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Lesson study in mathematics with TDS and RME as theoretical support: two cases from the European TIME project

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Abstract

Purpose – The authors investigate the use and potential of a theoretical combination of Realistic Mathematics Education (RME) and the Theory of Didactic Situation (TDS) to support Lesson Study (LS) in upper secondary mathematics.

Design/methodology/approach – Case study performed by university researchers, based on theoretical analysis and case studies based on documents and observation from lesson studies.

Findings – Even within a project lasting just about three years, teachers (with no preliminary experience of lesson study) engaged in lesson design based on the combination of theoretical perspectives from TDS and RME in ways that confirm the potential of that combination to enrich and focus teachers' professional development within the framework of LS . It is not clear to what extent the intensive and continued engagement of university researchers has been or would be essential for similar and longitudinal realizations of these potentials.

Practical implications – As current European frameworks seek to engage researchers and teachers in collaboration and exchange across countries, networking of major paradigms of research (like TDS and RME) and uses of them as supports for teachers' inquiry (like demonstrated in this paper) is of considerable institutional interest and potential impact on schools.

Social implications – Teachers' Inquiry in Mathematics Education (TIME) is a prerequisite for the development of Inquiry Based Mathematics Education, which in turn is required in many countries across the world, with the aim of fostering critical and competent citizens.

Originality/value – This combination of (major) mathematics education theories to support and enrich LS has not previously been investigated. While several aspects of adapting to LS Western contexts have been investigated in the past, including the inclusion of perspectives and tools from academic research, the role of university researchers is also quite open. While authors do not offer a systematic study of this role, authors examine how this role may involve development of new practical combinations of different, complementary theoretical tools, which indeed hold potential to support lesson study in a European context.

Keywords Lesson study, Realistic mathematics education, Theory of didactical situations, Networking of theories

Paper type Research paper

Introduction

An important challenge for individuals who initiate, guide and lead Lesson Study (LS) activities, is to support the study and research of teachers by fueling the processes of designing, observing and analyzing lessons with theoretical principles and results that go beyond the particular context and lesson. Choy (2016) argues how teachers need to have

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explicit foci if they are to learn from the process of LS. We investigate how this purpose may be served by a specific combination (in the sense of Prediger *et al.*, 2008) of Realistic Mathematics Education (RME) and the Theory of Didactical Situations (TDS), two major research paradigms originating in European Didactics of Mathematics.

We do so through two case studies based on data from the European project "TIME: Teachers' Inquiry in Mathematics Education" (TIME, 2021). TIME involves teams of university researchers and upper secondary teachers in four countries, including Denmark and the Netherlands. In the project, LS was introduced to all teams and implemented as a format for TIME. Drawing on elements from RME and TDS that highlight the importance of students' activity in learning mathematics, we developed templates and manuals related to the various aspects of lesson study, to be used in the ongoing design, experimentation and analysis of research lessons. In this sense, the project produces a concrete, practice-oriented combination of (some elements from) these two theoretical frameworks in order to support and guide the teachers' use of lesson study. This combination was made possible by the expertise of researcher teams within the theoretical frameworks (RME for the Dutch team and TDS for the Danish team).

In the actual implementation, teachers and researchers work closely together in each country. Researchers participate in some of the planning meetings and in observations and reflection meetings. As results from lesson studies are shared and discussed among the teams, we develop a stronger sense of how the two theoretical perspectives can be combined to support and sharpen different aspects of lesson study: lesson design, experimentation and observation, analysis of observations and the sharing of results with peers, for instance through practice reports (in the Japanese sense, cf. Miyakawa and Winsløw, 2019).

The two case studies presented illustrate this process of progressive combination as well as some of the obstacles. One case is from Denmark and one from the Netherlands. Analyzing both cases from both theoretical perspectives, we illustrate how the initial development, shaped mainly by one of the perspectives, could be enriched by more intensive use of the other perspective. A main outcome is that the two frameworks can largely provide *complementary* points of support, both in preliminary analysis of the mathematical knowledge to be taught, in lesson planning and in analyzing observations. Our main goal is to identify more precisely what these complementary points are, by analyzing each other's cases from the theoretical perspective that was, initially, less present in the practice reports. The paper will thus advance our overall hypotheses that lesson study can draw compatible, but different points of support from RME and TDS and that teachers (with a strong academic background) can benefit from these both while carrying out lesson studies and while sharing the outcomes with peers.

