathcal{C}}}(\mathcal{C})$ is equivalent to the ∞-category of commutative algebras in $\mathcal{C}$.

5.2.5.12. Example. — Recall the presentable symmetric monoidal ∞-categories $\mathcal{H}o$ and $\mathcal{S}p$ from Example 5.2.1.31. The suspension spectrum functor $\Sigma^{\infty}_{+}\colon \mathcal{H}o \to \mathcal{S}p$ is symmetric monoidal, see [HA, Proposition 4.8.2.18]. Let $\mathcal{O}^{\otimes} \to \mathcal{F}in_{*}$ be an ∞-operad. Then the functor Σ^{∞}_{+} induces an equivalence $\mathcal{A}lg_{\mathcal{O}}(\mathcal{S}p) \simeq \mathcal{A}lg_{\Sigma^{\infty}_{+}(\mathcal{O})}(\mathcal{S}p)$ of ∞-categories by Theorem 5.2.5.5.

5.2.5.13. Example. — Recall the ∞-operad $\mathcal{E}_n^{\otimes}$ from Example 5.2.1.13. The tower (5.1.0.2) of embeddings of the topological $\mathbf{E}_n$-operads induces the tower

$$\mathcal{E}_0 \hookrightarrow \mathcal{E}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{E}_n \hookrightarrow \mathcal{E}_{n+1} \cdots \tag{5.2.5.1}$$

of ∞-operads whose colimit in $\mathcal{O}pd(\mathcal{H}o)$ is equivalent to the commutative ∞-operad $\mathcal{C}om$, also denoted by $\mathcal{E}_{\infty}$. Therefore, we obtain by Proposition 5.2.4.15 a tower

$$\cdots \to \mathcal{A}lg_{\mathcal{E}_{n+1}}(\mathcal{C}) \to \mathcal{A}lg_{\mathcal{E}_n}(\mathcal{C}) \to \cdots \to \mathcal{A}lg_{\mathcal{E}_0}(\mathcal{C})$$

of ∞-categories whose inverse limit is the ∞-category $\mathcal{A}lg_{\mathcal{C}om}(\mathcal{C})$.

5.3. Operadic Koszul duality

Koszul duality is a classical notion in the theory of algebras and modules over a field [Pri70], which is generalised to the theory of operads and algebras over an operad by Ginzburg–Kapranov [GK94]. The goal of this expositional section is to present operadic Koszul duality between ∞-operads and ∞-cooperads, and discuss the relationship between the associated ∞-categories of algebras and coalgebras. We begin with a brief introduction of ∞-cooperads and coalgebras (see §5.3.1). Then we discuss the (Bar $\dashv$ Cobar)-adjunction between ∞-cooperads and ∞-operads (see §5.3.2). In the end we specialise to the context of *stable* presentable symmetric monoidal ∞-categories and give some examples of operadic Koszul duality from a historical point of view (see §5.3.3).

5.3.1. Coalgebras over an ∞-cooperad. — In this subsection we introduce the notion of ∞-cooperad with values in a presentable symmetric monoidal ∞-category.

5.3.1.1. Construction. — Let $p\colon \mathcal{O}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ be an ∞-operad and $q\colon \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be an $\mathcal{O}$-monoidal ∞-category. There exists an induced $\mathcal{O}$-monoidal structure on the opposite ∞-category $\mathcal{C}^{op}$, as we define in the following.

The cocartesian fibration $q\colon \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ corresponds to a functor $F\colon \mathcal{O}^{\otimes} \to \mathcal{C}\mathrm{at}_{\infty}$ satisfying the (∞-categorical) Segal condition (5.2.1.2). The composition

$$\mathcal{O}^{\otimes} \xrightarrow{F} \mathcal{C}\mathrm{at}_{\infty} \xrightarrow{(-)^{\mathrm{op}}} \mathcal{C}\mathrm{at}_{\infty}$$

of functors is equivalent to a cocartesian fibration

$$(q^{\vee})^{\mathrm{op}}\colon (\mathcal{C}^{\mathrm{op}})^{\otimes} \to \mathcal{O}^{\otimes},$$

where $(q^{\vee})^{\mathrm{op}}$ satisfies the Segal's condition (5.2.1.2) and $(\mathcal{C}^{\mathrm{op}})^{\otimes} \otimes_{\mathcal{O}^{\otimes}} \mathcal{O}^{\otimes}_{\langle 1\rangle} \simeq \mathcal{C}^{\mathrm{op}}$.(8) In other words, the functor $(q^{\vee})^{\mathrm{op}}$ exhibits $(\mathcal{C}^{\mathrm{op}})^{\otimes}$ as an $\mathcal{O}$-monoidal ∞-category whose underlying $\mathcal{O}$-monoidal ∞-category is equivalent to $\mathcal{C}^{\mathrm{op}}$.

5.3.1.2. Definition. — Let $p\colon \mathcal{O}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ be an ∞-operad and $q\colon \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a $\mathcal{O}$-monoidal ∞-category. An $\mathcal{O}$*-coalgebra* X in $\mathcal{C}$ is an $\mathcal{O}$-algebra in the opposite monoidal ∞-category $\mathcal{C}^{\mathrm{op}}$. Denote the ∞-category of $\mathcal{O}$-coalgebras in $\mathcal{C}$ by $\mathrm{co}\mathcal{A}\mathrm{lg}_{/\mathcal{O}}(\mathcal{C})$. In particular, there exists the following equivalence of ∞-categories

$$\mathrm{co}\mathcal{A}\mathrm{lg}_{/\mathcal{O}}(\mathcal{C}) \simeq \Big(\mathcal{A}\mathrm{lg}_{/\mathcal{O}}(\mathcal{C}^{\mathrm{op}})\Big)^{\mathrm{op}}.$$

5.3.1.3. Remark. — Unravelling the definition, an $\mathcal{O}$-coalgebra in $\mathcal{C}$ is an object X in $\mathcal{C}$ together with comultiplication maps $\mathcal{O}(r) \otimes X \to X^{\otimes r}$ for every $r \in \mathbb{N}$ which are compatible with each other up to coherent homotopy. Here $\otimes$ denotes the symmetric monoidal product of $\mathcal{C}$, see Remark 5.2.1.28.

(8)We use the notation from [BGN18, §1.1].

5.3.1.4. Example. — Let $\mathcal{C}^{\otimes} \to \mathcal{LM}^{\otimes}$ be a $\mathcal{LM}$-monoidal ∞-category exhibiting $\mathcal{C}_m$ as left-tensored over $\mathcal{C}_a$ (see Situation 5.2.2.7). The ∞-category $\mathrm{co}\mathcal{L}\mathcal{M}\mathrm{od}(\mathcal{M})$ of *left comodules* in $\mathcal{C}$ is defined as the ∞-category $(\mathcal{A}\mathrm{lg}_{/\mathcal{LM}}(\mathcal{C}^{\mathrm{op}}))^{\mathrm{op}}$. There exists a forgetful functor

$$\mathrm{forg}_m \colon \mathrm{co}\mathcal{L}\mathcal{M}\mathrm{od}(\mathcal{C}) \to \mathrm{co}\mathcal{A}\mathrm{lg}_{/\mathcal{A}\mathrm{ss}}(\mathcal{C}_a)$$

induced by the inclusion $\mathcal{A}\mathrm{ss}^{\otimes} \hookrightarrow \mathcal{LM}^{\otimes}$.

Let $B \in \mathrm{co}\mathcal{A}\mathrm{lg}_{/\mathcal{A}\mathrm{ss}}(\mathcal{C}_a)$ be a coalgebra in $\mathcal{C}_a$. The ∞-category $\mathrm{co}\mathcal{L}\mathcal{M}\mathrm{od}_{\mathrm{B}}(\mathcal{C})$ of *left B-comodules* in $\mathcal{C}$ is defined as

$$\mathrm{co}\mathcal{L}\mathcal{M}\mathrm{od}_{\mathrm{B}}(\mathcal{C}) \coloneqq \mathrm{co}\mathcal{L}\mathcal{M}\mathrm{od}(\mathcal{C}) \times_{\mathrm{co}\mathcal{A}\mathrm{lg}_{/\mathcal{A}\mathrm{ss}}(\mathcal{C}_a)} \{B\}.$$

5.3.1.5. Proposition. — *The ∞-category $\mathrm{co}\mathcal{L}\mathcal{M}\mathrm{od}_B(\mathcal{C}_m)$ admits small colimits. In particular, the forgetful functor*

$$\mathrm{forg}_m \colon \mathrm{co}\mathcal{L}\mathcal{M}\mathrm{od}(\mathcal{C}) \to \mathrm{co}\mathcal{A}\mathrm{lg}_{/\mathcal{A}\mathrm{ss}}(\mathcal{C})$$

creates small colimits: A diagram in $\mathrm{co}\mathcal{L}\mathcal{M}\mathrm{od}(\mathcal{C})$ is a colimit diagram if and only if its image in $\mathrm{co}\mathcal{A}\mathrm{lg}_{/\mathcal{A}\mathrm{ss}}(\mathcal{C})$ under forg_m is a colimit diagram.

Proof. — This follows from [HA, Proposition 4.2.3.1]. □

5.3.1.6. Proposition. — *In the situation of Example 5.3.1.4, assume that*

(i) *the ∞-category $\mathcal{C}_m$ is a presentable, and*

(ii) *the functor $B \otimes - \colon \mathcal{C}_m \to \mathcal{C}_m$ preserves κ-filtered colimits where κ is an uncountable regular cardinal such that $\mathcal{C}_m$ is κ-accessible.*

Then the ∞-category $\mathrm{co}\mathcal{L}\mathcal{M}\mathrm{od}_{\mathrm{B}}(\mathcal{M})$ is presentable.

Proof. — By [Pér22, Proposition 2.8] the ∞-category $\mathcal{L}\mathcal{M}\mathrm{od}(\mathcal{C})$ and $\mathrm{co}\mathcal{A}\mathrm{lg}_{/\mathcal{A}\mathrm{ss}}(\mathcal{C})$ are presentable ∞-categories. Then the theorem follows from [HTT, Theorem 5.5.3.13], which says that presentable ∞-categories are closed under small limits in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ and these limits can be calculated in the ∞-category $\mathcal{CAT}_{\infty}$ of ∞-categories. □

5.3.1.7. Situation. — From now on till §5.3.3 we work with a presentable symmetric monoidal ∞-category $\mathcal{C}$ (see Definition 5.2.4.2).

5.3.1.8. Definition. — An *∞-cooperad* with values in $\mathcal{C}$ is a coassociative coalgebra in the monoidal ∞-category $q_{\circledcirc} \colon \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C})^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes}$ of symmetric sequences (see Convention 5.2.4.9). Define the ∞-category of ∞-cooperads with values in $\mathcal{C}$ as

$$\mathrm{co}\mathcal{O}\mathrm{pd}(\mathcal{C}) \coloneqq \mathrm{co}\mathcal{A}\mathrm{lg}_{/\mathcal{A}\mathrm{ss}}\left(\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C})\right).$$

Recall the monoidal functor

$$\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}) \to \mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{C}),\ F \mapsto (\underline{X} \mapsto F \circledcirc \underline{X}),$$

from Proposition 5.2.4.10.(iii).

5.3.1.9. **Comonad.** — A *comonad* on $\mathcal{C}$ is a coassociative coalgebra object in the ∞-category $\mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{C})$. For an ∞-cooperad $\mathcal{L}$ with values in $\mathcal{C}$, the associated functor $\mathrm{T}_{\mathcal{L}} := \mathcal{L} \circledcirc \underline{(-)}$ is a comonad, by Corollary 5.2.2.2. Moreover, we have

$$\mathrm{T}_{\mathcal{L}}(X) \simeq \coprod_{r\geq 0} \left(\mathcal{L}(r) \otimes X^{\otimes r}\right)_{\mathfrak{S}_r}$$

for every object $X \in \mathcal{C}$, by Proposition 5.2.4.10.(ii).

5.3.1.10. **Comonads and comonadic adjunctions.** — Working in the opposite categories, we obtain the "comonadic version" of §5.2.3. For example, given an adjunction $F \dashv G$ of ∞-categories, the composition $F \circ G$ admits the structure of a comonad: The coassociative multiplication is given by the canonical natural transformation $F \circ G \circ F \circ G \to F \circ G$ and the counit map is given by the adjunction-counit. Moreover, the functor F factors through the ∞-category $\mathrm{co}\mathcal{L}\mathcal{M}\mathrm{od}_{F\circ G}(\mathcal{D})$ of left comodules over $F \circ G$. The reader can find a precise formulation of the comonadic Barr–Beck theorem in [BM23a, Theorem 4.5]

5.3.1.11. Definition. — Let $\mathcal{L}$ be an ∞-cooperad with values in $\mathcal{C}$. A *conilpotent divided power coalgebra* over $\mathcal{L}$ is a left comodule in $\mathcal{C}$ over $\mathrm{T}_{\mathcal{L}}$ (Example 5.3.1.4). Define the ∞-category of conilpotent divided power $\mathcal{L}$-coalgebras as

$$\mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{ndp}}_{\mathcal{L}}(\mathcal{C}) := \mathrm{co}\mathcal{L}\mathcal{M}\mathrm{od}_{F_{\mathcal{L}}}(\mathcal{C}).$$

5.3.1.12. Remark. — In the situation of Definition 5.3.1.11, a conilpotent divided power coalgebra is an object $X \in \mathcal{C}$ together with the comultiplication map

$$X \to \coprod_{r\geq 0} \left(\mathcal{L}(r) \otimes X^{\otimes r}\right)_{\mathfrak{S}_r} \tag{5.3.1.1}$$

which is coassociative and unital up to coherent homotopy. We explain now the added adjectives "conilpotent" and "divided power" in our definition of coalgebras.

Unprecisely speaking, there might be another notion of coalgebras over an ∞-cooperad $\mathcal{L}$. Recall that for an ∞-operad with values in $\mathcal{C}$ an $\mathcal{O}$-algebra is an object X together with the structure map

$$\coprod_{r\geq 0} \left(\mathcal{O}(r) \otimes X^{\otimes r}\right)_{\mathfrak{S}_r} \to X$$

satisfying certain compatibility conditions (see Definition 5.2.4.13). Following the intuition that "a coalgebra in $\mathcal{C}$ over $\mathcal{L}$ is an $\mathcal{L}$-algebra in the opposite category $\mathcal{C}^{\mathrm{op}}$" we would like to consider an $\mathcal{L}$-coalgebra as an object Y together with the structure map

$$Y \to \prod_{r\geq 0} \left(\mathcal{L}(r) \otimes Y^{\otimes r}\right)^{\mathfrak{S}_r}, \tag{5.3.1.2}$$

since colimits becomes limits and the arrow of morphisms are reversed in $\mathcal{C}^{\mathrm{op}}$. For example, ordinary-categorically, the formula (5.3.1.2) is sometimes used to give the definition for coalgebra over an cooperad, for example, see [LV, §5.7.3]. As another example, let $\mathcal{C}$ be the ∞-category of spectra and assume that $\mathcal{O}(r)$ is a finite spectrum for every $r \in \mathbb{N}$. Then an object $Z \in \mathrm{co}\mathcal{A}\mathrm{lg}_{/\mathcal{O}}(\mathcal{S}\mathrm{p})$ (see Definition 5.3.1.2) is an object in $\mathcal{C}$ together with the structure map

$$Z \to \prod_{r \geq 0} \left(\mathcal{O}(r)^\vee \otimes Z^{\otimes r}\right)^{\mathfrak{S}_r},$$

where $\mathcal{O}(r)^\vee$ denotes the Spanier–Whitehead dual of $\mathcal{O}(r)^\vee$. In particular, the symmetric sequence $(\mathcal{O}(r)^\vee)_{r\geq 0}$ forms an ∞-cooperad, as we will explain in ¶5.3.4.7. Back to a general presentable symmetric monoidal ∞-category $\mathcal{C}$, the technical problem to define an coalgebra over $\mathcal{L}$ as (5.3.1.2) is that the functor

$$\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}) \to \mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{C}), F \to \prod_{r \geq 0} \left(F(r) \otimes (-)^{\otimes r}\right)^{\mathfrak{S}_r}$$

is not oplax monoidal with respect to the composition product and the composition of endofunctors, i.e. it does not send an ∞-cooperad to a comonad.

In Definition 5.3.1.11 "conilpotent" refers to taking the coproducts and "divided power" refers to taking the orbit of the $\mathfrak{S}_r$-action in the structure map (5.3.1.1), as opposed to (5.3.1.2).

5.3.1.13. Proposition. — *Let $\mathcal{L}$ be an ∞-cooperad with values in $\mathcal{C}$.*

(i) *There exists a forgetful functor*

$$\mathrm{forg}_{\mathcal{L}} \colon \mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{ndp}}_{\mathcal{L}}(\mathcal{C}) \to \mathcal{C}$$

assigning to an $\mathcal{L}$-coalgebra its underlying object in $\mathcal{C}$.

(ii) *A morphism $u \colon \mathcal{L} \to \mathcal{K}$ of ∞-cooperads induces a functor*

$$u_* \colon \mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{ndp}}_{\mathcal{L}}(\mathcal{C}) \to \mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{ndp}}_{\mathcal{K}}(\mathcal{C}),$$

such that $\mathrm{forg}_{\mathcal{K}} \circ u_* \simeq \mathrm{forg}_{\mathcal{L}}$. Furthermore, the functor u_* preserves small colimits and admits a right adjoint $u^!$.

Proof. — This is the "coalgebra version" of Proposition 5.2.4.15. The construction of the functors $\mathrm{forg}_{\mathcal{L}}$ and u_* is similar to that for algebras, see Definition 5.2.4.13 and Proposition 5.2.4.15, where we work instead in the opposite ∞-categories. The fact that u_* preserves small colimits follows by applying [HA, Corollary 4.2.3.3] in the opposite ∞-categories. The existence of the right adjoint is due to the presentability of $\mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{ndp}}_{\mathcal{L}}(\mathcal{C})$ (see Proposition 5.3.1.6) and the Adjoint Functor Theorem [HTT, Corollary 5.5.2.9]. □

5.3.1.14. Definition. — A *coaugmentation* of an ∞-cooperad $\mathcal{L}$ with values in $\mathcal{C}$ is a morphism $\eta \colon \mathbb{1}^{\mathfrak{S}}_{\mathcal{C}} \to \mathcal{L}$ of ∞-cooperads An *augmented ∞-cooperad* is an ∞-cooperad together with an coaugmentation.

5.3.1.15. Example. — Let $\mathcal{L}$ and $\mathcal{K}$ be ∞-cooperads with values in $\mathcal{C}$, where $\mathcal{K}$ is coaugmented. There exist the following adjunctions for coalgebras, as analogues to Corollary 5.2.4.17, Example 5.2.4.20 and Example 5.2.4.21.

(i) The counit $\mathcal{L} \to \mathbb{1}_{\mathcal{C}}^{\mathfrak{S}}$ induces the (*forgetful* ⊣ *cofree*)-adjunction

$$\mathrm{forg}_{\mathcal{L}} \colon \mathrm{co}\mathcal{A}\mathrm{lg}_{\mathcal{L}}^{\mathrm{ndp}}(\mathcal{C}) \rightleftarrows \mathcal{C} : \mathrm{cofree}_{\mathcal{L}} .$$

(ii) The coaugmentation $\eta \colon \mathbb{1}_{\mathcal{C}}^{\mathfrak{S}} \to \mathcal{K}$ induces the (*trivial* ⊣ *primitive*)-adjunction

$$\mathrm{triv}_{\mathcal{K}} \colon \mathcal{C} \rightleftarrows \mathrm{co}\mathcal{A}\mathrm{lg}_{\mathcal{K}}^{\mathrm{ndp}}(\mathcal{C}) : \mathrm{prim}_{\mathcal{K}} .$$

(iii) The horizontal compositions forg ∘ triv and prim ∘ cofree in the diagram

$$\mathcal{C} \underset{\mathrm{prim}_{\mathcal{K}}}{\overset{\mathrm{triv}_{\mathcal{K}}}{\rightleftarrows}} \mathrm{co}\mathcal{A}\mathrm{lg}_{\mathcal{K}}^{\mathrm{ndp}}(\mathcal{C}) \underset{\mathrm{cofree}_{\mathcal{K}}}{\overset{\mathrm{forg}_{\mathcal{K}}}{\rightleftarrows}} \mathcal{C}$$

are equivalent to the identity functor of $\mathcal{C}$.

5.3.2. Bar and Cobar constructions. — The starting point of operadic Koszul duality is a Bar ⊣ CoBar adjunction between ∞-operads and ∞-cooperads, which induces an adjunction between the associated ∞-categories of algebras and coalgebras. We explain these adjunctions in this subsection.

5.3.2.1. Construction. — Let $\mathcal{O}\mathrm{pd}^{\mathrm{aug}}(\mathcal{C})$ (respectively $\mathrm{co}\mathcal{O}\mathrm{pd}^{\mathrm{coaug}}(\mathcal{C})$) denote the ∞-categories of augmented ∞-operads (respectively coaugmented ∞-cooperads) with values in $\mathcal{C}$. In [HA, Theorem 5.2.2.7] Lurie constructs a (Bar ⊣ Cobar)-adjunction between the ∞-category of augmented associative algebras and the ∞-category of coaugmented coassociative coalgebras in a monoidal ∞-category which admits geometric realisations and totalisations. Thus, we can apply the adjunction to $\mathcal{O}\mathrm{pd}^{\mathrm{aug}}(\mathcal{C}) \simeq \mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{/\mathcal{A}\mathrm{ss}}(\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}))$ and $\mathrm{co}\mathcal{O}\mathrm{pd}^{\mathrm{coaug}}(\mathcal{C}) \simeq \mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{coaug}}_{/\mathcal{A}\mathrm{ss}}(\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}))$, which leads to an adjunction

$$\mathrm{Bar} \colon \mathcal{O}\mathrm{pd}^{\mathrm{aug}}(\mathcal{C}) \rightleftarrows \mathrm{co}\mathcal{O}\mathrm{pd}^{\mathrm{coaug}}(\mathcal{C}) : \mathrm{Cobar} .$$

Let $\mathcal{O}$ be an augmented ∞-operad with values in $\mathcal{C}$. As shown in [HA, Proposition 5.2.2.5, Remark 5.2.2.8], the underlying symmetric sequence $\mathrm{forg}_{\mathrm{co}\mathcal{O}\mathrm{pd}}(\mathrm{Bar}(\mathcal{O}))$ of the ∞-cooperad $\mathrm{Bar}(\mathcal{O})$ is equivalent to the geometric realisation of the simplicial object

$$\mathrm{Bar}\left(\mathbb{1}_{\mathcal{C}}^{\mathfrak{S}}, \mathcal{O}, \mathbb{1}_{\mathcal{C}}^{\mathfrak{S}}\right) := \left(\cdots \mathcal{O} \circledcirc \mathcal{O} \; \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} \; \mathcal{O} \; \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} \; \mathbb{1}_{\mathcal{C}}^{\mathfrak{S}} \right)$$

in $\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C})$. Similarly, the underlying symmetric sequence $\mathrm{forg}_{\mathcal{O}\mathrm{pd}}(\mathrm{Cobar}(\mathcal{L}))$ of a coaugmented ∞-cooperad $\mathcal{L}$ is equivalent to the totalisation of the cosimplicial object

$$\mathrm{Cobar}\left(\mathbb{1}_{\mathcal{C}}^{\mathfrak{S}}, \mathcal{L}, \mathbb{1}_{\mathcal{C}}^{\mathfrak{S}}\right) := \left(\mathbb{1}_{\mathcal{C}}^{\mathfrak{S}} \; \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} \; \mathcal{L} \; \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} \; \mathcal{L} \circledcirc \mathcal{L} \cdots \right)$$

in $\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C})$. See [HA, p.826, c)] for some remarks regarding these explicit expressions.

5.3.2.2. Construction. — Recall that a monad on $\mathcal{C}$ is an associative algebra in the monoidal ∞-category $\mathcal{F}\mathrm{un}(\mathcal{C},\mathcal{C})$. A monad T is *augmented* if it is equipped with a morphism $T \to \mathrm{id}_{\mathcal{C}}$ of associative algebras in $\mathcal{F}\mathrm{un}(\mathcal{C},\mathcal{C})$. In the same manner, one can define the notion of *coaugmented comonads*. Let $\mathcal{M}\mathrm{onad}^{\mathrm{aug}}(\mathcal{C})$ (respectively $\mathrm{co}\mathcal{M}\mathrm{nd}^{\mathrm{coaug}}(\mathcal{C})$) denote the ∞-categories of augmented monads (respectively coaugmented comonads) with values in $\mathcal{C}$.

We can apply [HA, Theorem 5.2.2.7] again and obtain the ($\mathrm{Bar} \dashv \mathrm{Cobar}$)-adjunction

$$\mathrm{Bar}\colon \mathcal{M}\mathrm{onad}^{\mathrm{aug}}(\mathcal{C}) \rightleftarrows \mathrm{co}\mathcal{M}\mathrm{nd}^{\mathrm{coaug}}(\mathcal{C}) : \mathrm{Cobar}$$

between augmented monads and coaugmented comonads of $\mathcal{C}$. As in the case with ∞-operads and ∞-cooperads, see Construction 5.3.2.1, one can write down explicit formulas for the evaluations of Bar and Cobar on a given augmented monad and coaugmented comonad respectively.

5.3.2.3. Proposition. — *The following diagram of ∞-categories commutes:*

$$\begin{array}{ccc} \mathcal{O}\mathrm{pd}^{\mathrm{aug}}(\mathcal{C}) & \xrightarrow{\mathrm{Bar}} & \mathrm{co}\mathcal{O}\mathrm{pd}^{\mathrm{coaug}}(\mathcal{C}) \\ \downarrow{\scriptstyle \mathrm{T}_{(-)}} & & \downarrow{\scriptstyle \mathrm{T}_{(-)}} \\ \mathcal{M}\mathrm{onad}^{\mathrm{aug}}(\mathcal{C}) & \xrightarrow{\mathrm{Bar}} & \mathrm{co}\mathcal{M}\mathrm{nd}^{\mathrm{coaug}}(\mathcal{C}); \end{array} \tag{5.3.2.1}$$

recall the functor $\mathrm{T}_{(-)}$ *from Corollary 5.2.4.12 and ¶5.3.1.9.*

Proof. — The commutativity follows by the following arguments:

(i) The functor $\mathrm{T}_{(-)}$ preserves the property of being augmented/coaugmented, since the ∞-operad/∞-cooperad structure on the unit symmetric sequence $\mathbb{1}_{\mathcal{C}}$ is unique.

(ii) The functor $\mathrm{T}_{(-)}$ results from a colimit construction (as one can see from its formula), which commutes with the cocontinuous functors Bar in the rows. □

5.3.2.4. Proposition. — *Let* $\mathcal{O} \to \mathbb{1}_{\mathcal{C}}^{\mathfrak{S}}$ *be an augmented ∞-operad with values in* $\mathcal{C}$. *Recall the adjunction* ($\mathrm{indec}_{\mathcal{O}} \dashv \mathrm{triv}_{\mathcal{O}}$) *induced by the augmentation (see Example 5.2.4.20). The functor* $\mathrm{indec}_{\mathcal{O}}$ *factors through the ∞-category* $\mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{ndp}}_{\mathrm{Bar}(\mathcal{O})}(\mathcal{C})$, *given by the following commutative diagram*

$$\begin{array}{ccc} \mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C}) & \xrightarrow{\mathrm{indec}_{\mathcal{O}}} & \mathcal{C} \\ & {\scriptstyle \mathrm{Bar}_{\mathcal{O}}}\searrow \quad \nearrow{\scriptstyle \mathrm{forg}_{\mathrm{Bar}(\mathcal{O})}} & \\ & \mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{ndp}}_{\mathrm{Bar}(\mathcal{O})}(\mathcal{C}) & \end{array}$$

of ∞-categories.

Proof. — We will prove that the comonad $\mathrm{indec}_{\mathcal{O}} \circ \mathrm{triv}_{\mathcal{O}}$ is equivalent to the comonad $\mathrm{T}_{\mathrm{Bar}(\mathcal{O})}$. Then the factorisation follows from ¶5.3.1.10. By Proposition 5.2.4.18 we have

$$\begin{aligned}(\mathrm{indec}_{\mathcal{O}} \circ \mathrm{triv}_{\mathcal{O}})\,(X) &\simeq \varinjlim \mathrm{Bar}\left(\mathrm{free}_{\mathbb{1}^{\mathfrak{S}}_{\mathcal{C}}}, \mathrm{T}_{\mathcal{O}}, \mathrm{forg}_{\mathcal{O}}\,(\mathrm{triv}_{\mathcal{O}}(X))\right)\\ &\simeq \varinjlim \mathrm{Bar}\left(\mathrm{free}_{\mathbb{1}^{\mathfrak{S}}_{\mathcal{C}}}, \mathrm{T}_{\mathcal{O}}, \mathrm{free}_{\mathbb{1}^{\mathfrak{S}}_{\mathcal{C}}}(X)\right)\\ &\simeq \mathrm{T}_{\varinjlim \mathrm{Bar}(\mathbb{1}^{\mathfrak{S}}_{\mathcal{C}}, \mathcal{O}, \mathbb{1}^{\mathfrak{S}}_{\mathcal{C}})}(X)\\ &\simeq \mathrm{T}_{\mathrm{Bar}(\mathcal{O})}(X),\end{aligned}$$

where the second equivalence holds by $\mathrm{forg}_{\mathcal{O}} \circ \mathrm{triv}_{\mathcal{O}} \simeq \mathrm{id}_{\mathcal{C}}$ (see Example 5.2.4.21), and the third equivalence holds by calculation of the colimits and our assumption on $\mathcal{C}$ that the symmetric monoidal product of $\mathcal{C}$ commutes with small colimits in each variable (see Definition 5.2.4.2). □

5.3.2.5. Remark. — In other words, for every $\mathcal{O}$-algebra X in $\mathcal{C}$, its "indecomposables" $\mathrm{indec}_{\mathcal{O}}(X)$ admits the structure of a conilpotent divided power $\mathrm{Bar}(\mathcal{O})$-coalgebra. By Proposition 5.2.3.5., if there exists another comonad T of $\mathcal{C}$ such that $\mathrm{indec}_{\mathcal{O}}$ factors through the forgetful functor $\mathrm{co}\mathcal{L}\mathcal{M}\mathrm{od}_T(\mathcal{C}) \to \mathcal{C}$, then there exists an induced morphism $\mathrm{Bar}(\mathcal{O}) \to T$ of comonads, unique up to contractible choice.

5.3.2.6. Proposition. — *Let $\mathbb{1}^{\mathfrak{S}}_{\mathcal{C}} \to \mathcal{L}$ be an coaugmented ∞-cooperad with values in $\mathcal{C}$. Recall the adjunction $\mathrm{triv}_{\mathcal{L}} \dashv \mathrm{prim}_{\mathcal{L}}$ induces by the coaugmentation (see Example 5.3.1.15). The functor $\mathrm{prim}_{\mathcal{L}}$ factors through the ∞-category $\mathcal{A}\mathrm{lg}_{\mathrm{Cobar}(\mathcal{L})}(\mathcal{C})$, given by the following commutative diagram*

$$\begin{array}{ccc} \mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{ndp}}_{\mathcal{L}}(\mathcal{C}) & \xrightarrow{\mathrm{prim}_{\mathcal{L}}} & \mathcal{C} \\ & {\scriptstyle \mathrm{Cobar}_{\mathcal{L}}} \searrow \quad \nearrow {\scriptstyle \mathrm{forg}_{\mathrm{Cobar}(\mathcal{L})}} & \\ & \mathcal{A}\mathrm{lg}_{\mathrm{Cobar}(\mathcal{L})}(\mathcal{C}). & \end{array}$$

of ∞-categories.

Proof. — By Proposition 5.2.3.5 it suffices to show that there exists a morphism $\mathrm{T}_{\mathrm{Cobar}(\mathcal{L})} \to \mathrm{prim}_{\mathcal{L}} \circ \mathrm{triv}_{\mathcal{L}}$ of monads. By the same proof as in Proposition 5.2.4.18 we have

$$(\mathrm{prim}_{\mathcal{L}} \circ \mathrm{triv}_{\mathcal{L}})(X) \simeq \varprojlim \left(X \rightrightarrows \mathrm{T}_{\mathcal{L}}(X) \rightrightarrows (T_{\mathcal{L}} \circ \mathrm{T}_{\mathcal{L}})(X) \cdots \right).$$

In other words there exists the equivalence of monads below

$$\mathrm{prim}_{\mathcal{L}} \circ \mathrm{triv}_{\mathcal{L}} \simeq \mathrm{Cobar}\,(\mathrm{T}_{\mathcal{L}})\,.$$

Abbreviate the functor $\mathrm{T}(-)$ in the diagram (5.3.2.1) by T, we need to show that there exists a natural transformation

$$\mathrm{T}\circ\mathrm{Cobar}\to\mathrm{Cobar}\circ\mathrm{T}\,.$$

By adjunction and commutativity of (5.3.2.1) the existence of the above natural transformation is equivalent to the existence of one of the following natural transformations

$$\mathrm{Bar}\circ\mathrm{T}\circ\mathrm{Cobar}\to\mathrm{T}\;\text{ and }\;\mathrm{T}\circ\mathrm{Bar}\circ\mathrm{Cobar}\to\mathrm{T}\,.$$

Composing T with the counit natural transformation of the $(\mathrm{Bar}\dashv\mathrm{Cobar})$-adjunction gives the second natural transformation. □

5.3.2.7. Remark. — In the situation of Proposition 5.3.2.6 we don't have an equivalence of monads between $\mathrm{T}_{\mathrm{Cobar}(\mathcal{L})}$ and $\mathrm{prim}_{\mathcal{L}}\circ\mathrm{triv}_{\mathcal{L}}$, in contrary to Proposition 5.3.2.4. This is due to the fact that the symmetric monoidal product of $\mathcal{C}$ does not commute with totalisations of cosimplicial objects in general. One may consider just adding this extra assumption to $\mathcal{C}$. However, it is also not often possible to find concrete examples of presentable symmetric monoidal ∞-category whose symmetric monoidal product commutes with both geometric realisations *and* totalisations.

5.3.2.8. Proposition **(Francis–Gaitsgory).** — *Let $\mathcal{O}$ be an augmented ∞-operad with values in $\mathcal{C}$ and let $f\colon \mathrm{Bar}(\mathcal{O})\to\mathcal{L}$ be a morphism of coaugmented ∞-cooperads. Denote the morphism adjoint to f under the* $(\mathrm{Bar}\dashv\mathrm{Cobar})$*-adjunction by $g\colon\mathcal{O}\to\mathrm{Cobar}(\mathcal{L})$ (see (5.3.2.3)). Then the functor*

$$\mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C})\xrightarrow{\mathrm{Bar}_{\mathcal{O}}}\mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{ndp}}_{\mathrm{Bar}(\mathcal{O})}(\mathcal{C})\xrightarrow{f_*}\mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{ndp}}_{\mathcal{L}}(\mathcal{C})$$

is left adjoint to

$$\mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{ndp}}_{\mathcal{L}}(\mathcal{C})\xrightarrow{\mathrm{Cobar}_{\mathcal{L}}}\mathcal{A}\mathrm{lg}_{\mathrm{Cobar}(\mathcal{L})}(\mathcal{C})\xrightarrow{g^*}\mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C}).$$

Proof. — Let L denote the left adjoint to $\mathrm{Cobar}_{\mathcal{L}}$, which exists by the Adjoint Functor Theorem (see [HTT, Corollary 5.5.2.9]). Recall that the left adjoint to g^* is denoted by $g_!$ (see Proposition 5.2.4.16).

We show that the composition $L\circ g_!$ is equivalent to the functor $f_*\circ\mathrm{Bar}_{\mathcal{O}}$. Recall from the proof of Proposition 5.2.4.18 that every $\mathcal{O}$-algebra X is equivalent to a colimit of free $\mathcal{O}$-algebras. Thus it suffices to show that the functors $L\circ g_!$ and $f_*\circ\mathrm{Bar}_{\mathcal{O}}$ agree on their evaluations on free $\mathcal{O}$-algebras, since both of them preserve small colimits.

Propositions 5.3.2.4 and 5.3.2.6 and Examples 5.2.4.21 and 5.3.1.15 provide the following equivalences

$$L\circ(g_!\circ\mathrm{free}_{\mathcal{O}})\simeq L\circ\mathrm{free}_{\mathrm{Cobar}(\mathcal{L})}\simeq\mathrm{triv}_{\mathcal{L}}\;\text{ and}$$
$$f_*\circ(\mathrm{Bar}_{\mathcal{O}}\circ\mathrm{free}_{\mathcal{O}})\simeq f_*\circ\mathrm{triv}_{\mathrm{Bar}(\mathcal{O})}\simeq\mathrm{triv}_{\mathcal{L}},$$

of functors, which concludes the proof. See also [FG12, Corollary 3.3.13]. □

5.3.2.9. Example. — Let $\mathcal{O}$ be an augmented ∞-operad with values in $\mathcal{C}$. Then identity morphism $\mathrm{id}\colon \mathrm{Bar}(\mathcal{O}) \to \mathrm{Bar}(\mathcal{O})$ and its adjoint morphism $\delta\colon \mathcal{O} \to \mathrm{Cobar}(\mathrm{Bar}(\mathcal{O}))$ under the $(\mathrm{Bar} \dashv \mathrm{Cobar})$-adjunction induce an adjunction

$$\mathrm{Bar}_{\mathcal{O}}\colon \mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C}) \rightleftarrows \mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{ndp}}_{\mathrm{Bar}(\mathcal{O})}(\mathcal{C}) : \delta^* \circ \mathrm{Cobar}_{\mathrm{Bar}(\mathcal{O})}, \tag{5.3.2.2}$$

which is sometimes called the *Koszul duality* between $\mathcal{O}$-algebras and $\mathrm{Bar}(\mathcal{O})$-coalgebras. In §5.3.3 we will discuss the cases where δ^* is equivalent to the identity morphism, i.e. the right adjoint to $\mathrm{Bar}_{\mathcal{O}}$ is just given by $\mathrm{Cobar}_{\mathrm{Bar}(\mathcal{O})}$.

5.3.2.10. **Non-unital symmetric sequences.** — Let $\mathcal{F}\mathrm{in}^{\simeq}_{\geq 1}$ denotes the full ∞-subcategory of $\mathcal{F}\mathrm{in}^{\simeq}$ (see Definition 5.2.4.5) whose objects are finite sets $\underline{n}$ for natural numbers $n \geq 1$. The canonical inclusion $i\colon \mathcal{F}\mathrm{in}^{\simeq}_{\geq 1} \hookrightarrow \mathcal{F}\mathrm{in}^{\simeq}$ induces a functor

$$i^*\colon \mathcal{F}\mathrm{un}(\mathcal{F}\mathrm{in}^{\simeq}, \mathcal{C}) \to \mathcal{F}\mathrm{un}(\mathcal{F}\mathrm{in}^{\simeq}_{\geq 1}, \mathcal{C}).$$

The functor i^* admits a *two-sided adjoint* (both a left and right adjoint to i^*)

$$(-)^+\colon \mathcal{F}\mathrm{un}(\mathcal{F}\mathrm{in}^{\simeq}_{\geq 1}, \mathcal{C}) \to \mathcal{F}\mathrm{un}(\mathcal{F}\mathrm{in}^{\simeq}, \mathcal{C}),\ F \mapsto F^+$$

obtained by the left/right Kan extension, where $F^+(r) \simeq F(r)$ for every $r \geq 1$ and $F^+(0)$ is equivalent to the initial object of $\mathcal{C}$.