Background

The recognition and categorization of scholarly work in the field of education varies to some extent among cultures, societies and institutions, even if globalization has led to more international collaboration and communication in this field. In particular, this holds for the connections and distinctions between "professional knowledge" developed by and for teachers in a given school institution and "scientific knowledge" developed by researchers from another (university type) institution. Miyakawa and Winsløw (2009) take the cases of LS in Japan and Didactical Engineering based on TDS as paradigmatic examples, considering both similarities and differences. Clivaz (2015) addresses the same two constructs and identifies potentials in a dialogue between the two frameworks.

One important difference is the role and nature of *theory*. In Japan, LS and similar activities draw on, and contribute to, different kinds of theory, some of which are by now known internationally, such as *open-ended approach* (Nohda, 2000). These theories are about principles and methods of teaching and involve specialized terminology that helps teachers communicate precisely about teaching designs, including key aspects of school mathematics. They are also central in teacher education and in the (often close) collaboration between university and schoolteachers (Miyakawa and Winsløw, 2019). By contrast, theories in mathematics education research as carried out in Western countries like France are usually developed by and for researchers – whether they do "experimental" (as in Didactic Engineering and TDS) or descriptive-analytic studies that do not involve classroom interventions. Of course, modalities of "action research" and the like also exist in the West, so the difference outlined is only partial.

When LS in the West is often initiated and supported by university researchers, it is not surprising that they draw on theoretical perspectives from their scholarly work in order to make sense of – and in – the activity (cf. Winsløw *et al.*, 2018). Using theory to make sense *of* LS takes LS as a research object to be analyzed with theoretical tools. For instance, Miyakawa and Winsløw (2019), categorizes LS as an element of a wider paradidactic infrastructure in the sense of the Anthropological Theory of the Didactic. Scholarly theory can also be used to make sense *within* the LS activity. In particular, the term "learning study" is frequently used to designate a LS with the deliberate use of variation theory (Pang and Ling, 2012). For involved researchers, a LS can be a practice-based illustration of theory, and it highlights the teachers' influence in the implementation of theory (Clivaz, 2015).

Here we are concerned with the combined use, within a LS activity, of two theoretical frameworks namely RME and TDS, which are both central to the authors' research. In our collaboration with teachers and other researchers in the European project Mathematics Education – Relevant Interesting and Applicable (MERIA project, 2019), the two theories had already become central to shared design and analysis of teaching situations. LS was introduced in the subsequent project TIME, as a work format which we estimated could transfer the main initiative in lesson design from researchers to teachers, while still drawing on the theoretical tools from MERIA, now in the setting of LS activities. Both the work of design and analysis of observations heavily involved researchers who were already familiar with these theoretical tools; however, the TIME project included at larger number of teachers who were new to both frameworks. Still, in the TIME project, as teacher teams were to lead the design of lessons (and, in particular, the problems to be studied there); they were largely in charge of handling the theoretical tools.

Theoretical framework

The combination of theoretical frameworks for this study originates from the MERIA project. Roughly, in this combination, elements of TDS serve to structure lesson plans and analyze observations, while principles for task design are drawn from RME (Winsløw, 2017).

TDS studies classroom situations that support students in developing mathematical knowledge. A key component of designing such situations is the notion of the didactical milieu. The milieu is the environment, including problem situations and artifacts to use and manipulate, with which the student interacts to obtain new knowledge. When preparing a lesson, teachers design an appropriate milieu for the students' development of new knowledge. The main types of situations defined in TDS are: devolution, action, formulation, validation and institutionalization (Brousseau, 1997). Within MERIA and TIME, these were interpreted as successive phases of a lesson. In the devolution phase, the students are handed over a problem. In the action phase, they work on the problem and in the following formulation phase, they share their findings. Next, personally developed knowledge is validated against the milieu by comparing and discussing strategies and ideally becomes closer to what can be regarded as institutional knowledge (Winsløw, 2017). Most situations are didactical and the teacher actively orchestrates the activities. The action phase is adidactical, since the teacher is not expected to intervene in the students' activity.