One can check that $(-)^+$ is fully faithful and the ∞-subcategory $\mathcal{F}\mathrm{un}(\mathcal{F}\mathrm{in}^{\simeq}_{\geq 1}, \mathcal{C})$ is closed under forming composition products. Thus $\mathcal{F}\mathrm{un}(\mathcal{F}\mathrm{in}^{\simeq}_{\geq 1}, \mathcal{C})$ becomes a monoidal ∞-subcategory of $\mathcal{F}\mathrm{un}(\mathcal{F}\mathrm{in}^{\simeq}, \mathcal{C})$ and the functor $(-)^+$ becomes a monoidal functor. The ∞-category of *non-unital symmetric sequences* is defined as

$$\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}_{\geq 1}(\mathcal{C}) \coloneqq \mathcal{F}\mathrm{un}\left(\mathcal{F}\mathrm{un}^{\simeq}_{\geq 1}, \mathcal{C}\right) \subseteq \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}).$$

By [HA, Corollary 7.3.2.7] the functor i^* is lax and oplax monoidal. Thus it induces functors on the ∞-categories of associative algebras (∞-operads) and on the ∞-categories of coassociative coalgebras (∞-cooperads), respectively.

5.3.2.11. Definition. — Let $\mathcal{O}$ be an ∞-operad with values in $\mathcal{C}$.

(i) We say $\mathcal{O}$ is *non-unital* if its underlying symmetric sequence is equivalent to an object of $\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}_{\geq 1}(\mathcal{C})$.

(ii) The *deunitalisation* $\mathcal{O}^{\mathrm{nu}}$ of $\mathcal{O}$ is defined as the non-unital ∞-operad

$$\mathcal{O}^{\mathrm{nu}} \coloneqq (i^*(\mathcal{O}))^+.$$

(iii) We say $\mathcal{O}$ is *reduced* if $\mathcal{O}$ is non-unital and $\mathcal{O}(1) \simeq \mathbb{1}_{\mathcal{C}}$.

(iv) The ∞-category of $\mathcal{O}^{\mathrm{nu}}$-algebras is denoted by $\mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{O}}(\mathcal{C})$.

In the same way we can make these definitions and notations for ∞-cooperads.

5.3.2.12. Remark. — Assume that $\mathcal{O}$ is a *unital* ∞-operad in $\mathcal{C}$, i.e. $\mathcal{O}(0) \simeq \mathbb{1}_{\mathcal{C}}$. An algebra X over $\mathcal{O}$ is equipped with a "unit map" $\mathcal{O}(0) \to X$. Informally speaking, we think about algebras over $\mathcal{O}^{\mathrm{nu}}$ as an "$\mathcal{O}$-algebra" without the unit map.

5.3.2.13. Remark. — Note that a reduced ∞-operad (respectively ∞-cooperad) is (co)augmented. For a reduced ∞-operad (respectively ∞-cooperad) $\mathcal{M}$ with values in $\mathcal{C}$, its associated monad (respectively comonad) $\mathrm{T}_{\mathcal{M}}$ is of the form

$$\mathrm{T}_{\mathcal{M}}(X) \simeq \coprod_{r \geq 1} \left(\mathcal{M}(r) \otimes X^{\otimes r}\right)_{\mathfrak{S}_r}.$$

Indeed, $\mathcal{M}(0) \otimes X$ is equivalent to the initial object since the symmetric monoidal product of $\mathcal{C}$ preserves small colimits in each variable. For the initial object $I \in \mathcal{C}$, the coproduct $I \coprod X$ is equivalent to X for every $X \in \mathcal{C}$.

5.3.3. Koszul duality. — In this subsection we discuss Koszul duality between ∞-operads and ∞-cooperads with values in a stable presentable symmetric monoidal ∞-category.

5.3.3.1. Definition. — An ∞-category $\mathcal{C}^{\otimes}$ together with a functor $q\colon \mathcal{C}^{\otimes} \to \mathcal{C}\mathrm{om}^{\otimes}$ is a *stable symmetric monoidal* ∞-category if

(i) $q\colon \mathcal{C}^{\otimes} \to \mathcal{C}\mathrm{om}^{\otimes}$ is a symmetric monoidal ∞-category,
(ii) $\mathcal{C}^{\otimes}_{\langle m \rangle}$ is a stable ∞-category for $m = 1$ (and thus for all $m \geq 1$), and
(iii) the symmetric monoidal product of $\mathcal{C}$ is exact in each variable.

We say $\mathcal{C}$ is the *underlying stable symmetric monoidal ∞-category.* Furthermore, we call $q\colon \mathcal{C}^{\otimes} \to \mathcal{C}\mathrm{om}^{\otimes}$ is *presentable stable symmetric monoidal* if it satisfies in addition to (i)-(iii) the hypotheses in Definition 5.2.4.2. Again by abuse of notation we conflate $\mathcal{C}^{\otimes} \to \mathcal{C}\mathrm{om}^{\otimes}$ and $\mathcal{C}$ whenever convenient.

5.3.3.2. Situation. — For the rest of this section we work with a presentable *stable* symmetric monoidal ∞-category $\mathcal{C}$.

***5.3.3.3. Theorem* (Heuts).** — *In Situation 5.3.3.2 the* (Bar ⊣ Cobar)*-adjunction in Construction 5.3.2.1 restricts to an equivalence between the ∞-categories of reduced ∞-operads and the ∞-categories of reduced ∞cooperads with values in $\mathcal{C}$.*

5.3.3.4. Remark. — We include Theorem 5.3.3.3 for the completeness of our exposition on ∞-categorical operadic Koszul duality. Similar results in the ordinary category of differential graded chain complexes over a commutative ring and in certain model category of spectra can be found in [Fre04] and [Chi05], respectively. We learnt about Theorem 5.3.3.3 from communications with Gijs Heuts and we read the proof in a draft of the manuscript. For a report of the work [Heu], see [Heu20a].

We use this theorem for Proposition 5.3.3.6, and the latter is not necessary for later applications (because the existence of such an adjunction as in Proposition 5.3.3.6 is already given by Proposition 5.3.2.8).

5.3.3.5. Definition. — Let $\mathcal{O}$ be a reduced ∞-operad and $\mathcal{L}$ be a reduced ∞-cooperad with values in $\mathcal{C}$. We call $\mathrm{K}(\mathcal{O}) \coloneqq \mathrm{Bar}(\mathcal{O})$ the *Koszul dual ∞-cooperad* of $\mathcal{O}$ and $\mathrm{K}(\mathcal{L}) \coloneqq \mathrm{Cobar}(\mathcal{L})$ the *Koszul dual ∞-operad* of $\mathcal{L}$.

5.3.3.6. Proposition. — *Let $\mathcal{O}$ be a reduced ∞-operad with values in $\mathcal{C}$. The* $\mathrm{Bar}_{\mathcal{O}}$ *and* $\mathrm{Cobar}_{\mathrm{K}(\mathcal{O})}$ *functor defined in Propositions 5.3.2.4 and 5.3.2.6, respectively, form an adjunction*

$$\mathrm{Bar}_{\mathcal{O}} \colon \mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C}) \rightleftarrows \mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{ndp}}_{\mathrm{K}(\mathcal{O})}(\mathcal{C}) : \mathrm{Cobar}_{\mathrm{K}(\mathcal{O})}\,. \tag{5.3.3.1}$$

Proof. — This is due to Proposition 5.3.2.8 and Theorem 5.3.3.3. □

***5.3.3.7.* Spectral ∞-operads.** — A *spectral ∞-operad* (respectively ∞-cooperad) is an ∞-operad (respectively ∞-cooperad) with values in the presentable stable symmetric monoidal ∞-category $\mathcal{S}\mathrm{p}$ of spectra (see Example 5.2.1.31).

The ∞-category $\mathcal{S}\mathrm{p}$ is the presentable stable ∞-category freely generated by the sphere spectrum $\mathbb{S}$, see [HA, Corollary 1.4.4.6]. Thus, for every presentable stable symmetric monoidal ∞-category $\mathcal{C}$, there is a symmetric monoidal functor $F \colon \mathcal{S}\mathrm{p} \to \mathcal{C}$ in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$, unique up to contractible choice; F is determined by its evaluation $F(\mathbb{S}) \simeq \mathbb{1}_{\mathcal{C}}$. By Proposition 5.2.5.1 we obtain induced functors

$$F \colon \mathcal{O}\mathrm{pd}(\mathcal{S}\mathrm{p}) \to \mathcal{O}\mathrm{pd}(\mathcal{C}) \text{ and}$$
$$F \colon \mathrm{co}\mathcal{O}\mathrm{pd}(\mathcal{S}\mathrm{p}) \to \mathrm{co}\mathcal{O}\mathrm{pd}(\mathcal{C}).$$

Let $\mathcal{O}$ be a spectral ∞-operad and let $\mathcal{L}$ be a spectral ∞-cooperad. The ∞-category $\mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C})$ of *$\mathcal{O}$-algebras in $\mathcal{C}$* is defined as the ∞-category $\mathcal{A}\mathrm{lg}_{F(\mathcal{O})}(\mathcal{C})$. Similarly, the ∞-category $\mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{ndp}}_{\mathcal{L}}(\mathcal{C})$ of *conilpotent divided power $\mathcal{L}$-coalgebras in $\mathcal{C}$* is defined as the ∞-category $\mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{ndp}}_{F(\mathcal{O})}(\mathcal{C})$. Assuming in addition that $\mathcal{O}$ is augmented, we obtain an equivalence $(F \circ \mathrm{Bar})(\mathcal{O}) \simeq (\mathrm{Bar} \circ F)(\mathcal{O})$ of spectral ∞-cooperads, since both functors preserve small colimits. Thus in this situation the adjunction (5.3.2.2) is still valid.

5.3.4. Examples of operadic Koszul duality. — We present in this subsection several important examples of operadic Koszul duality. For this purpose we need to introduce suspensions of spectral ∞-operads. Recall that we work with a stable presentable symmetric monoidal ∞-category $\mathcal{C}$.

5.3.4.1. Construction **(Suspension an ∞-operad).** — For the analogous ordinary categorical construction for operads, see [GJ94; AK14; CS22]. We learnt the ∞-categorical construction from [Hei18], which we explain now.

The suspension functor $\Sigma_{\mathcal{C}}$ on $\mathcal{C}$ induces the suspension functor $\Sigma^{\mathfrak{S}}_{\mathcal{C}}$ on $\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C})$, given on objects by applying $\Sigma_{\mathcal{C}}$ pointwise, i.e. $\Sigma^{\mathfrak{S}}_{\mathcal{C}}(F) \simeq \Sigma_{\mathcal{C}} \circ F$ for a symmetric sequence F. For simplicity, we make the following abbreviation in this construction:

$$\Sigma \coloneqq \Sigma_{\mathcal{C}} \quad \text{and} \quad \Sigma^{\mathfrak{S}} \coloneqq \Sigma^{\mathfrak{S}}_{\mathcal{C}} \quad \text{and} \quad \mathbb{1}^{\mathfrak{S}} \coloneqq \mathbb{1}^{\mathfrak{S}}_{\mathcal{C}}$$

By calculation with the explicit formula (5.2.4.3) of the composition product we obtain

$$\Sigma^{\mathfrak{S}}(\mathbb{1}^{\mathfrak{S}}) \circledcirc (\Sigma^{\mathfrak{S}})^{-1}(\mathbb{1}^{\mathfrak{S}}) \simeq (\Sigma^{\mathfrak{S}})^{-1}(\mathbb{1}^{\mathfrak{S}}) \circledcirc \Sigma^{\mathfrak{S}}(\mathbb{1}^{\mathfrak{S}}) \simeq \mathbb{1}^{\mathfrak{S}}.$$

Under the monoidal equivalence (5.2.4.2)

$$\mathcal{F}\mathrm{un}^{\otimes}_{\mathcal{P}\mathrm{r}^{\mathrm{L}},\mathcal{C}}(\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}), \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C})) \xrightarrow{\mathrm{ev}} \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}),$$

the symmetric sequence $\Sigma^{\mathfrak{S}}(\mathbb{1}^{\mathfrak{S}})$ corresponds to an equivalence s_1 in the functor ∞-category $\mathcal{F}\mathrm{un}^{\otimes}_{\mathcal{P}\mathrm{r}^{\mathrm{L}},\mathcal{C}}(\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}), \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}))$, and an inverse equivalence s_1^{-1} is given by the symmetric sequence $(\Sigma^{\mathfrak{S}})^{-1}(\mathbb{1}^{\mathfrak{S}})$. By conjugation s_1 induces a *monoidal auto-equivalence* of $\mathcal{F}\mathrm{un}^{\otimes}_{\mathcal{P}\mathrm{r}^{\mathrm{L}},\mathcal{C}}(\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}), \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}))$ and thus of $\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C})$, illustrated by the following commutative diagram

$$\begin{array}{ccc}
\mathcal{F}\mathrm{un}^{\otimes}_{\mathcal{P}\mathrm{r}^{\mathrm{L}},\mathcal{C}}(\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}), \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C})) & \xrightarrow[\mathrm{ev}]{\sim} & \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}) \\
{\scriptstyle s_1\circ(-)\circ s_1^{-1}}\downarrow & & \downarrow{\scriptstyle (\Sigma^{\mathfrak{S}})^{-1}(\mathbb{1}^{\mathfrak{S}})\circledcirc(-)\circledcirc\Sigma^{\mathfrak{S}}(\mathbb{1}^{\mathfrak{S}})} \\
\mathcal{F}\mathrm{un}^{\otimes}_{\mathcal{P}\mathrm{r}^{\mathrm{L}},\mathcal{C}}(\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}), \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C})) & \xrightarrow[\mathrm{ev}]{\sim} & \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}).
\end{array}$$

of ∞-categories.

Recall the monoidal functor $\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}) \to \mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{C})$ from Proposition 5.2.4.10, which sends $\Sigma^{\mathfrak{S}}(\mathbb{1}^{\mathfrak{S}})$ to the suspension functor $\Sigma \colon \mathcal{C} \to \mathcal{C}$. A functor $F \in \mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{C})$ is *reduced* if it preserves the zero object of $\mathcal{C}$. Denote the full ∞-subcategory of reduced functors by $\mathcal{F}\mathrm{un}^{\mathrm{red}}(\mathcal{C}, \mathcal{C})$. Restricting to non-unital symmetric sequences (see ¶5.3.2.10) we obtain the following commutative diagram

$$\begin{array}{ccc}
\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}))_{\geq 1} & \longrightarrow & \mathcal{F}\mathrm{un}^{\mathrm{red}}(\mathcal{C}, \mathcal{C}) \\
{\scriptstyle (\Sigma^{\mathfrak{S}})^{-1}(\mathbb{1}^{\mathfrak{S}})\circledcirc(-)\circledcirc\Sigma^{\mathfrak{S}}(\mathbb{1}^{\mathfrak{S}})}\downarrow & & \downarrow{\scriptstyle \Sigma^{-1}\circ(-)\circ\Sigma} \\
\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}))_{\geq 1} & \longrightarrow & \mathcal{F}\mathrm{un}^{\mathrm{red}}(\mathcal{C}, \mathcal{C}).
\end{array} \tag{5.3.4.1}$$

of ∞-categories.

5.3.4.2. Definition. — Let F be a symmetric sequence in $\mathcal{C}$. Define the *operadic suspension* ΣF and *operadic desuspension* $\Sigma^{-1}F$ of F as the symmetric sequences

$$\Sigma F \coloneqq (\Sigma^{\mathfrak{S}}_{\mathcal{C}})^{-1}(\mathbb{1}^{\mathfrak{S}}) \circledcirc F \circledcirc (\Sigma^{\mathfrak{S}}_{\mathcal{C}})(\mathbb{1}^{\mathfrak{S}})$$
$$\Sigma^{-1} F \coloneqq (\Sigma^{\mathfrak{S}}_{\mathcal{C}})(\mathbb{1}^{\mathfrak{S}}) \circledcirc F \circledcirc (\Sigma^{\mathfrak{S}}_{\mathcal{C}})^{-1}(\mathbb{1}^{\mathfrak{S}})$$

Similarly, the *functional suspension* $\Sigma\widetilde{F}$ and *functional desuspension* $\Sigma^{-1}\widetilde{F}$ of a functor $\widetilde{F} \in \mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{C})$ is defined as

$$\Sigma\widetilde{F} \coloneqq \Sigma_{\mathcal{C}}^{-1} \circ \widetilde{F} \circ \Sigma_{\mathcal{C}}$$
$$\Sigma^{-1}\widetilde{F} \coloneqq \Sigma_{\mathcal{C}} \circ \widetilde{F} \circ \Sigma_{\mathcal{C}}^{-1}$$

5.3.4.3. Corollary. — *The operadic suspension functors in Definition 5.3.4.2 induces to an auto-equivalence on the ∞-category of ∞-operads (respectively ∞-cooperads) with values in $\mathcal{C}$.*

Proof. — By Construction 5.3.4.1 the operation Σ on $\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C})$ is a monoidal auto-equivalence, with monoidal inverse given by Σ^{-1}. □

5.3.4.4. Remark. — In the situation of Definition 5.3.4.2, explicit calculations of the composition products gives an equivalence

$$(\Sigma^z F)(r) \simeq \Sigma_{\mathcal{C}}^{zr-z} F(r)$$

in $\mathcal{C}$, for every integer z and every natural number $r \geq 0$.

5.3.4.5. Proposition. — *Let $\mathcal{O}$ be an non-unital ∞-operad with values in $\mathcal{C}$. There exists an equivalence between the ∞-categories $\mathcal{A}\mathrm{lg}_{\Sigma\mathcal{O}}(\mathcal{C})$ and $\mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C})$ given by taking the suspensions of the underlying objects, illustrated by the following commutative diagram of ∞-categories*

$$\begin{array}{ccc}
\mathcal{A}\mathrm{lg}_{\Sigma\mathcal{O}}(\mathcal{C}) & \xrightarrow{\sim} & \mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C}) \\
{\scriptstyle \mathrm{forg}_{\Sigma\mathcal{O}}}\downarrow & & \downarrow{\scriptstyle \mathrm{forg}_{\mathcal{O}}} \\
\mathcal{C} & \xrightarrow[\Sigma_{\mathcal{C}}]{\sim} & \mathcal{C}.
\end{array}$$

Proof. — Recall the $\mathcal{LM}$-monoidal ∞-category $q\colon \mathcal{M}(\mathcal{C})^{\otimes} \to \mathcal{LM}^{\otimes}$ exhibiting $\mathcal{C}$ as left-tensored over $\mathcal{F}\mathrm{un}(\mathcal{C},\mathcal{C})$ (see Example 5.2.2.12). The functor q corresponds to a functor $\mathcal{LM}^{\otimes} \to \mathcal{CAT}_{\infty}$ of ∞-categories by straightening. The functional suspension Σ of $\mathcal{F}\mathrm{un}(\mathcal{C},\mathcal{C})$ and the suspension $\Sigma_{\mathcal{C}}$ of $\mathcal{C}$ fit in the commutative diagram

$$\begin{array}{ccc}
\mathcal{F}\mathrm{un}(\mathcal{C},\mathcal{C}) \times \mathcal{C} & \xrightarrow{\mathrm{ev}} & \mathcal{C} \\
{\scriptstyle (\Sigma^{-1}(-))\times\Sigma_{\mathcal{C}}}\downarrow & & \downarrow{\scriptstyle \Sigma_{\mathcal{C}}} \\
\mathcal{F}\mathrm{un}(\mathcal{C},\mathcal{C}) \times \mathcal{C} & \xrightarrow{\mathrm{ev}} & \mathcal{C},
\end{array}$$

which induces a natural transformation of the functor $\mathcal{LM}^{\otimes} \to \mathcal{CAT}_{\infty}$ to itself. Recall that $\Sigma^{-1}(-)$ denotes the functional suspension of functors. Thus, we obtain a $\mathcal{LM}$-monoidal functor $\phi\colon \mathcal{M}(\mathcal{C})^{\otimes} \to \mathcal{M}(\mathcal{C})^{\otimes}$, illustrated by the following commutative diagram of ∞-categories

$$\begin{array}{ccc}
\mathcal{M}(\mathcal{C})^{\otimes} & \xrightarrow[\phi]{\sim} & \mathcal{M}(\mathcal{C})^{\otimes} \\
& {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle q} & \\
& \mathcal{LM}^{\otimes}\,. &
\end{array}$$

Therefore, it induces the following commutative diagram

$$\begin{array}{ccccccc} \mathcal{L}\mathrm{Mod}(\mathcal{M}(\mathcal{C})) & \xrightarrow[\phi_*]{\sim} & \mathcal{L}\mathrm{Mod}(\mathcal{M}(\mathcal{C})) & \quad & (\Sigma F, M) & \longmapsto & (F, \Sigma_{\mathcal{C}}(M)) \\ \downarrow{\scriptstyle \mathrm{forg}_m} & & \downarrow{\scriptstyle \mathrm{forg}_m} & & \downarrow & & \downarrow \\ \mathcal{A}\mathrm{lg}_{/\mathcal{A}\mathrm{ss}}(\mathcal{C}) & \xrightarrow[\phi_*]{\sim} & \mathcal{A}\mathrm{lg}_{/\mathcal{A}\mathrm{ss}}(\mathcal{C}), & & \Sigma F & \longmapsto & F, \end{array} \tag{5.3.4.2}$$

where we illustrated the assignments on the underlying objects on the right side.

By Definition 5.3.4.2 and the diagram (5.3.4.1) there exits a natural equivalence $\mathrm{T}_{\Sigma\mathcal{O}} \simeq \Sigma\mathrm{T}_{\mathcal{O}}$ between the associated monads of the ∞-operads $\Sigma\mathcal{O}$ and $\mathcal{O}$ (Here we use the non-unital assumption on the ∞-operads). Recall that we have

$$\mathcal{A}\mathrm{lg}_{\Sigma\mathcal{O}}(\mathcal{C}) \simeq \mathcal{L}\mathrm{Mod}(\mathcal{M}(\mathcal{C})) \times_{\mathcal{A}\mathrm{lg}_{/\mathcal{A}\mathrm{ss}}(\mathcal{C})} \{\Sigma\mathrm{T}_{\mathcal{O}}\}, \text{ and}$$
$$\mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C}) \simeq \mathcal{L}\mathrm{Mod}(\mathcal{M}(\mathcal{C})) \times_{\mathcal{A}\mathrm{lg}_{/\mathcal{A}\mathrm{ss}}(\mathcal{C})} \{\mathrm{T}_{\mathcal{O}}\},$$

see Definitions 5.2.2.10 and 5.2.4.13. Then the commutative diagram (5.3.4.2) induces an equivalence

$$\phi_* \colon \mathcal{A}\mathrm{lg}_{\Sigma\mathcal{O}}(\mathcal{C}) \xrightarrow{\sim} \mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C})$$

of the fibres ∞-categories of the vertical arrows. Finally, by the definition of the forgetful functors (see Definition 5.2.2.10) we obtain the following commutative diagram

$$\begin{array}{ccc} \mathcal{A}\mathrm{lg}_{\Sigma\mathcal{O}}(\mathcal{C}) & \xrightarrow[\phi_*]{\sim} & \mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C}) \\ \downarrow{\scriptstyle \mathrm{forg}_{\Sigma\mathcal{O}}} & & \downarrow{\scriptstyle \mathrm{forg}_{\mathcal{O}}} \\ \mathcal{C} & \xrightarrow[\Sigma_{\mathcal{C}}]{\sim} & \mathcal{C}. \end{array}$$

of ∞-categories. □

5.3.4.6. Remark. — We use the convention that the notation ΣF, where F is a symmetric sequence, *always* refers to the operadic suspension (see Definition 5.3.4.2).

***5.3.4.7.* Spectral operads.** — The examples of spectral ∞-operads and operadic Koszul duality that we will explain next are obtained from results carried out in the context of model categories. Therefore, let us say a few words about the comparison between model categorical operads and ∞-operads. We learnt this from [Bra17, §4, Appendix D].

Let **Sp** denote the cofibrantly generated model category of S-modules [EKMM]. Its underlying ∞-category is equivalent to the ∞-category $\mathcal{S}\mathrm{p}$ of spectra. The ordinary categorical Day convolution endows the category $\mathbf{SymSeq}(\mathbf{Sp}) := \mathbf{Fun}(\mathbf{Fin}^{\cong}, \mathbf{Sp})$ of symmetric sequences in **Sp** a symmetric monoidal model structure, which is induced from that of **Sp**. Denote the subcategory of fibrant and cofibrant objects of $\mathbf{SymSeq}(\mathbf{Sp})$ (with the just-defined model structure) by $\mathbf{SymSeq}(\mathbf{Sp})^{\circ}$, and the set of weak equivalences in $\mathbf{SymSeq}(\mathbf{Sp})$ by W. Then the underlying ∞-category $\mathrm{N}(\mathbf{SymSeq}(\mathbf{Sp})^{\circ})[W^{-1}]$, see [HA, Definition 1.3.4.15], is equivalent to the ∞-category $\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{S}\mathrm{p})$.

One can define the a strict composition product between two symmetric sequences in $\mathbf{Sp}$. This induces a binary operation on the underlying ∞-category $\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{S}\mathrm{p})$, which is equivalent to the composition product $\circledcirc$ defined in Construction 5.2.4.8. With this strict composition product and the symmetric sequence $\underline{\mathbb{S}}$, the category $\mathbf{SymSeq}(\mathbf{Sp})$ becomes a monoidal category. Furthermore, there exists the following functor of monoidal ∞-categories:

$$\Phi\colon\ \mathrm{N}(\mathbf{SymSeq}(\mathbf{Sp})^{\circ})[W^{-1}] \to \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{S}\mathrm{p})$$

An operad (respectively cooperad) with values in $\mathbf{Sp}$ is defined as an associative algebra (respectively coassociative coalgebra) in the monoidal category $\mathbf{SymSeq}(\mathbf{Sp})$. Most of the definitions we give for ∞-operads/∞-cooperads can be adapted to operads/cooperads with values in $\mathbf{SymSeq}(\mathbf{Sp})$, such as being augmented, reduced and non-unital (these are defined on the underlying symmetric sequences). The category $\mathbf{Opd}^{\mathrm{red}}(\mathbf{Sp})$ of reduced operads with values in $\mathbf{Sp}$ admits a cofibrantly-generated simplicial model category structure, where weak equivalences are arity-wise weak equivalences of spectra. The functor Φ induces a functor

$$\Phi\colon\ \mathcal{N}(\mathbf{Opd}^{\mathrm{red}}(\mathbf{Sp})) \to \mathcal{O}\mathrm{pd}^{\mathrm{red}}(\mathcal{S}\mathrm{p}),$$

of ∞-categories, where $\mathcal{N}$ denotes the simplicial nerve (see Example 1.1.1.1) and $\mathcal{O}\mathrm{pd}^{\mathrm{red}}(\mathcal{S}\mathrm{p})$ denotes the ∞-category of reduced ∞-operads with values in $\mathcal{S}\mathrm{p}$. See [Bra17, p.81] for more details.

Using this functor, one can construct many examples of spectral ∞-operads from operad with values in (the model category) of spectra. In [Chi05] Ching defines the (Bar $\dashv$ Cobar)-adjunction between reduced operads and reduced cooperads with values in $\mathbf{Sp}$. Under the functor Φ it is shown that Ching's (Bar $\dashv$ Cobar)-adjunction is compatible with our ∞-categorical (Bar $\dashv$ Cobar)-adjunction (see Construction 5.3.2.1), see [Bra17, Lemma 5.4.14, Proposition 5.4.19].

Let $\mathbf{L}$ be a reduced cooperad with values in $\mathbf{Sp}$. It is shown in [Chi05, Lemma 6.1] that one can define a reduced operad $\mathbf{L}^{\vee}$ with values in $\mathbf{Sp}$ by taking the arity-wise Spanier–Whitehead dual, i.e.

$$\mathbf{L}^{\vee}(r) := \mathbb{M}\mathrm{ap}(\mathbf{L}(r), \mathbb{S})$$

for every $r \geq 1$, where $\mathbb{M}\mathrm{ap}$ denotes the mapping spectrum. Let $\mathbf{O}$ be a reduced operad with values in $\mathbf{Sp}$. Denote the reduced cooperad obtained from $\mathbf{O}$ via the (ordinary categorical) Bar construction by $\mathrm{Bar}(\mathbf{O})$. Denote

$$\mathcal{O} := \Phi(\mathbf{O}) \in \mathcal{O}\mathrm{pd}^{\mathrm{red}}(\mathcal{S}\mathrm{p}).$$

Define the symmetric sequence $\underline{\mathrm{Bar}(\mathcal{O})}^\vee \in \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{S}\mathrm{p})$ by applying arity-wise the Spanier–Whitehead dual functor to the underlying symmetric sequence $\underline{\mathrm{Bar}(\mathcal{O})}$ of the ∞-cooperad $\mathrm{Bar}(\mathcal{O})$. One can check by the explicit construction of Φ that $\underline{\mathrm{Bar}(\mathcal{O})}^\vee$ is equivalent to the underlying symmetric sequence of $\Phi(\mathrm{Bar}(\mathbf{O})^\vee)$, which in turn equips the symmetric sequence $\underline{\mathrm{Bar}(\mathcal{O})}^\vee$ the structure of an ∞-operad with values in $\mathcal{S}\mathrm{p}$. Thus we denote

$$\mathrm{Bar}(\mathcal{O})^\vee := \Phi(\mathrm{Bar}(\mathbf{O})^\vee) \in \mathcal{O}\mathrm{pd}(\mathcal{S}\mathrm{p}).$$

5.3.4.8. Definition. — Given two operads $\mathbf{O}$ and $\mathbf{P}$ with values in $\mathbf{Sp}$, we say $\mathbf{O}$ and $\mathbf{P}$ are *Koszul dual* if there exists an (model categorical) equivalence

$$\mathrm{Bar}(\mathbf{O})^\vee \simeq \mathbf{P}\,.$$

Denote $\mathcal{P} := \Phi(\mathbf{P})$. We say that the ∞-operads $\mathcal{O}$ and $\mathcal{P}$ are *Koszul dual* if

$$\mathrm{Bar}(\mathcal{O})^\vee \simeq \mathcal{P}.$$

5.3.4.9. Example. — Consider the non-unital commutative operad $\mathbf{Com}^{\mathrm{nu}}$ with values in $\mathbf{Sp}$, given by $\mathbf{Com}^{\mathrm{nu}}(r) := \mathbb{S}$ for every $r \geq 1$. In particular, we obtain an equivalence

$$\mathcal{C}\mathrm{om}^{\mathrm{nu}} \simeq \Phi(\mathbf{Com}^{\mathrm{nu}})$$

of ∞-operads with values in $\mathcal{S}\mathrm{p}$, see Example 5.2.5.11.

5.3.4.10. Example. — Recall the Lie operad $\mathbf{Lie}$ from Example 5.1.0.19. In this example we construct the (shifted) *spectral Lie ∞-operad* $\mathcal{L}\mathrm{ie}$ such that

$$\mathbf{Lie}(r) \cong \widetilde{\mathrm{H}}_0((\Sigma\mathcal{L}\mathrm{ie})(r);\mathbb{Z})$$

in every arity $r \in \mathbb{N}$.

For the set $\underline{n} = \{1, 2, \ldots, n\}$, we can define the poset $\widehat{\Pi}_n$ of partitions of $\underline{n}$: An element λ of $\widehat{\Pi}_n$ is an equivalence relation on $\underline{n}$. Two equivalence relations $\lambda \leq \lambda'$ if λ is finer than λ', i.e. if $x \sim_{\lambda'} y$ then $x \sim_\lambda y$. The minimal equivalence relation $\widehat{0}$ is given by $x \sim y$ if $x = y$, and the maximal equivalence relation $\widehat{1}$ is given by $x \sim y$ for every pari (x, y) of elements in $\underline{n}$. We define the subset

$$\Pi_n := \widehat{\Pi}_n \setminus \left\{\widehat{0}, \widehat{1}\right\} \subseteqq \widehat{\Pi}_n.$$

It inherits the poset structure from $\widehat{\Pi}_n$. We consider the poset Π_n as a category. It is shown in [AB21, §4.6] that there exists an isomorphism of $\mathfrak{S}_r$-representations

$$\begin{aligned} \mathbf{Lie}(n) &\cong \mathrm{sgn}_{\mathfrak{S}_r} \otimes \widetilde{\mathrm{H}}^{r-1}\left(\Sigma(|\Pi_r|^\diamond)\,;\mathbb{Z}\right) \\ &\cong \widetilde{\mathrm{H}}_0\left(\mathrm{Map}\left(\mathbb{S}^1, (\mathbb{S}^1)^r\right) \otimes_{\mathcal{S}\mathrm{p}} \left(\Sigma(|\Pi_r|^\diamond)\right)^\vee;\mathbb{Z}\right). \end{aligned} \tag{5.3.4.3}$$

In the above isomorphisms,

(i) $\mathrm{sgn}_{\mathfrak{S}_r}$ denotes the sign representation of $\mathfrak{S}_r$,

(ii) $\otimes$ in the first isomorphism denotes the tensor product of $\mathfrak{S}_r$-representations,
(iii) $(-)^\diamond$ denotes the unreduced suspension of a homotopy type,[9]
(iv) $\mathbb{S}^1 := \Sigma_{\mathcal{S}\mathrm{p}}\mathbb{S}$, and
(v) $(-)^\vee$ denotes the Spanier–Whitehead dual.

It is shown in [AB21] that the symmetric sequence $\mathbf{sLie} = \big((\Sigma(|\Pi_r|^\diamond))^\vee\big)_{r\geq 1}$ forms an operad with values in $\mathbf{Sp}$.

The *spectral Lie* ∞*-operad* $\mathcal{L}\mathrm{ie}$ is defined as $\Phi(\mathbf{sLie})$. In particular, for every $r \geq 1$, we have $\mathcal{L}\mathrm{ie}(r) := (\Sigma(|\Pi_r|^\diamond))^\vee$ and $\mathcal{L}\mathrm{ie}(0) = 0$ (the zero spectrum). By Remark 5.3.4.4 and (5.3.4.3) we obtain the isomorphism $\mathbf{Lie}(r) \cong \widetilde{\mathrm{H}}_0\left(\Sigma\mathcal{L}\mathrm{ie}(r)\right).$ of abelian groups for every $r \geq 0$ by calculation.

5.3.4.11. Definition. — A *spectral Lie algebra* in a presentable stable symmetric monoidal ∞-category $\mathcal{C}$ is an algebra over the spectral Lie ∞-operad $\mathcal{L}\mathrm{ie}$.

5.3.4.12. Remark. — Alternatively the spectral Lie ∞-operad is defined as the derivatives, in the sense of Goodwillie calculus (see ¶6.2.2.10), of the identity functor of the ∞-category $\mathcal{H}\mathrm{o}_*$ of pointed homotopy types, see [Chi05]. Furthermore we refer the interested reader to [Cam20, Proposition 5.2] and [Kja18, §3.2] for a discussion about the Lie bracket and the Jacobi identity relation for spectral Lie algebras.

5.3.4.13. Example. — Koszul duality between Lie algebras and cocommutative coalgebras can be traced back to Quillen's work on rational homotopy theory [QuiHA]. In loc. cit. Quillen shows that the Chevalley–Eilenberg functor CE induces a Quillen equivalence between a model category of connected differential graded (dg) Lie algebras and a model category of simply connected dg-cocommutative coalgebras over the rational numbers. The functor CE is an example of the functor $\mathrm{Bar}_{\mathcal{O}}$ in (5.3.3.1). Later in [Moo71] the adjunction (5.3.3.1) between dg-Lie algebras and dg-cocommutative coalgebras over a field of characteristic other than 2 is established.

In characteristic zero situations, Ginzburg and Kapranov show that the Lie operad and the cocommutative cooperad are Koszul dual, where they use the theory of quadratic operads [GK94, Theorem 2.1.11]. This duality is generalised to the Lie operad and cocommutative cooperad over an arbitrary commutative ring in [Fre04, Fact 6.2]. In [Chi05, Corollary 8.8] Ching defines the Koszul duality between the spectral Lie operad $\mathbf{sLie}$ and the spectral cocommutative cooperad $\mathbf{Com}^{\mathrm{nu}}$.[10]

Therefore by Definition 5.3.4.8 we obtain the Koszul duality

$$\mathrm{Bar}(\mathcal{C}\mathrm{om}^{\mathrm{nu}})^\vee \simeq \mathcal{L}\mathrm{ie}$$

between the spectral Lie ∞-operad and the non-unital commutative ∞-operad.

(9) Model a homotopy types by a CW complex X, we have $X^\diamond = X \times \mathrm{I}/(X \times \{0\} \sqcup X \times \{1\})$

(10) The spectral Lie operad is defined as the linear dual of the Bar construction of the cocommutative cooperad in [Chi05]

5.3.4.14. Example. — One of the first appearances of the notion of Koszul duality for (associative) algebras was in [Pri70], where the notion of a *Koszul algebra* and the *Koszul dual coalgebra* over a field is introduced [Pri70, §3]. The (Bar $\dashv$ Cobar)-adjunction between dg-algebras and dg-coalgebras over a commutative ring appears in [Moo71]. In [GK94, 2.1.11] Ginzburg–Kapranov prove the Koszul duality of the associative operad with itself [GK94, Theorem 2.1.11], known as the *self-Koszul duality*. See [Chi05, Example 4.8] for a more recent demonstration of this example of Koszul duality.

5.3.4.15. Example. — The self-Koszul duality of the associative operad generalises to self-Koszul duality of the $\mathcal{E}_n$ ∞-operad (see Example 5.2.1.13) for every natural number $n \geq 1$. Recall the homology operad $\mathrm{H}_\bullet(\mathbf{E}_n; R)$ with coefficient in a commutative ring R from Example 5.1.0.24. In [GJ94, Theorem 3.1] Getzler and Jones show the self-Koszul duality $\mathrm{Bar}\,(\mathrm{H}_\bullet(\mathbf{E}_n; k))^\vee \simeq \Sigma^{-n}\mathrm{H}_\bullet(\mathbf{E}_n; k)$ for every natural number $n \geq 1$ over a field k of characteristic zero. This is upgraded to a self-Koszul duality for the $\mathbf{E}_n$ operad with values in the category $\mathbf{Ch}_\mathbb{Z}$ of chain complexes over the integers by Fresse [Fre11].