In RME, mathematics is interpreted as a human and constructive activity guided by task situations that are "realistic" for the learners (Freudenthal, 1991). A situation is "realistic" when the involved tasks and artifacts are meaningful for the target audience. Task design can be based on a didactical phenomenology, which consists of a search for phenomena or contexts that beg to be mathematized by the new knowledge and offer starting points for a process of mathematization (Freudenthal, 1983). This implies that the situation or milieu are meaningful to the students and supports them in developing means for organizing and solving the problem. This mathematization process involves "horizontal" and "vertical" elements (Treffers, 1987). Horizontal mathematization refers to the process where the real-world situation is approached through mathematical means. Vertical mathematization refers to a reorganization of emerging, personal and informal mathematical conceptions and procedures by more formal and abstract means.

In the research literature we can find only a little about LS involving TDS or RME separately, and the combination is probably entirely new. Concerning TDS and LS, Bahn and Winsløw (2019) report on a relatively large-scale experiment with TDS-based LS in Danish primary school mathematics, where it turned out that TDS became mostly a tool for the researcher, who found *few explicit instances of teachers making use of TDS tools* (p. 97). In a similar project involving RME-based LS in Namibian primary school mathematics, Peters (2016) observed more positive results and attitudes in terms of teachers' relationship with the theoretical elements. Of course, circumstances and designs of these relatively isolated studies differed too much to conclude anything from this difference. Certainly, such results depend both on general contexts and on the ways in which the "teaching end" (in the sense of Gascón and Nicolas, 2017) of the theories are presented to teachers. The combination of TDS and RME proposed in MERIA (Winsløw, 2017) made the theories apply to different aspects of lesson planning: the preliminary analysis of mathematical knowledge and the problem design was mainly based on RME, while the "staging" of the problem through a structured lesson script was based on TDS. The two aspects certainly interact for instance in considerations about the milieu and how it could support students' work with the problem, or when reflecting on observations of situations of action and formulation, where students mathematize both horizontally and vertically.

Research purpose and research questions

In the TIME project, we faced the challenge of teachers' and researchers' different academic and theoretical background and in particular their varying familiarity with the theoretical frameworks. Some of the teachers have worked the theories previously, though most teachers were new to RME, TDS or even both. This naturally leads to our research question: *How could teachers develop a combined and consistent use of both RME and TDS when engaging in LS and what is the potential of this combination for their professional development through LS?*

Methodology

Initially, all teachers in the TIME project were given an introduction to the TIME project, LS (Bašić, 2020b) and inquiry-based mathematics education based on RME and TDS, as practiced in MERIA (Winsløw, 2017). Together with the introduction to LS, the teachers were provided with the TIMEplate, which includes the template for the lesson plan, a recap of the main ideas of LS and a template for writing practice reports (Bašić, 2020a). The main activities in the project are performing LS cycles by LS groups of participating teacher teams at partner upper secondary schools in each country. During the project, the teachers leading each team have met (mainly online) and shared ideas for lessons, got feedback, shared points from observations and discussed what they have learned from their LS in each team. Each lesson was implemented twice (or more), with adjustments between each implementation, based on observations and reflection meetings. The teams plan their research lessons and ask for inputs from researchers, if they find it necessary. At least one researcher acts as “knowledgeable other” during the first implementation where the entire team observes. During the second implementation, all national researchers participate if possible and other local teacher teams observe the lesson as well. Based on this entire cycle, the group writes a practice report in English, which is peer reviewed by another TIME team—from a different country—before the report is published at the TIME project web page (TIME, 2021). Furthermore, the practice report is translated into the local language of the group and shared nationally in teacher journals. In the template for lesson plans (TIME, 2021), teachers fill in short description of Target knowledge, Broader goals, Prerequisite mathematical knowledge, Grade level, Time, Required materials and Problem (for students to work on in the lesson). Furthermore, the teachers complete a script of the lesson that describes different phases of the lesson. The description of each phase includes the approximate duration, “Teacher’s action incl. instructions” and “Expected students” actions’ In addition, for each phase there is a blank space for “Observations from implementation” (Bašić, 2020a). There is no explicit mention of RME or TDS in the template or in the handbook on LS (Bašić, 2020b), and teachers were not obliged to use their methods and notions, although they were introduced to the combination (from MERIA) outlined above.