Recently, it is shown that the previous result also holds for spectral $\mathbf{E}_n$ operad, which is the operad with values in $\mathbf{Sp}$ obtained by applying the suspension spectrum functor arity-wise, for every natural number $n \geq 1$, see [CS22]. Translating the result in loc. cit. to ∞-categorical settings using ¶5.3.4.7, we obtain the self-Koszul duality of the non-unital spectral $\mathcal{E}_n$ ∞-operad. In other words, there exists an equivalence

$$\mathrm{Bar}\,(\mathcal{E}_n^{\mathrm{nu}})^\vee \simeq \Sigma^{-n}\mathcal{E}_n^{\mathrm{nu}}$$

of spectral ∞-operads for every natural number $n \geq 1$; here $\mathcal{E}_n$ denotes $\Sigma^\infty_+(\mathcal{E}_n)$, see Example 5.2.5.12.

CHAPTER 6

Higher enveloping algebras in monochromatic layers

6.1. Higher enveloping algebras of spectral Lie algebras

Let $\mathcal{C}$ be a presentable stable symmetric monoidal ∞-category. Recall that a commutative algebra in $\mathcal{C}$ has an underlying $\mathcal{E}_n$-algebra structure for every $n \in \mathbb{N}$, induced by the tower (5.2.5.1)

$$\mathcal{E}_0 \hookrightarrow \mathcal{E}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{E}_n \hookrightarrow \mathcal{E}_{n+1} \cdots \hookrightarrow \mathcal{C}\mathrm{om}$$

of ∞-operad inclusions. In this section we are interested in the Koszul dual picture of this. We construct the following commutative diagram in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$

$$\begin{array}{c} \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{C}) \\ \swarrow \quad \downarrow{\scriptstyle \mathrm{U}_n} \quad \searrow \quad \searrow{\scriptstyle \mathrm{U}_1} \quad \searrow \\ \cdots \longrightarrow \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_n}(\mathcal{C}) \xrightarrow[\mathrm{B}_n]{} \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_{n-1}}(\mathcal{C}) \xrightarrow[\mathrm{B}_{n-1}]{} \cdots \longrightarrow \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_1}(\mathcal{C}) \xrightarrow[\mathrm{B}_1]{} \mathcal{C}, \end{array} \tag{6.1.0.1}$$

exhibiting a relationship between spectral Lie algebras and non-unital (or augmented) $\mathcal{E}_n$-algebras in $\mathcal{C}$, originally due to [Knu18]. Although the results in this section are not original, we provide a different construction of the above commutative diagram than that in [Knu18], using the self Koszul duality of the $\mathcal{E}_n$ ∞-operads (see Example 5.3.4.15). Our construction allows more explicit presentations of the functors U_n and B_n, which are useful in later applications. In the next sections we will investigate in concrete situations whether the ∞-category $\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{C})$ is equivalent to the inverse limit, taken in the ∞-category $\mathcal{CAT}_\infty$ of ∞-categories, of the lower horizontal tower in the above diagram.

6.1.0.1. Situation. — In this section we use $\mathcal{E}_n$-algebras in various ∞-categories and not all of them are stable ∞-categories. We do not write explicit each time about in which ∞-categories the $\mathcal{E}_n$ ∞-operad takes value, because of Theorem 5.2.5.5, Example 5.2.5.12 and ¶5.3.3.7. For example, when we write the operadic suspension $\Sigma^{-n}\mathcal{E}_n$, we implicitly mean that we are considering the $\mathcal{E}_n$ ∞-operad in a stable ∞-category, as this is our situation where we defined the operadic suspension (see §5.3.4).

Let $\mathcal{C}$ be a fixed presentable stable symmetric monoidal ∞-category throughout this section. Denote the suspension functor of $\mathcal{C}$ by $\Sigma_{\mathcal{C}}$ and the loop functor of $\mathcal{C}$ by $\Omega_{\mathcal{C}}$; they are auto-equivalences of $\mathcal{C}$.

6.1.0.2. Construction. — We give a construction of the diagram (6.1.0.1) using Koszul duality. Recall that the Koszul dual of the non-unital commutative ∞-operad $\mathcal{C}\mathrm{om}^{\mathrm{nu}}$ is the spectral Lie ∞-operad $\mathcal{L}\mathrm{ie}$ (see Example 5.3.4.13), and the Koszul dual of $\mathcal{E}_n^{\mathrm{nu}}$ is the n-fold operadic desuspension $\Sigma^{-n}\mathcal{E}_n^{\mathrm{nu}}$ of itself (see Example 5.3.4.15). Thus, the tower (5.2.5.1) induces the following tower of ∞-operads

$$\mathcal{L}\mathrm{ie} \cdots \to \Sigma^{-(n+1)}\mathcal{E}_{n+1}^{\mathrm{nu}} \xrightarrow{c_n^{n+1}} \Sigma^{-n}\mathcal{E}_n^{\mathrm{nu}} \to \cdots \to \Sigma^{-1}\mathcal{E}_1^{\mathrm{nu}} \to \mathcal{E}_0^{\mathrm{nu}}, \tag{6.1.0.2}$$

by taking the Koszul dual. In particular, we have $\mathcal{L}\mathrm{ie} \simeq \varprojlim \Sigma^{-n}\mathcal{E}_n^{\mathrm{nu}}$ from the definition of Koszul duality (see Definition 5.3.3.5 and Proposition 5.3.2.3). Denote the induced morphism $c_n \colon \mathcal{L}\mathrm{ie} \to \Sigma^{-n}\mathcal{E}_n^{\mathrm{nu}}$.

For each $n \in \mathbb{N}$, the canonical morphism c_n induces a commutative diagram

$$\begin{array}{ccccc}
\mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_n}(\mathcal{C}) & \xrightarrow{\sim} & \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\Sigma^{-n}\mathcal{E}_n}(\mathcal{C}) & \xrightarrow{(c_n)^*} & \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{C}) \\
\downarrow{\scriptstyle \mathrm{forg}_{\mathcal{E}_n^{\mathrm{nu}}}} & & \downarrow{\scriptstyle \mathrm{forg}_{\Sigma^{-n}\mathcal{E}_n^{\mathrm{nu}}}} & & \downarrow{\scriptstyle \mathrm{forg}_{\mathcal{L}\mathrm{ie}}} \\
\mathcal{C} & \underset{\Sigma_{\mathcal{C}}^n}{\xrightarrow{\sim}} & \mathcal{C} & \underset{\mathrm{id}}{\xrightarrow{\sim}} & \mathcal{C}
\end{array} \tag{6.1.0.3}$$

in $\mathcal{P}\mathrm{r}^{\mathrm{R}}$, by Proposition 5.2.4.15 and Proposition 5.3.4.5. The cocontinuous functor U_n is defined as the left adjoint to the composition of the upper row in (6.1.0.3). In particular, it fits in the following commutative diagram

$$\begin{array}{ccc}
\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{C}) & \xrightarrow{\mathrm{U}_n} & \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_n}(\mathcal{C}) \\
\uparrow{\scriptstyle \mathrm{free}_{\mathcal{L}\mathrm{ie}}} & & \uparrow{\scriptstyle \mathrm{free}_{\mathcal{E}_n^{\mathrm{nu}}}} \\
\mathcal{C} & \underset{\Omega_{\mathcal{C}}^n}{\xrightarrow{\sim}} & \mathcal{C}.
\end{array} \tag{6.1.0.4}$$

in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. Similarly, for each $n \in \mathbb{N}$, the morphism

$$c_n^{n+1} \colon \Sigma^{-(n+1)}\mathcal{E}_{n+1}^{\mathrm{nu}} \to \Sigma^{-n}\mathcal{E}_n^{\mathrm{nu}}$$

induces the following commutative diagram in $\mathcal{P}\mathrm{r}^{\mathrm{R}}$

$$\begin{array}{ccccccc}
\mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_n}(\mathcal{C}) & \xrightarrow{\sim} & \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\Sigma^{-n}\mathcal{E}_n}(\mathcal{C}) & \xrightarrow{(c_n^{n+1})^*} & \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\Sigma^{-(n+1)}\mathcal{E}_{n+1}}(\mathcal{C}) & \xrightarrow{\sim} & \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_{n+1}}(\mathcal{C}) \\
\downarrow{\scriptstyle \mathrm{forg}_{\mathcal{E}_n^{\mathrm{nu}}}} & & \downarrow{\scriptstyle \mathrm{forg}} & & \downarrow{\scriptstyle \mathrm{forg}} & & \downarrow{\scriptstyle \mathrm{forg}_{\mathcal{E}_{n+1}^{\mathrm{nu}}}} \\
\mathcal{C} & \underset{\Sigma_{\mathcal{C}}^n}{\xrightarrow{\sim}} & \mathcal{C} & \underset{\mathrm{id}}{\xrightarrow{\sim}} & \mathcal{C} & \underset{\Omega_{\mathcal{C}}^{n+1}}{\xrightarrow{\sim}} & \mathcal{C}
\end{array}$$

The cocontinuous functor B_n is defined as the left adjoint to the composition of upper row in the above diagram. Thus, it also fits in the following commutative diagram

$$\begin{array}{ccc} \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_{n+1}}(\mathcal{C}) & \xrightarrow{\mathrm{B}_n} & \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_n}(\mathcal{C}) \\ {\scriptstyle \mathrm{free}^{\mathrm{nu}}_{\mathcal{E}_n}}\uparrow & & \uparrow{\scriptstyle \mathrm{free}^{\mathrm{nu}}_{\mathcal{E}_n}} \\ \mathcal{C} & \underset{\Sigma_{\mathcal{C}}}{\xrightarrow{\sim}} & \mathcal{C} \end{array} \tag{6.1.0.5}$$

in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. Assembling the functors together gives the desired commutative diagram (6.1.0.1) in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$.

6.1.0.3. **Comparison map.** — The commutative diagram (6.1.0.1) in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ induces an adjunction

$$\mathrm{U}_\infty \colon \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{C}) \rightleftarrows \varprojlim_n \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_n}(\mathcal{C}) : \mathrm{T}_\infty \,.$$

by the universal property of ∞-categorical limits. We are interested in the question of how close this adjunction comes to being an equivalence of ∞-categories, for various choices of the ∞-category $\mathcal{C}$.

6.1.0.4. Construction **(Knudsen).** — Before the work [CS22] on Koszul duality of the spectral $\mathcal{E}_n$ ∞-operad, Knudsen already constructed the functor U_n using the theory of constructible cosheaves and the theory of factorisation homology, see [Knu18]. Let us give a brief summary of the approach his approach.

It is shown that the $\mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_n}(\mathcal{C})$ is equivalent to the ∞-category of spectral Lie algebras in a symmetric monoidal ∞-category $(\mathcal{D}, \otimes_{\amalg})$ of constructible cosheaves with values in $\mathcal{C}$, and $\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{C})$ is equivalent to the symmetric monoidal ∞-category of spectral Lie algebras in the symmetric monoidal ∞-category $(\mathcal{D}, \otimes_{\cup})$. The functor U_n is induced by the identity natural transformation $(\mathcal{D}, \otimes_{\amalg}) \to (\mathcal{D}, \otimes_{\cup})$ of the underlying ∞-category $\mathcal{D}$. See [Knu18, §3.3] for more details.

6.1.0.5. Remark. — We can consider the functor U_1 as the universal enveloping algebra functor, which assigns to a free Lie algebra over a field k a free associative algebra over k. By (6.1.0.4) the functor U_1 assigns to a free spectral Lie algebra $\mathrm{free}_{\mathcal{L}\mathrm{ie}}(X)$ generated by an object $X \in \mathcal{C}$ the free non-unital associative algebra $\mathrm{free}_{\mathcal{E}_1}(\Omega_{\mathcal{C}}(X))$ generated by the object $\Omega_{\mathcal{C}}(X)$. One may expect the image $\mathrm{U}_1\,(\mathrm{free}_{\mathcal{L}\mathrm{ie}}(X))$ to be $\mathrm{free}_{\mathcal{E}_1}(X)$ instead of $\mathrm{free}_{\mathcal{E}_1}(\Omega_{\mathcal{C}}(X))$, by the property of the classical universal enveloping algebra functor. Here we have a degree shift $(\Omega_{\mathcal{C}})$ because of our choice of the definition of the spectral Lie ∞-operad $\mathcal{L}\mathrm{ie}$: The homology of $\mathcal{L}\mathrm{ie}$ is isomorphic to the *shifted* Lie operad (Example 5.3.4.10).

Because of this analogy, we call U_n the *higher enveloping algebra functor*, following the convention of [Knu18].

6.1.0.6. Question. — Both the $\mathcal{E}_n$ ∞-operad and the spectral Lie ∞-operad relate to configuration spaces of points in Euclidean spaces, see for example [Aro06, Proposition 2.1]. Is there a more geometric way to construct the morphism $\Sigma^n\mathcal{L}\mathrm{ie} \to \mathcal{E}_n$ of spectral ∞-operads?

6.1.0.7. **Augmented algebras and coalgebras.** — To give a more explicit construction of the functors U_n and B_n, we need the notion of *augmented algebras* over an ∞-operad.

A *unital* ∞-operad with values in $\mathcal{H}\mathrm{o}$ is an ∞-operad $\mathcal{O}$ such that $\mathcal{O}(0)$ is equivalent to the symmetric monoidal unit pt of $\mathcal{H}\mathrm{o}$. Let $\mathcal{D}$ be a symmetric monoidal ∞-category. Define the ∞-overcategory $\mathcal{D}_{/\mathbb{1}_\mathcal{D}}$ of objects over the symmetric monoidal unit $\mathbb{1}_\mathcal{D}$ of $\mathcal{D}$ via the following pullback diagram

$$\begin{array}{ccc} \mathcal{D}_{/\mathbb{1}_\mathcal{D}} & \longrightarrow & \Delta^0 \\ \downarrow{\scriptstyle i} & \lrcorner & \downarrow{\scriptstyle \mathbb{1}_\mathcal{D}} \\ \mathcal{F}\mathrm{un}(\Delta^1, \mathcal{D}) & \xrightarrow{\text{target}} & \mathcal{D} \end{array}$$

of ∞-categories. There exists a symmetric monoidal structure on the ∞-category $\mathcal{D}_{/\mathbb{1}_\mathcal{D}}$ such that the canonical functor

$$S\colon \mathcal{D}_{/\mathbb{1}_\mathcal{D}} \to \mathcal{F}\mathrm{un}(\Delta^1, \mathcal{D}) \xrightarrow{\text{source}} \mathcal{D}$$

is symmetric monoidal, see [HA, Theorem 2.2.2.4, Remark 2.2.2.5]. Informally speaking, given two objects $X \to \mathbb{1}_\mathcal{D}$ and $Y \to \mathbb{1}_\mathcal{D}$ of $\mathcal{D}_{/\mathbb{1}_\mathcal{D}}$, their symmetric monoidal product in $\mathcal{D}_{/\mathbb{1}_\mathcal{D}}$ is given by

$$X \otimes_\mathcal{D} Y \to \mathbb{1}_\mathcal{D} \otimes_\mathcal{D} \mathbb{1}_\mathcal{D} \xrightarrow{\sim} \mathbb{1}_\mathcal{D}.$$

The ∞-category $\mathcal{A}\mathrm{lg}^{\mathrm{aug}}_\mathcal{O}(\mathcal{D})$ of *augmented $\mathcal{O}$-algebras* in $\mathcal{D}$ is defined as the ∞-overcategory $\mathcal{A}\mathrm{lg}_\mathcal{O}(\mathcal{D})_{/\mathbb{1}_\mathcal{D}}$. Moreover, the symmetric monoidal functor $\mathcal{D}_{/\mathbb{1}_\mathcal{D}} \to \mathcal{D}$ induces a functor

$$S_\mathcal{O}\colon \mathcal{A}\mathrm{lg}_\mathcal{O}(\mathcal{D}_{/\mathbb{1}_\mathcal{D}}) \to \mathcal{A}\mathrm{lg}_\mathcal{O}(\mathcal{D}).$$

By the universal property of the ∞-overcategory $\mathcal{A}\mathrm{lg}_\mathcal{O}(\mathcal{D})_{/\mathbb{1}_\mathcal{D}}$ and the construction of the symmetric monoidal structure on $\mathcal{D}_{/\mathbb{1}_\mathcal{D}}$, the functor $S_\mathcal{O}$ induces an equivalence

$$\mathcal{A}\mathrm{lg}_\mathcal{O}(\mathcal{D}_{/\mathbb{1}_\mathcal{D}}) \to \mathcal{A}\mathrm{lg}_\mathcal{O}(\mathcal{D})_{/\mathbb{1}_\mathcal{D}},$$

see [HA, Definition 2.2.2.1 and Notation 2.2.2.3]. Whenever we denote $\mathcal{A}\mathrm{lg}_\mathcal{O}(\mathcal{D}_{/\mathbb{1}_\mathcal{D}})$ by $\mathcal{A}\mathrm{lg}^{\mathrm{aug}}_\mathcal{O}(\mathcal{C})$, we denote the forgetful functor $\mathcal{A}\mathrm{lg}_\mathcal{O}\left(\mathcal{D}_{/\mathbb{1}_\mathcal{D}}\right) \to \mathcal{D}_{/\mathbb{1}}$ by

$$\mathrm{forg}^{\mathrm{aug}}_\mathcal{O}\colon \mathcal{A}\mathrm{lg}^{\mathrm{aug}}_\mathcal{O}(\mathcal{D}) \to \mathcal{D}_{/\mathbb{1}}.$$

In a similar manner, one can define a *coaugmented coalgebra* in $\mathcal{D}$ over a *unital ∞-cooperad* $\mathcal{L}$ with values in $\mathcal{H}\mathrm{o}$ (that is $\mathcal{L}(0) \simeq \mathrm{pt}$).

6.1.0.8. Definition. — Let $\mathcal{O}$ be a unital ∞-operad with values in $\mathcal{H}o$. The *augmentation ideal* functor is defined as the following composition

$$\widetilde{\mathrm{I}}_{\mathcal{O}}\colon \mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{\mathcal{O}}(\mathcal{C}) \xrightarrow{\mathrm{forg}^{\mathrm{aug}}_{\mathcal{O}}} \mathcal{C}_{/\mathbb{1}_{\mathcal{C}}} \xrightarrow{\mathrm{fib}} \mathcal{C}$$

where fib denotes the composition $\mathcal{C}_{/\mathbb{1}_{\mathcal{C}}} \to \mathcal{F}\mathrm{un}(\Delta^1, \mathcal{C}) \xrightarrow{\mathrm{fibre}} \mathcal{C}$, assigning to an object $X \to \mathbb{1}_{\mathcal{C}}$ of $\mathcal{C}_{/\mathbb{1}_{\mathcal{C}}}$ its fibre (recall that $\mathcal{C}$ is stable).

6.1.0.9. Proposition. — *In the situation of Definition 6.1.0.8 the augmentation ideal functor admits a left adjoint $\widetilde{\mathrm{A}}_{\mathcal{O}}$, called the* trivial augmentation functor. *Furthermore, the adjunction $\widetilde{\mathrm{A}}_{\mathcal{O}} \dashv \widetilde{\mathrm{I}}_{\mathcal{O}}$ is monadic, and its associated monad is given by*

$$\widetilde{\mathrm{I}}_{\mathcal{O}} \circ \widetilde{\mathrm{A}}_{\mathcal{O}}\colon \mathcal{C} \to \mathcal{C},\ X \mapsto \coprod_{r\geq 1} \mathcal{O}(r) \otimes_{\mathfrak{S}_r} X^{\otimes r}$$

where $\otimes$ denotes the tensor product of $\mathcal{C}$.

Proof. — We abbreviate the symmetric monoidal unit $\mathbb{1}_{\mathcal{C}}$ of $\mathcal{C}$ by $\mathbb{1}$ in this proof. Recall the unit ∞-operad $\mathcal{E}_0$ (Example 5.2.1.7). The functor $\mathrm{I}_{\mathcal{O}}$ admits the following factorisation

$$\begin{array}{ccccc} \mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{\mathcal{O}}(\mathcal{C}) & \xrightarrow{\mathrm{forg}_{\mathcal{O}}} & \mathcal{C}_{/\mathbb{1}} & \xrightarrow{\mathrm{fib}} & \mathcal{C} \simeq \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_0}(\mathcal{C}) \\ & \searrow^{u^*} & \uparrow{\scriptstyle \mathrm{forg}_{\mathcal{E}_0}} & \nearrow^{\simeq}_{\mathrm{I}_0} & \\ & & \mathcal{A}\mathrm{lg}_{\mathcal{E}_0}(\mathcal{C}_{/\mathbb{1}}) \simeq \mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{\mathcal{E}_0}(\mathcal{C}). & & \end{array}$$

The functors in the diagram are defined as follows:

(i) The functor u^* is induced by the morphism $u\colon \mathcal{E}_0 \to \mathcal{O}$ of ∞-operads. The left triangle commutes because it is induced by the morphisms $\mathcal{T}\mathrm{riv} \to \mathcal{E}_n \to \mathcal{O}$ of ∞-operads.

(ii) The functor I_0 is an equivalence because $\mathcal{C}$ is stable; an inverse of I_0 is given by $X \mapsto X \oplus \mathbb{1}$.

Thus, the functor left adjoint to $\widetilde{\mathrm{I}}_{\mathcal{O}}$ is given by composing the inverse of I_0 with functor left adjoint to u^*. Since I_0 is an equivalence, it suffices to show that u^* is monadic, which follows from [HA, Proposition 4.7.3.22].

Note that the functor fib is right adjoint to the functor $\mathcal{C} \to \mathcal{C}_{/\mathbb{1}}$ sending an object X to the zero morphism $X \to \mathbb{1}$. The formula of the monad $\widetilde{\mathrm{I}}_{\mathcal{O}} \circ \widetilde{\mathrm{A}}_{\mathcal{O}}$ follows from the computation and the formula for the monad $\mathrm{forg}_{\mathcal{O}} \circ \mathrm{free}_{\mathcal{O}}$ (see Proposition 5.2.4.10). □

6.1.0.10. Corollary. — *In the situation of Proposition 6.1.0.9 there exists the following commutative diagram of ∞-categories:*

$$\begin{array}{ccc} \mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{\mathcal{O}}(\mathcal{C}) & \xrightarrow{\sim} & \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{O}}(\mathcal{C}) \\ {\scriptstyle \mathrm{forg}^{\mathrm{aug}}_{\mathcal{O}}}\downarrow & & \downarrow{\scriptstyle \mathrm{forg}_{\mathcal{O}^{\mathrm{nu}}}} \\ \mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{\mathcal{E}_0}(\mathcal{C}) & \xrightarrow[\mathrm{I}_0]{\sim} & \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_0}(\mathcal{C}). \end{array} \tag{6.1.0.6}$$

Proof. — The associated monad of the monadic adjunction in Proposition 6.1.0.9 is equivalent to the monad $\mathrm{forg}_{\mathcal{O}^{\mathrm{nu}}} \circ \mathrm{free}_{\mathcal{O}^{\mathrm{nu}}}$, see Remark 5.3.2.13. See also [HA, Proposition 7.3.4.5]. □

6.1.0.11. Notation. — In the situation of Corollary 6.1.0.10, we denote the functor left adjoint to $\mathrm{forg}^{\mathrm{aug}}_{\mathcal{O}}$ by $\mathrm{free}^{\mathrm{aug}}_{\mathcal{O}}$. The upper horizontal equivalence in (6.1.0.6) is denoted by

$$\mathrm{A}_{\mathcal{O}} \colon \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{O}}(\mathcal{C}) \overset{\sim}{\rightleftarrows} \mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{\mathcal{O}}(\mathcal{C}) : \mathrm{I}_{\mathcal{O}}$$

On the underlying objets $\mathrm{A}_{\mathcal{O}}$ is given by the inverse

$$\mathrm{A}_0 \colon \mathcal{C} \simeq \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_0}(\mathcal{C}) \to \mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{\mathcal{E}_0}(\mathcal{C}), \ X \mapsto X \oplus \mathbb{1}_{\mathcal{C}}.$$

to the functor I_0.

6.1.0.12. Remark. — Let $\mathcal{L}$ be a unital ∞-cooperad with values in $\mathcal{H}$o. In the same way, one can construct the augmentation ideal functor $\widetilde{\mathrm{I}}_{\mathcal{L}} \colon \mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{\mathcal{L}}(\mathcal{C}) \to \mathcal{C}$ functor, which is right adjoint to the trivial augmentation functor $\widetilde{\mathrm{A}}_{\mathcal{L}}$. By the same arguments this adjunction induces an equivalence

$$\mathrm{I}_{\mathcal{L}} \colon \mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{\mathcal{L}}(\mathcal{C}) \rightleftarrows \mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{L}}(\mathcal{C}) : \mathrm{A}_{\mathcal{L}}$$

of ∞-categories.

6.1.0.13. Theorem. — *Let $\mathcal{D}$ be a symmetric monoidal ∞-category admitting geometric realisations of simplicial objects and totalisations of cosimplicial objects. Then for every natural number $n \geq 1$, we obtain the following statements:*

(i) *There exists a cocontinuous functor, called the* iterated Bar construction,

$$\mathrm{Bar}_n \colon \mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{\mathcal{E}_n}(\mathcal{D}) \to \mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{\mathcal{E}_0}(\mathcal{D}).$$

(ii) *Under the equivalence*

$$\mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{\mathcal{E}_n}(\mathcal{D}) \simeq \big(\mathcal{A}\mathrm{lg}_{\mathcal{E}_n}(\mathcal{D})\big)_{/\mathbb{1}_{\mathcal{D}}} \simeq \mathcal{A}\mathrm{lg}_{\mathcal{E}_n}(\mathcal{D}_{/\mathbb{1}_{\mathcal{D}}}),$$

the functor Bar_n *is equivalent to the composition*

$$\mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{\mathcal{E}_n}(\mathcal{D}) \simeq \mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{\mathcal{E}_1}\Big(\mathcal{A}\mathrm{lg}_{\mathcal{E}_{n-1}}(\mathcal{D})\Big) \xrightarrow{\mathrm{Bar}_1} \mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{\mathcal{E}_{n-1}}(\mathcal{D}) \xrightarrow{\mathrm{Bar}_{n-1}} \mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{\mathcal{E}_0}(\mathcal{D}).$$

(iii) *The functor* Bar_n *induces an adjunction*

$$\mathrm{Bar}_n \colon \mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{\mathcal{E}_n}(\mathcal{D}) \rightleftarrows \mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{\mathcal{E}_n}(\mathcal{D}) : \mathrm{Cobar}_n\,.$$

Assume now that for every weakly contractible simplicial set K, the ∞-category $\mathcal{D}$ admits K-indexed colimits and the symmetric monoidal product of $\mathcal{D}$ preserves K-colimits in each variable. Then we have

(iv) *for every object $X \in \mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{\mathcal{E}_0}(\mathcal{D})$ an equivalence*

$$\mathrm{Bar}_n\left(\mathrm{free}^{\mathrm{aug}}_{\mathcal{E}_n}(X)\right) \simeq \Sigma^n_{\mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{\mathcal{E}_0}(\mathcal{D})}(X).$$

Assume furthermore that $\mathcal{D}$ is stable, then

(v) *we have the following commutative diagram of ∞-categories:*

$$\begin{array}{ccc} \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_n}(\mathcal{D}) & \xrightarrow[\sim]{\mathrm{A}_{\mathcal{E}_n}} & \mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{\mathcal{E}_n}(\mathcal{D}) \\ {\scriptstyle \Sigma^n_{\mathcal{D}} \circ \mathrm{indec}_{\mathcal{E}_n^{\mathrm{nu}}}} \downarrow & & \downarrow {\scriptstyle \mathrm{Bar}_n} \\ \mathcal{D} & \xrightarrow[\sim]{\mathrm{A}_0} & \mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{\mathcal{E}_0}(\mathcal{D}) \end{array} \tag{6.1.0.7}$$

Proof. — See [HA, §5.2]. □

6.1.0.14. Corollary. — *The functor* $\mathrm{B}_n\colon \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_{n+1}}(\mathcal{C}) \to \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_n}(\mathcal{C})$ *from diagram* (6.1.0.5) *is equivalent to the following composition*

$$\mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_{n+1}}(\mathcal{D}) \xrightarrow{\mathrm{A}_{\mathcal{E}_n}} \mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{\mathcal{E}_{n+1}}(\mathcal{C}) \simeq \mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{\mathcal{E}_1}\left(\mathcal{A}\mathrm{lg}_{\mathcal{E}_n}(\mathcal{C})\right) \xrightarrow{\mathrm{Bar}_1} \mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{\mathcal{E}_n}(\mathcal{C}) \xrightarrow{\mathrm{I}_{\mathcal{E}_n}} \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_n}(\mathcal{D}).$$

In other words, under the equivalence between the augmented and the non-unital $\mathcal{E}_n$-algebras, we can regard B_n as the Bar construction.

Proof. — Using the commutative diagram (6.1.0.7) one can check that the composition also fits in the upper row of diagram (6.1.0.5). □

***6.1.0.15. Theorem* (Knudsen).** — *For every spectral Lie algebra $L \in \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{C})$, the object $(\mathrm{A}_0 \circ \mathrm{indec}_{\mathcal{L}\mathrm{ie}})(\Omega^n_{\mathcal{L}\mathrm{ie}}(L)) \in \mathcal{C}$ admits the structure of an augmented $\mathcal{E}_n$-algebra and the functor U_n satisfies*

$$\mathrm{forg}_{\mathcal{E}_n}\left(\mathrm{A}_{\mathcal{E}_n} \circ \mathrm{U}_n\right) \simeq \left(\mathrm{A}_0 \circ \mathrm{indec}_{\mathcal{L}\mathrm{ie}}\right)\left(\Omega^n_{\mathcal{L}\mathrm{ie}}(L)\right).$$

Proof. — See [Knu18, Theorem A and Theorem B]. □

6.1.0.16. Situation. — From now on we consider $\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{C})$ as a pointed symmetric monoidal ∞-category where the symmetric monoidal structure is cartesian. The zero object of $\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{C})$ is the zero object $\mathbb{0}_{\mathcal{C}}$ of $\mathcal{C}$ equipped with the trivial Lie algebra structure. It is also the symmetric monoidal unit of $\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{C})$. Thus we obtain equivalences

$$\mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{\mathcal{E}_0}\left(\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{C})\right) \simeq \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{C})$$
$$\mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{\mathcal{E}_n}\left(\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{C})\right) \simeq \mathcal{A}\mathrm{lg}_{\mathcal{E}_n}\left(\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{C})\right)$$

of ∞-categories.

6.1.0.17. — Fix a prime number p and recall the p-local telescope spectrum $\mathrm{T}(h)$ of height h from ¶1.2.0.13. In our later application we show that the functor

$$\mathrm{indec}^{+}_{\mathcal{L}\mathrm{ie}} \coloneqq A_0 \circ \mathrm{indec}_{\mathcal{L}\mathrm{ie}}\colon \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{C}) \to \mathcal{C} \xrightarrow{\sim} \mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{\mathcal{E}_0}(\mathcal{C})$$

is symmetric monoidal, if $\mathcal{C}$ is the ∞-category $\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$ of $\mathrm{T}(h)_\bullet$-local spectra with the standard symmetric monoidal structure of spectra, see Proposition 6.2.2.9. Furthermore, we conjecture that the functor $\mathrm{indec}^{+}_{\mathcal{L}\mathrm{ie}}$ is symmetric monoidal for any presentable stable symmetric monoidal ∞-category $\mathcal{C}$. Assuming that this conjecture holds, the functor $\mathrm{indec}^{+}_{\mathcal{L}\mathrm{ie}}$ induces a functor

$$\mathrm{indec}^{+,n}_{\mathcal{L}\mathrm{ie}}\colon \mathcal{A}\mathrm{lg}_{\mathcal{E}_n}\left(\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{C})\right) \to \mathcal{A}\mathrm{lg}_{\mathcal{E}_n}\left(\mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{\mathcal{E}_0}(\mathcal{C})\right) \simeq \mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{\mathcal{E}_n}(\mathcal{C})$$

for every natural number n. Then this would give an equivalence

$$\mathrm{U}_n \simeq \mathrm{I}_{\mathcal{E}_n} \circ \mathrm{indec}^{+,n}_{\mathcal{L}\mathrm{ie}} \tag{6.1.0.8}$$

of functors. In the following we explain some steps towards showing that the functor $\mathrm{indec}^{+}_{\mathcal{L}\mathrm{ie}}$ is symmetric monoidal, and prove the equivalence (6.1.0.8) assuming the conjecture. In later sections we will specialise to the situation of $\mathrm{T}(h)_{\bullet}$-local spectra.

***6.1.0.18.* Chevalley–Eilenberg functor.** — Recall that the indecomposable functor $\mathrm{indec}_{\mathcal{L}\mathrm{ie}}$ factors through the ∞-category $\mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{nu,ndp}}_{\mathcal{C}\mathrm{om}}(\mathcal{C})$ of nilpotent divided power non-unital cocommutative coalgebras:

$$\begin{array}{ccc} \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{C}) & \xrightarrow{\quad \mathrm{indec}_{\mathcal{L}\mathrm{ie}} \quad} & \mathcal{C} \\ & {\scriptstyle \mathrm{Bar}_{\mathcal{L}\mathrm{ie}}} \searrow \quad \nearrow {\scriptstyle \mathrm{forg}_{\mathcal{C}\mathrm{om}}} & \\ & \mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{nu,ndp}}_{\mathcal{C}\mathrm{om}}(\mathcal{C}) & \end{array}$$

by Proposition 5.3.2.4, since we have the Koszul duality $\mathrm{Bar}(\mathcal{L}\mathrm{ie})^{\vee} \simeq \mathcal{C}\mathrm{om}^{\mathrm{nu}}$. In [Heu] Heuts defines a product-preserving natural transformation

$$\mathrm{forg}_{\mathrm{nil}}\colon\ \mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{aug,ndp}}_{\mathcal{C}\mathrm{om}} \to \mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{aug,dp}}_{\mathcal{C}\mathrm{om}}(\mathcal{C})$$

where one should think of the target ∞-category as follows: an object in $\mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{aug,dp}}_{\mathcal{C}\mathrm{om}}(\mathcal{C})$ is an object of $\mathcal{C}$ together with a "comultiplication"

$$X \to \prod_{r=0}^{\infty} \left(X^{\otimes r}\right)_{\mathfrak{S}_r}.$$

The functor $\mathrm{forg}_{\mathrm{nil}}$ is induced by the natural transformation

$$\coprod_{r=0}^{\infty} \left((-)^{\otimes r}\right)_{\mathfrak{S}_r} \to \prod_{r=0}^{\infty} \left((-)^{\otimes r}\right)_{\mathfrak{S}_r}.$$

The *Chevalley–Eilenberg functor* is defined as the composition

$$\mathrm{CE}\colon\ \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{C}) \xrightarrow{\mathrm{Bar}_{\mathcal{L}\mathrm{ie}}} \mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{nu,ndp}}_{\mathcal{C}\mathrm{om}} \xrightarrow{\sim} \mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{aug,ndp}}_{\mathcal{C}\mathrm{om}} \xrightarrow{\mathrm{forg}_{\mathrm{nil}}} \mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{aug,dp}}_{\mathcal{C}\mathrm{om}}(\mathcal{C}).$$

Denote $\mathrm{forg}^{\mathrm{aug,dp}}_{\mathcal{C}\mathrm{om}}\colon \mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{aug,dp}}_{\mathcal{C}\mathrm{om}}(\mathcal{C}) \to \mathcal{C}$. Then we obtain the following equivalence of functors.

$$\mathrm{forg}^{\mathrm{aug,dp}}_{\mathcal{C}\mathrm{om}} \circ \mathrm{CE} \simeq \mathrm{A}_{\mathcal{C}} \circ \mathrm{indec}_{\mathcal{L}\mathrm{ie}}$$

Therefore, to prove that $\mathrm{indec}^{+}_{\mathcal{L}\mathrm{ie}} \simeq \mathrm{A}_{\mathcal{C}} \circ \mathrm{indec}_{\mathcal{L}\mathrm{ie}}$ is symmetric monoidal, one can show that CE preserves products, that the ∞-category $\mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{aug,dp}}_{\mathcal{C}\mathrm{om}}(\mathcal{C})$ admits a cartesian monoidal structure and that $\mathrm{forg}^{\mathrm{aug,dp}}_{\mathcal{C}\mathrm{om}}$ is symmetric monoidal.

6.1.0.19. Proposition. — *Let $n \in \mathbb{N}$ be a natural number. The n-fold loop functor $\Omega^n_{\mathcal{L}\mathrm{ie}}$ of the ∞-category $\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{C})$ admits the following factorisation:*

$$\begin{array}{ccc} \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{C}) & \xrightarrow{\Omega^n_{\mathcal{L}\mathrm{ie}}} & \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{C}) \\ & {\scriptstyle \widetilde{\Omega}^n_{\mathcal{L}\mathrm{ie}}} \searrow \quad \nearrow {\scriptstyle \mathrm{forg}_{\mathcal{E}_n}} & \\ & \mathcal{A}\mathrm{lg}_{\mathcal{E}_n}\left(\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{C})\right). & \end{array} \tag{6.1.0.9}$$

Furthermore, the functor $\widetilde{\Omega}^n_{\mathcal{L}\mathrm{ie}}$ is an equivalence of ∞-categories.

Proof. — This is a consequence of the following more general results Propositions 6.1.0.20 and 6.1.0.21 □

6.1.0.20. Proposition. — *Let $\mathcal{D}$ be a pointed ∞-category admitting small limits. Consider $\mathcal{D}$ as a cartesian symmetric monoidal ∞-category. For $n \in \mathbb{N}$ denote the n-fold iterated loop functor on $\mathcal{D}$ by $\Omega^n_{\mathcal{D}}$.*

Then the object $\Omega^n_{\mathcal{D}}(X)$ is an $\mathcal{E}_n$-algebra in $\mathcal{D}$, for every object $X \in \mathcal{D}$.