In order to determine if the teachers develop a combined and consistent use of both RME and TDS when engaging in LS and identify further potentials of this merger for professional development, we need to study the practice developed throughout the project. Therefore, we draw on specific case studies and data from those. We draw on teachers’ lesson plans and practice reports (Axelsen, 2021; Boss-Reus *et al.*, 2021), our notes and pictures from observations and communications with the teams involved in the two case studies presented below. When analyzing our data we identify when the teachers explicitly or implicitly refer to or draw on theoretical constructs from RME or TDS mentioned in the section on theoretical frameworks. As part of the results presented below, we discuss how the data can be interpreted as explicit or implicit uses of the theoretical frameworks.

When comparing and contrasting our two case studies to identify potential (rather than actual) uses of TDS and RME, we draw on methodology from networking theory (Prediger *et al.*, 2008). In particular, we follow the strategies for “coordinating and combining”, which is typically used “for a networked understanding of an *empirical phenomenon* or a piece of data” (p. 172). Concretely, we carry out what has been coined a *parallel analysis* of lesson plans, observation notes and practice reports. Hence, after our presentation of each case, we reflect on the case from an RME and a TDS perspective, respectively, while of course also noting the elements actually present in the teams’ own work. Studying, in this way, the design, implementation of and reflection about the lesson, is the basis for our outlines of how the theories have been and could be, combined in the two cases. Based on this analysis we discuss how the theories could complement each other in the context of LS and how they actually supported teachers’ professional development.

Results–Case 1

The first case is a lesson on trigonometry, designed by a team of Dutch pre-university secondary school teachers and two university members of the TIME-project (Boss-Reus *et al.*, 2021). The learning goal of the lesson was for students to relate four central situations in which the sine function occurs: right triangles, the calculator, the unit circle and sinusoids.

The design of the 90 min lesson involves two main tasks. For the first task, groups of four students receive a CD (disc) marked by one dot near the edge. The teacher demonstrates the motion of the dot when the CD is rotated and raises the issue of how the height of the dot varies during this circular motion. The first task for the students is as follows: "Make a graph of the height of the dot with respect to the rotational angle, on a sheet of paper" (see Plate 1). Groups are allowed two attempts. In between the first and second attempt, they hang their first result on a clothing line (a wire on which washed clothes are hung to dry) in the classroom, where they can all view each other's graphs. Then the students are invited to discuss all first outcomes, which include saw tooth shaped graphs, graphs consisting of two semi-circles and other varieties, correct and incorrect. These intermediate results give rise to a short classroom discussion, to fix certain crucial conventions: where to measure the angle and the height? Where does the dot begin? Inspired by their peers, groups may choose a different approach in their second attempt, following the agreed conventions. After all second attempts have also been shared on the clothing line, selected students present their groups' result, followed by a classroom discussion on the questions: which graph is most correct and why? What do we learn from the incorrect ones? One subtle point is that the height responds less to a change in angle when the dot is at the top or bottom. Finally, the teacher compares the best solutions to a sine graph made in GeoGebra and projected onto a screen.