Proof. — Consider the pointed ∞-groupoid S^0 modelled by the (pointed) zero dimensional sphere. For every object $X \in \mathcal{D}^{\mathrm{op}}$, consider the pointed constant functor $c_X \colon \mathrm{S}^0 \to \mathcal{D}$ sending the remaining vertex of S^0 to X. Since $\mathcal{H}\mathrm{o}_*$ is the free pointed ∞-category generated by S^0 under small pointed colimits, we obtain an induced functor $(X^-)^{\mathrm{op}} \colon \mathcal{H}\mathrm{o}_* \to \mathcal{D}^{\mathrm{op}}$ by (the ∞-categorical) left Kan extension, illustrated by the following diagram

$$\begin{array}{ccc} \mathrm{S}^0 & \xrightarrow{c_X} & \mathcal{D}^{\mathrm{op}} \\ \big\downarrow & \nearrow {\scriptstyle (X^-)^{\mathrm{op}}} & \\ \mathcal{H}\mathrm{o}_* \simeq \mathcal{F}\mathrm{un}^{\mathrm{red}}\left((\mathrm{S}^0)^{\mathrm{op}}, \mathcal{H}\mathrm{o}_*\right), & & \end{array}$$

of ∞-categories, where $\mathcal{F}\mathrm{un}^{\mathrm{red}}(-,-)$ denotes the full ∞-subcategory of functors that preserves zero objects.[(1)]

By construction the functor $(X^-)^{\mathrm{op}}$ preserves small colimits. Denote the induced functor $\mathcal{H}\mathrm{o}_*^{\mathrm{op}} \to \mathcal{D}$ by X^-. Then X^- preserves small limits. By definition of the left Kan extension the evaluation X^Y is equivalent to the limit in $\mathcal{C}$ of the constant diagram $Y \to \mathcal{C}$ sending every vertex of Y to the object X. Thus X^- is symmetric monoidal with respect to the cocartesian symmetric monoidal structure on $\mathcal{H}\mathrm{o}_*^{\mathrm{op}}$ and the cartesian symmetric monoidal structure on $\mathcal{D}$, see [HA, Corollary 2.4.1.8]. In particular, we obtain the following functor induced by X^-, for every $n \in \mathbb{N}$:

$$\mathrm{co}\mathcal{A}\mathrm{lg}_{\mathcal{E}_n}(\mathcal{H}\mathrm{o}_*)^{\mathrm{op}} \simeq \mathcal{A}\mathrm{lg}_{\mathcal{E}_n}(\mathcal{H}\mathrm{o}_*^{\mathrm{op}}) \to \mathcal{A}\mathrm{lg}_{\mathcal{E}_n}(\mathcal{D}),$$

(1) For more details about the ∞-categorical Kan extension, see [HTT, §4.3]

Note that the n-dimensional sphere S^n is a $\mathcal{E}_n$-coalgebra, see [MW19] and [KSV97]. Thus the object X^{S^n} admits the structure of a $\mathcal{E}_n$-algebra. We complete the proof by observing that

$$X \simeq (X^{\mathrm{S}^0})^{\mathrm{op}} \text{ and } \Omega^n_{\mathcal{D}}(X) \simeq X^{\mathrm{S}^n}$$

for every $n \geq 1$, because $(X^{-})^{\mathrm{op}}$ preserves small colimits and $\mathrm{S}^n \simeq \Sigma^n_{\mathcal{H}o_*}(\mathrm{S}^0)$. □

6.1.0.21. Proposition. — *Let $\mathcal{O}$ be a non-unital ∞-operad with values in $\mathcal{C}$. Consider $\mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C})$ as a symmetric monoidal ∞-category equipped with the cartesian symmetric monoidal structure. For $n \in \mathbb{N}$ the n-fold loop functor $\Omega^n_{\mathcal{O}}$ on $\mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C})$ admits the following factorisation*

$$\begin{array}{ccc} \mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C}) & \xrightarrow{\quad\Omega^n_{\mathcal{O}}\quad} & \mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C}) \\ & {\scriptstyle \widetilde{\Omega}^n_{\mathcal{O}}}\searrow \quad \nearrow{\scriptstyle \mathrm{forg}_{\mathcal{E}_n}} & \\ & \mathcal{A}\mathrm{lg}_{\mathcal{E}_n}(\mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C}))\,. & \end{array}$$

Moreover, we have that

(i) *The functor $\widetilde{\Omega}^n_{\mathcal{O}}$ admits a left adjoint, which is equivalent to the Bar construction Bar_n from Theorem 6.1.0.13.(i).*

(ii) *The adjunction $\mathrm{Bar}_n \dashv \widetilde{\Omega}^n_{\mathcal{O}}$ exhibits an equivalence*

$$\mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C}) \simeq \mathcal{A}\mathrm{lg}_{\mathcal{E}_n}(\mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C}))$$

of ∞-categories.

Proof. — The $n = 0$ case holds because $\mathcal{C}$ is pointed. We consider the $n \geq 1$ case in the following.

We obtain the factorisation from Proposition 6.1.0.20 and the functor $\widetilde{\Omega}^n_{\mathcal{O}}$ admits a left adjoint by construction.

To see (i), note that the functor $\widetilde{\Omega}^n_{\mathcal{O}}$ is equivalent to the Cobar construction

$$\mathrm{Cobar}_n\colon \mathrm{co}\mathcal{A}\mathrm{lg}_{\mathcal{E}_n}(\mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C})) \simeq \mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C}) \to \mathcal{A}\mathrm{lg}_{\mathcal{E}_n}(\mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C}))$$

from Theorem 6.1.0.13.(iii). Indeed, recall that the symmetric monoidal structure on $\mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C})$ is assumed to be cartesian. Thus for every $n \geq 1$ we have

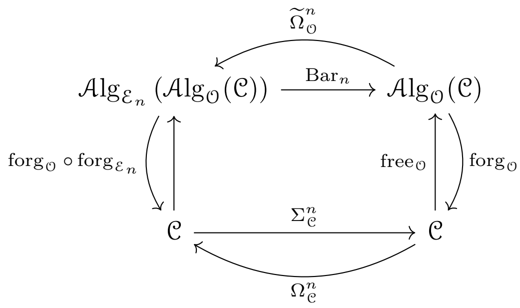
$$\mathrm{co}\mathcal{A}\mathrm{lg}_{\mathcal{E}_n}(\mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C})) \simeq \mathrm{co}\mathcal{A}\mathrm{lg}_{\mathcal{E}_1}(\mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C})) \simeq \mathcal{C}$$

by [HA, Example 3.2.4.4] and [HA, Proposition 2.4.3.9]. Moreover, by [HA, Example 5.2.2.4], we have in this case

$$(\mathrm{forg}_{\mathcal{E}_n} \circ \mathrm{Cobar}_n)(X) \simeq \Omega^n_{\mathcal{O}}.$$

Therefore, the left adjoint to $\widetilde{\Omega}^n_{\mathcal{O}}$ is equivalent to Bar_n.

As for (ii), we consider the following diagram



(6.1.0.10)

where the straight arrows commute among each other and the curved arrows commute among each other. We show that the unit and counit of the $(\mathrm{Bar}_n \dashv \widetilde{\Omega}^n_{\mathcal{O}})$-adjunction are both equivalences.

Since the forgetful functors are conservative, it suffices to show that

$$\mathrm{forg}_{\mathcal{O}} \circ \mathrm{forg}_{\mathcal{E}_n} \to \mathrm{forg}_{\mathcal{O}} \circ \mathrm{forg}_{\mathcal{E}_n} \circ \widetilde{\Omega}^n_{\mathcal{O}} \circ \mathrm{Bar}_n, \text{ and}$$
$$\mathrm{forg}_{\mathcal{O}} \circ \mathrm{Bar}_n \circ \widetilde{\Omega}^n_{\mathcal{O}} \to \mathrm{forg}_{\mathcal{O}}$$

are equivalences. Both of these equivalences follow if we provide an equivalence

$$\mathrm{forg}_{\mathcal{O}} \circ \mathrm{Bar}_n \simeq \Sigma^n_{\mathcal{C}} \circ \mathrm{forg}_{\mathcal{O}} \circ \mathrm{forg}_{\mathcal{E}_n}, \tag{6.1.0.11}$$

which indeed holds by the following arguments:

(i) It suffices to show this for $n = 1$ because Bar_n is n-fold iterated Bar construction (see Theorem 6.1.0.13).

(ii) The forgetful functors commute with geometric realisations, and $\mathrm{Bar}_1(X)$ is equivalent to the geometric realisation of $\mathrm{Bar}(\mathbb{0}_{\mathcal{C}}, \mathrm{forg}_{\mathcal{E}_n}(X), \mathbb{0}_{\mathcal{C}})$ in $\mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C})$.

(iii) Consider $\mathcal{C}$ as a cocartesian symmetric monoidal ∞-category. By [HA, Corollary 2.4.3.10] we have an equivalence $\mathcal{C} \simeq \mathcal{A}\mathrm{lg}_{\mathcal{E}_\infty}(\mathcal{C})$ of ∞-categories. Moreover, the geometric realisation of $\mathrm{Bar}\left(\mathbb{0}_{\mathcal{C}}, (\mathrm{forg}_{\mathcal{O}} \circ \mathrm{forg}_{\mathcal{E}_n})(X), \mathbb{0}_{\mathcal{C}}\right)$ becomes equivalent to the Bar construction of the commutative algebra $(\mathrm{forg}_{\mathcal{O}} \circ \mathrm{forg}_{\mathcal{E}_n})(X)$ in $\mathcal{C}$. Therefore, we obtain the equivalence in $\mathcal{C}$

$$\varinjlim \left(\mathrm{Bar}\left(\mathbb{0}_{\mathcal{C}}, (\mathrm{forg}_{\mathcal{O}} \circ \mathrm{forg}_{\mathcal{E}_n})(X), \mathbb{0}_{\mathcal{C}}\right)\right) \simeq \Sigma_{\mathcal{C}}((\mathrm{forg}_{\mathcal{O}} \circ \mathrm{forg}_{\mathcal{E}_n})(X)).$$

by [HA, Example 5.2.2.4]. □

6.1.0.22. Proposition. — *Recall the notations from ¶6.1.0.17. Assume that the functor* $\mathrm{indec}^+_{\mathcal{L}\mathrm{ie}}$ *is symmetric monoidal. Let* $\mathrm{U}_{n,+}$ *be the composition*

$$\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{C}) \xrightarrow{\widetilde{\Omega}^n_{\mathcal{L}\mathrm{ie}}} \mathcal{A}\mathrm{lg}_{\mathcal{E}_n}(\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{C})) \xrightarrow{\mathrm{indec}^{+,n}_{\mathcal{L}\mathrm{ie}}} \mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{\mathcal{E}_n}(\mathcal{C}).$$

Then there exists the following equivalence of functors:

$$\mathrm{U}_n \simeq \mathrm{I}_{\mathcal{E}_n} \circ \mathrm{U}_{n,+}\,.$$

Proof. — It suffices to check that the evaluation of the two functors on free spectral Lie algebras are equivalent, by [HA, Proposition 4.7.3.14], i.e. $\mathrm{I}_{\mathcal{E}_n} \circ \mathrm{U}_{n,+}$ also fits in the upper row of the diagram (6.1.0.4). Equivalently by adjunction, we show that the right adjoint of $\mathrm{I}_{\mathcal{E}_n} \circ \mathrm{U}_{n,+}$ fits in the upper row in the commutative diagram (6.1.0.3).

The functor right adjoint to the symmetric monoidal functor $\mathrm{indec}^{+}_{\mathcal{L}\mathrm{ie}}$ is

$$\mathrm{triv}^{+}_{\mathcal{L}\mathrm{ie}} := \mathrm{triv}_{\mathcal{L}\mathrm{ie}} \circ \mathrm{I}_0,$$

which is lax symmetric monoidal by [HA, Corollary 7.3.2.7]. Thus, it also induces a functor $\mathrm{triv}^{+,n}_{\mathcal{L}\mathrm{ie}}$ which is the right adjoint to $\mathrm{indec}^{+,n}_{\mathcal{L}\mathrm{ie}}$. An inverse of $\widetilde{\Omega}^{n}_{\mathcal{L}\mathrm{ie}}$ is given by the Bar construction Bar_n, see Proposition 6.1.0.19. Thus, we obtain the following commutative diagram

$$\begin{array}{ccccccc}
\mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_n}(\mathcal{C}) & \xrightarrow[\simeq]{\mathrm{A}_{\mathcal{E}_n}} & \mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{\mathcal{E}_n}(\mathcal{C}) & \xrightarrow{\mathrm{triv}^{+,n}_{\mathcal{L}\mathrm{ie}}} & \mathcal{A}\mathrm{lg}_{\mathcal{E}_n}(\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{C})) & \xrightarrow{\mathrm{Bar}_n} & \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{C}) \\
\downarrow{\scriptstyle \mathrm{forg}^{\mathrm{nu}}_{\mathcal{E}_n}} & & \downarrow{\scriptstyle \mathrm{forg}^{\mathrm{aug}}_{\mathcal{E}_n}} & & \downarrow{\scriptstyle \mathrm{forg}} & & \downarrow{\scriptstyle \mathrm{forg}_{\mathcal{L}\mathrm{ie}}} \\
\mathcal{C} & \xrightarrow{\mathrm{A}_0} & \mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{\mathcal{E}_0}(\mathcal{C}) & \xrightarrow{\mathrm{I}_0} & \mathcal{C} & \xrightarrow{\Sigma^n_{\mathcal{C}}} & \mathcal{C},
\end{array} \tag{6.1.0.12}$$

where the composition $\mathrm{Bar}_n \circ \mathrm{triv}^{+,n}_{\mathcal{L}\mathrm{ie}} \circ \mathrm{A}_{\mathcal{E}_n}$ of the functors in the upper row is right adjoint to $\mathrm{I}_{\mathcal{E}_n} \circ \mathrm{U}_{n,+}$. The diagram (6.1.0.12) commutes because all the three small squares commutes:

(i) The first square commutes by Corollary 6.1.0.10,
(ii) The second commutes because $\mathrm{triv}^{+}_{\mathcal{L}\mathrm{ie}}$ is lax monoidal, and
(iii) The third square commutes by (6.1.0.11). □

6.1.0.23. Notation. — Let $\mathrm{T}_{n,+} := \mathrm{Bar}_n \circ \mathrm{triv}^{+,n}_{\mathcal{L}\mathrm{ie}}$ denote the right adjoint to $\mathrm{U}_{n,+}$. The right adjoint T_n to U_n is given by $\mathrm{T}_n \simeq \mathrm{T}_{n,+} \circ \mathrm{A}_{\mathcal{E}_n}$.

6.2. Spectral Lie algebras in monochromatic layers

Let h be a natural number. For the rest of this chapter we explain the behaviour of the adjunction

$$\mathrm{U}_\infty\colon \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}) \rightleftarrows \varprojlim_n \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_n}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}) : \mathrm{T}_\infty, \tag{6.2.0.1}$$

where $\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$ is the ∞-category of $\mathrm{T}(h)_\bullet$-local spectra, induced by the diagram (6.1.0.1) when $\mathcal{C} = \mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$. In this section we focus mainly on the $h = 0$ case; recall that $\mathcal{S}\mathrm{p}_{\mathrm{T}(0)}$ is the ∞-category $\mathcal{S}\mathrm{p}_{\mathbb{Q}}$ of rational spectra. Using the formality theorem of the rational $\mathcal{E}_n$ ∞-operads we show that the above adjunction exhibits an equivalence between $\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathbb{Q}})$ and $\varprojlim_n \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_n}(\mathcal{S}\mathrm{p}_{\mathbb{Q}})$, see Theorem 6.2.1.17. We can not find a written account of this theorem in the literature, but the result is probably known to the experts. In §6.2.2 we record some prerequisites about the ∞-category $\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$ for $h \geq 1$, as a preparation for later applications.

6.2.1. Lie algebras and $\mathcal{E}_n$-algebras in characteristic 0.

— The goal of this subsection is to show that the functor U_∞ in (6.2.0.1) is an equivalence of ∞-categories when $h = 0$. We begin by recalling some basic notions from the theory of derived ∞-categories.

6.2.1.1. Situation. — Let R be a commutative ring. Denote the (ordinary) category of chain complexes of R-modules by $\mathbf{Ch}_R$. The category $\mathbf{Ch}_R$ admits a projective model structure where the weak equivalences are quasi-isomorphisms and the fibrations are degree-wise surjective morphisms of chain complexes, see [HA, Proposition 7.1.2.8].[(2)] Let $A_\bullet$ and $B_\bullet$ be two objects of $\mathbf{Ch}_R$. The tensor product $A \otimes B$ is a chain complex with

$$(A \otimes B)_n = \oplus_{p+q=n}(A_p \otimes_R B_q),$$

see [HA, Remark 1.2.3.21]. With these structures $\mathbf{Ch}_R$ becomes a symmetric monoidal combinatorial model category, see [HA, Proposition 7.1.2.11]. Furthermore, $\mathbf{Ch}_R$ is a simplicial enriched category by the Dold–Kan correspondence, see [HA, Construction 1.3.1.13], but it is *not* a simplicial model category with the prescribed model structure, see [HA, Warning 1.3.5.4]. Let $\mathbf{Ch}^\circ_R$ denote the full subcategory of cofibrant objects (which are by definition also fibrant) of $\mathbf{Ch}_R$. Denote the set of quasi-isomorphisms in $\mathbf{Ch}^\circ_R$ by W°.

6.2.1.2. Definition. — In Situation 6.2.1.1 define the *derived ∞-category* $\mathcal{D}(R)$ as the underlying ∞-category $\mathrm{N}(\mathbf{Ch}^\circ_R)[(W^\circ)^{-1}]$ of the model category $\mathbf{Ch}_R$, where N denotes the nerve of an ordinary category.

---

(2) A morphism of chain complexes is a quasi-isomorphism if it induces an isomorphism on homology groups in every degree.

6.2.1.3. Remark. — One can also define $\mathcal{D}(R)$ equivalently as a localisation of the ∞-category of the differential graded nerve of $\mathbf{Ch}_R$, see [HA, Definition 1.3.5.8, Proposition 1.3.5.15].

6.2.1.4. Proposition. — *In Situation 6.2.1.1 the ∞-category $\mathcal{D}(R)$ is a presentable stable symmetric monoidal ∞-category.*

Sketch. — We outline the ideas and references of the proof. The symmetric monoidal structure inherits from the symmetric monoidal structure of $\mathbf{Ch}_R$ (and thus $\mathbf{Ch}_R^{\circ}$), see [HA, Example 4.1.7.6]. The ∞-category $\mathcal{D}(R)$ is a ∞-subcategory of the stable ∞-category of differential graded nerve of $\mathbf{Ch}_R$, see [HA, §1.3.1], which implies the stability of $\mathcal{D}(R)$, see [HA, Proposition 1.3.5.9]. The presentability of $\mathcal{D}(R)$ is proven using the following three ingredients [HA, Proposition 1.3.4.22]:

(i) Every combinatorial model category $\mathbf{C}$ is Quillen equivalent to a simplicial combinatorial model category $\widetilde{\mathbf{C}}$, see [HA, Corollary 1.2].

(ii) Under the categorical equivalence from (i), the underlying ∞-category of $\mathbf{C}$ [HA, Definition 1.3.4.15] and the underlying ∞-category of $\widetilde{\mathbf{C}}$ are equivalent as ∞-categories, see [HA, Theorem 1.3.4.20].

(iii) The underlying ∞-category of a combinatorial simplicial model category is presentable, see [HTT, Proposition 3.7.6]. □

6.2.1.5. Proposition. — *Let R be a commutative ring. There exists a symmetric monoidal functor $\mathrm{Sing}_R\colon \mathcal{H}\mathrm{o} \to \mathcal{D}(R)$ in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ sending a homotopy type X to its R-valued singular chains $\mathrm{Sing}_R(X)$.*

Proof. — Recall that $\mathrm{H}R$ denotes the Eilenberg–MacLane spectrum with $\pi_\bullet^{\mathrm{st}}(\mathrm{H}R) \cong R$. Consider the following sequence

$$F\colon \mathcal{H}\mathrm{o} \xrightarrow{\Sigma^\infty_+} \mathcal{S}\mathrm{p} \xrightarrow{-\otimes \mathrm{H}R} \mathcal{M}\mathrm{od}_{\mathrm{H}R} \xrightarrow{\sim} \mathcal{D}(R)$$

of symmetric monoidal functors in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ where

(i) $\mathcal{M}\mathrm{od}_{\mathrm{H}R}$ denotes the ∞-category of $\mathrm{H}R$-modules, and

(ii) the third equivalence is given by [HA, Theorem 7.1.2.13].

The R-valued singular chain functor $\mathrm{Sing}_R(-)$ is symmetric monoidal (Eilenberg–Zilber theorem), and it preserves small colimits. Thus it is equivalent to the functor F. □

6.2.1.6. Corollary. — *The functor $\mathrm{Sing}_R(-)$ in Proposition 6.2.1.5 induces the following functor*

$$\mathrm{Sing}_R\colon \mathcal{O}\mathrm{pd}\,(\mathcal{H}\mathrm{o}) \to \mathcal{O}\mathrm{pd}(\mathcal{D}(R)).$$

of the ∞-categories of ∞-operads.

Proof. — This is by Proposition 5.2.5.1 and Remark 5.2.5.2 □

6.2.1.7. Proposition. — *Let $\mathcal{O}$ be an ∞-operad. The functor* Sing_R *induces the following equivalence of ∞-categories:*

$$\mathcal{A}\mathrm{lg}_{\mathcal{O}}\left(\mathcal{D}(R)\right) \simeq \mathcal{A}\mathrm{lg}_{\mathrm{Sing}_R(\mathcal{O})}\left(\mathcal{D}(R)\right).$$

Proof. — This is a corollary of Theorem 5.2.5.5 and Proposition 6.2.1.5. □

6.2.1.8. Construction. — Let k be a field. The graded homology of a k-valued chain complex can be regarded as a chain complex with zero differentials. The homology functor $\mathrm{H}_\bullet(-)\colon \mathcal{D}(k) \to \mathcal{D}(k)$ is symmetric monoidal, by the Künneth Theorem. By Proposition 5.2.5.1 and Remark 5.2.5.2 it induces a functor $\mathrm{H}_\bullet(-)$ on the ∞-category of ∞-operads with values in $\mathcal{D}(k)$.

Let $\mathcal{O}$ be an ∞-operad. We denote the induced ∞-operad $\mathrm{H}_\bullet\left(\mathrm{Sing}_k(\mathcal{O})\right)$ with values in $\mathcal{D}(k)$ by $\mathrm{H}_\bullet(\mathcal{O};k)$. In particular, for every $r \geq 0$, we have an isomorphism

$$\mathrm{H}_\bullet(\mathcal{O};k)(r) \cong \mathrm{H}_\bullet(\mathcal{O}(r);k)$$

of graded k-vector spaces.

6.2.1.9. Definition. — Let k be a field and let $\mathcal{O}$ be an ∞-operad with values in $\mathcal{D}(k)$.

(i) We say $\mathcal{O}$ is *formal* if there exists an equivalence $\mathcal{O} \simeq \mathrm{H}_\bullet(\mathcal{O})$ in $\mathcal{O}\mathrm{pd}\left(\mathcal{D}(k)\right)$.

(ii) Let $f\colon \mathcal{O} \to \mathcal{P}$ be a morphism of ∞-operads with values in $\mathcal{D}(k)$, where $\mathcal{O}$ and $\mathcal{P}$ are formal. We say f is *formal* if there exists a commutative diagram

$$\begin{array}{ccc} \mathcal{O} & \xrightarrow{\simeq} & \mathrm{H}_\bullet(\mathcal{O};k) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_*} \\ \mathcal{P} & \xrightarrow{\simeq} & \mathrm{H}_\bullet(\mathcal{P};l) \end{array}$$

of ∞-operads with values in $\mathcal{D}(k)$.

***6.2.1.10.* Rectification.** — In the following we will use results about $\mathcal{E}_n$-algebras which were proven in a model category of $\mathbf{E}_n$-algebras in chain complexes over a field of characteristic 0. Thus we need the following rectification theorem by Haugseng [Hau19].

Let k be a field of characteristic 0. For an operad $\mathbf{O}$ with values in the model category $\mathbf{Ch}_k$ with the projective model structure, there exists an equivalence

$$\left(\mathcal{A}\mathrm{lg}_{\mathbf{O}}\left(\mathbf{Ch}_k\right)\right)[\mathrm{W}^{-1}] \simeq \mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{D}(k))$$

of ∞-categories where

(i) $\mathcal{A}\mathrm{lg}_{\mathcal{O}}\left(\mathbf{Ch}_k\right)$ denotes the model category of $\mathbf{O}$-algebras in $\mathbf{Ch}_k$ with an induced model structure by the model structure of $\mathbf{Ch}_k$.

(ii) W denotes the set of weak equivalences in the model category $\mathcal{A}\mathrm{lg}_{\mathcal{O}}\left(\mathbf{Ch}_k\right)$, and

(iii) $\mathcal{O}$ denotes the (arity-wise) image of $\mathbf{O}$ under the localisation $\mathbf{Ch}_k \to \mathcal{D}(k)$.

In [Hau19] Haugseng uses another model for ∞-operads with values in a symmetric monoidal ∞-category than ours, see [Hau19, Theorem 4.10, Corollary 4.11]. However, the associated monads of his ∞-operads are equivalent to ours. Thus, the the associated ∞-category of $\mathcal{O}$-algebras are also equivalent.

***6.2.1.11. Theorem* (Fresse–Willwacher).** —

(i) *The ∞-operad $\mathrm{Sing}_{\mathbb{Q}}(\mathcal{E}_n)$ with values in $\mathcal{D}(\mathbb{Q})$ is formal, for every $n \geq 0$.*

(ii) *Let $i_n^{n+k} \colon \mathcal{E}_n \to \mathcal{E}_{n+k}$ denote the composition of the morphisms*

$$\mathcal{E}_n \to \mathcal{E}_{n+1} \to \cdots \mathcal{E}_{n+k}$$

of ∞-operads in (5.2.5.1). For every $n \geq 0$ and every $k \geq 2$, the induced morphism $\mathrm{Sing}_{\mathbb{Q}}\left(i_n^{n+k}\right)$ in $\mathcal{O}\mathrm{pd}\left(\mathcal{D}(\mathbb{Q})\right)$ is formal.

Proof. — See [FW20, Theorem B' and Theorem D'] □

6.2.1.12. Proposition. — *Recall the commutative ∞-operad $\mathcal{C}\mathrm{om}_{\mathcal{D}(\mathbb{Q})} \in \mathcal{O}\mathrm{pd}\left(\mathcal{D}(\mathbb{Q})\right)$ from Example 5.2.5.11.*

(i) *For every natural number n, there exist morphisms $\iota_c \colon \mathcal{C}\mathrm{om}_{\mathcal{D}(\mathbb{Q})} \hookrightarrow \mathrm{H}_\bullet\left(\mathcal{E}_n;\mathbb{Q}\right)$ and $\pi_c \colon \mathrm{H}_\bullet\left(\mathcal{E}_n;\mathbb{Q}\right) \twoheadrightarrow \mathcal{C}\mathrm{om}_{\mathcal{D}(\mathbb{Q})}$ of ∞-operads with values in $\mathcal{D}(\mathbb{Q})$. For every $n \geq 2$, we have $\pi_c \circ \iota_c = \mathrm{id}$.*

(ii) *For every $n \geq 1$, there exists the following commutative diagram*

$$\begin{array}{ccc}
\mathrm{H}_\bullet(\mathcal{E}_n;\mathbb{Q}) & \overset{\pi}{\twoheadrightarrow} & \mathcal{C}\mathrm{om}_{\mathcal{D}(\mathbb{Q})} \\
{\scriptstyle (i_n^{n+1})_*}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\
\mathrm{H}_\bullet(\mathcal{E}_{n+1};\mathbb{Q}) & \underset{\iota}{\hookleftarrow} & \mathcal{C}\mathrm{om}_{\mathcal{D}(\mathbb{Q})}
\end{array}$$

of ∞-operads with values in $\mathcal{D}(\mathbb{Q})$.

Proof. — This is a translation of Theorem 5.1.0.25 to ∞-categorical language. □

6.2.1.13. Corollary. — *For every $n \geq 0$ and every $k \geq 2$, the induced map $\mathrm{Sing}_{\mathbb{Q}}(i_n^{n+k})$ of ∞-operads with values in $\mathcal{D}(\mathbb{Q})$ factors through the commutative ∞-operad $\mathcal{C}\mathrm{om}_{\mathcal{D}(\mathbb{Q})}$. More precisely, there exists the commutative diagram*

$$\begin{array}{ccccc}
\mathrm{Sing}_{\mathbb{Q}}\left(\mathcal{E}_n\right) & \xrightarrow{\sim} & \mathrm{H}_\bullet(\mathcal{E}_n;\mathbb{Q}) & \overset{\pi}{\twoheadrightarrow} & \mathcal{C}\mathrm{om}_{\mathcal{D}(\mathbb{Q})} \\
{\scriptstyle (i_n^{n+k})_*}\downarrow & & \downarrow{\scriptstyle (i_n^{n+k})_*} & & \downarrow{\scriptstyle \mathrm{id}} \\
\mathrm{Sing}_{\mathbb{Q}}\left(\mathcal{E}_{n+k}\right) & \xleftarrow{\sim} & \mathrm{H}_\bullet(\mathcal{E}_{n+k};\mathbb{Q}) & \underset{\iota}{\hookleftarrow} & \mathcal{C}\mathrm{om}_{\mathcal{D}(\mathbb{Q})}\,.
\end{array}$$

of ∞-operads with values in $\mathcal{D}(k)$.

Proof. — This is follows from Theorem 6.2.1.11 and Proposition 6.2.1.12. □

6.2.1.14. Remark. — If we replace all the ∞-operads in Proposition 6.2.1.12 and Corollary 6.2.1.13 by their non-unital versions, the statements still hold and the proof are the same.

6.2.1.15. Notation. — Let $\mathcal{L}\mathrm{ie}_{\mathbb{Q}}$ denote the ∞-operad with values in $\mathcal{D}(\mathbb{Q})$ which is defined as follows: Recall that there exists a unique symmetric monoidal functor $F\colon \mathcal{S}\mathrm{p} \to \mathcal{D}(R)$ in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. It induces a functor $F\colon \mathcal{O}\mathrm{pd}(\mathcal{S}\mathrm{p}) \to \mathcal{O}\mathrm{pd}(\mathcal{D}(\mathbb{Q}))$. The ∞-operad $\mathcal{L}\mathrm{ie}_{\mathbb{Q}}$ is the image of the spectral Lie ∞-operad $\mathcal{L}\mathrm{ie}$ under the functor F.

By ¶5.3.3.7 the ∞-operad $\mathcal{L}\mathrm{ie}_{\mathbb{Q}}$ is Koszul dual to the non-unital cocommutative ∞-cooperad $\mathcal{C}\mathrm{om}^{\mathrm{nu}}_{\mathcal{D}(\mathbb{Q})}$.

6.2.1.16. Corollary. — *For every $n \geq 0$ and every $k \geq 2$, the Koszul dual morphism*

$$\mathrm{Sing}_{\mathbb{Q}}(c_n^{n+k})\colon \Sigma^{-(n+k)}\mathrm{Sing}_{\mathbb{Q}}(\mathcal{E}_{n+k}) \to \Sigma^{-n}\mathrm{Sing}_{\mathbb{Q}}(\mathcal{E}_n)$$

of $\mathrm{Sing}_{\mathbb{Q}}(i_n^{n+k})$ *factors through* $\mathcal{L}\mathrm{ie}_{\mathbb{Q}}$. *More precisely, there exists the following commutative diagram*

$$\begin{array}{ccc} \Sigma^{-n}\mathrm{Sing}_{\mathbb{Q}}(\mathcal{E}_n^{\mathrm{nu}}) & \longleftarrow & \mathcal{L}\mathrm{ie}_{\mathbb{Q}} \\ {\scriptstyle (c_n^{n+k})_*}\uparrow & & \uparrow{\scriptstyle \mathrm{id}} \\ \Sigma^{-n-k}\mathrm{Sing}_{\mathbb{Q}}(\mathcal{E}_{n+k}^{\mathrm{nu}}) & \longrightarrow & \mathcal{L}\mathrm{ie}_{\mathbb{Q}} \end{array}$$

of ∞-operads with values in $\mathcal{D}(\mathbb{Q})$.

Proof. — We apply the Koszul duality functor to the outer commutative diagram in Corollary 6.2.1.13. □

6.2.1.17. Theorem. — *The cocontinuous functor*

$$\mathrm{U}_\infty\colon \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathbb{Q}}) \to \varprojlim \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_n}(\mathcal{S}\mathrm{p}_{\mathbb{Q}})$$

in the adjunction (6.2.0.1) *is an equivalence.*

Proof. — It is equivalent to show that

$$\mathrm{U}_\infty\colon \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{D}(\mathbb{Q})) \to \varprojlim_n \mathcal{A}\mathrm{lg}_{\mathcal{E}_n}(\mathcal{D}(\mathbb{Q}))$$

is an equivalence under the equivalence $\mathcal{S}\mathrm{p}_{\mathbb{Q}} \simeq \mathcal{M}\mathrm{od}_{\mathrm{H}\mathbb{Q}} \simeq \mathcal{D}(\mathbb{Q})$ of symmetric monoidal ∞-categories. Throughout the whole proof we also use the identifications

$$\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{D}(R)) \simeq \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}_{\mathbb{Q}}}(\mathcal{D}(R))$$

by Notation 6.2.1.15 and

$$\mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{D}(R)) \simeq \mathcal{A}\mathrm{lg}_{\mathrm{Sing}_{\mathbb{Q}}(\mathcal{O})}(\mathcal{D}(R))$$

for an ∞-operad $\mathcal{O} \in \mathcal{O}\mathrm{pd}(\mathcal{H}\mathrm{o})$ by Proposition 6.2.1.7.

Recall that the functor $\mathrm{B}_n\colon \mathcal{A}\mathrm{lg}_{\mathcal{E}_n}(\mathcal{D}(\mathbb{Q})) \to \mathcal{A}\mathrm{lg}_{\mathcal{E}_{n-1}}(\mathcal{D}(\mathbb{Q}))$ with $n \in \mathbb{N}$ in the inverse limit diagram is induced by the morphisms $c_n^{n+1}\colon \Sigma^{-(n+1)}\mathcal{E}_{n+1} \to \Sigma^{-n}\mathcal{E}_n$ of

∞-operads (see (6.1.0.2)). Abbreviate the left adjoint to the induced functor $(c_n^{n+1})_*$ on algebras by $c^!$. Denote the composition $\mathrm{B}_{n+k} \circ \mathrm{B}_{n+k-1} \circ \cdots \circ \mathrm{B}_{n+1}$ by $\mathrm{B}_{n+k,n}$. By Corollary 6.2.1.16, we obtain the following commutative diagrams

$$\begin{array}{ccc}
\mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_{n+k}}(\mathcal{D}(\mathbb{Q})) & \xrightarrow{\mathrm{B}_{n+k,n}} & \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_n}(\mathcal{D}(\mathbb{Q})) \\
\downarrow \simeq & & \uparrow \simeq \\
\mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\Sigma^{-n-k}\mathcal{E}_{n+k}}(\mathcal{D}(\mathbb{Q})) & \xrightarrow{c^!} & \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\Sigma^{-n}\mathcal{E}_n}(\mathcal{D}(\mathbb{Q})) \\
\downarrow & & \uparrow \\
\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{D}(\mathbb{Q})) & \xrightarrow{\mathrm{id}} & \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{D}(\mathbb{Q})).
\end{array}$$

of ∞-categories, for every $n \geq 0$ and every $k \geq 2$. This gives the commutative diagram of ∞-categories below:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_{n+2}}(\mathcal{D}(\mathbb{Q})) & \xrightarrow{\mathrm{B}_{n+2,n}} & \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_n}(\mathcal{D}(\mathbb{Q})) & \xrightarrow{\mathrm{B}_{n,n-2}} & \cdots & \xrightarrow{\mathrm{B}_{2,0}} & \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_0}(\mathcal{D}(\mathbb{Q})) \\
\downarrow \quad \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & & \nearrow & \downarrow \mathrm{id} \\
\cdots \xrightarrow{\mathrm{id}} & \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{D}(\mathbb{Q})) & \xrightarrow{\mathrm{id}} & \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{D}(\mathbb{Q})) & \xrightarrow{\mathrm{id}} & \cdots & \xrightarrow{\mathrm{id}} & \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_0}(\mathcal{D}(\mathbb{Q})).
\end{array}$$

Therefore, the inverse limit of the upper row is equivalent to the inverse limit of lower horizontal rows, and the latter is equivalent to $\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{D}(\mathbb{Q}))$ because the diagram becomes constant after the first arrow. ☐

6.2.1.18. — Let $\mathcal{C}$ be the derived ∞-category $\mathcal{D}(k)$ of a field k of characteristic 0. Then we can also interpret the diagram (6.1.0.1) in terms of deformation theory and formal moduli problems, see [CG21; DAGX; BM23a] for an introduction for some introductions for the relationship between Koszul duality and deformation theory. It is shown in [DAGX; Pri10] that the ∞-category $\mathcal{M}\mathrm{oduli}_k$ of (commutative) moduli problems over k is equivalent to the ∞-category $\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{D}(k))$ of spectral Lie algebras in $\mathcal{D}(k)$. Furthermore, the ∞-category $\mathcal{M}\mathrm{oduli}_{\mathcal{E}_n,k}$ of $\mathcal{E}_n$-moduli problems is equivalent to the ∞-category $\mathcal{A}\mathrm{lg}_{\mathcal{E}_n}(\mathcal{D}(k))$. The ∞-operad inclusion $\mathcal{E}_n \hookrightarrow \mathcal{C}\mathrm{om}$ induces a cocontinuous functor $\mathcal{M}\mathrm{oduli}_k \to \mathcal{M}\mathrm{oduli}_{\mathcal{E}_n,k}$ by left Kan extension, which is equivalent to the functor U_n. Therefore, another interpretation of Theorem 6.2.1.17 is that the ∞-category $\mathcal{M}\mathrm{oduli}_k$ of commutative formal Moduli problems over k is equivalent to the inverse limit of the ∞-category $\mathcal{M}\mathrm{oduli}_{\mathcal{E}_n,k}$ of $\mathcal{E}_n$-formal Moduli problems over k.

Furthermore, the association between formal moduli problems and Lie algebras is generalised to positive and mixed characteristic situations, see [BM23a]. This might help with understanding the functor U_∞ in the case where $\mathcal{C}$ is the ∞-category of module spectra over a field of positive characteristic.

6.2.2. Monochromatic layers and homotopy types. — Fix a prime number p and a natural number $h \geq 1$ throughout this subsection. Recall the ∞-category $\mathcal{H}\mathrm{o}_{v_h}$ of the localisation of $\mathcal{H}\mathrm{o}$ at the set of v_h-periodic equivalences, see Definition 3.4.0.6. This ∞-category relates closely with its stable counterpart $\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$, which is equivalent to the localisation of the ∞-category $\mathcal{S}\mathrm{p}_{(p)}$ of p-local spectra at the set of v_h-periodic equivalences, via spectral Lie algebras. In this short subsection we review this relationship and document several prerequisites of $\mathrm{T}(h)_\bullet$-local spectra for later use. The main references for this section are [Heu20b] and [Heu21].

***6.2.2.1. Theorem* (Quillen [Qui69]).** — *The ∞-category $\mathcal{H}\mathrm{o}_{\mathbb{Q}}^{\geq 2}$ of pointed simply connected rational homotopy types is equivalent to the ∞-category $\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{D}(\mathbb{Q}))^{\geq 1}$ of connected differential graded Lie algebras over $\mathbb{Q}$.*

6.2.2.2. Remark. — Note that there exists an equivalence

$$\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{D}(\mathbb{Q}))^{\geq 1} \simeq \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathbb{Q}})^{\geq 1}$$

of ∞-categories.