Now that the class has studied a unit circle (the CD) and a sinusoid (the constructed graph), the second main question is: "How can we express the height h in terms of the angle α?" To support this task, students receive a worksheet with juxtaposed unit circle and sine graph. In the unit circle, an example dot is placed and its height is marked with an "h". The challenge for students is to construct a right triangle in the unit circle and to apply the notion of sine as a proportion: $\sin \alpha = \frac{h}{1}$. Only a few groups manage to apply this insight to solve the task. The activity is followed by a presentation by some groups and a classroom discussion. Finally, the teacher institutionalizes (Brousseau, 1997, p. 193) the definition of sine as a y-coordinate in the unit circle. Additional tasks are to compute values of sine using the unit circle and sine graph, for the angles 90°, 135° and 210°, to find out how the cosine graph arises from the unit circle.

Case 1 from an RME-perspective

In Dutch textbooks, the sine and cosine function are in 10th grade defined in the unit circle, after being first encountered in right triangles. Students find it challenging to move between these different aspects of trigonometric functions. Therefore the teachers "opted for an

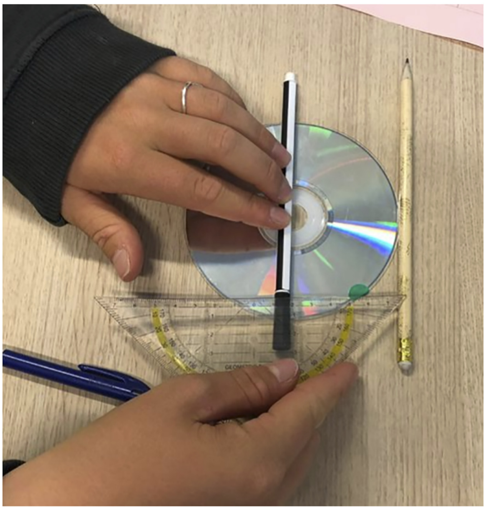

Plate 1.
A student measuring the height of the dot on a CD

approach with old-school means in which the learner has everything in hand–literally - developing the circle and graph perspectives through inquiry and relating these to the triangle perspective" (Boss-Reus *et al.*, 2021, p. 4). They developed the approach through circular motion using CD's. The first task is very accessible to students: rotating a CD and measuring – or at least estimating – angle and height are familiar actions. In this way, the teachers took into account that this situation is "realistic". The first task is set in a context that promotes the process of horizontal mathematization: the motion of the dot is approached through mathematical means like angle and height. The circular trajectory of the dot on the disc is mathematized as a covariation between these two quantities, leading to a periodic graph on the provided coordinate system. Moreover, from an RME-perspective, it is not surprising that students at first try to apply the graphs that are familiar and meaningful to them: lines (combined to a saw tooth) and collated parabolas and semicircles.

For the second task, the teachers have chosen to further mathematize the initial mathematical situation of the unit circle and height graph. The task is to express the height as a function of the angle. This is an instance of vertical mathematization, using mathematical symbols h, α and a mathematical function $h(\alpha) = \sin\alpha$. To solve the task the students need to introduce a right triangle within the unit circle and apply $\sin\alpha = \frac{\text{opposite}}{\text{hypothenuse}}$. The former is challenging, since α is considered a variable in the unit circle, and therefore the right triangle is not considered fixed. Moreover, the latter is challenging as well, since the students are most familiar with applying this formula when a right triangle with fixed lengths is given.

In the practice report, the teachers write about the second task: "Here we come to the heart of the whole lesson, which is to integrate the different perspectives to the sine: triangles, circle and graph. Our idea and hope is that the experience of their own inquiry, a joint discussion, and the confirmation by the teacher will lead to a stronger anchoring of that cohesion" (Boss-Reus *et al.*, 2021, p. 5). This remark stresses the importance of involving students in mathematizing processes for a better anchoring of mathematical concepts and procedures.

Case 1 from a TDS-perspective

The first thing to notice in this case is that no explicit use of TDS is found in the practice report (Boss-Reus *et al.*, 2021). This suggests that the teachers have not used its models or terminology when preparing their lesson and reflecting on the outcomes, even though two of the involved teachers were familiar with TDS from the MERIA-project. At first, the teachers were not enthusiastic about the TIMEplate or TDS-phases suggested in MERIA. They would rather use their usual lesson plan setup, which they considered more brief and easier to communicate to their colleagues. However, as is demonstrated below, some ideas from TDS did eventually find a place in their work.