***6.2.2.3.* The ∞-category $\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$.** — The construction of the telescope spectrum $\mathrm{T}(h)$ can be regarded as a generalisation of one construction of the rational Eilenberg–MacLane spectrum $\mathrm{H}\mathbb{Q}$, given by inverting the degree p self-map of the p-local sphere spectrum, see ¶1.2.0.13. Recall the finite localisation functor

$$\mathrm{L}_h^{\mathrm{f}} \colon \mathcal{S}\mathrm{p} \to \mathcal{L}_h^{\mathrm{f}}\left(\mathcal{S}\mathrm{p}_{(p)}\right)$$

from ¶3.4.1.1. Let F_h be a (p-local) finite spectrum of type h. The ∞-category $\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$ is compactly generated with a generator given by $\mathrm{L}_h^{\mathrm{f}}(F_h)$, by the Thick Subcategory Theorem (see Theorem 1.2.0.9). Note that any v_h self-map $v_h \colon \Sigma^d F_h \to F_h$ becomes an equivalence $\mathrm{L}_h^{\mathrm{f}}(v_h)$ in $\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$, since it is a v_h-periodic equivalence.

***6.2.2.4. Theorem* (Heuts [Heu21]).** — *There exists an equivalence*

$$\mathcal{H}\mathrm{o}_{v_h} \simeq \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}).$$

of ∞-categories.

6.2.2.5. — In the proof of the theorem there are the following two main steps.

(i) The Bousfield–Kuhn functor adjunction (see Theorem 3.4.1.6)

$$\Theta_h \colon \mathcal{S}\mathrm{p}_{\mathrm{T}(h)} \rightleftarrows \mathcal{H}\mathrm{o}_{v_h} : \Phi_h$$

is monadic, see [EHMM19].

(ii) The monad $\Phi \circ \Theta$ is equivalent to the arity-wise $\mathrm{T}(h)_\bullet$-localisation of the monad $\mathrm{T}_{\mathcal{L}\mathrm{ie}}$ associated with spectral Lie ∞-operad, see [Heu21].

Under the equivalence $\mathcal{H}\mathrm{o}_{\mathbb{Q}}^{\geq 2} \xrightarrow{\sim} \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{D}(\mathbb{Q}))^{\geq 1}$ the rational homotopy groups of a rational homotopy type are isomorphic, up to a degree shift, to the homology groups of the underlying chain complex of the corresponding rational differential graded Lie algebra. The analogy to this in higher height h is given by the following commutative diagram

$$\begin{tikzcd} \mathcal{H}\mathrm{o}_{v_h} \arrow[rr, "\sim", "\Phi'_h"'] \arrow[dr, "\Phi_h"'] & & \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}) \arrow[dl, "\mathrm{forg}_{\mathcal{L}\mathrm{ie}}"] \\ & \mathcal{S}\mathrm{p}_{\mathrm{T}(h)}, & \end{tikzcd}$$

obtained by the monadicity of the adjunction $\Theta_h \dashv \Phi_h$: One can "calculate" the v_h-periodic homotopy groups of $X \in \mathcal{H}\mathrm{o}_{v_h}$ from the stable homotopy groups of the underlying spectrum of the associated spectral Lie algebra $\Phi'_h(X)$, see Theorem 3.4.1.5.

***6.2.2.6.* Stabilisation of an ∞-category.** — A functor between ∞-categories is *excisive* if it sends pushout diagrams to pullback diagrams (given that both exist in the source and target ∞-categories, respectively).

Let $\mathcal{D}$ be an ∞-category that admits finite limits. The *stabilisation* $\mathcal{S}\mathrm{p}(\mathcal{D})$ of $\mathcal{D}$ is the stable ∞-category

$$\mathcal{S}\mathrm{p}(\mathcal{D}) := \mathcal{E}\mathrm{xc}_*(\mathcal{H}\mathrm{o}_*^{\mathrm{fin}}, \mathcal{D})$$

of reduced (preserves terminal objects) excisive functors from the ∞-category $\mathcal{H}\mathrm{o}_*^{\mathrm{fin}}$ of pointed finite homotopy types to $\mathcal{D}$, see [HA, §1.4.2].

Let $\mathcal{D}_* := \mathcal{D}_{\mathrm{pt}/}$ denote the ∞-category of pointed objects of $\mathcal{D}$. The forgetful functor $\mathcal{D}_* \to \mathcal{D}$ induces an equivalence $\mathcal{S}\mathrm{p}(\mathcal{D}) \simeq \mathcal{S}\mathrm{p}(\mathcal{D}_*)$ of their stabilisations, using the fact that a stable ∞-category is pointed, see [HA, Remark 1.4.2.18]. Moreover, there is an equivalence

$$\mathcal{S}\mathrm{p}(\mathcal{D}_*) \simeq \varprojlim \left(\cdots \xrightarrow{\Omega_{\mathcal{D}}} \mathcal{D}_* \xrightarrow{\Omega_{\mathcal{D}}} \mathcal{D}_* \right)$$

of stable ∞-categories, see [HA, Proposition 1.4.2.24]. Thus, we think of an object in $\mathcal{S}\mathrm{p}(\mathcal{D}_*)$ as a sequence $(X_n)_{n\geq 0}$ of objects of $\mathcal{C}$ such that $\Omega_{\mathcal{D}}(X_{n+1}) \simeq X_n$ for every $n \geq 0$. From this we obtain $\mathcal{S}\mathrm{p}(\mathcal{H}\mathrm{o}_*) \simeq \mathcal{S}\mathrm{p}$.

The stabilisation $\mathcal{S}\mathrm{p}(\mathcal{D})$ is equipped with a canonical functor

$$\Omega^\infty_{\mathcal{D}} \colon \mathcal{S}\mathrm{p}(\mathcal{D}) \to \mathcal{D}.$$

Depending on which definitions one uses for the stabilisation, one can consider this functor either as

(i) the evaluation of an excisive functor at $\mathrm{pt} \in \mathcal{H}\mathrm{o}_*^{\mathrm{fin}}$, or

(ii) the functor assigning to $(X_n)_{n\geq 0}$ the object X_0, if $\mathcal{D}$ is pointed.

The functor $\Omega^\infty_{\mathcal{D}}$ satisfies the universal property that it is the terminal finite limit preserving functor from a stable ∞-category to $\mathcal{D}$, see [HA, Corollary 1.4.2.23].

If $\mathcal{D}$ is in addition presentable, the functor $\Omega^\infty_{\mathcal{D}}$ admits a left adjoint, denoted by $\Sigma^\infty_{\mathcal{D}}$. Sometimes we abbreviate $\Sigma^\infty_{\mathcal{D}}$ and $\Omega^\infty_{\mathcal{D}}$ by Σ^∞ and Ω^∞, respectively.

***6.2.2.7. Theorem* (Basterra–Mandell, Lurie).** — *Let* $\mathcal{O}$ *be a non-unital* ∞*-operad with values in a presentable stable symmetric monoidal* ∞*-category* $\mathcal{C}$*. The* $(\mathrm{indec}_{\mathcal{O}} \dashv \mathrm{triv}_{\mathcal{O}})$*-adjunction (see Example 5.2.4.20) exhibits* $\mathcal{C}$ *as the stabilisation of the* ∞*-category* $\mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C})$ *of* $\mathcal{O}$*-algebras in* $\mathcal{C}$*, i.e. we can consider* $\mathrm{indec}_{\mathcal{O}} \dashv \mathrm{triv}_{\mathcal{O}}$ *equivalently as the* $(\Sigma^{\infty}_{\mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C})} \dashv \Omega^{\infty}_{\mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C})})$*-adjunction.*

Proof. — See also [HA, Theorem 7.3.4.7]. We also refer the reader to [Heu20b, Theorem 4.3] for a sketch of the proof. □

***6.2.2.8.* Stabilisation of** $\mathcal{H}\mathrm{o}_{v_h}$**.** — Under the equivalence of Theorem 6.2.2.4 the two adjunctions

$$\mathcal{S}\mathrm{p}_{\mathrm{T}(h)} \underset{\mathrm{forg}}{\overset{\mathrm{free}}{\rightleftarrows}} \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}) \underset{\mathrm{triv}}{\overset{\mathrm{indec}}{\rightleftarrows}} \mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$$

associated with $\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})$ correspond to the following two adjunctions

$$\mathcal{S}\mathrm{p}_{\mathrm{T}(h)} \underset{\Phi_h}{\overset{\Theta_h}{\rightleftarrows}} \mathcal{H}\mathrm{o}_{v_h} \underset{\Omega^{\infty}_{v_h}}{\overset{\Sigma^{\infty}_{v_h}}{\rightleftarrows}} \mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$$

associated with $\mathcal{H}\mathrm{o}_{v_h}$, where the adjunction $\Sigma^{\infty}_{v_h} \dashv \Omega^{\infty}_{v_h}$ exhibits $\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$ as the stabilisation of $\mathcal{H}\mathrm{o}_{v_h}$. In particular, we have $\Sigma^{\infty}_{v_h} \simeq \mathrm{L}_{\mathrm{T}(h)} \circ \Sigma^{\infty}_{\mathcal{H}\mathrm{o}_*}$, see [Heu21, §3.3]

6.2.2.9. Proposition. — *Recall the augmentation ideal functor* I_0 *from Corollary 6.1.0.10. The functor*

$$\mathrm{indec}^{+}_{\mathcal{L}\mathrm{ie}} \colon \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}) \xrightarrow{\mathrm{indec}} \mathcal{S}\mathrm{p}_{\mathrm{T}(h)} \xrightarrow{\mathrm{I}_0} \mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{\mathcal{E}_0}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})$$

is symmetric monoidal with respect to the standard symmetric monoidal structure on $\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$ *and the cartesian monoidal structure on* $\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})$.

Proof. — The functor $\Sigma^{\infty}_{+,v_h} := \mathrm{L}_{\mathrm{T}(h)} \circ \Sigma^{\infty}_{+} \colon \mathcal{H}\mathrm{o}_{v_h} \to \mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$ is symmetric monoidal with respect to the cartesian monoidal structure on $\mathcal{H}\mathrm{o}_{v_h}$ (given by taking the product of the underlying pointed homotopy types) and the standard symmetric monoidal structure on $\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$ given by the smash product of the underlying spectra, because each functor in the composition does, see [HA, Proposition 2.2.1.9].

Furthermore, we can lift Σ^{∞}_{+,v_h} to a symmetric monoidal functor

$$\Sigma^{\infty}_{+,v_h} \colon \mathcal{H}\mathrm{o}_{v_h} \to \mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{\mathcal{E}_0}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)});$$

every object $X \in \mathcal{H}\mathrm{o}_{v_h}$ is pointed and the functor Σ^{∞}_{+,v_h} assigns to the canonical maps $\mathrm{pt} \to X \to \mathrm{pt}$ the following morphisms

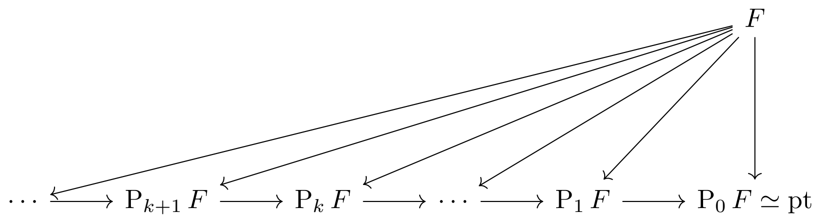
$$\mathrm{L}_{\mathrm{T}(h)}(\mathbb{S}) \to \left(\mathrm{L}_{\mathrm{T}(h)} \circ \Sigma^{\infty}_{+}\right)(X) \simeq \left(\mathrm{L}_{\mathrm{T}(h)} \circ \Sigma^{\infty}\right)(X) \oplus \mathrm{L}_{\mathrm{T}(h)}(\mathbb{S}) \to \mathrm{L}_{\mathrm{T}(h)}(\mathbb{S}),$$

where the $\mathrm{T}(h)_{\bullet}$-local sphere spectrum $\mathrm{L}_{\mathrm{T}(h)}(\mathbb{S})$ is the symmetric monoidal unit of $\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$. In particular, this shows that $\mathrm{I}_0 \circ \Sigma^{\infty}_{+,v_h} \simeq \Sigma^{\infty}_{v_h}$. Under the equivalence $\mathcal{H}\mathrm{o}_{v_h} \simeq \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})$, the functor Σ^{∞}_{+,v_h} corresponds to $\mathrm{indec}^{+}_{\mathcal{L}\mathrm{ie}}$. □

***6.2.2.10.* Goodwillie calculus.** — Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor where

(i) $\mathcal{C}$ admits finite colimits and final objects,
(ii) $\mathcal{D}$ admits finite limits and sequential colimits, and
(iii) sequential colimits commute with finite limits in $\mathcal{D}$.

Using the theory of Goodwillie calculus [Goo03], one can construct the following commutative diagram



of functors from $\mathcal{C}$ to $\mathcal{D}$ where the functor $\mathrm{P}_k F$ is the best approximation of F by "k-polynomial" functors for every $k \geq 0$. Very informally speaking, one can consider the lower horizontal tower as an approximation of the functor F, analogue to the Taylor approximation of a function.

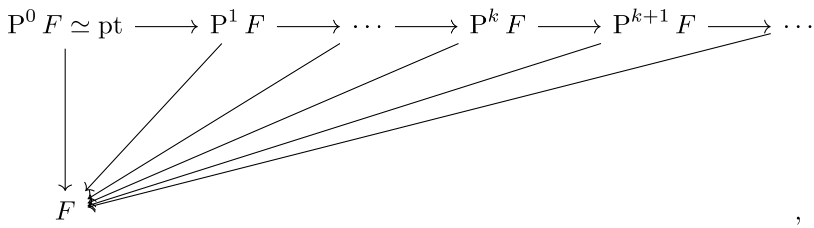
This diagram, or the lower horizontal tower, is known as the *Goodwillie tower* of F. In particular, we have

$$\mathrm{P}_1(F) \simeq \varinjlim_{n\geq 0} \Omega^n_{\mathcal{D}} \circ F \circ \Sigma^n_{\mathcal{C}},$$

known as the "linear approximation" of F, see [HA, Example 6.1.1.28]. For $\mathcal{C} = \mathcal{D}$ and $F = \mathrm{id}$ we see that $\mathrm{P}_1(F) \simeq \Omega^\infty_{\mathcal{C}} \Sigma^\infty_{\mathcal{C}}$, which relates closely to the stabilisation of $\mathcal{C}$. We refer the reader to [Goo03; HA] for a detailed introduction of this theory and to [Kuh07] for nice applications.

***6.2.2.11.* Dual Goodwillie calculus of Endofunctors of $\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$.** — Let F be the functor in ¶6.2.2.10. Because of the hypotheses (i)-(iii) in ¶6.2.2.10, one can not simply construct a Goodwillie tower of the functor $F^{\mathrm{op}}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$: For example, the opposite category $\mathcal{D}^{\mathrm{op}}$ does not satisfy the conditions (ii) and (iii) in general.

However, if $\mathcal{D}$ is stable and $\mathcal{C}$ admits finite limits and initial objects, then F^{op} does fulfil the hypotheses (i)-(iii) in ¶6.2.2.10. Assume that we work in this situation. Then the *dual Goodwillie tower* of F is defined as the Goodwillie tower of $F^{\mathrm{op}}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$, illustrated by the commutative diagram in the ∞-category $\mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{D})$ below:



,

This is called the *dual Goodwillie tower* of F. Similarly, we have the following *colinear approximation*

$$\mathrm{P}^1(F) \simeq \varprojlim_{n\geq 0} \Sigma^n_{\mathcal{C}} \circ F \circ \Omega^n_{\mathcal{C}}.$$

For $k \in \mathbb{N}$, we say

(i) F is *k-polynomial* if $\mathrm{P}^k F \simeq F$, and

(ii) F is *k-homogeneous* if $\mathrm{P}^k F \simeq F$ and $\mathrm{P}^i F = \mathrm{pt}$ for $i \leq k$.

The Dual Goodwillie tower of a functor $F\colon \mathcal{S}\mathrm{p}_{\mathrm{T}(h)} \to \mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$ has particular nice properties, see [Heu21, §4.1, Appendix B] for more details. We need the following one lemma for our latter applications.

6.2.2.12. Lemma. — *Consider a functor $F\colon \mathcal{S}\mathrm{p}_{\mathrm{T}(h)} \to \mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$ satisfying the following property: There exists a sequence $(F_j)_{j\geq 1}$ of endofunctors of $\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$ such that*

(i) $F \simeq \coprod_{j=1}^{\infty} F_j$ and

(ii) F_j is j-homogeneous for every $j \geq 1$.

Then for every $k \geq 1$, the natural map

$$\coprod_{j=1}^{k} F_j \to \mathrm{P}^k F$$

is an equivalence of functors.

Proof. — See [Heu21, Lemma 4.6]. The proof uses a "uniform nilpotence" result for $\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$, which uses the fact that the Tate construction in $\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$ vanishes; this does not hold for a general stable ∞-category, not even for $\mathcal{S}\mathrm{p}$, see [Heu21, Lemma B.4]. □

6.3. Higher enveloping algebras in positive chromatic heights

We fix a prime number p and a natural number $h \geq 1$ throughout this section. Recall the adjunction (6.2.0.1)

$$\mathrm{U}_\infty \colon \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}) \rightleftarrows \varprojlim_n \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_n}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}) : \mathrm{T}_\infty .$$

Based on Theorem 6.2.1.17 and ¶6.2.2.3, we conjecture that U_∞ (and thus T_∞) is an equivalence of ∞-categories. In this section we prove that U_∞ is fully faithful, which is original work.

6.3.0.1. Theorem. — *The functor*

$$\mathrm{U}_\infty \colon \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}) \to \varprojlim \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_n}\left(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}\right)$$

is fully faithful.

***6.3.0.2.* Proof strategy.** — We explain our proof strategy first before going into detailed arguments. To prove the fully faithfulness of U_∞, it is equivalent to show that the unit natural transformation

$$\mathrm{id} \to \mathrm{T}_\infty \circ \mathrm{U}_\infty$$

of the adjunction $\mathrm{U}_\infty \dashv \mathrm{T}_\infty$ is an equivalence, i.e. we need to show that the evaluation $L \to (\mathrm{T}_\infty \circ \mathrm{U}_\infty)(L)$ is an equivalence for every spectral Lie algebra $L \in \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})$. By [HTT, Proposition 1.2.4.1] it suffices to show that for every $M \in \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})$, the induced map

$$\mathcal{M}\mathrm{ap}_{\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})}(M, L) \to \mathcal{M}\mathrm{ap}_{\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})}(M, (\mathrm{T}_\infty \circ \mathrm{U}_\infty)(L)) \tag{6.3.0.1}$$

on mapping spaces is an equivalence in $\mathcal{H}\mathrm{o}$.

Let V_h be any pointed finite complex of type h. Recall that $\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$ is generated under small colimits by the $\mathrm{L}^{\mathrm{f}}_h\,\Sigma^\infty V_h$, see ¶6.2.2.3. Furthermore, every spectral Lie algebra is equivalent to a sifted colimit of free spectral Lie algebras. Thus, to verify that (6.3.0.1) is an equivalence for every spectral Lie algebra M, it suffices to prove it for $M = \mathrm{free}_{\mathcal{L}\mathrm{ie}}(\mathrm{L}^{\mathrm{f}}_h\,\Sigma^\infty V_h)$. Using series of adjunctions and that $\mathrm{T}_\infty \circ \mathrm{U}_\infty$ preserves finite limits (Proposition 6.3.0.8), we reduce the problem to proving that there exists an equivalence

$$L^{V_h} \xrightarrow{\sim} (\mathrm{T}_\infty \circ \mathrm{U}_\infty)\left(L^{V_h}\right) \tag{6.3.0.2}$$

for every $L \in \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})$, where L^{V_h} denotes the limit of the constant diagram $V_h \to \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})$ sending every vertex of V_h to L (Proposition 6.3.0.6). As the last step, we choose a finite complex V of type h which admits a v_h self-map and show that (6.3.0.2) is an equivalence in the case $V_h = V$ (Proposition 6.3.0.13), using the fact that L^V is a trivial spectral Lie algebra (Corollary 6.3.0.12).

6.3.0.3. Proposition. — *The functor* U_∞ *is fully faithful if and only if there exists an equivalence* $\mathrm{T}_\infty \circ \mathrm{U}_\infty \simeq \mathrm{id}$ *of functors.*

Proof. — See [JohSE, Lemma 1.1.1]. □

6.3.0.4. Situation. — Let V be a finite complex of type h together with a v_h self-map $v\colon \Sigma^d V \to V$.

6.3.0.5. Definition. — For $L \in \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})$, define the $\mathrm{T}(h)_\bullet$-local spectral Lie algebra L^V as the limit of the constant diagram $V \to \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})$ sending every point of V to L.

By construction the endo-functor $(-)^V$ of $\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})$ is right adjoint to the *copower functor* $V \otimes (-)$, which assigns to an object $L \in \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})$ the colimit $V \otimes L$ of the just-mentioned constant diagram $V \to \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})$.

6.3.0.6. Proposition. — *Consider unit natural transformation* $\eta\colon \mathrm{id} \to \mathrm{T}_\infty \circ \mathrm{U}_\infty$ *of the adjunction* $\mathrm{U}_\infty \dashv \mathrm{T}_\infty$. *The following statements are equivalent:*

(i) *The natural transformation* η *is an equivalence of functors.*

(ii) *The natural transformation* η *induces the equivalence*

$$(\mathrm{T}_\infty \circ \mathrm{U}_\infty)((-)^V) \simeq (-)^V.$$

of functors below

6.3.0.7. — Recall the notations from §6.1. The $(\mathrm{U}_\infty \dashv \mathrm{T}_\infty)$-adjunction is induced by the adjunctions $\mathrm{U}_n \dashv \mathrm{T}_n$ where, for every $n \geq 1$,

$$\begin{aligned}\mathrm{U}_n\colon \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}) &\to \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_n}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}) \\ L &\mapsto (\mathrm{I}_{\mathcal{E}_n} \circ \mathrm{indec}^{+,n}_{\mathcal{L}\mathrm{ie}} \circ \widetilde{\Omega}^n_{\mathcal{L}\mathrm{ie}})(L) \\ \mathrm{T}_n\colon \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_n}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}) &\to \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}) \\ K_n &\mapsto (\mathrm{Bar}_n \circ \mathrm{triv}^{+,n}_{\mathcal{L}\mathrm{ie}} \circ \mathrm{A}_{\mathcal{E}_n})(K_n),\end{aligned}$$

by Proposition 6.1.0.22, since we showed in Proposition 6.2.2.9 that the functor $\mathrm{indec}^+_{\mathcal{L}\mathrm{ie}}$ is symmetric monoidal. Denote the right adjoint to the "non-unital" Bar construction B_n by C_n, see Corollary 6.1.0.14. The family of natural transformations

$$\mathrm{T}_{n+1} \circ \mathrm{U}_{n+1} \to \mathrm{T}_{n+1} \circ (\mathrm{C}_{n+1} \circ \mathrm{B}_{n+1}) \circ \mathrm{U}_{n+1} \to \mathrm{T}_n \circ \mathrm{U}_n$$

for $n \geq 0$ induces an equivalence

$$\begin{aligned}\mathrm{T}_\infty \circ \mathrm{U}_\infty &\simeq \varprojlim (\mathrm{T}_n \circ \mathrm{U}_n) \\ &\simeq \varprojlim_n \left(\mathrm{Bar}_n \circ \mathrm{triv}^{n,+}_{\mathcal{L}\mathrm{ie}} \circ \mathrm{indec}^{n,+}_{\mathcal{L}\mathrm{ie}} \circ \Omega^n_{\mathcal{L}\mathrm{ie}}\right)\end{aligned}$$

of functors, where the limits are taken in ∞-category of endofunctors of $\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})$ and the first equivalence holds by the universal property of the limit.

To prove Proposition 6.3.0.6, we need to use the following property of the functor $\mathrm{T}_\infty \circ \mathrm{U}_\infty$.

6.3.0.8. Proposition. — *The functor* $\mathrm{T}_\infty \circ \mathrm{U}_\infty$ *preserves finite limits.*

6.3.0.9. Notation. — In the proof of Proposition 6.3.0.8, we make the following simplification of notations.

(i) Abbreviate the forgetful functor $\mathrm{forg}_{\mathcal{E}_n}$ by forg^n.

(ii) Abbreviate the trivial augmentation $\mathrm{A}_{\mathcal{E}_n}$ and augmentation ideal $\mathrm{I}_{\mathcal{E}_n}$ functor by A_n and I_n, respectively.

(iii) Abbreviate the suspension and loop functor of $\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$ by Σ and Ω, respectively.

(iv) For $m, n \in \mathbb{N} \cup \{\infty\}$ with $m > n$, let forg^m_n denote the forgetful functor

$$\mathcal{A}\mathrm{lg}_{\mathcal{E}_m}(\mathcal{D}) \to \mathcal{A}\mathrm{lg}_{\mathcal{E}_n}(\mathcal{D}).$$

for a symmetric monoidal ∞-category $\mathcal{D}$, induces by the morphism $i^m_n\colon\, \mathcal{E}_n \to \mathcal{E}_m$ of ∞-operads.

(v) Recall the n-fold loop functor

$$\Omega^n_{\mathcal{L}\mathrm{ie}}\colon\, \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}) \to \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})$$

and its factorisation

$$\widetilde{\Omega}^n_{\mathcal{L}\mathrm{ie}}\colon\, \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}) \to \mathcal{A}\mathrm{lg}_{\mathcal{E}_n}\left(\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})\right)$$

from Proposition 6.1.0.19. In particular, we have

$$\Omega^n_{\mathcal{L}\mathrm{ie}} = \mathrm{forg}^n \circ \widetilde{\Omega}^n_{\mathcal{L}\mathrm{ie}}.$$

Proof of Proposition 6.3.0.8. — Since the forgetful functor $\mathrm{forg}_{\mathcal{L}\mathrm{ie}}$ is conservative, it suffices to prove that the composition

$$\mathrm{forg}_{\mathcal{L}\mathrm{ie}} \circ \mathrm{T}_\infty \circ \mathrm{U}_\infty\colon\, \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}) \to \mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$$

preserves finite limits. Moreover, it is then equivalent to show that $\mathrm{forg}_{\mathcal{L}\mathrm{ie}} \circ \mathrm{T}_\infty \circ \mathrm{U}_\infty$ preserves the zero object and commutes with the loop functors, because the target ∞-category is stable, see [HTT, Corollary 4.4.2.5] and [Heu21, Lemma 3.9].

Since every single functor in the composition preserves the zero object, the composition also does. We write the functor $\mathrm{forg}_{\mathcal{L}\mathrm{ie}} \circ \mathrm{T}_\infty \circ \mathrm{U}_\infty$ explicitly as

$$\begin{aligned}
\mathrm{forg}_{\mathcal{L}\mathrm{ie}} \circ \mathrm{T}_\infty \circ \mathrm{U}_\infty &\simeq \mathrm{forg}_{\mathcal{L}\mathrm{ie}} \circ \varprojlim_n (\mathrm{T}_n \circ \mathrm{U}_n) \\
&\simeq \varprojlim_n ((\mathrm{forg}_{\mathcal{L}\mathrm{ie}} \circ \mathrm{T}_n) \circ \mathrm{U}_n) \\
&\simeq \varprojlim_n \left(\Sigma^n \circ \mathrm{forg}^{\mathrm{nu}}_{\mathcal{E}_n} \circ \mathrm{U}_n\right) \\
&\simeq \varprojlim_n \left(\Sigma^n \circ \mathrm{forg}^{\mathrm{nu}}_{\mathcal{E}_n} \circ \mathrm{I}_n \circ \mathrm{indec}^{n,+}_{\mathcal{L}\mathrm{ie}} \circ \widetilde{\Omega}^n_{\mathcal{L}\mathrm{ie}}\right)
\end{aligned} \tag{6.3.0.3}$$

where the second equivalence holds because $\mathrm{forg}_{\mathcal{L}\mathrm{ie}}$ commutes with small limits, and the third equivalence holds by (6.1.0.3). For every $L \in \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})$, we obtain

$$\begin{aligned}
&(\mathrm{forg}_{\mathcal{L}\mathrm{ie}} \circ \mathrm{T}_\infty \circ \mathrm{U}_\infty)(\Omega_{\mathcal{L}\mathrm{ie}}(L)) \\
&\simeq \varprojlim_n \left(\Sigma^n \circ \mathrm{forg}_n^{\mathrm{nu}} \circ \mathrm{I}_n \circ \mathrm{indec}_{\mathcal{L}\mathrm{ie}}^{+,n} \circ \widetilde{\Omega}_{\mathcal{L}\mathrm{ie}}^n\right)(\Omega_{\mathcal{L}\mathrm{ie}}(L)) \\
&\overset{(a)}{\simeq} \varprojlim_n \left(\Sigma^n \circ \mathrm{forg}_n^{\mathrm{nu}} \circ \mathrm{I}_n \circ \mathrm{indec}_{\mathcal{L}\mathrm{ie}}^{+,n} \circ \mathrm{forg}_n^{n+1} \circ \widetilde{\Omega}_{\mathcal{L}\mathrm{ie}}^{n+1}\right)(L) \\
&\overset{(b)}{\simeq} \varprojlim_n \left(\Sigma^n \circ \mathrm{forg}_n^{\mathrm{nu}} \circ \mathrm{I}_n \circ \mathrm{forg}_n^{n+1} \circ \mathrm{indec}_{\mathcal{L}\mathrm{ie}}^{+,n+1} \circ \widetilde{\Omega_{\mathcal{L}\mathrm{ie}}^{n+1}}\right)(L) \\
&\overset{(c)}{\simeq} \Omega\left(\varprojlim_n \left(\Sigma^{n+1} \circ \mathrm{forg}_{n+1}^{\mathrm{nu}} \circ \mathrm{I}_{n+1} \circ \mathrm{indec}_{\mathcal{L}\mathrm{ie}}^{+,n+1} \circ \widetilde{\Omega}_{\mathcal{L}\mathrm{ie}}^{n+1}\right)(L)\right) \\
&\simeq \Omega((\mathrm{forg}_{\mathcal{L}\mathrm{ie}}\, \mathrm{T}_\infty\, \mathrm{U}_\infty)(X)) \,.
\end{aligned}$$

The equivalence (a) holds because of the equivalence

$$\mathrm{forg}_n^{n+1} \circ \widetilde{\Omega}_{\mathcal{L}\mathrm{ie}}^{n+1} \simeq \widetilde{\Omega}_{\mathcal{L}\mathrm{ie}}^{n} \circ \mathrm{forg}^1 \circ \widetilde{\Omega}_{\mathcal{L}\mathrm{ie}},$$

since every functor here is a right adjoint. The equivalence (b) holds because the functor $\mathrm{indec}_{\mathcal{L}\mathrm{ie}}^{+}$ is symmetric monoidal, and thus induces the following commutative diagram

$$\begin{array}{ccc}
\mathcal{A}\mathrm{lg}_{\mathcal{E}_m}\left(\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})\right) & \xrightarrow{\mathrm{indec}_{\mathcal{L}\mathrm{ie}}^{+,m}} & \mathcal{A}\mathrm{lg}_{\mathcal{E}_m}\left(\mathcal{A}\mathrm{lg}_{\mathcal{E}_0}^{\mathrm{aug}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})\right) \simeq \mathcal{A}\mathrm{lg}_{\mathcal{E}_m}^{\mathrm{aug}}\left(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}\right) \\
\Big\downarrow \mathrm{forg}_n^m & & \Big\downarrow \mathrm{forg}_n^m \\
\mathcal{A}\mathrm{lg}_{\mathcal{E}_n}\left(\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})\right) & \xrightarrow[\mathrm{indec}_{\mathcal{L}\mathrm{ie}}^{+,n}]{} & \mathcal{A}\mathrm{lg}_{\mathcal{E}_n}\left(\mathcal{A}\mathrm{lg}_{\mathcal{E}_0}^{\mathrm{aug}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})\right) \simeq \mathcal{A}\mathrm{lg}_{\mathcal{E}_n}^{\mathrm{aug}}\left(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}\right),
\end{array}$$

for every $m, n \in \mathbb{N} \cup \{\infty\}$ with $m > n$. The equivalence (c) holds by

$$\mathrm{forg}_n^{\mathrm{nu}} \circ \mathrm{I}_n \circ \mathrm{forg}_n^{n+1} \simeq \mathrm{forg}_{n+1}^{\mathrm{nu}} \circ \mathrm{I}_{n+1},$$

following from Corollary 6.1.0.10. ☐

6.3.0.10. Corollary. — *For every spectral Lie algebra $L \in \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})$, there exists an natural equivalence*

$$(\mathrm{T}_\infty\, \mathrm{U}_\infty)(L^V) \simeq ((\mathrm{T}_\infty\, \mathrm{U}_\infty)(L))^V$$

of spectral Lie algebras.

Proof. — The object L^V is obtained by a finite limit construction. ☐

Proof of Proposition 6.3.0.6. — It is obvious that (i) implies (ii). We show that (ii) implies (i). Abbreviate the mapping space $\mathcal{M}\mathrm{ap}_{\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})}$ by $\mathcal{M}\mathrm{ap}_{\mathcal{L}\mathrm{ie}}$ in this proof.

As we explained in ¶6.3.0.2, it suffices to prove that the unit natural transformation of the adjunction $\mathrm{U}_\infty \dashv \mathrm{T}_\infty$ induces an equivalence

$$\begin{array}{c} \mathcal{M}\mathrm{ap}_{\mathcal{L}\mathrm{ie}}\left(\mathrm{free}_{\mathcal{L}\mathrm{ie}}\left(\mathrm{L}_h^{\mathrm{f}}\,\Sigma^\infty V\right),\ L\right) \\ \big\downarrow \simeq \\ \mathcal{M}\mathrm{ap}_{\mathcal{L}\mathrm{ie}}\left(\mathrm{free}_{\mathcal{L}\mathrm{ie}}\left(\mathrm{L}_h^{\mathrm{f}}\,\Sigma^\infty V\right),\ (\mathrm{T}_\infty \circ \mathrm{U}_\infty)(L)\right) \end{array} \tag{6.3.0.4}$$

for every $L \in \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})$. Then we have

$$\begin{aligned} &\mathcal{M}\mathrm{ap}_{\mathcal{L}\mathrm{ie}}\left(\mathrm{free}_{\mathcal{L}\mathrm{ie}}\left(\mathrm{L}_h^{\mathrm{f}}\,\Sigma^\infty V\right),\ L\right) \\ \simeq\, &\mathcal{M}\mathrm{ap}_{\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}}\left(\mathrm{L}_h^{\mathrm{f}}\,\Sigma^\infty V,\ \mathrm{forg}_{\mathcal{L}\mathrm{ie}}\, L\right) \\ \simeq\, &\mathcal{M}\mathrm{ap}_{\mathcal{S}\mathrm{p}}\left(\Sigma^\infty V,\ \mathrm{forg}_{\mathcal{L}\mathrm{ie}}\, L\right) \\ \simeq\, &\mathcal{M}\mathrm{ap}_{\mathcal{S}\mathrm{p}}\left(V \otimes \mathbb{S},\ \mathrm{forg}_{\mathcal{L}\mathrm{ie}}\, L\right) \\ \simeq\, &\mathcal{M}\mathrm{ap}_{\mathcal{S}\mathrm{p}}\left(\mathbb{S},\ (\mathrm{forg}_{\mathcal{L}\mathrm{ie}}\, L)^V\right) \\ \simeq\, &\mathcal{M}\mathrm{ap}_{\mathcal{S}\mathrm{p}}\left(\mathbb{S},\ \mathrm{forg}_{\mathcal{L}\mathrm{ie}}\,(L^V)\right), \end{aligned} \tag{6.3.0.5}$$

where the first equivalence holds by adjunction free $\dashv$ forg, the second holds by the universal property of the localisation $\mathrm{L}_n^{\mathrm{f}}$, the third and fourth hold by the copower–power adjunction $(- \otimes V) \dashv (-)^V$ and the last equivalence holds because the forgetful functor preserves small limits.

Similarly for the target mapping space, we have

$$\begin{array}{c} \mathcal{M}\mathrm{ap}_{\mathcal{L}\mathrm{ie}}\left(\mathrm{free}_{\mathcal{L}\mathrm{ie}}\left(\mathrm{L}_h^{\mathrm{f}}\,\Sigma^\infty V\right),\ (\mathrm{T}_\infty \circ \mathrm{U}_\infty)(L)\right) \\ \big\downarrow \simeq \\ \mathcal{M}\mathrm{ap}_{\mathcal{S}\mathrm{p}}\left(\mathbb{S},\ \mathrm{forg}_{\mathcal{L}\mathrm{ie}}\left(((\mathrm{T}_\infty \circ \mathrm{U}_\infty)(L))^V\right)\right) \end{array}$$

Thus, the morphism (6.3.0.4) is equivalent to following morphism

$$\mathcal{M}\mathrm{ap}_{\mathcal{S}\mathrm{p}}\left(\mathbb{S},\ \mathrm{forg}_{\mathcal{L}\mathrm{ie}}\,(L^V)\right) \to \mathcal{M}\mathrm{ap}_{\mathcal{S}\mathrm{p}}\left(\mathbb{S},\ \mathrm{forg}_{\mathcal{L}\mathrm{ie}}\left(((\mathrm{T}_\infty \circ \mathrm{U}_\infty)(L))^V\right)\right),$$

which is indeed an equivalence for every $L \in \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})$, by assumption (ii) and Corollary 6.3.0.10. □

6.3.0.11. Proposition. — *For every $L \in \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})$, the spectral Lie algebra L^V is an infinite loop object of the ∞-category $\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})$.*

Proof. — Recall the v_h periodic self-map $v\colon \Sigma^d V \to V$ from Situation 6.3.0.4, which becomes an equivalence $(\mathrm{L}_h^{\mathrm{f}} \circ \Sigma^\infty)(v)$ in $\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$, see ¶6.2.2.3. Thus, we expect the following equivalence in $\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})$.

Claim. The map v induces an equivalence $L^v\colon L^V \to L^{\Sigma^d V}$ of spectral Lie algebras.

The equivalence $v_*\colon \mathrm{L}_h^{\mathrm{f}}\,\Sigma^{\infty+d}V \to \mathrm{L}_h^{\mathrm{f}}\,\Sigma^{\infty}V$ in $\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$ induces an equivalence

$$\mathrm{Map}_{\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}}\left(\mathrm{L}_h^{\mathrm{f}}\,\Sigma^{\infty}V,\ \mathrm{forg}_{\mathcal{L}\mathrm{ie}}\,L\right) \xrightarrow{\sim} \mathrm{Map}_{\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}}\left(\mathrm{L}_h^{\mathrm{f}}\,\Sigma^{\infty+d}V,\ \mathrm{forg}_{\mathcal{L}\mathrm{ie}}\,L\right).$$

With the same arguments as we proved the equivalences in (6.3.0.5) we obtain by adjunctions an equivalence

$$\mathrm{forg}_{\mathcal{L}\mathrm{ie}}\left(L^V\right) \to \mathrm{forg}_{\mathcal{L}\mathrm{ie}}\left(L^{\Sigma^d V}\right),$$

of spectra. The claim follows by the fact that the functor $\mathrm{forg}_{\mathcal{L}\mathrm{ie}}$ is conservative.

Considering the $(\Sigma \dashv \Omega)$-adjunction of pointed homotopy types, we show the following claims.

Claim. There exists a natural equivalence

$$\Omega^d_{\mathcal{L}\mathrm{ie}}(L^V) \simeq L^{\Sigma^d V}$$

of spectral Lie algebras.