Most of the TDS-characteristics are implicit in the design and not fully elaborated. For the first task, the teachers have designed an inquiry-based material milieu against which the relevant results can be produced: a CD with a dot, pen and paper, measuring tools. For the second task, the students are handed out a paper with the unit circle and the sine graph. We can consider this an elaboration of the milieu, which facilitates or even prompts certain student actions. Let us point out two design choices made to foster the formulation (and potentially the validation, cf. Brousseau, 1997 p. 89) phase. First, the teachers experiment with group sizes to foster further conversation about hypotheses regarding graphs and strategies (Boss-Reus *et al.*, 2021, p. 7). Second, the teachers organize the sharing of answers by asking the students to hang their solutions (written on paper) on a clothing line in classroom, which form the basis of the plenary discussion of group results. The latter was inspired by another lesson design from the project, where it facilitated to make the central points in formulation and validation visible to all students (Bašić, 2020a, p. 13). The institutionalization of the first task was planned as the teacher connecting the best student solutions with a sine graph in

GeoGebra presented on a screen. It is not evident how the validation and institutionalization are prepared by the teachers to include other student answers into the shared process of inquiry. The second task is also followed by episodes that we can identify as action and formulation.

As the LS progressed, the teachers developed some appreciation of the TIMEplate and the various types of situations as a means to support the development of an inquiry-based lesson. After the first implementation, the university partners pointed out that opportunities were missed for students to validate the different graphs students had drawn. The teachers realized that more time was needed to be dedicated to a phase where visible good ideas and misconceptions were discussed. Moreover, they realized the challenge of connecting the students' results to the learning goals and how anticipating various students' strategies could help the teacher prepare for this. As such, in retrospect after the lesson was taught several times, the structuring of the lesson as a sequence of different situations (with different ends) began to make sense to the teachers and the benefit of TDS was better understood.

Results–Case 2

The second case, from Denmark, concerns the lesson named "Ringsted Hill" (Axelsen, 2021). The team included one teacher who participated in the MERIA project. The others are new to TDS and RME. The problem is about the construction of a ski slope (Plate 2).

The lesson was structured around two questions: *how to design and draw the skiing slope and how to make a mathematical description of the drawn slope*?

Students immediately start designing the slope (Axelsen, 2021). They relate to "real world" conditions (such as the need for users to stop at the end of the slope) and they sketch their slopes as graphs of functions. The students are struggling to describe their slope algebraically, so that different pieces "meet". All groups but one had at least two "pieces" in their designs during first presentations, including a horizontal line segment at the end. Only few groups produced an algebraic description of the slope. Those who managed to do so used the digital graphing device *Nspire* (commonly used in this school). In a second version of the lesson, students were handed out a manual for using *Nspire*. This led students to spend less time on their initial drawings much less and move on to work with *Nspire*. Students who were not used to this, were stuck in technical difficulties and delivered less developed answers, compared to the students in the first implementation.

The company "Curves for all" has been asked to design a car park house in the corn silo of the old steam mill in Ringsted (left picture). At the same time, a ski slope should be made from the roof, as a new attraction in Ringsted, inspired by Copenhill (right picture). The building is 36 m tall and, the ski slope must extend maximally 45m from the building. How should the company design the ski slope?

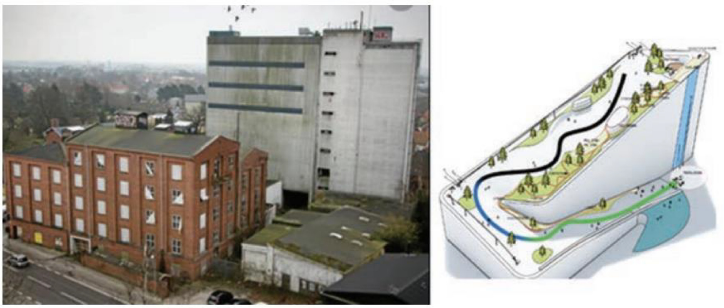

Plate 2.
The pictures handed out to the students of the grain storage and Copenhill (Axelsen, 2021, p. 16)