Let S^d denote the d-dimensional sphere. The following sequence of natural equivalences proves the claim:

$$L^{\Sigma^d V} \simeq \varprojlim_{\Sigma^d V} L \simeq \varprojlim_{\mathrm{S}^d} \varprojlim_{V} L \simeq \varprojlim_{\mathrm{S}^d}(L^V) \simeq \Omega^d_{\mathcal{L}\mathrm{ie}}(L^V). \qquad \square$$

6.3.0.12. Corollary. — *For every $L \in \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})$, the spectral Lie algebra L^V is trivial, i.e. it lies in the image of the trivial spectral Lie algebra functor* $\mathrm{triv}_{\mathcal{L}\mathrm{ie}}$ *(see Example 5.2.4.20). In particular, there exists a natural equivalence*

$$L^V \simeq (\mathrm{triv}_{\mathcal{L}\mathrm{ie}} \circ \mathrm{forg}_{\mathcal{L}\mathrm{ie}})(L^V)$$

in the ∞-category $\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})$.

Proof. — Recall that the adjunction $\mathrm{indec}_{\mathcal{L}\mathrm{ie}} \dashv \mathrm{triv}_{\mathcal{L}\mathrm{ie}}$ exhibits $\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$ as the stabilisation of the ∞-category $\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})$ (Theorem 6.2.2.7). In other words, the image of $\mathrm{triv}_{\mathcal{L}\mathrm{ie}}$ is exactly the set of infinite loop objects of $\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})$, see ¶6.2.2.6. So by Proposition 6.3.0.11, the spectral Lie algebra L^V is contained in the image of $\mathrm{triv}_{\mathcal{L}\mathrm{ie}}$. Thus we have

$$L^V \simeq (\mathrm{triv}_{\mathcal{L}\mathrm{ie}} \circ \mathrm{forg}_{\mathcal{L}\mathrm{ie}})(L^V),$$

because $\mathrm{forg}_{\mathcal{L}\mathrm{ie}} \circ \mathrm{triv}_{\mathcal{L}\mathrm{ie}} \simeq \mathrm{id}_{\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})}$ by Example 5.2.4.21. $\square$

6.3.0.13. Proposition. — *There exists a natural equivalence*

$$(\mathrm{T}_\infty \circ \mathrm{U}_\infty)\left(L^V\right) \simeq L^V$$

of $\mathrm{T}(h)_\bullet$-local spectral Lie algebras, for every $L \in \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})$.

Proof. — Since the forgetful functor is conservative, it is enough to show that

$$(\mathrm{forg}_{\mathcal{L}\mathrm{ie}} \circ \mathrm{T}_\infty \circ \mathrm{U}_\infty)\left(L^V\right) \simeq \mathrm{forg}_{\mathcal{L}\mathrm{ie}}\left(L^V\right).$$

We will use the abbreviations for notations as in Notation 6.3.0.9. Recall the expression of $\mathrm{forg}_{\mathcal{L}\mathrm{ie}} \circ \mathrm{T}_\infty \circ \mathrm{U}_\infty$ from (6.3.0.3). We have

$$\begin{aligned}
&\left(\mathrm{forg}_{\mathcal{L}\mathrm{ie}} \circ \mathrm{T}_\infty \circ \mathrm{U}_\infty\right)\left(L^V\right)\\
\simeq{}& \varprojlim_n \left(\Sigma^n \circ \mathrm{forg}_n^{\mathrm{nu}} \circ \mathrm{I}_n \circ \mathrm{indec}_{\mathcal{L}\mathrm{ie}}^{+,n} \circ \widetilde{\Omega}^n_{\mathcal{L}\mathrm{ie}}\right)\left(L^V\right)\\
\overset{\text{(i)}}{\simeq}{}& \varprojlim_n \left(\Sigma^n \circ \mathrm{forg}_n^{\mathrm{nu}} \circ \mathrm{I}_n \circ \mathrm{indec}_{\mathcal{L}\mathrm{ie}}^{+,n} \circ \widetilde{\Omega}^n_{\mathcal{L}\mathrm{ie}} \circ \mathrm{triv}_{\mathcal{L}\mathrm{ie}}\right)\left(\mathrm{forg}_{\mathcal{L}\mathrm{ie}}\left(L^V\right)\right)\\
\overset{\text{(ii)}}{\simeq}{}& \varprojlim_n \left(\Sigma^n \circ \mathrm{forg}_n^{\mathrm{nu}} \circ \mathrm{I}_n \circ \mathrm{indec}_{\mathcal{L}\mathrm{ie}}^{+,n} \circ \mathrm{triv}_{\mathcal{L}\mathrm{ie}}^{+,n} \circ \mathrm{A}_n \circ \Omega^n\right)\left(\mathrm{forg}_{\mathcal{L}\mathrm{ie}}\left(L^V\right)\right)\\
\overset{\text{(iii)}}{\simeq}{}& \varprojlim_n \left(\Sigma^n \circ \mathrm{forg}_n^{\mathrm{nu}} \circ \mathrm{I}_n \circ \left(\mathrm{A}_0 \circ \mathrm{indec}_{\mathcal{L}\mathrm{ie}} \circ \mathrm{triv}_{\mathcal{L}\mathrm{ie}} \circ \mathrm{I}_0\right)^n \circ \mathrm{A}_n \circ \Omega^n\right)\left(\mathrm{forg}_{\mathcal{L}\mathrm{ie}}\left(L^V\right)\right)\\
\overset{\text{(iv)}}{\simeq}{}& \varprojlim_n \left(\Sigma^n \circ \mathrm{Sym} \circ \Omega^n\right)\left(\mathrm{forg}_{\mathcal{L}\mathrm{ie}}\left(L^V\right)\right)\\
\overset{\text{(v)}}{\simeq}{}& \left(\mathrm{P}^1\left(\mathrm{Sym}\right)\right)\left(\mathrm{forg}_{\mathcal{L}\mathrm{ie}} L^V\right)\\
\overset{\text{(vi)}}{\simeq}{}& \mathrm{forg}_{\mathcal{L}\mathrm{ie}}\left(L^V\right).
\end{aligned}$$

The reasons for the equivalences are given below:

(i) is by Corollary 6.3.0.12.

(ii) holds by the equivalence $\widetilde{\Omega}^n_{\mathcal{L}\mathrm{ie}} \circ \mathrm{triv}_{\mathcal{L}\mathrm{ie}} \simeq \mathrm{triv}_{\mathcal{L}\mathrm{ie}}^{+,n} \circ \mathrm{A}_n \circ \Omega^n$, which can be checked by applying the forgetful functor forg^n on both sides, since forg^n is conservative.

(iii) holds by the equivalence

$$\mathrm{indec}_{\mathcal{L}\mathrm{ie}}^{+} \circ \mathrm{triv}_{\mathcal{L}\mathrm{ie}}^{+} \simeq \mathrm{A}_0 \circ \mathrm{indec}_{\mathcal{L}\mathrm{ie}} \circ \mathrm{triv}_{\mathcal{L}\mathrm{ie}} \circ \mathrm{I}_0,$$

and $\left(\mathrm{A}_0 \circ \mathrm{indec}_{\mathcal{L}\mathrm{ie}} \circ \mathrm{triv}_{\mathcal{L}\mathrm{ie}} \circ \mathrm{I}_0\right)^n$ denotes the functor on the ∞-category of augmented $\mathcal{E}_n$-algebras, induced by the composition $\mathrm{A}_0 \circ \mathrm{indec}_{\mathcal{L}\mathrm{ie}} \circ \mathrm{triv}_{\mathcal{L}\mathrm{ie}} \circ \mathrm{I}_0$.

(iv) is a change of notation

$$\mathrm{Sym} := \mathrm{indec}_{\mathcal{L}\mathrm{ie}} \circ \mathrm{triv}_{\mathcal{L}\mathrm{ie}} = \coprod_{r \geq 1} \left((-)^{\otimes r}\right)_{\mathfrak{S}_r},$$

by the Koszul duality between the spectral Lie ∞-operad and the non-unital cocommutative ∞-cooperad, see Proposition 5.3.2.4.

(v) holds by the construction of the first colinear approximation $\mathrm{P}^1(F)$ of F in the dual Goodwillie calculus tower, see ¶6.2.2.11.

(vi) holds because we have $\mathrm{P}^1\,\mathrm{Sym} \simeq \mathrm{id}_{\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}}$ by Lemma 6.2.2.12. ☐

Proof of Theorem 6.3.0.1. — This is a consequence of Proposition 6.3.0.13, Proposition 6.3.0.6 and Proposition 6.3.0.3, as we explained in the proof strategy ¶6.3.0.2. ☐

Recall the ∞-category $\mathcal{S}\mathrm{p}_{\mathrm{K}(h)}$ of $\mathrm{K}(h)_\bullet$-local spectra, where $\mathrm{K}(h)$ is the p-local Morava K-theory spectrum.

6.3.0.14. Corollary. — *The functor*

$$\mathrm{U}_\infty\colon \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{K}(h)}) \to \varprojlim \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_n}\left(\mathcal{S}\mathrm{p}_{\mathrm{K}(h)}\right),$$

obtained from the diagram (6.1.0.1)*, is fully faithful.*

Proof. — The ∞-category $\mathcal{S}\mathrm{p}_{\mathrm{K}(h)}$ is the reflective localisation of $\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$ at the set of $\mathrm{K}(h)_\bullet$-equivalences of (p-local) spectra. Thus one can prove the corollary using the same proof strategy ¶6.3.0.2 as for Theorem 6.3.0.1: Every statement holds after replacing $\mathrm{T}(h)_\bullet$ by $\mathrm{K}(h)$. In particular,

(i) By [HTT, Corollary 5.5.7.3] the ∞-category $\mathcal{S}\mathrm{p}_{\mathrm{K}(h)}$ is compactly generated by one compact object $\mathrm{L}_{\mathrm{K}(h)}\,\Sigma^\infty V_h$.

(ii) Proposition 6.3.0.8 holds with $\mathrm{K}(h)$ in place of $\mathrm{T}(h)$: In the original proof we uses that $\mathrm{indec}^+_{\mathcal{L}\mathrm{ie}}$ is symmetric monoidal. This is still true in the $\mathrm{K}(h)_\bullet$-local case, since the localisation functor $\mathcal{S}\mathrm{p}_{\mathrm{T}(h)} \to \mathcal{S}\mathrm{p}_{\mathrm{K}(h)}$ is product preserving, see [HA, Lemma 1.4.4.7].

□

CHAPTER 7

Costabilisation of v_h-periodic homotopy types

7.1. The costabilisation of an ∞-category

In this section we introduce the theory of costabilisation, the dual notion of stabilisation (¶6.2.2.6).

7.1.0.1. Definition. — Let $\mathcal{C}$ be an ∞-category admits finite colimits. The *costabilisation* $\mathrm{co}\mathcal{S}\mathrm{p}(\mathcal{C})$ of $\mathcal{C}$ is the stabilisation $\mathcal{S}\mathrm{p}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ of the opposite ∞-category $\mathcal{C}^{\mathrm{op}}$. An object of $\mathrm{co}\mathcal{S}\mathrm{p}(\mathcal{C})$ is called a *cospectrum object* of $\mathcal{C}$.

7.1.0.2. Example. — Here are two rather elementary examples.

(i) The costabilisation of the ∞-category $\mathcal{H}\mathrm{o}$ of homotopy types is trivial. Let $F\colon \mathcal{H}\mathrm{o}_*^{\mathrm{fin}} \to \mathcal{H}\mathrm{o}^{\mathrm{op}}$ be a reduced excisive functor. Then we know $F(\mathrm{pt}) = \emptyset$, the terminal object in $\mathcal{H}\mathrm{o}^{\mathrm{op}}$. Let $F(\mathrm{S}^0) = X$. Then the pointed map $\mathrm{pt} \to \mathrm{S}^0$ is sent to $X \to \emptyset$ in $\mathcal{H}\mathrm{o}$ under F, which implies that X is also the empty set. More generally, let $\mathcal{C}$ be an ∞-category with strict initial objects[(1)], then $\mathrm{co}\mathcal{S}\mathrm{p}(\mathcal{C})$ is trivial by the same arguments.

(ii) Let $\mathcal{C}$ be a stable ∞-category. Then we have

$$\mathrm{co}\mathcal{S}\mathrm{p}(\mathcal{C}) = \left(\mathcal{E}\mathrm{xc}_*(\mathcal{H}\mathrm{o}_*^{\mathrm{fin}}, \mathcal{C}^{\mathrm{op}})\right)^{\mathrm{op}} \overset{(a)}{\simeq} (\mathcal{F}\mathrm{un}^{\mathrm{rex}}(\mathcal{H}\mathrm{o}_*, \mathcal{C}^{\mathrm{op}}))^{\mathrm{op}} \overset{(b)}{\simeq} (\mathcal{C}^{\mathrm{op}})^{\mathrm{op}},$$

where $\mathcal{F}\mathrm{un}^{\mathrm{rex}}$ denotes the ∞-category of right exact functors. The equivalence (a) holds because pullbacks and pushouts in a stable ∞-category coincide, see [HA, Proposition 1.1.3.4], and a functor preserves finite colimits if it preserves the initial object and pushouts [HTT, Corollary 4.4.2.5]. The equivalence (b) holds because $\mathcal{H}\mathrm{o}_*^{\mathrm{fin}}$ the ∞-category freely generated by the pt under finite colimits, see [HA, Remark 1.4.2.6].

7.1.0.3. Proposition. — *In the situation of Definition 7.1.0.1 the costabilisation $\mathrm{co}\mathcal{S}\mathrm{p}(\mathcal{C})$ of $\mathcal{C}$ is a stable ∞-category.*

(1) An initial object I is strict if any morphism mapping into I is an equivalence

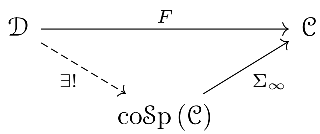
Proof. — The stabilisation of an ∞-category admitting finite limits is stable, see [HA, Proposition 1.4.2.17]. The opposite ∞-category of a stable ∞-category is stable, see [HA, Remark 1.1.1.3]. □

7.1.0.4. Notation. — In the situation of Definition 7.1.0.1, let $\Sigma^{\mathcal{C}}_{\infty}\colon \mathrm{co}\mathcal{S}\mathrm{p}(\mathcal{C}) \to \mathcal{C}$ denote the opposite of the functor $\Omega^{\infty}_{\mathcal{C}^{\mathrm{op}}}\colon \mathcal{S}\mathrm{p}(\mathcal{C}^{\mathrm{op}}) \to \mathcal{C}^{\mathrm{op}}$ (see ¶6.2.2.6). We abbreviate $\Sigma^{\mathcal{C}}_{\infty}$ by Σ_{∞} if it is clear from the context which ∞-category $\mathcal{C}$ we are working with.

7.1.0.5. Proposition. — *The functor $\Sigma_{\infty}\colon \mathrm{co}\mathcal{S}\mathrm{p}(\mathcal{C}) \to \mathcal{C}$ preserves finite colimits.*

Proof. — By [HA, Remark 1.4.2.3] the functor $\Omega^{\infty}_{\mathcal{C}^{\mathrm{op}}}$ preserves finite limits, since finite limits of functors are computed point-wise. Thus the functor $\Sigma^{\mathcal{C}}_{\infty} \simeq (\Omega^{\infty}_{\mathcal{C}})^{\mathrm{op}}$ preserves finite colimits. □

7.1.0.6. Proposition. — *Let $\mathcal{C}$ be an ∞-category admits finite colimits. The costabilisation $\mathrm{co}\mathcal{S}\mathrm{p}(\mathcal{C})$ together with the functor $\Sigma_{\infty}\colon \mathrm{co}\mathcal{S}\mathrm{p}(\mathcal{C}) \to \mathcal{C}$ satisfies the following universal property: Any finite colimits preserving functor F from a stable ∞-category $\mathcal{D}$ to $\mathcal{C}$ factors through $\Sigma_{\infty}\colon \mathrm{co}\mathcal{S}\mathrm{p}(\mathcal{C}) \to \mathcal{C}$, uniquely up to contractible choice. We illustrate the universal property by the following commutative diagrams.*



Proof. — The functor $\Omega^{\infty}_{\mathcal{C}^{\mathrm{op}}}$ satisfies the universal property that it is the terminal finite limits preserving functor from a stable ∞-category to $\mathcal{C}^{\mathrm{op}}$, see [HA, Corollary 1.4.2.23]. Taking the opposite ∞-categories, we obtain the universal property of $\Sigma_{\infty}(\mathcal{C})$. □

7.1.0.7. Proposition. — *Let $\mathcal{C}$ be a pointed ∞-category admitting finite colimits. There exists an equivalence*

$$\mathrm{co}\mathcal{S}\mathrm{p}(\mathcal{C}) \simeq \varprojlim\left(\cdots \xrightarrow{\Sigma_{\mathcal{C}}} \mathcal{C} \xrightarrow{\Sigma_{\mathcal{C}}} \mathcal{C}\right), \tag{7.1.0.1}$$

of stable ∞-categories, where the inverse limit is take in the ∞-category $\mathcal{CAT}_{\infty}$ of (not necessarily small) ∞-categories.

Proof. — The opposite ∞-category $\mathcal{C}^{\mathrm{op}}$ is pointed and admits finite limits. By [HA, Proposition 1.4.2.24], we have

$$\begin{aligned}\mathcal{S}\mathrm{p}\,(\mathcal{C}^{\mathrm{op}}) &\simeq \varprojlim\left(\cdots \xrightarrow{\Omega_{\mathcal{C}^{\mathrm{op}}}} \mathcal{C}^{\mathrm{op}} \xrightarrow{\Omega_{\mathcal{C}^{\mathrm{op}}}} \mathcal{C}^{\mathrm{op}}\right) \\ &\simeq \left(\varprojlim\left(\cdots \xrightarrow{\Sigma_{\mathcal{C}}} \mathcal{C} \xrightarrow{\Sigma_{\mathcal{C}}} \mathcal{C}\right)\right)^{\mathrm{op}}.\end{aligned}$$

Then the proposition follows from the definition of the costabilisation. □

7.1.0.8. Remark. — Let $\mathcal{C}$ be a pointed ∞-category admitting finite colimits. An object of the ∞-category

$$\varprojlim\left(\cdots \xrightarrow{\Sigma_{\mathcal{C}}} \mathcal{C} \xrightarrow{\Sigma_{\mathcal{C}}} \mathcal{C}\right)$$

is a sequence $(X_i)_{i\geq 0}$ of objects in $\mathcal{C}$ such that $X_i \simeq \Sigma_{\mathcal{C}}(X_{i+1})$ for every $i \geq 0$. In particular, we see that

(i) for every $i \geq 0$ the object $X_i \in \mathcal{C}$ admits infinite desuspensions in $\mathcal{C}$, and

(ii) the functor $\Sigma^{\mathcal{C}}_{\infty}$ is equivalent to the canonical functor

$$\varprojlim\left(\cdots \xrightarrow{\Sigma_{\mathcal{C}}} \mathcal{C} \xrightarrow{\Sigma_{\mathcal{C}}} \mathcal{C}\right) \to \mathcal{C},\ (X_i)_{i\geq 0} \mapsto X_0,$$

under the equivalence (7.1.0.1).

7.1.0.9. Corollary. — *The costabilisation of the ∞-category $\mathcal{H}\mathrm{o}_*$ of pointed homotopy types is trivial.*

Proof. — The suspension functor increases the connectivity of pointed homotopy types. In particular, any pointed homotopy types that admits infinite desuspensions is contractible. □

7.1.0.10. Remark. — Let $\mathcal{C}$ be an ∞-category admits finite limits and finite colimits. Denote the final object of $\mathcal{C}$ by $t_{\mathcal{C}}$. Let $\mathcal{C}_* := \mathcal{C}_{t_{\mathcal{C}}/}$ denote the ∞-category of pointed objects of $\mathcal{C}$, see [HTT, Definition 7.2.2.1]. In the situation of stabilisation, the canonical forgetful functor $\mathcal{C}_* \to \mathcal{C}$ induces an equivalence $\mathcal{S}\mathrm{p}(\mathcal{C}_*) \simeq \mathcal{S}\mathrm{p}(\mathcal{C})$ of stable ∞-categories. However, the costabilisations $\mathrm{co}\mathcal{S}\mathrm{p}(\mathcal{C}_*)$ and $\mathrm{co}\mathcal{S}\mathrm{p}(\mathcal{C})$ are in general not equivalent ∞-categories. For example, let $\mathcal{C}$ be $\mathcal{H}\mathrm{o}^{\mathrm{op}}$. We have $(\mathcal{H}\mathrm{o}^{\mathrm{op}})_* \simeq \{\emptyset\}$. Thus, the costabilisation $\mathrm{co}\mathcal{S}\mathrm{p}((\mathcal{H}\mathrm{o}^{\mathrm{op}})_*)$ is trivial, whereas $\mathrm{co}\mathcal{S}\mathrm{p}(\mathcal{H}\mathrm{o}^{\mathrm{op}}) \simeq \mathcal{S}\mathrm{p}(\mathcal{H}\mathrm{o})^{\mathrm{op}} \simeq \mathcal{S}\mathrm{p}^{\mathrm{op}}$ is not trivial.

7.1.0.11. Proposition. — *Let $\mathcal{C}$ be a pointed presentable ∞-category.*

(i) *The costabilisation* $\mathrm{co}\mathcal{S}\mathrm{p}(\mathcal{C})$ *is presentable.*

(ii) *The functor $\Sigma^{\mathcal{C}}_{\infty}\colon \mathrm{co}\mathcal{S}\mathrm{p}(\mathcal{C}) \to \mathcal{C}$ admits a right adjoint $\Omega^{\mathcal{C}}_{\infty}$.*

(iii) *Let $\mathcal{D}$ be a stable presentable ∞-category. An exact functor $F\colon \mathcal{D} \to \mathrm{co}\mathcal{S}\mathrm{p}(\mathcal{C})$ admits a right adjoint if and only if $\Sigma_{\infty}\circ F\colon \mathcal{D} \to \mathcal{C}$ admits a right adjoint.*

Proof. — This is the analogue statement for costabilisation as [HA, Proposition 1.4.4.4] for stabilisation. Since $\mathcal{C}$ is presentable, the functor $\Sigma_{\mathcal{C}}$ admits a right adjoint $\Omega_{\mathcal{C}}$, see [HA, Remark 1.1.2.8]. The inverse limit

$$\varprojlim\left(\cdots \xrightarrow{\Sigma_{\mathcal{C}}} \mathcal{C} \xrightarrow{\Sigma_{\mathcal{C}}} \mathcal{C}\right)$$

in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ is preserved under the inclusion $\mathcal{P}\mathrm{r}^{\mathrm{L}} \hookrightarrow \mathcal{CAT}_{\infty}$, see [HTT, Proposition 5.5.3.13]. Thus $\mathrm{co}\mathcal{S}\mathrm{p}(\mathcal{C})$ is an object of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ and $\Sigma^{\mathcal{C}}_{\infty}$ is a morphism in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$, by Proposition 7.1.0.7. These proved (i) and (ii). Property (iii) follows from the universal property of the costabilisation $\Sigma_{\infty}\colon \mathrm{co}\mathcal{S}\mathrm{p}(\mathcal{C}) \to \mathcal{C}$ in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. □

7.1.0.12. Corollary. — *Let $\mathcal{C}$ and $\mathcal{D}$ be pointed presentable ∞-categories and assume that $\mathcal{D}$ is stable. Let $G\colon \mathcal{C} \to \mathcal{D}$ be a functor that admits a left adjoint. Then G factors through $\Omega_\infty\colon \mathcal{C} \to \mathrm{co}\mathcal{S}\mathrm{p}(\mathcal{C})$, uniquely up to contractible choice. We illustrate this universal property of $\Omega_\infty^{\mathcal{C}}$ by the following commutative diagram.*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\quad G \quad} & \mathcal{D} \\ & {\scriptstyle \Omega_\infty} \searrow \quad \nearrow {\scriptstyle \exists !} & \\ & \mathrm{co}\mathcal{S}\mathrm{p}(\mathcal{C}). & \end{array}$$

Proof. — This follows from Proposition 7.1.0.6 by adjunctions. □

7.1.0.13. Proposition. — *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories admitting finite colimits and $F\colon \mathcal{C} \to \mathcal{D}$ be a functor preserving initial objects.*

(i) *If F is right exact, then F induces an exact functor*

$$\mathbb{F}\colon\ \mathrm{co}\mathcal{S}\mathrm{p}(\mathcal{C}) \to \mathrm{co}\mathcal{S}\mathrm{p}(\mathcal{D})$$

of stable ∞-categories.

(ii) *Assuming that $\mathcal{C}$ and $\mathcal{D}$ are pointed and $F \circ \Sigma_{\mathcal{C}} \simeq \Sigma_{\mathcal{D}} \circ F$, then F induces an exact functor $\mathbb{F}$ of their costabilisations.*

(iii) *In the situation of (i) and (ii), if F is fully faithful, the induced functor $\mathbb{F}$ is also fully faithful.*

Proof. — (i) Since F is right exact, it induces a right exact functor

$$\mathbb{F}\colon\ \mathcal{E}\mathrm{xc}_*\left(\mathcal{H}\mathrm{o}_*^{\mathrm{fin}}, \mathcal{C}^{\mathrm{op}}\right)^{\mathrm{op}} \to \mathcal{E}\mathrm{xc}_*\left(\mathcal{H}\mathrm{o}_*^{\mathrm{fin}}, \mathcal{D}^{\mathrm{op}}\right)^{\mathrm{op}},$$

between stable ∞-categories, since limits of (excisive) functors are computed pointwise, see [HA, Remark 1.4.2.3]. By [HA, Proposition 1.1.4.1], the functor $\mathbb{F}$ is exact.

(ii) Let $G\colon\ \mathcal{H}\mathrm{o}_*^{\mathrm{fin}} \to \mathcal{C}^{\mathrm{op}}$ be a reduced and excisive functor. We show that $F^{\mathrm{op}} \circ G$ is also reduced and excisive, where F^{op} denotes the canonical functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ associated to F.

Let P be a pushout diagram in $\mathcal{H}\mathrm{o}_*^{\mathrm{fin}}$, so we have $G(P)$ is a pullback diagram in $\mathcal{C}^{\mathrm{op}}$. By [Heu21, Lemma 3.9] $(\Omega_{\mathcal{D}^{\mathrm{op}}} \circ F^{\mathrm{op}})(G(P))$ is a pullback diagram in $\mathcal{D}^{\mathrm{op}}$. Since $\mathcal{E}\mathrm{xc}_*\left(\mathcal{H}\mathrm{o}_*^{\mathrm{fin}}, \mathcal{D}^{\mathrm{op}}\right)$ is stable and its limits are computed pointwise $\mathcal{D}^{\mathrm{op}}$, we have that $F(G(P))$ is a pullback diagram in $\mathcal{D}^{\mathrm{op}}$. Similarly, we can show that $\mathbb{F}$ is right exact and thus exact.

(iii) Fully faithfulness of $\mathbb{F}$ is checked pointwise on the induced map on mapping spaces in $\mathcal{C}^{\mathrm{op}}$ and $\mathcal{D}^{\mathrm{op}}$, which can be verified by the fully faithfulness of F. □

7.1.0.14. — Let $\mathcal{C}$ be an ∞-category which admits finite colimits. The ∞-category $\mathrm{co}\mathcal{E}\mathrm{xc}_*(\mathcal{H}\mathrm{o}_*, \mathcal{C})$ of reduced coexcisive functors from $\mathcal{H}\mathrm{o}_*$ to $\mathcal{C}$ is a stable ∞-category, since it is equivalent to the ∞-category $\mathcal{E}\mathrm{xc}_*((\mathcal{H}\mathrm{o}_*)^{\mathrm{op}}, \mathcal{C}^{\mathrm{op}})$, which is stable by [HA, Proposition 1.4.2.16]. Evaluation at a fixed object $X \in \mathcal{C}$ gives a functor $\mathrm{ev}_X\colon \operatorname{co}\mathcal{E}\mathrm{xc}_*(\mathcal{H}\mathrm{o}_*, \mathcal{C}) \to \mathcal{C}$, which is colimit preserving since the colimit of functors are computed pointwise. Thus, by universal property of costabilisation, we obtain the following commutative diagram

$$\begin{array}{ccc} \operatorname{co}\mathcal{E}\mathrm{xc}_*(\mathcal{H}\mathrm{o}_*, \mathcal{C})^{\mathrm{op}} & \xrightarrow{\quad \mathrm{ev}_X \quad} & \mathcal{C} \\ & {\scriptstyle (\mathcal{M}\mathrm{ap}_*(-,X))^*} \searrow \qquad \nearrow {\scriptstyle \mathrm{ev}_{S^0}} & \\ & \operatorname{co}\mathcal{S}\mathrm{p}(\mathcal{C}) \simeq \mathcal{E}\mathrm{xc}_*\left(\mathcal{H}\mathrm{o}_*^{\mathrm{fin}}, \mathcal{C}^{\mathrm{op}}\right)^{\mathrm{op}} & \end{array}$$

where the functor $(\mathcal{M}\mathrm{ap}(-,X))^*$ is given by composing with the functor

$$\mathcal{M}\mathrm{ap}(-,X)\colon\ \mathcal{H}\mathrm{o}_*^{\mathrm{fin}} \to \mathcal{H}\mathrm{o}_*^{\mathrm{op}},\ V \mapsto \mathcal{M}\mathrm{ap}(V,X).$$

7.2. Examples of costabilisations

Let $\mathcal{C}$ be a pointed presentable stable symmetric monoidal ∞-category and let $\mathcal{O}$ be a reduced ∞-operad with values in $\mathcal{C}$ (see Definition 5.3.2.11.(iii)). The ∞-category $\mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C})$ of $\mathcal{O}$-algebras is pointed: The zero object of $\mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C})$ is the zero object of $\mathcal{C}$ endowed with a trivial $\mathcal{O}$-algebra structure. Recall that the free $\mathcal{O}$-algebra functor $\mathrm{free}_{\mathcal{O}} \colon \mathcal{C} \to \mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C})$ is cocontinuous, i.e. small colimits preserving. Since $\mathcal{C}$ is stable, the suspension functor $\Sigma_{\mathcal{C}}$ of $\mathcal{C}$ gives an autoequivalence of $\mathcal{C}$, with the inverse given by the loop functor $\mathcal{C}$. Denote the suspension endofunctor of $\mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C})$ by $\Sigma_{\mathcal{O}}$, which admits a right adjoint $\Omega_{\mathcal{O}}$. Thus, for every object $X \in \mathcal{C}$, we have

$$\begin{aligned}
\mathrm{free}_{\mathcal{O}}(X) &\simeq \mathrm{free}_{\mathcal{O}}\left(\Sigma_{\mathcal{C}}\Omega_{\mathcal{C}}X\right) \\
&\simeq \Sigma_{\mathcal{O}}(\mathrm{free}_{\mathcal{O}}(\Omega_{\mathcal{C}}X)) \\
&\simeq \Sigma_{\mathcal{O}}^2\left(\mathrm{free}_{\mathcal{O}}(\Omega_{\mathcal{C}}^2X)\right) \\
&\simeq \cdots \\
&\simeq \Sigma_{\mathcal{O}}^{\infty}(\mathrm{free}_{\mathcal{O}}(\Omega_{\mathcal{C}}^{\infty}X)) .
\end{aligned}$$

This implies that every free $\mathcal{O}$-algebra admits infinite desuspensions in $\mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C})$, which corresponds by Proposition 7.1.0.7 to a cospectrum object of $\mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C})$. Therefore, we expect that the costabilisation of $\mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C})$ is non-trivial, cf. Example 7.1.0.2. In the following we are going to present some examples of this kind.

***7.2.0.1. Proposition* (Gaitsgory–Rozenblyum).** — *Let $\mathcal{O}$ be a reduced ∞-operad with values in the ∞-category $\mathcal{S}\mathrm{p}_{\mathbb{Q}}$ of rational spectra. The loop functor $\Omega_{\mathcal{O}} \colon \mathcal{A}\mathrm{lg}_{\mathcal{O}}\left(\mathcal{S}\mathrm{p}_{\mathbb{Q}}\right) \to \mathcal{A}\mathrm{lg}_{\mathcal{O}}\left(\mathcal{S}\mathrm{p}_{\mathbb{Q}}\right)$ admits the following factorisation*

$$\begin{array}{ccc}
\mathcal{A}\mathrm{lg}_{\mathcal{O}}\left(\mathcal{S}\mathrm{p}_{\mathbb{Q}}\right) & \xrightarrow{\Omega_{\mathcal{O}}} & \mathcal{A}\mathrm{lg}_{\mathcal{O}}\left(\mathcal{S}\mathrm{p}_{\mathbb{Q}}\right) \\
{\scriptstyle \mathrm{forg}_{\mathcal{O}}}\big\downarrow & & \big\uparrow{\scriptstyle \mathrm{triv}_{\mathcal{O}}} \\
\mathcal{S}\mathrm{p}_{\mathbb{Q}} & \xrightarrow[\Omega_{\mathcal{S}\mathrm{p}_{\mathbb{Q}}}]{} & \mathcal{S}\mathrm{p}_{\mathbb{Q}}
\end{array} \tag{7.2.0.1}$$

Proof. — See [GR, Chapter 6, Proposition 1.7.2]. □

7.2.0.2. Corollary. — *In the situation of Proposition 7.2.0.1, the suspension functor $\Sigma_{\mathcal{O}} \colon \mathcal{A}\mathrm{lg}_{\mathcal{O}}\left(\mathcal{S}\mathrm{p}_{\mathbb{Q}}\right) \to \mathcal{A}\mathrm{lg}_{\mathcal{O}}\left(\mathcal{S}\mathrm{p}_{\mathbb{Q}}\right)$ admits a factorisation*

$$\begin{array}{ccc}
\mathcal{A}\mathrm{lg}_{\mathcal{O}}\left(\mathcal{S}\mathrm{p}_{\mathbb{Q}}\right) & \xrightarrow{\Sigma_{\mathcal{O}}} & \mathcal{A}\mathrm{lg}_{\mathcal{O}}\left(\mathcal{S}\mathrm{p}_{\mathbb{Q}}\right) \\
{\scriptstyle \mathrm{indec}_{\mathcal{O}}}\big\downarrow & & \big\uparrow{\scriptstyle \mathrm{free}_{\mathcal{O}}} \\
\mathcal{S}\mathrm{p}_{\mathbb{Q}} & \xrightarrow[\Sigma_{\mathcal{S}\mathrm{p}_{\mathbb{Q}}}]{} & \mathcal{S}\mathrm{p}_{\mathbb{Q}} .
\end{array}$$

Proof. — Taking the left adjoints of functors in (7.2.0.1) gives the desired result. □

7.2.0.3. Corollary. — *Let $\mathcal{O}$ be a reduced ∞-operad with values in $\mathcal{S}\mathrm{p}_{\mathbb{Q}}$. The ∞-category $\mathcal{S}\mathrm{p}_{\mathbb{Q}}$ together with the free $\mathcal{O}$-algebra functor $\mathrm{free}_{\mathcal{O}}\colon \mathcal{S}\mathrm{p}_{\mathbb{Q}} \to \mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C})$ is equivalent to the costabilisation of $\mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C})$.*

Proof. — Abbreviate $\Sigma_{\mathcal{S}\mathrm{p}_{\mathbb{Q}}}$ by Σ. By Corollary 7.2.0.2 we obtain the following commutative diagram

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{\Sigma_{\mathcal{O}}} & \mathcal{A}\mathrm{lg}_{\mathcal{O}}\left(\mathcal{S}\mathrm{p}_{\mathbb{Q}}\right) & \xrightarrow{\Sigma_{\mathcal{O}}} & \mathcal{A}\mathrm{lg}_{\mathcal{O}}\left(\mathcal{S}\mathrm{p}_{\mathbb{Q}}\right) & \xrightarrow{\Sigma_{\mathcal{O}}} & \mathcal{A}\mathrm{lg}_{\mathcal{O}}\left(\mathcal{S}\mathrm{p}_{\mathbb{Q}}\right) \\
\uparrow{\scriptstyle \mathrm{free}_{\mathcal{O}}} & \searrow^{\Sigma \circ\, \mathrm{indec}_{\mathcal{O}}} & \uparrow{\scriptstyle \mathrm{free}_{\mathcal{O}}} & \searrow^{\Sigma \circ\, \mathrm{indec}_{\mathcal{O}}} & \uparrow{\scriptstyle \mathrm{free}_{\mathcal{O}}} & \searrow^{\Sigma \circ\, \mathrm{indec}_{\mathcal{O}}} & \uparrow{\scriptstyle \mathrm{free}_{\mathcal{O}}} \\
\cdots & \xrightarrow[\Sigma]{} & \mathcal{S}\mathrm{p}_{\mathbb{Q}} & \xrightarrow[\Sigma]{} & \mathcal{S}\mathrm{p}_{\mathbb{Q}} & \xrightarrow[\Sigma]{} & \mathcal{S}\mathrm{p}_{\mathbb{Q}},
\end{array}$$

in $\mathcal{CAT}_{\infty}$. Thus the functor

$$\mathrm{co}\mathcal{S}\mathrm{p}(\mathrm{free}_{\mathcal{L}\mathrm{ie}})\colon \mathcal{S}\mathrm{p}_{\mathbb{Q}} \simeq \mathrm{co}\mathcal{S}\mathrm{p}\left(\mathcal{S}\mathrm{p}_{\mathbb{Q}}\right) \xrightarrow{\sim} \mathrm{co}\mathcal{S}\mathrm{p}\left(\mathcal{A}\mathrm{lg}_{\mathcal{O}}\left(\mathcal{S}\mathrm{p}_{\mathbb{Q}}\right)\right),$$

induced by $\mathrm{free}_{\mathcal{O}}$, from the inverse limit of the lower row to that of the upper row is an equivalence of ∞-categories. In other words, we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{co}\mathcal{S}\mathrm{p}\left(\mathcal{A}\mathrm{lg}_{\mathcal{O}}\left(\mathcal{S}\mathrm{p}_{\mathbb{Q}}\right)\right) & \xrightarrow{\Sigma^{\infty}} & \mathcal{A}\mathrm{lg}_{\mathcal{O}}\left(\mathcal{S}\mathrm{p}_{\mathbb{Q}}\right) \\
{\scriptstyle \mathrm{co}\mathcal{S}\mathrm{p}(\mathrm{free}_{\mathcal{L}\mathrm{ie}})}\uparrow{\scriptstyle \simeq} & & \uparrow{\scriptstyle \mathrm{free}_{\mathcal{O}}} \\
\mathrm{co}\mathcal{S}\mathrm{p}\left(\mathcal{S}\mathrm{p}_{\mathbb{Q}}\right) & \xrightarrow[\Sigma_{\infty}]{\sim} & \mathcal{S}\mathrm{p}_{\mathbb{Q}}
\end{array}$$

of ∞-categories. From the above diagram we see that $\mathrm{free}_{\mathcal{O}}$ is equivalent to the lower horizontal arrow Σ_{∞}. □

7.2.0.4. — In the next section we show that the costabilisation $\mathrm{co}\mathcal{S}\mathrm{p}(\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}))$ of the ∞-category $\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})$ of $\mathrm{T}(h)_{\bullet}$-local spectral Lie algebras is equivalent to $\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$, for every $h \geq 1$, generalising Corollary 7.2.0.3 in the case where $\mathcal{O}$ is the spectral Lie ∞-operad $\mathcal{L}\mathrm{ie}$. In particular, this implies that the costabilisation of the ∞-category $\mathcal{H}\mathrm{o}_{v_h}$ of v_h-periodic homotopy types is non-trivial, in contrast to the costabilisation of $\mathcal{H}\mathrm{o}_{*}$, cf. Example 7.1.0.2.

***7.2.0.5. Theorem* (Heuts–Land [HL]).** — *Let $\mathcal{C}$ be a presentable symmetric monoidal ∞-category. The n-fold suspension functor $\Sigma^n_{\mathcal{E}_n}\colon \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_n}(\mathcal{C}) \to \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_n}(\mathcal{C})$ admits the following factorisation*

$$\begin{array}{ccc}
\mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_n}(\mathcal{C}) & \xrightarrow{\Sigma^n_{\mathcal{E}_n}} & \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_n}(\mathcal{C}) \\
{\scriptstyle \mathrm{indec}_{\mathcal{E}^{\mathrm{nu}}_n}}\downarrow & & \uparrow{\scriptstyle \mathrm{free}_{\mathcal{E}^{\mathrm{nu}}_n}} \\
\mathcal{C} & \xrightarrow[\Sigma^n_{\mathcal{C}}]{} & \mathcal{C}.
\end{array}$$

7.2.0.6. Remark. — We learnt about Theorem 7.2.0.5 from communications with Gijs Heuts and we checked the proof in a draft of the manuscript. We use this theorem for Corollary 7.2.0.7, Proposition 7.3.0.3, Theorem 7.3.0.4, and Corollary 7.3.0.7.

7.2.0.7. Corollary. — *In the situation of Theorem 7.2.0.5, the stable ∞-category* $\mathrm{co}\mathcal{S}\mathrm{p}\,(\mathcal{C})$ *together with the functor* $\mathrm{co}\mathcal{S}\mathrm{p}(\mathcal{C}) \xrightarrow{\Sigma_\infty} \mathcal{C} \xrightarrow{\mathrm{free}_{\mathcal{E}_n^{\mathrm{nu}}}} \mathcal{A}\mathrm{lg}_{\mathcal{E}_n}(\mathcal{C})$ *is equivalent to the costabilisation of* $\mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_n}(\mathcal{C})$.

Proof. — We can use the same proof strategy as the proof of Corollary 7.2.0.3, where we replace the 1-fold suspension functor by the n-fold suspension functor. In particular, we obtain the following commutative diagram

$$\begin{array}{ccc} \mathrm{co}\mathcal{S}\mathrm{p}\left(\mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_n}(\mathcal{C})\right) & \xrightarrow{\Sigma_\infty} & \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_n}(\mathcal{C}) \\ {\scriptstyle \mathrm{co}\mathcal{S}\mathrm{p}(\mathrm{free}_{\mathcal{E}_n^{\mathrm{nu}}})}\Big\uparrow{\scriptstyle \simeq} & & \Big\uparrow{\scriptstyle \mathrm{free}_{\mathcal{E}_n^{\mathrm{nu}}}} \\ \mathrm{co}\mathcal{S}\mathrm{p}(\mathcal{C}) & \xrightarrow[\Sigma_\infty]{} & \mathcal{C} \end{array}$$

of ∞-categories, which gives an equivalence $\mathrm{free}_{\mathcal{E}_n^{\mathrm{nu}}} \circ \Sigma_\infty^{\mathcal{C}} \simeq \Sigma_\infty^{\mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_n}(\mathcal{C})}$ of functors. □

7.2.0.8. — We cannot simply dualise the argument for Theorem 6.2.2.7 to give a general statement about the costabilisation of the ∞-category $\mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C})$ of $\mathcal{O}$-algebras in a stable symmetric monoidal ∞-category $\mathcal{C}$. For simplicity, let $\mathcal{O}$ be an ∞-operad with values in $\mathcal{H}\mathrm{o}$. Almost equivalently, we can consider the problem of determining the costabilisation of $\mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C})$ as the problem of determining the stabilisation of $\mathrm{co}\mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C}) \simeq \mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$. In the situation of Theorem 6.2.2.7 the proof uses two ingredients:

(i) The adjunction induced by the stabilisation of a monadic adjunction is monadic, see [HA, Example 4.7.3.10].

(ii) The linear approximation (in the context of Goodwillie calculus) of the functor $\mathrm{forg}_{\mathcal{O}} \circ \mathrm{free}_{\mathcal{O}}$ satisfies

$$\mathrm{P}_1\left(\coprod_{r\geq 0} \mathcal{O}(r) \otimes_{\mathfrak{S}_r} X^{\otimes r}\right) \simeq \mathcal{O}(1) \otimes X.$$

None of the two facts generalise to the dual/opposite setting in general. In particular, we would have to consider the Goodwillie calculus tower for the divided power functor $(\mathcal{O}(r) \otimes (-)^{\otimes r})^{\mathfrak{S}_r}$, which is non-trivial, in contrary to the Goodwillie calculus tower of $(\mathcal{O}(r) \otimes (-)^{\otimes r})_{\mathfrak{S}_r}$ (this functor is r-homogeneous).

However, if we work with the ∞-category $\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$ of $\mathrm{T}(h)_\bullet$-local spectra, recall that we do have similar formula of the co-linear approximation $\mathrm{P}^1(F)$ of certain endofunctors of $\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$, see Lemma 6.2.2.12. This is exactly one of the important ingredients we use to investigate the costabilisation of $\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})$ in the next section.

7.3. The costabilisation of v_h-periodic homotopy types

From now on we fix a prime number h and a natural number $h \geq 1$. In this section we prove that the costabilisation of the ∞-category $\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})$ of $\mathrm{T}(h)_\bullet$-local spectral Lie algebras is equivalent to the ∞-category of $\mathrm{T}(h)$-local spectra, see Theorem 7.3.0.4. Under the equivalence $\mathcal{H}\mathrm{o}_{v_h} \simeq \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})$ (see Theorem 6.2.2.4), the ∞-category $\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$ is also equivalent to the costabilisation of $\mathcal{H}\mathrm{o}_{v_h}$. As a corollary we provide a universal property of the Bousfield–Kuhn functor, see Corollary 7.3.0.7. This section is original work.

7.3.0.1. Situation. — Let $\mathcal{O}$ be a reduced ∞-operad with values in the ∞-category $\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$ of $\mathrm{T}(h)_\bullet$-local spectra. Recall the free–forgetful adjunction

$$\mathrm{free}_{\mathcal{O}} \colon \mathcal{S}\mathrm{p}_{\mathrm{T}(h)} \rightleftarrows \mathcal{A}\mathrm{lg}_{\mathcal{O}}\left(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}\right) : \mathrm{forg}_{\mathcal{O}}\,.$$

Since the functor $\mathrm{free}_{\mathcal{O}}$ preserves small colimits, the adjunction induces an adjunction

$$\mathbb{F}\mathrm{ree}_{\mathcal{O}} \colon \mathcal{S}\mathrm{p}_{\mathrm{T}(h)} \rightleftarrows \mathrm{co}\mathcal{S}\mathrm{p}\left(\mathcal{A}\mathrm{lg}_{\mathcal{O}}\left(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}\right)\right) : \mathbb{G}_{\mathcal{O}},$$

on costabilisations, by Proposition 7.1.0.13.

7.3.0.2. Proposition. — *In Situation 7.3.0.1 the functor $\mathbb{F}\mathrm{ree}_{\mathcal{O}}$ is fully faithful.*

Proof. — By Proposition 6.3.0.3 it suffices to show that there is an equivalence $\mathbb{G}_{\mathcal{O}} \circ \mathbb{F}\mathrm{ree}_{\mathcal{O}} \simeq \mathrm{id}$. Abbreviate the the suspension and the loop functor of the ∞-category $\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$ by Σ and Ω, respectively.

For every object $E \in \mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$, we have

$$\begin{aligned}(\mathbb{G}_{\mathcal{O}} \circ \mathbb{F}\mathrm{ree}_{\mathcal{O}})(E) &\simeq \varprojlim_n \left(\Sigma^n \circ \mathrm{forg}_{\mathcal{O}} \circ \mathrm{free}_{\mathcal{O}} \circ \Omega^n\right)(E) \\ &\simeq \mathrm{P}^1\left(\coprod_{i\geq 1} \left(\mathcal{O}(n) \otimes E^{\otimes i}\right)_{\mathfrak{S}_i}\right) \\ &\simeq E.\end{aligned}$$

The first equivalence holds by the construction of $\mathbb{F}\mathrm{ree}_{\mathcal{O}}$ and $\mathbb{G}_{\mathcal{O}}$. The second equivalence holds by the construction of the colinear approximation, in the sense of dual Goodwillie calculus, of the endofunctors $\mathrm{forg}_{\mathcal{O}} \circ \mathrm{free}_{\mathcal{O}}$ of $\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$. The last equivalence holds by Lemma 6.2.2.12. □

7.3.0.3. Proposition. — *In Situation 7.3.0.1, let $\mathcal{O}$ be the spectral Lie ∞-operad $\mathcal{L}\mathrm{ie}$. Then the functor $\mathbb{F}\mathrm{ree}_{\mathcal{L}\mathrm{ie}}$ is an equivalence of stable ∞-categories.*

Proof. — Recall the fully faithful functor functor (see Theorem 6.3.0.1)

$$\mathrm{U}_\infty \colon \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}) \to \varprojlim \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_n}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}),$$

induced by the commutative diagram (6.1.0.1). Let

$$\mathrm{pr}_0\colon \varprojlim_n \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_n}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}) \to \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_0}\left(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}\right) \simeq \mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$$

denote the canonical map in the inverse limit diagram. Note that we have

$$\mathrm{indec}_{\mathcal{L}\mathrm{ie}} \simeq \mathrm{U}_0 = \mathrm{pr}_0 \circ \mathrm{U}_\infty\,.$$

Since $\mathrm{indec}_{\mathcal{L}\mathrm{ie}} \circ \mathrm{free}_{\mathcal{L}\mathrm{ie}} \simeq \mathrm{id}$ (see Example 5.2.4.21), the following composition

$$\mathcal{S}\mathrm{p}_{\mathrm{T}(h)} \xrightarrow{\mathrm{free}_{\mathcal{L}\mathrm{ie}}} \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}) \xrightarrow{\mathrm{U}_\infty} \varprojlim_n \mathcal{A}\mathrm{lg}_{\mathcal{E}_n}\left(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}\right) \xrightarrow{\mathrm{pr}_0} \mathcal{S}\mathrm{p}_{\mathrm{T}(h)} \tag{7.3.0.1}$$

of cocontinuous functors is equivalent to identity functor of $\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$. Thus the following composition of functors

$$\begin{array}{ccc} \mathcal{S}\mathrm{p}_{\mathrm{T}(h)} \xrightarrow{\mathbb{F}\mathrm{ree}_{\mathcal{L}\mathrm{ie}}} & \mathrm{co}\mathcal{S}\mathrm{p}\left(\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})\right) & \\ & \downarrow{\scriptstyle \mathrm{co}\mathcal{S}\mathrm{p}(\mathrm{U}_\infty)} & \\ & \mathrm{co}\mathcal{S}\mathrm{p}\left(\varprojlim_n \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_n}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})\right) & \xrightarrow{\mathrm{co}\mathcal{S}\mathrm{p}(\mathrm{pr}_0)} \mathcal{S}\mathrm{p}_{\mathrm{T}(h)} \end{array}$$

on costabilisations, induced by (7.3.0.1), is equivalent to the identity functor of $\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$ as well. Thus, to show that $\mathbb{F}\mathrm{ree}_{\mathcal{L}\mathrm{ie}}$ is an equivalence of ∞-categories, it suffices to show that $\mathrm{co}\mathcal{S}\mathrm{p}(\mathrm{pr}_0)$ is an equivalence, since the functor $\mathrm{co}\mathcal{S}\mathrm{p}(\mathrm{U}_\infty)$ is fully faithful by Proposition 7.1.0.13.

Claim. The functor $\mathrm{co}\mathcal{S}\mathrm{p}(\mathrm{pr}_0)$ is an equivalence of ∞-categories.

Consider the following commutative diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_{n+1}}\left(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}\right) & \xrightarrow{\mathrm{B}_{n+1}} & \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_n}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}) & \xrightarrow{\mathrm{B}_n} & \cdots & \xrightarrow{\mathrm{B}_1} & \mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_0}\left(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}\right) \\ \uparrow & & \uparrow{\scriptstyle \mathrm{free}^{\mathrm{nu}}_{\mathcal{E}_{n+1}}} & & \uparrow{\scriptstyle \mathrm{free}^{\mathrm{nu}}_{\mathcal{E}_n}} & & \uparrow & & \uparrow{\scriptstyle \mathrm{id}\,\simeq} \\ \cdots & \xrightarrow{\Sigma} & \mathcal{S}\mathrm{p}_{\mathrm{T}(h)} & \xrightarrow{\Sigma} & \mathcal{S}\mathrm{p}_{\mathrm{T}(h)} & \xrightarrow{\Sigma} & \cdots & \xrightarrow{\Sigma} & \mathcal{S}\mathrm{p}_{\mathrm{T}(h)} \end{array}$$

in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ obtained from (6.1.0.5). It induces the commutative diagram below on costabilisations:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \mathrm{co}\mathcal{S}\mathrm{p}\left(\mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_n}\left(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}\right)\right) & \longrightarrow & \cdots & \longrightarrow & \mathrm{co}\mathcal{S}\mathrm{p}\left(\mathcal{A}\mathrm{lg}^{\mathrm{nu}}_{\mathcal{E}_0}\left(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}\right)\right) \\ \simeq\uparrow & & \uparrow{\scriptstyle \mathbb{F}\mathrm{ree}_{\mathcal{E}_n}\,\simeq} & & \uparrow\simeq & & \uparrow{\scriptstyle \mathbb{F}\mathrm{ree}_{\mathcal{E}_0}\,\simeq} \\ \cdots & \xrightarrow{\sim} & \mathcal{S}\mathrm{p}_{\mathrm{T}(h)} & \underset{\Sigma}{\xrightarrow{\sim}} & \cdots & \underset{\Sigma}{\xrightarrow{\sim}} & \mathcal{S}\mathrm{p}_{\mathrm{T}(h)}, \end{array}$$

where the vertical arrows are equivalences by Corollary 7.2.0.7 Then the claim follows by the induced map between the colimits of the rows and from the commutativity of the second diagram. □

7.3.0.4. Theorem. — *The stable ∞-category $\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$ together with the functor* $\mathrm{free}_{\mathcal{L}\mathrm{ie}}$ *is equivalent to the costabilisation of the ∞-category $\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})$.*

Proof. — The functor $\mathrm{free}_{\mathcal{L}\mathrm{ie}}$ induces a commutative diagram

$$\begin{array}{ccc} \mathcal{S}\mathrm{p}_{\mathrm{T}(h)} & \xrightarrow[\Sigma_\infty]{\sim} & \mathcal{S}\mathrm{p}_{\mathrm{T}(h)} \\ {\scriptstyle \mathbb{F}\mathrm{ree}_{\mathcal{L}\mathrm{ie}}}\downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \mathrm{free}_{\mathcal{L}\mathrm{ie}}} \\ \mathrm{co}\mathcal{S}\mathrm{p}\left(\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})\right) & \xrightarrow[\Sigma_\infty]{} & \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}), \end{array}$$

of ∞-categories; here the left vertical arrow is an equivalence by Proposition 7.3.0.3, and the upper horizontal arrow is an equivalence because $\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}$ is stable. This commutative diagram exhibits an equivalence between the functor $\mathrm{free}_{\mathcal{L}\mathrm{ie}}$ and the canonical functor $\Sigma_\infty\colon \mathrm{co}\mathcal{S}\mathrm{p}\left(\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})\right) \to \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)})$. □

7.3.0.5. Corollary. — *The stable ∞-category $\mathcal{S}\mathrm{p}_{\mathrm{K}(h)}$ together with the functor* $\mathrm{free}_{\mathcal{L}\mathrm{ie}}$ *is equivalent to the costabilisation of the ∞-category $\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{K}(h)})$.*

Proof. — By Corollary 6.3.0.14 the result of Proposition 7.3.0.3 still holds, after replacing $\mathrm{T}(h)$ by $\mathrm{K}(h)$. Thus the proof strategy of Theorem 7.3.0.4 works as well in the $\mathrm{K}(h)_\bullet$-local setting. □

7.3.0.6. — Recall that under the equivalence $\mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}) \simeq \mathcal{H}\mathrm{o}_{v_h}$ (see Theorem 6.2.2.4) the free–forgetful adjunction of spectral Lie algebras corresponds to the Bousfield–Kuhn adjunction $\Theta_h\colon \mathcal{S}\mathrm{p}_{\mathrm{T}(h)} \rightleftarrows \mathcal{H}\mathrm{o}_{v_h} : \Phi_h$, see ¶6.2.2.8.

Thus Theorem 7.3.0.4 implies that the functor Θ_h is equivalent to the costabilisation of $\mathcal{H}\mathrm{o}_h$. This gives a universal property of Θ_h and Φ_h.

7.3.0.7. Corollary. —

(i) *Let $\mathcal{C}$ be a stable ∞-category. Then composing with the functor Θ_h induces an equivalence*

$$\mathcal{F}\mathrm{un}^{\mathrm{rex}}(\mathcal{C}, \mathcal{S}\mathrm{p}_{\mathrm{T}(h)}) \xrightarrow{\sim} \mathcal{F}\mathrm{un}^{\mathrm{rex}}(\mathcal{C}, \mathcal{H}\mathrm{o}_{v_h}),$$

of ∞-categories, where $\mathcal{F}\mathrm{un}^{\mathrm{rex}}$ denotes the ∞-category of right exact functors, i.e. functors that preserve finite colimits.

(ii) *Let $\mathcal{D}$ be a presentable stable ∞-category. Composing with the Bousfield–Kuhn functor Φ_h induces an equivalence*

$$\mathcal{F}\mathrm{un}^{R}(\mathcal{S}\mathrm{p}_{\mathrm{T}(h)}, \mathcal{D}) \xrightarrow{\sim} \mathcal{F}\mathrm{un}^{R}(\mathcal{H}\mathrm{o}_{v_h}, \mathcal{D}),$$

of ∞-categories, where $\mathcal{F}\mathrm{un}^R$ denote the ∞-category of functors that are accessible and preserves small limits, i.e. functors admitting left adjoints.

Proof. — This is a consequence of Theorem 7.3.0.4, Proposition 7.1.0.6 and Corollary 7.1.0.12. □

APPENDIX A

Further details for Chapter 5

A.1. The explicit formula of the composition product

A.1.0.1. Situation. — In this section we verify the equivalence (5.2.4.3). Let F and G be symmetric sequences in a presentable symmetric monoidal ∞-category $\mathcal{C}$. We show that there exists an equivalence

$$(F \circledcirc G)(r) \simeq \coprod_{n \geq 0} \left(\coprod_{\underline{r} = \sqcup_{i=1}^{n} S_i} F(n) \otimes_{\mathfrak{S}_n} (\otimes_{i=1}^{n} G(S_i)) \right) \tag{A.1.0.1}$$

in $\mathcal{C}$ for every natural number r, where $\circledcirc$ denotes the composition product (see Construction 5.2.4.8).

***A.1.0.2.* Day convolution.** — Let F_1 and F_2 be symmetric sequences in $\mathcal{C}$. The *Day convolution product* $F_1 \circledast F_2$ is a symmetric sequence in $\mathcal{C}$ such that

$$(F_1 \circledast F_2)(r) \simeq \varinjlim_{\underline{r} \cong S_1 \sqcup S_2} F_1(S_1) \otimes F_2(S_2) \tag{A.1.0.2}$$

for every $r \in \mathbb{N}$, where $\otimes$ denotes the symmetric monoidal product of $\mathcal{C}$. The Day convolution product endows the ∞-category $\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C})$ of symmetric sequences with the structure of a symmetric monoidal ∞-category, where the symmetric monoidal unit the symmetric sequence $\underline{\mathbb{1}}_{\mathcal{C}}$ (see Example 5.2.4.7). See [HA, §2.2.6] for a more detailed introduction of the Day convolution.

Recall the unit symmetric sequence $\mathbb{1}_{\mathcal{C}}^{\mathfrak{S}}$ in $\mathcal{C}$ from Example 5.2.4.7. By calculation the n-fold Day convolution $(\mathbb{1}_{\mathcal{C}}^{\mathfrak{S}})^{\circledast n}$ of $\mathbb{1}_{\mathcal{C}}^{\mathfrak{S}}$ with itself, where $n \geq 1$, is given by

$$((\mathbb{1}_{\mathcal{C}}^{\mathfrak{S}})^{\circledast n})(r) \simeq \begin{cases} \mathbb{1}_{\mathcal{C}}, & \text{if } r = n, \\ \text{the initial object of } \mathcal{C}, & \text{otherwise.} \end{cases}$$

We define $(\mathbb{1}_{\mathcal{C}}^{\mathfrak{S}})^{\circledast 0} := \underline{\mathbb{1}}_{\mathcal{C}}$.

A.1.0.3. Proposition. — *Let F be a symmetric sequence in $\mathcal{C}$. Recall the construction $\underline{(-)}$ of symmetric sequences from Example 5.2.4.7. Then there exists an equivalence*

$$F \simeq \coprod_{n\in\mathbb{N}} ((\mathbb{1}_{\mathcal{C}}^{\mathfrak{S}})^{\circledast n}) \circledast \underline{F(n)}$$

of symmetric sequences in $\mathcal{C}$, where the coproduct is taken in $\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C})$.

Proof. — Using (A.1.0.2) we obtain for every $n \in \mathbb{N}$

$$((\mathbb{1}_{\mathcal{C}}^{\mathfrak{S}})^{\circledast n} \circledast \underline{F(n)})(r) \simeq \begin{cases} F(n), & \text{if } r = n, \\ \text{an initial object}, & \text{otherwise.} \end{cases}$$

for every $n \in \mathbb{N}$. Then the statement follows because the small colimits in the functor category $\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C})$ is calculated pointwise. □

Proof of (A.1.0.1). — Recall that the composition product is induced by the following equivalence

$$\begin{aligned} \mathrm{ev}\colon \mathcal{F}\mathrm{un}^{\otimes}_{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathcal{C}/}}(\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}), \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C})) &\xrightarrow{\sim} \mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C}) \\ \overline{G} \circ \overline{F} &\mapsto F \circledcirc G \end{aligned}$$

of ∞-categories, given by evaluation at the unit symmetric sequence $\mathbb{1}_{\mathcal{C}}^{\mathfrak{S}}$, that is

$$F = \widetilde{F}(\mathbb{1}_{\mathcal{C}}^{\mathfrak{S}}) \text{ and } G = \widetilde{G}(\mathbb{1}_{\mathcal{C}}^{\mathfrak{S}}).$$

Thus we need to calculate the evaluation $\left(\overline{G} \circ \overline{F}\right)(\mathbb{1}_{\mathcal{C}}^{\mathfrak{S}})$:

$$\begin{aligned} (\widetilde{G} \circ \widetilde{F})(\mathbb{1}_{\mathcal{C}}^{\mathfrak{S}}) &= \widetilde{G}(F) \\ &\overset{(i)}{\simeq} \widetilde{G}\left(\coprod_{n\in\mathbb{N}} ((\mathbb{1}_{\mathcal{C}}^{\mathfrak{S}})^{\circledast n}) \circledast \underline{F(n)}\right) \\ &\overset{(ii)}{\simeq} \coprod_{n\in\mathbb{N}} \widetilde{G}\left((\mathbb{1}_{\mathcal{C}}^{\mathfrak{S}})^{\circledast n} \circledast \underline{F(n)}\right) \\ &\overset{(iii)}{\simeq} \coprod_{n\in\mathbb{N}} \widetilde{G}\left((\mathbb{1}_{\mathcal{C}}^{\mathfrak{S}})^{\circledast n}\right) \circledast \widetilde{G}\left(\underline{F(n)}\right) \\ &\overset{(iv)}{\simeq} \coprod_{n\in\mathbb{N}} \left(\widetilde{G}\left(\mathbb{1}_{\mathcal{C}}^{\mathfrak{S}}\right)\right)^{\circledast n} \circledast \underline{F(n)} \\ &\overset{(v)}{\simeq} \coprod_{n\in\mathbb{N}} G^{\circledast n} \circledast \underline{F(n)}. \end{aligned}$$

Here are the arguments for the above equivalences:

(i) This follows from Proposition A.1.0.3,

(ii) This holds because $\overline{G}$ preserves small colimits,

(iii) This holds since $\widetilde{G}$ is a symmetric monoidal endofunctor of $\mathcal{S}\mathrm{ym}\mathcal{S}\mathrm{eq}(\mathcal{C})$ with respect to the Day convolution products.

(iv) This holds because $\overline{G}$ preserves small colimits and the full ∞-subcategory $\mathcal{C}$ of $\mathcal{S}ym\mathcal{S}eq(\mathcal{C})$ (via the $\underline{(-)}$ functor).

(v) This is by definition.

Calculating the Day convolution products of G with itself, we obtain the equivalence

$$\left(\coprod_{n\in\mathbb{N}} G^{\circledast n} \circledast \underline{F(n)}\right)(r) \simeq \coprod_{n\geq 0}\left(\coprod_{\underline{r}=\sqcup_{i=1}^{n} S_i} F(n) \otimes_{\mathfrak{S}_n} (\otimes_{i=1}^{n} G(S_i))\right)$$

in $\mathcal{C}$ for every $r \in \mathbb{N}$. □

A.2. ∞-categories of algebras over an ∞-operad with values in $\mathcal{H}o$

The goal of this section is to prove Theorem 5.2.5.5. We repeat the situations and the statement of the theorem here for convenience.

A.2.0.1. Situation. — Let $p\colon \mathcal{O}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ together with an essentially surjective morphism $\Delta^0 \to \mathcal{O}^{\otimes}_{\langle 1\rangle}$ be a one-coloured ∞-operad. Let c denote the image of Δ^0 in $\mathcal{O}^{\otimes}_{\langle 1\rangle}$, which is considered as the colour of $\mathcal{O}^{\otimes}$. For every $r \in \mathbb{N}$, recall

(i) the equivalence (Remark 5.2.1.4)

$$R_r\colon \mathcal{O}^{\otimes}_{\langle r\rangle} \xrightarrow{\sim} \left(\mathcal{O}^{\otimes}_{\langle 1\rangle}\right)^{\times r}$$

of ∞-categories, induced by the sequence $(\rho_i)_{i\geq 1}$ of morphisms of pointed finite sets (see Definition 5.1.1.5), and

(ii) the morphism $f_r\colon \langle r\rangle \to \langle 1\rangle$ of pointed finite sets satisfying $f_r^{-1}(\mathrm{pt}) = \mathrm{pt}$.

For every $r \in \mathbb{N}$, fix an inverse Q_r of R_r and define the ∞-groupoid

$$\mathcal{O}(r) \coloneqq \mathcal{M}\mathrm{ap}^{f_r}_{\mathcal{O}^{\otimes}}\left(Q_r(c^{\times r}), c\right)$$

of morphisms lifting f_r (see Definition 5.2.1.2). Note that $\mathcal{O}(r)$ admits a $\mathfrak{S}_r$-action induced by the permutation group action on $\langle r\rangle$. We call

$$\mathcal{O} \coloneqq \left(\mathcal{O}(r)\right)_{r\geq 0} \in \mathcal{S}ym\mathcal{S}eq(\mathcal{H}o)$$

is the *underlying symmetric sequence* of the ∞-operad $\mathcal{O}^{\otimes}$.

A.2.0.2. Situation. — Let $\mathcal{C}$ be a presentable symmetric monoidal ∞-category. Then there exists a symmetric monoidal functor $F\colon \mathcal{H}o \to \mathcal{C}$ in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$, unique up to contractible choice, since $\mathcal{H}o$ is the free presentable ∞-category generated by a point.

***A.2.0.3. Theorem* (Theorem 5.2.5.5).** — *In Situations A.2.0.1 and A.2.0.2, we obtain the following statements:*

(i) *The symmetric sequence $\mathcal{O}$ admits the structure of an associative algebra objects in the monoidal ∞-category $q_{\circledcirc}\colon \mathcal{S}ym\mathcal{S}eq(\mathcal{H}o)^{\otimes} \to \mathcal{A}ss^{\otimes}$.*

(ii) *Consider $\mathcal{O}$ as an ∞-operad with values in $\mathcal{H}\mathrm{o}$ by (ii). There exists an equivalence*

$$\mathcal{A}\mathrm{lg}_{\mathcal{O}/\mathcal{C}\mathrm{om}}(\mathcal{C}) \simeq \mathcal{L}\mathcal{M}\mathrm{od}_{\mathrm{T}_{F(\mathcal{O})}}(\mathcal{C})$$

of ∞-categories, cf. Definitions 5.2.1.24 and 5.2.4.13.

Proof. — We will explain in Construction A.2.0.6 that the (free $\dashv$ forg)-adjunction of $\mathcal{A}\mathrm{lg}_{\mathcal{O}/\mathcal{C}\mathrm{om}}(\mathcal{C})$ is monadic and show that the associated monad is of the form

$$\mathcal{C} \to \mathcal{C},\ X \mapsto \coprod_{r\geq 0} \left(\mathcal{O}(r) \otimes X^{\otimes r}\right)_{\mathfrak{S}_r}.$$

where $\mathcal{O}(r) \otimes X^{\otimes r}$ denotes the tensor of $X^{\otimes r}$ with $\mathcal{O}(r)$, i.e. the colimit of the constant diagram $\mathcal{O}(r) \to \mathcal{C}$ mapping to the object $X^{\otimes r}$, see [HTT, Corollary 4.4.4.9].

By [GHK22, §3.2], which shows an equivalence between the ∞-category of "analytic monads" of $\mathcal{H}\mathrm{o}$ and the ∞-category of symmetric sequences in $\mathcal{H}\mathrm{o}$, we conclude that the symmetric sequence $\mathcal{O} = (\mathcal{O}(r))_{r\geq 0}$ admits the structure of an ∞-operad with values in $\mathcal{H}\mathrm{o}$ (Definition 5.2.4.11). Thus, by Proposition 5.2.5.1, we can consider the symmetric sequence $F(\mathcal{O})$ as an ∞-operad with values in $\mathcal{C}$.

Finally, we have

$$\begin{aligned}
\mathrm{T}_{F(\mathcal{O})}(X) &\simeq \coprod_{r\geq 0} \left(F(\mathcal{O}(r)) \otimes_{\mathcal{C}} X^{\otimes r}\right)_{\mathfrak{S}_r} \\
&\simeq \coprod_{r\geq 0} \left(F(\mathcal{O}(r) \times \mathrm{pt}) \otimes_{\mathcal{C}} X^{\otimes r}\right)_{\mathfrak{S}_r} \\
&\simeq \coprod_{r\geq 0} \left(\varinjlim_{\mathcal{O}(r)} (F(\mathrm{pt})) \otimes_{\mathcal{C}} X^{\otimes r}\right)_{\mathfrak{S}_r} \\
&\simeq \coprod_{r\geq 0} \left(\varinjlim_{\mathcal{O}(r)} (\mathbb{1}_{\mathcal{C}} \otimes_{\mathcal{C}} X^{\otimes r})\right)_{\mathfrak{S}_r} \\
&\simeq \coprod_{r\geq 0} \left(\mathcal{O}(r) \otimes X^{\otimes r}\right)_{\mathfrak{S}_r},
\end{aligned}$$

where we obtain the last three equivalences by the assumption that F is symmetric monoidal and preserves small colimits and the symmetric monoidal product $\otimes_{\mathcal{C}}$ of $\mathcal{C}$ preserves small colimits in each variable. □

A.2.0.4. Proposition. — *In Situation A.2.0.1 let $q\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes}$ be an $\mathcal{O}$-monoidal ∞-category which is compatible with small colimits (see Definition 5.2.4.1). The forgetful functor*

$$\mathrm{forg}_{\mathcal{O}}\colon \mathcal{A}\mathrm{lg}_{/\mathcal{O}}(\mathcal{D}) \to \mathcal{F}\mathrm{un}\left(\mathcal{O}^{\otimes}_{\langle 1\rangle}, \mathcal{D}\right) \simeq \mathcal{D},$$

defined by restricting a functor $\mathcal{O}^{\otimes} \to \mathcal{D}^{\otimes}$ to the ∞-subcategory $\mathcal{O}^{\otimes}_{\langle 1\rangle}$, admits a left adjoint $\mathrm{free}_{\mathcal{O}}$. Furthermore, the adjunction $\mathrm{free}_{\mathcal{O}} \dashv \mathrm{forg}_{\mathcal{O}}$ is monadic.

Proof. — The existence of the adjunction is due to [HA, Corollary 3.1.3.5, Example 3.1.3.6]. The monadicity of the adjunction can be proven using the Barr–Beck theorem, see Theorem 5.2.3.3. The hypotheses of the Barr–Beck theorem are satisfied by [HA, Lemma 3.2.2.6, Proposition 3.2.3.1]. See also [HA, Example 4.7.3.11]. □

A.2.0.5. Corollary. — *In Situations A.2.0.1 and A.2.0.2 the forgetful functor*

$$\mathrm{forg}_{\mathcal{O}}\colon \mathcal{A}\mathrm{lg}_{\mathcal{O}/\,\mathcal{C}\mathrm{om}}(\mathcal{C}) \to \mathcal{F}\mathrm{un}(\mathcal{O}^{\otimes}_{\langle 1\rangle}, \mathcal{C}) \simeq \mathcal{C},$$

defined by restricting a functor to the ∞-subcategory $\mathcal{O}^{\otimes}_{\langle 1\rangle}$*, admits a left adjoint* $\mathrm{free}_{\mathcal{O}}$ *and the adjunction is monadic.*

Proof. — We abbreviate $\mathcal{O}^{\otimes}_{\langle 1\rangle}$ by $\mathcal{O}$ in this proof. Recall that there is the following pullback diagram

$$\begin{array}{ccc} (\mathcal{O}\otimes_{\mathcal{C}\mathrm{om}}\mathcal{C})^{\otimes} & \xrightarrow{\mathrm{pr}} & \mathcal{C}^{\otimes} \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle r} \\ \mathcal{O}^{\otimes} & \longrightarrow & \mathcal{C}\mathrm{om}^{\otimes}\,. \end{array} \tag{A.2.0.1}$$

inducing an equivalence

$$\mathcal{A}\mathrm{lg}_{/\mathcal{O}}\,(\mathcal{O}\otimes_{\mathcal{C}\mathrm{om}}\mathcal{C}) \xrightarrow{\sim} \mathcal{A}\mathrm{lg}_{\mathcal{O}/\,\mathcal{C}\mathrm{om}}(\mathcal{C}),$$

see Proposition 5.2.1.27. The statement of the corollary follows by considering the following commutative diagram

$$\begin{array}{ccc} \mathcal{A}\mathrm{lg}_{/\mathcal{O}}\,(\mathcal{O}\otimes_{\mathcal{C}\mathrm{om}}\mathcal{C}) & \xrightarrow{\sim} & \mathcal{A}\mathrm{lg}_{\mathcal{O}/\,\mathcal{C}\mathrm{om}}(\mathcal{C}) \\ {\scriptstyle \mathrm{forg}_{\mathcal{O}}}\downarrow & & \downarrow{\scriptstyle \mathrm{forg}_{\mathcal{O}}} \\ \mathcal{O}\otimes_{\mathcal{C}\mathrm{om}}\mathcal{C} & \xrightarrow[\mathrm{pr}]{\sim} & \mathcal{C} \end{array} \tag{A.2.0.2}$$

of ∞-categories: The left vertical functor belongs to a monadic adjunction by Proposition A.2.0.4. Thus, the right vertical functor also admits a left adjoint, and the resulting adjunction is monadic. □

A.2.0.6. Construction. — In the situation of Corollary A.2.0.5 we would like to show that the monad $\mathrm{forg}_{\mathcal{O}} \circ \mathrm{free}_{\mathcal{O}}$ of $\mathcal{C}$ is of the form

$$\mathcal{C} \to \mathcal{C},\ X \mapsto \coprod_{r\geq 0} \big(\mathcal{O}(r)\otimes X^{\otimes r}\big)_{\mathfrak{S}_r},$$

where $\mathcal{O}(r)\otimes X^{\otimes r}$ denotes the tensor of $X^{\otimes r}$ by $\mathcal{O}(r)$. For this purpose, we begin with recalling an explicit construction of the functor $\mathrm{free}_{\mathcal{O}}$ from [HA, Construction 3.1.3.9].

We continue to abbreviate $\mathcal{O}^{\otimes}_{\langle 1\rangle}$ by $\mathcal{O}$ in this construction. Note that [HA, Construction 3.1.3.9] works with $\mathcal{O}$-monoidal ∞-categories. Thus to apply it in the situation of Corollary A.2.0.5, we again make use of the $\mathcal{O}$-monoidal ∞-category defined via the pullback diagram (A.2.0.1). We denote the $\mathcal{O}$-monoidal category by

$$q\colon \mathcal{D}^{\otimes} := \mathcal{O}^{\otimes} \times_{\mathcal{C}\mathrm{om}^{\otimes}} \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}.$$

In particular, the underlying $\mathcal{O}$-monoidal category $\mathcal{D} \simeq \mathcal{O} \otimes_{\mathcal{C}\mathrm{om}} \mathcal{C} \simeq \mathcal{C}$, since $\mathcal{O}^{\otimes}$ is one-coloured. For an object $X \in \mathcal{C}$, denote

$$\widetilde{X} := (c, X) \in \mathcal{D},$$

which is an object of $\mathcal{D}$ mapped to X under the functor pr. By [HA, Proposition 3.1.3.13] we have

$$\mathrm{forg}_{\mathcal{O}} \circ \mathrm{free}_{\mathcal{O}} \colon \mathcal{D} \to \mathcal{D},\ \widetilde{X} \mapsto \coprod_{r \geq 0} \mathrm{Sym}^{r}_{\mathcal{O}}(\widetilde{X}), \tag{A.2.0.3}$$

where $\mathrm{Sym}^{r}_{\mathcal{O}}(-)$ is a certain colimit construction which we recall now (see also [HA, Proposition 3.1.3.9]).

Let $f\colon \mathcal{T}\mathrm{riv}^{\otimes} \to \mathcal{O}^{\otimes}$ be the map of ∞-operads sending $\langle 1 \rangle$ to c. Let $r \in \mathbb{N}$. Define a full ∞-subcategory $\mathcal{P}(r)$ of $\mathcal{T}\mathrm{riv}^{\otimes} \otimes_{\mathcal{O}^{\otimes}} \mathcal{O}^{\otimes}_{/c}$ as follows: An object in $\mathcal{P}(r)$ is a pair $(\langle r \rangle, f(\langle r \rangle) \to c)$ of the object $\langle r \rangle$ of $\mathcal{T}\mathrm{riv}^{\otimes}$ and an object $f(\langle r \rangle) \to c$ of $\mathcal{O}^{\otimes}_{/c}$ such that $f(\langle r \rangle) \to c$, considered as a morphism in $\mathcal{O}^{\otimes}$, is a lift (under p) of the morphism f_r of pointed finite sets. Note that $\mathcal{P}(r)$ is a Kan complex since morphisms in $\mathcal{P}(r)$ is induced by the automorphisms $\mathfrak{S}_r$ of the pointed set $\langle r \rangle$. Thus projecting to the first component induces a functor

$$\pi\colon \mathcal{P}(r) \to \mathrm{N}(\mathfrak{S}_r),$$

where $\mathrm{N}(\mathfrak{S}_r)$ denotes the nerve of the groupoid $\mathfrak{S}_r$. The fibre of the functor/morphism π is equivalent to the ∞-groupoid $\mathcal{O}(r)$ by construction. Here we consider $\mathrm{N}(\mathfrak{S}_r)$ as the ∞-subcategory of $\mathcal{T}\mathrm{riv}^{\otimes}_{\langle r \rangle}$: It has a single object $\langle r \rangle$ and the morphisms are bijections of $\langle r \rangle$. Thus we obtain a natural transformation

$$H\colon \mathcal{P}(r) \times \Delta^1 \to \mathcal{O}^{\otimes}$$

between $h_0\colon\ f|_{\mathrm{N}(\mathfrak{S}_r)} \circ \pi$ and the constant map $h_1\colon \mathcal{P}(r) \to \mathcal{O}^{\otimes}_{\langle 1 \rangle} \hookrightarrow \mathcal{O}^{\otimes}$ mapping to the vertex c. The object $\widetilde{X} \in \mathcal{D}$ determines a lift

$$\overline{f}_{\widetilde{X}}\colon\ \mathcal{T}\mathrm{riv}^{\otimes} \to \mathcal{D}^{\otimes}$$

of f satisfying $\overline{f}_{\widetilde{X}}(1) = \widetilde{X}$. Since q is cocartesian, we obtain a q-cocartesian natural transformation[(1)]

$$\overline{H}\colon \mathcal{P}(r) \times \Delta^1 \to \mathcal{D}^{\otimes}$$

between $\overline{h}_0 = \overline{f}_{\widetilde{X}}\Big|_{\mathrm{N}\,\mathfrak{S}_r} \circ \pi$ and a morphism $\overline{h}_1\colon \mathcal{P}(r) \to \mathcal{D} \hookrightarrow \mathcal{D}^{\otimes}$, unique up to contractible choice.

(1) The natural transformation $\overline{H}$ is equivalent to a morphism $\mathcal{P}(r) \to \mathcal{M}\mathrm{ap}(\Delta^1, \mathcal{D}^{\otimes})$. We say $\overline{H}$ is q-cocartesian if the essential image of the latter map are q-cocartesian edges in $\mathcal{D}^{\otimes}$ lifting $f(\langle r \rangle) \to c$.

The object $\mathrm{Sym}^r_{\mathcal{O}}(\widetilde{X})$ is defined as the colimit of $\overline{h}_1$.

Recall that a q-cocartesian lift of $f(\langle r\rangle) \to c$ corresponds to a morphism $\Delta^1 \to \mathcal{D}^{\otimes}$ in $\mathcal{D}^{\otimes}$ represented by

$$(\widetilde{X}_1, \ldots, \widetilde{X}_r) \to \widetilde{X}_1 \otimes_{\mathcal{D}} \cdots \otimes_{\mathcal{D}} \widetilde{X}_r$$

using Straightening, see Remark 5.2.1.21, Example 5.2.1.23 and Remark 5.2.1.28. Thus the following composition

$$\mathcal{P}(r) \xrightarrow{\pi} \mathrm{N}(\mathfrak{S}_r) \xrightarrow{\overline{f}_{\widetilde{X}}} \mathcal{D}^{\otimes}_{\langle r\rangle} \xrightarrow{\otimes_{\mathcal{D}}} \mathcal{D}$$

is a candidate for $\overline{h}_1$. The induced diagram

$$h_X \colon \mathcal{P}(r) \xrightarrow{q} \mathrm{N}(\mathfrak{S}_r) \to \mathcal{D}^{\otimes}_{\langle r\rangle} \to \mathcal{D} \xrightarrow{\mathrm{pr}} \mathcal{C}.$$

maps every vertex of $\mathcal{P}(r)$ to $X^{\otimes_{\mathcal{C}} r}$. Denote the colimit of the diagram h_X in $\mathcal{C}$ by $\mathrm{Sym}^r_{\mathcal{O}}(X)$. Together with (A.2.0.3), the pullback diagram (A.2.0.2) relating $\mathcal{O}$-algebras objects in $\mathcal{D}$ and $\mathcal{C}$ implies the following equivalence in $\mathcal{C}$:

$$(\mathrm{forg}_{\mathcal{O}} \circ \mathrm{free}_{\mathcal{O}})(X) = \coprod_{r \geq 0} \mathrm{Sym}^r_{\mathcal{O}}(X). \tag{A.2.0.4}$$

We can compute $\mathrm{Sym}^r_{\mathcal{O}}(X)$ by left Kan extensions. Consider the following diagram

$$\begin{array}{ccc}
\mathcal{P}(r) & \xrightarrow{\quad h_X \quad} & \mathcal{C} \\
\downarrow{\scriptstyle \pi} & \nearrow_{h'_X} & \\
\mathrm{N}(\mathfrak{S}_r) & & \\
\downarrow & \nearrow_{h''_X} & \\
\mathrm{pt} & &
\end{array}$$

where h''_X is the left Kan extension of h_X along the constant functor from $\mathcal{P}(r)$ to a point. Thus there exists an equivalence

$$h''_X(\mathrm{pt}) \simeq \mathrm{Sym}^r_{\mathcal{O}}(X).$$

To calculate h''_X, we first construct the left Kan extension h'_X of h_X along π, and then consider the left Kan extension of h'_X along the constant functor to the point. By [HTT, Definition 4.3.3.2] we know that $h'_X(\langle r\rangle)$ is equivalent to the colimit of the induced diagram

$$\mathcal{P}(r) \times_{\mathrm{N}(\mathfrak{S}_r)} (\mathrm{N}(\mathfrak{S}_r))_{/\langle r\rangle} \to \mathcal{P}(r) \xrightarrow{h_X} \mathcal{C}. \tag{A.2.0.5}$$

Using the equivalence $\Delta^0 \xrightarrow{\sim} (\mathrm{N}\,\mathfrak{S}_r)_{/\langle r\rangle}$ mapping to the vertex $\mathrm{id}_{\langle r\rangle}$, the diagram (A.2.0.5) is equivalent to the constant diagram below

$$\mathcal{O}(r) \simeq \mathcal{P}(r) \times_{\mathrm{N}(\mathfrak{S}_r)} \Delta^0 \to \mathcal{P}(r) \xrightarrow{h_X} \mathcal{C}.$$

Therefore we obtain the equivalence

$$h'_X(\langle r\rangle) \simeq \varinjlim_{\mathcal{O}(r)} X^{\otimes_{\mathcal{C}} r} \simeq \mathcal{O}(r) \otimes X^{\otimes_{\mathcal{C}} r}$$

by the definition of tensor [HTT, Corollary 4.4.4.9]. Now, left Kan extending h'_X by the constant functor to pt is the same as taking the colimit of h'_X, which is equivalent to taking the orbit of the $\mathfrak{S}_r$-action. Therefore, we have

$$\mathrm{Sym}^r_{\mathcal{O}}(X) \simeq \left(\mathcal{O}(r) \otimes X^{\otimes_{\mathcal{C}} r}\right)_{\mathfrak{S}_r},$$

for every $r \in \mathbb{N}$. Together with (A.2.0.4) we obtain the desired equivalence

$$(\mathrm{forg}_{\mathcal{O}} \circ \mathrm{free}_{\mathcal{O}})(X) \simeq \coprod_{r\geq 0} \left(\mathcal{O}(r) \otimes X^{\otimes_{\mathcal{C}} r}\right)_{\mathfrak{S}_r}.$$

in $\mathcal{C}$, for every object X of $\mathcal{C}$.

Summaries

Samenvatting

In dit proefschrift bestuderen we onstabiele v_h-periodieke homotopietheorie, waarbij h een natuurlijk getal is; hier gebruiken we de term 'onstabiel' voor de homotopietheorie van topologische ruimten. Het werk bestaat uit twee delen. In deel I geven we een gedetailleerde uiteenzetting van de grondslagen van onstabiele v_h-periodieke homotopietheorie, verscherpen we een bestaand resultaat over v_h-periodieke equivalenties van H-ruimten, en formuleren we concrete vragen en vermoedens voor toekomstig onderzoek. Dit expositorische gedeelte volgt papers van Bousfield, Dror Farjoun en Heuts en heeft als doel de centrale noties en stellingen van onstabiele lokalisaties op één plek te verzamelen, met een focus op onstabiele periodieke homotopietheorie. Het doel van deel II is om onstabiele v_h-periodieke verschijnselen te begrijpen vanuit het gezichtspunt van Lie-algebra's in de stabiele v_h-periodieke homotopie categorie. We analyseren de costabilisatie van v_h-periodieke homotopie types en verkrijgen een universele eigenschap van de Bousfield–Kuhn functor.

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Curriculum Vitae

Yuqing Shi was born in Ezhou, Hubei Province, China, on November 22nd 1994. After finishing high school in 2012 she enrolled in Wuhan University with a major in statistics and studied there for one year.

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Bibliography

[AB21] G. Z. Arone and D. L. B. Brantner. “The action of Young subgroups on the partition complex”. *Publ. Math., Inst. Hautes Étud. Sci.* 133 (2021), pp. 47–156. DOI: 10.1007/s10240-021-00123-7.

[Ada66] J. F. Adams. “On the groups $J(X)$. IV”. *Topology* 5 (1966), pp. 21–71. DOI: 10.1016/0040-9383(66)90004-8.

[AdaSH] J. F. Adams. *Stable Homotopy and Generalised Homology.* Chicago Lectures in Mathematics. University of Chicago Press, Chicago, Ill.-London, 1974.

[AK14] G. Arone and M. Kankaanrinta. “The sphere operad”. *Bull. Lond. Math. Soc.* 46.1 (2014), pp. 126–132. DOI: 10.1112/blms/bdt077.

[Aro06] G. Arone. “A note on the homology of Σ_n, the Schwartz genus, and solving polynomial equations”. In: *An Alpine Anthology of Homotopy Theory. Proceedings of the 2nd Arolla Conference on Algebraic Topology, Arolla, Switzerland, August 24–29, 2004.* American Mathematical Society (AMS), Providence, RI, 2006, pp. 1–10. DOI: 10.1090/conm/399/07509.

[Bal20] P. Balmer. “A guide to tensor-triangular classification”. In: *Handbook of homotopy theory.* CRC Press, Boca Raton, FL, 2020, pp. 145–162. DOI: 10.1201/9781351251624-4.

[Bar17] C. Barwick. “Spectral Mackey functors and equivariant algebraic K-theory. I.” *Adv. Math.* 304 (2017), pp. 646–727. DOI: 10.1016/j.aim.2016.08.043.

[Bar18] C. Barwick. “From operator categories to higher operads”. *Geom. Topol.* 22.4 (2018), pp. 1893–1959. DOI: 10.2140/gt.2018.22.1893.

[BD03] A. J. Berrick and E. Dror Farjoun. “Fibration and nullifications”. *Isr. J. Math.* 135 (2003), pp. 205–220. DOI: 10.1007/BF02776058.

[BGN18] C. Barwick, S. Glasman, and D. Nardin. “Dualizing cartesian and cocartesian fibrations”. *Theory Appl. Categ.* 33 (2018), pp. 67–94.

[BHLS23] R. Burklund, J. Hahn, I. Levy, and T. Schlank. *Applications of algebraic K-theory to a problem in topology*. Talk at "A Panorama of Homotopy Theory. A Conference in Honour of Mike Hopkins". June 2023. URL: https://www.maths.ox.ac.uk/groups/topology/panorama-homotopy-theory.

[BK] A. K. Bousfield and D. M. Kan. *Homotopy Limits, Completions and Localizations*. Vol. 304. Lect. Notes Math. Springer-Verlag, Berlin-New York, 1972.

[BM07] C. Berger and I. Moerdijk. "Resolution of coloured operads and rectification of homotopy algebras". In: *Categories in algebra, geometry and mathematical physics*. Vol. 431. Contemp. Math. Amer. Math. Soc., Providence, RI, 2007, pp. 31–58. DOI: 10.1090/conm/431/08265.

[BM23a] A. Beuckelmann and I. Moerdijk. *A small catalogue of E_n-operads*. Version 2. June 17, 2023. arXiv: 2212.03842.

[BM23b] L. Brantner and A. Mathew. *Deformation theory and partition Lie algebras*. Version 3. Jan. 17, 2023. arXiv: 1904.07352.

[BorHCA] F. Borceux. *Handbook of Categorical Algebra. Volume 1: Basic Category Theory*. Paperback reprint of the hardback edition 1994. Vol. 50. Encycl. Math. Appl. Cambridge University Press, Cambridge, 2008.

[Bou01] A. K. Bousfield. "On the telescopic homotopy theory of spaces". *Trans. Am. Math. Soc.* 353.6 (2001), pp. 2391–2426. DOI: 10.1090/S0002-9947-00-02649-0.

[Bou05] A. K. Bousfield. "On the 2-primary v_1-periodic homotopy groups of spaces". *Topology* 44.2 (2005), pp. 381–413. DOI: 10.1016/j.top.2004.10.003.

[Bou75] A. K. Bousfield. "The localization of spaces with respect to homology". *Topology* 14 (1975), pp. 133–150. DOI: 10.1016/0040-9383(75)90023-3.

[Bou79a] A. K. Bousfield. "The Boolean algebra of spectra". *Comment. Math. Helv.* 54 (1979), pp. 368–377. DOI: 10.1007/BF02566281.

[Bou79b] A. K. Bousfield. "The localization of spectra with respect to homology". *Topology* 18 (1979), pp. 257–281. DOI: 10.1016/0040-9383(79)90018-1.

[Bou82] A. K. Bousfield. "On homology equivalences and homological localizations of spaces". *Am. J. Math.* 104 (1982), pp. 1025–1042. DOI: 10.2307/2374082.

[Bou94] A. K. Bousfield. "Localization and periodicity in unstable homotopy theory." *J. Am. Math. Soc.* 7.4 (1994), pp. 831–873. DOI: 10.2307/2152734.

[Bou96] A. K. Bousfield. “Unstable localization and periodicity”. In: *Algebraic Topology: New Trends in Localization and Periodicity. Barcelona Conference on Algebraic Topology (BCAT), Sant Feliu de Guíxols, Spain, June 1-7, 1994*. Birkhäuser, Basel, 1996, pp. 33–50. DOI: 10.1007/978-3-0348-9018-2_3.

[Bou97] A. K. Bousfield. “Homotopical localizations of spaces”. *Am. J. Math.* 119.6 (1997), pp. 1321–1354. DOI: 10.1353/ajm.1997.0036.

[Bou99a] A. K. Bousfield. “On $K(n)$-equivalences of spaces”. In: *Homotopy Invariant Algebraic Structures. A Conference in Honor of J. Michael Boardman. AMS Special Session on Homotopy Theory, Baltimore, MD, USA, January 7–10, 1998*. American Mathematical Society, Providence, RI, 1999, pp. 85–89. DOI: 10.1090/conm/239/03598.

[Bou99b] A. K. Bousfield. “The K-theory localizations and v_1-periodic homotopy groups of H-spaces”. *Topology* 38.6 (1999), pp. 1239–1264. DOI: 10.1016/S0040-9383(98)00052-4.

[BP66] E. H. jr. Brown and F. P. Peterson. “A spectrum whose $\mathbb{Z}_p$ cohomology is the algebra of reduced p-th powers”. *Topology* 5 (1966), pp. 149–154. DOI: 10.1016/0040-9383(66)90015-2.

[Bra17] L. Brantner. “The Lubin–Tate theory of spectral Lie algebras”. PhD thesis. Harvard University, 2017.

[Cam20] O. A. Camarena. “The mod 2 homology of free spectral Lie algebras”. *Trans. Am. Math. Soc.* 373.9 (2020), pp. 6301–6319. DOI: 10.1090/tran/8131.

[Cas95] C. Casacuberta. “Anderson localization from a modern point of view”. In: *The Čech Centennial. A Conference on Homotopy Theory Dedicated to Eduard Čech on the Occasion of His 100th Birthday, June 22-26, 1993, Northeastern University, Boston, MA, USA*. American Mathematical Society, Providence, RI, 1995, pp. 35–44. DOI: 10.1090/conm/181/02028.

[CG05] C. Casacuberta and J. J. Gutiérrez. “Homotopical localizations of module spectra”. *Trans. Am. Math. Soc.* 357.7 (2005), pp. 2753–2770. DOI: 10.1090/S0002-9947-04-03552-4.

[CG21] D. Calaque and J. Grivaux. “Formal moduli problems and formal derived stacks”. In: *Derived Algebraic Geometry*. Paris: Société Mathématique de France (SMF), 2021, pp. 85–145.

[CH20] H. Chu and R. Haugseng. “Enriched ∞-operads”. *Adv. Math.* 361 (2020). Id/No 106913, p. 85. DOI: 10.1016/j.aim.2019.106913.

[Chi05] M. Ching. "Bar constructions for topological operads and the Goodwillie derivatives of the identity". *Geom. Topol.* 9 (2005), pp. 833–933. DOI: 10.2140/gt.2005.9.833.

[Chi12] M. Ching. "Bar-cobar duality for operads in stable homotopy theory". *J. Topol.* 5.1 (2012), pp. 39–80. DOI: 10.1112/jtopol/jtr027.

[CP93] C. Casacuberta and G. Peschke. "Localizing with respect to self-maps of the circle". *Trans. Am. Math. Soc.* 339.1 (1993), pp. 117–140. DOI: 10.2307/2154211.

[CS22] M. Ching and P. Salvatore. "Koszul duality for topological E_n-operads". *Proc. Lond. Math. Soc. (3)* 125.1 (2022), pp. 1–60. DOI: 10.1112/plms.12442.

[CSY22] S. Carmeli, T. M. Schlank, and L. Yanovski. "Ambidexterity in chromatic homotopy theory". *Invent. Math.* 228.3 (2022), pp. 1145–1254. DOI: 10.1007/s00222-022-01099-9.

[DAGX] J. Lurie. *Derived algebraic geometry X: Formal moduli problems.* Version November 05, 2011. URL: https://www.math.ias.edu/~lurie/papers/DAG-X.pdf.

[Dav03] D. M. Davis. "Representation types and 2-primary homotopy groups of certain compact Lie groups". *Homology Homotopy Appl.* 5.1 (2003), pp. 297–324. DOI: 10.4310/HHA.2003.v5.n1.a13.

[Dav91] D. M. Davis. "The v_1-periodic homotopy groups of $\mathrm{SU}(n)$ at odd primes". *J. Lond. Math. Soc., II. Ser.* 43.3 (1991), pp. 529–544. DOI: 10.1112/jlms/s2-43.3.529.

[Dav95] D. M. Davis. "Computing v_1-periodic homotopy groups of spheres and some compact Lie groups". In: *Handbook of Algebraic Topology.* Amsterdam: North-Holland, 1995, pp. 993–1048. DOI: 10.1016/B978-044481779-2/50021-3.

[DD09] W. G. Dwyer and E. Dror Farjoun. "Localization and cellularization of principal fibrations". In: *Alpine Perspectives on Algebraic Topology. Third Arolla Conference on Algebraic Topology, Arolla, Switzerland, August 18–24, 2008.* American Mathematical Society (AMS), Providence, RI, 2009, pp. 117–124. DOI: 10.1090/conm/504/09878.

[DH14] W. Dwyer and K. Hess. "The Boardman-Vogt tensor product of operadic bimodules". In: *Algebraic topology: Applications and New directions. Proceedings of the Stanford Symposium on Algebraic Topology, Stanford University, Stanford, CA, USA, July 23–27, 2012.* American Mathematical Society (AMS), Providence, RI, 2014, pp. 71–98. DOI: 10.1090/conm/620/12352.

[DixEA] J. Dixmier. *Enveloping Algebras*. Vol. 11. Grad. Stud. Math. American Mathematical Society, Providence, RI, 1996. DOI: 10.1090/gsm/011.

[DK] J. F. Davis and P. Kirk. *Lecture Notes in Algebraic Topology.*

[DM87] W. G. Dwyer and G. Mislin. "On the homotopy type of the components of $\mathrm{map}_*(BS^3, BS^3)$". In: *Algebraic topology, Barcelona, 1986*. Vol. 1298. Lecture Notes in Math. Springer, Berlin, 1987, pp. 82–89. DOI: 10.1007/BFb0083001.

[Dro71] E. Dror Farjoun. "A generalization of the Whitehead theorem." In: *Symposium on Algebraic Topology (Battelle Seattle Res. Center, Seattle, Wash., 1971).* Lecture Notes in Math., Vol. 249, Springer, Berlin, 1971, pp. 13–22.

[DroCS] E. Dror Farjoun. *Cellular Spaces, Null Spaces and Homotopy Localization.* Vol. 1622. Springer-Verlag, Berlin, 1995. DOI: 10.1007/BFb0094429.

[DS95] E. Dror Farjoun and J. H. Smith. "Homotopy localization nearly preserves fibrations". *Topology* 34.2 (1995), pp. 359–375. DOI: 10.1016/0040-9383(94)00025-G.

[Dun88] G. Dunn. "Tensor product of operads and iterated loop spaces". *J. Pure Appl. Algebra* 50.3 (1988), pp. 237–258. DOI: 10.1016/0022-4049(88)90103-X.

[EHMM19] R. Eldred, G. Heuts, A. Mathew, and L. Meier. "Monadicity of the Bousfield-Kuhn functor". *Proc. Am. Math. Soc.* 147.4 (2019), pp. 1789–1796. DOI: 10.1090/proc/14331.

[EKMM] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. *Rings, Modules, and Algebras in Stable Homotopy Theory. With An Appendix by M. Cole.* Vol. 47. Math. Surv. Monogr. American Mathematical Society, Providence, RI, 1997. DOI: 10.1090/surv/047.

[FG12] J. Francis and D. Gaitsgory. "Chiral Koszul duality". *Sel. Math., New Ser.* 18.1 (2012), pp. 27–87. DOI: 10.1007/s00029-011-0065-z.

[FHT] Y. Felix, S. Halperin, and J.-C. Thomas. *Rational Homotopy Theory.* Vol. 205. Grad. Texts Math. Springer, New York, NY, 2001. DOI: 10.1007/978-1-4613-0105-9.

[Fre04] B. Fresse. "Koszul duality of operads and homology of partition posets". In: *Homotopy theory: Relations with Algebraic Geometry, Group Cohomology, and Algebraic K-theory. Papers from the International Conference on Algebraic Topology, Northwestern University, Evanston, IL, USA, March 24–28, 2002.* American Mathematical Society (AMS), Providence, RI, 2004, pp. 115–215. DOI: 10.1090/conm/346/06287.

[Fre11] B. Fresse. "Koszul duality of E_n-operads". *Sel. Math., New Ser.* 17.2 (2011), pp. 363–434. DOI: 10.1007/s00029-010-0047-6.

[FreHO] B. Fresse. *Homotopy of Operads and Grothendieck–Teichmüller Groups. Part 1: The Algebraic Theory and Its Topological Background*. Vol. 217. Math. Surv. Monogr. American Mathematical Society (AMS), Providence, RI, 2017. DOI: 10.1090/surv/217.1.

[FW20] B. Fresse and T. Willwacher. "The intrinsic formality of E_n-operads". *J. Eur. Math. Soc. (JEMS)* 22.7 (2020), pp. 2047–2133. DOI: 10.4171/JEMS/961.

[GHK22] D. Gepner, R. Haugseng, and J. Kock. "∞-operads as analytic monads". *Int. Math. Res. Not.* 2022.16 (2022), pp. 12516–12624. DOI: 10.1093/imrn/rnaa332.

[GJ94] E. Getzler and J. Jones. *Operads, homotopy algebra and iterated integrals for double loop spaces*. Version 1. Mar. 8, 1994. arXiv: hep-th/9403055.

[GK94] V. Ginzburg and M. Kapranov. "Koszul duality for operads". *Duke Math. J.* 76.1 (1994), pp. 203–272. DOI: 10.1215/S0012-7094-94-07608-4.

[Goo03] T. G. Goodwillie. "Calculus. III: Taylor series". *Geom. Topol.* 7 (2003), pp. 645–711. DOI: 10.2140/gt.2003.7.645.

[GR] D. Gaitsgory and N. Rozenblyum. *A Study in Derived Algebraic Geometry. Volume II: Deformations, Lie Theory and Formal Geometry*. Vol. 221. Math. Surv. Monogr. American Mathematical Society (AMS), Providence, RI, 2017. DOI: 10.1090/surv/221.1.

[HA] J. Lurie. *Higher algebra*. Version September, 2017. URL: https://www.math.ias.edu/~lurie/papers/HA.pdf.

[Har18] Harvard Thursday Seminar. *Unstable chromatic homotopy theory*. 2018. URL: https://www.math.ias.edu/~lurie/Thursday2017.html (visited on 07/13/2023).

[HatAT] A. Hatcher. *Algebraic Topology*. Cambridge University Press, Cambridge, 2002.

[Hau19] R. Haugseng. *Algebra for enriched ∞-operads*. Version 1. Sept. 22, 2019. arXiv: 1909.10042.

[Hau21] R. Haugseng. "Segal spaces, spans, and semicategories". *Proc. Am. Math. Soc.* 149.3 (2021), pp. 961–975. DOI: 10.1090/proc/15197.

[Hau22] R. Haugseng. "∞-operads via symmetric sequences". *Math. Z.* 301.1 (2022), pp. 115–171. DOI: 10.1007/s00209-021-02881-w.

[Hei18] H. Heine. "Restricted L_∞ algebras". PhD thesis. Universität Osnabrück, 2018.

[Hes99] K. Hess. "A history of rational homotopy theory". In: *History of Topology*. Elsevier, Amsterdam, 1999, pp. 757–796. DOI: 10.1016/B978-044482375-5/50028-6.

[Heu] G. Heuts. *Koszul duality and a conjecture of Francis–Gaitsgory*. In preparation.

[Heu20a] G. Heuts. “Koszul duality and a conjecture of Francis–Gaitsgory”. In: *Topologie*. Vol. 17. Oberwolfach Reports. European Mathematical Society, 2020, pp. 1095–1098. DOI: 10.4171/OWR/2020/21.

[Heu20b] G. Heuts. “Lie algebra models for unstable homotopy theory”. In: *Handbook of Homotopy Theory*. CRC Press, Boca Raton, FL, 2020, p. 42. DOI: 10.1201/9781351251624.

[Heu21] G. Heuts. “Lie algebras and v_n-periodic spaces”. *Ann. Math. (2)* 193.1 (2021), pp. 223–301. DOI: 10.4007/annals.2021.193.1.3.

[Hil82] P. Hilton. “Nilpotency in group theory and topology”. *Publ., Secc. Mat., Univ. Autòn. Barc.* 26.3 (1982), pp. 47–78.

[Hir19] P. S. Hirschhorn. “The Quillen model category of topological spaces”. *Expo. Math.* 37.1 (2019), pp. 2–24. DOI: 10.1016/j.exmath.2017.10.004.

[HL] G. Heuts and M. Land. *Cochain on spheres and formality of $\mathbb{E}_n$-algebras*. In preparation.

[HM22] G. Heuts and I. Moerdijk. *Simplicial and Dendroidal Homotopy Theory*. Vol. 3. Ergeb. Math. Grenzgeb., 3. Folge. Springer, 2022. DOI: 10.1007/978-3-031-10447-3.

[HopCOC] M. J. Hopkins. *Complex oriented cohomology theories and the language of stacks*. 1999. URL: https://people.math.rochester.edu/faculty/doug/otherpapers/coctalos.pdf (visited on 07/13/2023).

[Hov04] M. Hovey. “Homotopy theory of comodules over a Hopf algebroid”. In: *Homotopy theory: Relations with Algebraic Geometry, Group Cohomology, and Algebraic K-theory. Papers from the International Conference on Algebraic Topology, Northwestern University, Evanston, IL, USA, March 24–28, 2002*. American Mathematical Society (AMS), Providence, RI, 2004, pp. 261–304. DOI: 10.1090/conm/346/06291.

[HS98] M. J. Hopkins and J. H. Smith. “Nilpotence and stable homotopy theory II”. *Ann. Math. (2)* 148.1 (1998), pp. 1–49. DOI: 10.2307/120991.

[HTT] J. Lurie. *Higher Topos Theory*. Vol. 170. Princeton University Press, Princeton, NJ, 2009. DOI: 10.1515/9781400830558.

[JohSE] P. T. Johnstone. *Sketches of an Elephant. A Topos Theory Compendium. I*. Vol. 43. Oxf. Logic Guides. Clarendon Press, Oxford, 2002.

[Joy02] A. Joyal. “Quasi-categories and Kan complexes”. *J. Pure Appl. Algebra* 175.1-3 (2002), pp. 207–222. DOI: 10.1016/S0022-4049(02)00135-4.

[Kai71] P. C. Kainen. “Universal coefficient theorems for generalized homology and stable cohomotopy”. *Pac. J. Math.* 37 (1971), pp. 397–407. DOI: 10.2140/pjm.1971.37.397.

[Kel05] G. M. Kelly. “On the operads of J.P. May”. *Repr. Theory Appl. Categ.* 2005.13 (2005), pp. 1–13.

[Kja18] J. J. Kjaer. “On the odd primary homology of free algebras over the spectral Lie operad”. *J. Homotopy Relat. Struct.* 13.3 (2018), pp. 581–597. DOI: 10.1007/s40062-017-0194-y.

[Knu18] B. Knudsen. “Higher enveloping algebras”. *Geom. Topol.* 22.7 (2018), pp. 4013–4066. DOI: 10.2140/gt.2018.22.4013.

[KocB] S. O. Kochman. *Bordism, Stable Homotopy and Adams Spectral Sequences.* Vol. 7. Fields Inst. Monogr. American Mathematical Society (AMS), Providence, RI, 1996. DOI: 10.1090/fim/007.

[KSV97] J. Klein, R. Schwänzl, and R. M. Vogt. “Comultiplication and suspension”. *Topology Appl.* 77.1 (1997), pp. 1–18. DOI: 10.1016/S0166-8641(96)00103-4.

[Kuh07] N. J. Kuhn. “Goodwillie towers and chromatic homotopy: an overview”. In: *Proceedings of the Nishida Fest Conferences on Homotopy Theory, Kinosaki, Japan, July 28–August 1 2003 and August 4–8, 2003. Dedicated to Gôrô Nishida on the Occasion of His 60th Birthday.* Geometry & Topology Publications, Coventry, 2007, pp. 245–279. DOI: 10.2140/gtm.2007.10.245.

[Kuh08] N. J. Kuhn. “A guide to telescopic functors”. *Homology Homotopy Appl.* 10.3 (2008), pp. 291–319.

[LanII] M. Land. *Introduction to Infinity-categories.* Compact Textb. Math. Cham: Birkhäuser, 2021. DOI: 10.1007/978-3-030-61524-6.

[Law20a] T. Lawson. “E_n-spectra and Dyer-Lashof operations”. In: *Handbook of Homotopy Theory.* CRC Press, Boca Raton, FL, 2020, pp. 793–849. DOI: 10.1201/9781351251624-19.

[Law20b] T. Lawson. *An introduction to Bousfield localization.* Version 1. Feb. 10, 2020. arXiv: 2002.03888.

[LS01] L. Langsetmo and D. Stanley. “Nondurable K-theory equivalence and Bousfield localization”. *K-Theory* 24.4 (2001), pp. 397–410. DOI: 10.1023/A:1014083003668.

[LurCHT] J. Lurie. *Chromatic homotopy theory.* 2010. URL: https://www.math.ias.edu/~lurie/252x.html (visited on 07/13/2023).

[LV] J.-L. Loday and B. Vallette. *Algebraic Operads.* Vol. 346. Grundlehren Math. Wiss. Springer, Berlin, 2012. DOI: 10.1007/978-3-642-30362-3.

[MacCWM] S. Mac Lane. *Categories for the Working Mathematician*. 2nd ed. Vol. 5. Grad. Texts Math. Springer, New York, NY, 1998.

[Mah82] M. Mahowald. "The image of J in the EHP sequence". *Ann. Math. (2)* 116 (1982), pp. 65–112. DOI: 10.2307/2007048.

[MayGIL] J. P. May. *The Geometry of Iterated Loop Spaces*. Vol. 271. Lect. Notes Math. Springer-Verlag, Berlin-New York, 1972.

[Mit85] S. A. Mitchell. "Finite complexes with $A(n)$-free cohomology". *Topology* 24 (1985), pp. 227–246. DOI: 10.1016/0040-9383(85)90057-6.

[MM65] J. W. Milnor and J. C. Moore. "On the structure of Hopf algebras". *Ann. Math. (2)* 81 (1965), pp. 211–264. DOI: 10.2307/1970615.

[Moo71] J. C. Moore. "Differential homological algebra". In: *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1*. Gauthier-Villars, Paris, 1971, pp. 335–339.

[MP] J. P. May and K. Ponto. *More Concise Algebraic Topology*. University of Chicago Press, Chicago, IL, 2012.

[MRW77] H. R. Miller, D. C. Ravenel, and W. S. Wilson. "Periodic phenomena in the Adams-Novikov spectral sequence". *Ann. Math. (2)* 106 (1977), pp. 469–516. DOI: 10.2307/1971064.

[MW19] J. M. Moreno-Fernández and F. Wierstra. *Iterated suspensions are coalgebras over the little disks operad*. Version 1. Sept. 24, 2019. arXiv: 1909.11043.

[NeiAMU] J. Neisendorfer. *Algebraic Methods in Unstable Homotopy Theory*. Vol. 12. New Math. Monogr. Cambridge University Press, Cambridge, 2010. DOI: 10.1017/CBO9780511691638.

[NSS15] T. Nikolaus, U. Schreiber, and D. Stevenson. "Principal ∞-bundles: General theory". Version 3. *J. Homotopy Relat. Struct.* 10.4 (2015), pp. 749–801. DOI: 10.1007/s40062-014-0083-6.

[Pér22] M. Péroux. "The coalgebraic enrichment of algebras in higher categories". *J. Pure Appl. Algebra* 226.3 (2022). Id/No 106849, p. 11. DOI: 10.1016/j.jpaa.2021.106849.

[PetFG] E. Peterson. *Formal Geometry and Bordism Operations. With Appendices by Nathaniel Stapleton and Michael Hopkins*. Vol. 177. Camb. Stud. Adv. Math. Cambridge University Press, Cambridge, 2019. DOI: 10.1017/9781108552165.

[Pri10] J. P. Pridham. "Unifying derived deformation theories". *Adv. Math.* 224.3 (2010), pp. 772–826. DOI: 10.1016/j.aim.2009.12.009.

[Pri70] S. B. Priddy. "Koszul resolutions". *Trans. Am. Math. Soc.* 152 (1970), pp. 39–60. DOI: 10.2307/1995637.

[PstFHC] P. Pstrągowski. *Finite height chromatic homotopy theory.* 2021. URL: https://people.math.harvard.edu/~piotr/252y_notes.pdf (visited on 07/13/2023).

[Qui69] D. Quillen. "Rational homotopy theory". *Ann. Math. (2)* 90 (1969), pp. 205–295. DOI: 10.2307/1970725.

[QuiHA] D. G. Quillen. *Homotopical Algebra.* Vol. 43. Lect. Notes Math. Springer-Verlag, Berlin-New York, 1967. DOI: 10.1007/BFb0097438.

[Rav84] D. C. Ravenel. "Localization with respect to certain periodic homology theories". *Am. J. Math.* 106 (1984), pp. 351–414. DOI: 10.2307/2374308.

[ReuFLA] C. Reutenauer. *Free Lie Algebras.* Vol. 7. Lond. Math. Soc. Monogr., New Ser. Clarendon Press, Oxford, 1993.

[RotITG] J. J. Rotman. *An Introduction to the Theory of Groups.* 4th ed. Vol. 148. Grad. Texts Math. Springer-Verlag, New York, NY, 1995. DOI: 10.1007/978-1-4612-4176-8.

[RW80] D. C. Ravenel and W. S. Wilson. "The Morava K-theories of Eilenberg-MacLane spaces and the Conner-Floyd conjecture". *Am. J. Math.* 102 (1980), pp. 691–748. DOI: 10.2307/2374093.

[Seg74] G. Segal. "Categories and cohomology theories". *Topology* 13 (1974), pp. 293–312. DOI: 10.1016/0040-9383(74)90022-6.

[Ser53] J.-P. Serre. "Groupes d'homotopie et classes de groupes abéliens". *Ann. Math. (2)* 58 (1953), pp. 258–294. DOI: 10.2307/1969789.

[Smi70] L. Smith. "On realizing complex bordism modules. Applications to the stable homotopy of spheres". *Am. J. Math.* 92 (1970), pp. 793–865. DOI: 10.2307/2373397.

[SulGT] D. Sullivan. *Geometric Topology. Localization, Periodicity and Galois Symmetry. The 1970 MIT Notes. Edited by A. Ranicki.* Ed. by A. Ranicki. Vol. 8. K-Monogr. Math. Springer, Dordrecht, 2005.

[Tai98] J.-y. Tai. "On f-localization functors and connectivity". In: *Stable and Unstable Homotopy (Toronto, ON, 1996).* Vol. 19. Fields Inst. Commun. Amer. Math. Soc., Providence, RI, 1998, pp. 285–298. DOI: 10.1007/s002880050159.

[Tho93] R. D. Thompson. "A relation between K-theory and unstable homotopy groups with an application to $B\Sigma_p$". In: *Algebraic topology, Oaxtepec 1991. Proceedings of an international conference on algebraic topology, held July 4-11, 1991 in Oaxtepec, Mexico.* American Mathematical Society, Providence, RI, 1993, pp. 431–439. DOI: 10.1090/conm/146/01238.

[Tri] T. H. Trimble. *Notes on the Lie operad.* URL: https://math.ucr.edu/home/baez/trimble/ (visited on 07/13/2023).

[Wan15] G. Wang. "Unstable chromatic homotopy theory". PhD thesis. Massachusetts Institute of Technology, 2015.

[YauCO] D. Yau. *Colored Operads*. Vol. 170. Grad. Stud. Math. American Mathematical Society (AMS), Providence, RI, 2016. DOI: 10.1090/gsm/170.

[Yos75] Z.-i. Yosimura. "Universal coefficient sequences for cohomology theories of CW-spectra". *Osaka J. Math.* 12 (1975), pp. 305–323.

[Zab87] A. Zabrodsky. "Maps between classifying spaces". In: *Algebraic topology and algebraic K-theory*. Vol. 113. Ann. of Math. Stud. Princeton Univ. Press, Princeton, NJ, 1987, pp. 228–246.