The lesson was supposed to motivate the notion of piecewise defined functions and the corresponding notation

$$f(x) = \begin{cases} \dots \\ \dots \\ \dots \end{cases}$$

With different versions of the lesson, the teachers explored what strategies students could develop to solve the problem, how this depended on whether the students had been exposed to differential calculus and how the use of different tools could support or hinder mathematically sound solutions. Students were handed out A3 grid paper with an empty coordinate system. They were also allowed to use *Nspire*. The teachers had developed a table in which observers could record observed combinations between students producing one of 6 anticipated mathematical solutions (e.g. combining linear and quadratic functions) and their usage of one or more 7 available tools (including *Nspire*). The table was used both for observation and for orchestration of the formulation situations.

Case 2 from a TDS perspective

The target knowledge of the lesson is to describe a skiing slope in terms of a piecewise defined function. The milieu (including the problem) has the potential to lead students to formulate properties akin to continuity, smoothness and intersection points of functions and especially to broaden their notion of functions beyond those defined by a simple formula. The properties reflect real life, such as experiencing the slope as unpleasant, if it is not smooth or at least continuous at all points. The material milieu allowed the students to explore the problem. Functions, whose "graph shape" the students were familiar with, became important resources for the second task.

After the first implementation, the teachers concluded that "the devolutions of the didactic milieu to the students should have made the difference between the design question and the mathematical description more clear" (Axelsen, 2021, p. 16). The teachers consider that students did not recognize the two questions as being different, and noted that students working with *Nspire* had more developed answers by the end of the lesson. Therefore, the manual for *Nspire* and a list of known functions was produced for the second research lesson, as an enrichment of the milieu surrounding the second task. Observations suggest that the distinction became clearer but that some students were distracted by technical aspects of *Nspire* use.

The situations were explicitly named according to TDS in the lesson plans. This focused teachers' observation and reflection on certain elements in the lesson. Throughout the TIME project, special attention was given to the contribution of each situation and how the situations influence students' learning. From the practice report (Axelsen, 2021) we see that the teachers were particularly aware of the role of precise and complete devolution and how this affects the students' work in action and formulation phases.

The action phases took longer than planned, since all groups struggled to improve their solutions. Only few of the expected strategies from the table were present. Therefore, it was difficult for the teacher to orchestrate the subsequent formulation and validation. Thus, in relation to the first task, students were asked to sketch their ideas on the A3 grid paper that could then be affixed to the blackboard and presented by each group. The validation phase was initiated by checking the dimensions of the building and the distance to surroundings. Whole class sharing of formulae and *Nspire* work was less successful. Axelsen (2021) notes how visible student productions are essential for the formulation situation and for further work on students' modeled slopes. Still, during the validation, some mathematical questions were formulated, such as how to find intersections of curves and how to achieve "smoothness".

Finally, the teachers planned the institutionalization phase, where students' answers are related to share official knowledge regarding the problem worked on. Often the teachers choose to plan this phase with a slide show. However, it proved difficult to improvise clear connections between what was formulated and validated in the lesson, and the slide show planned to institutionalize the notion of piecewise functions.

Case 2 from an RME-perspective

From an RME-perspective, we analyze case 2 with respect to horizontal versus vertical mathematization and a didactical phenomenology. Students were presented with photos as in Plate 2 and teachers expected that students would make some transitional steps from this 3-dimensional real situation to a 2-dimensional problem situation with functions. This transition can be described as a horizontal mathematization. In RME, it is acknowledged that this is an important process in itself. In this task, it involves several choices for the students. It is not so clear how this process took shape in classroom and each group. In the practice report, the teachers write that "there should be a sharper line between the design part of the skiing slope and the actual description of the slope mathematically" (Axelsen, 2021, p. 16). Consequently, in the second implementation teachers added a table of mathematical functions and a suggested digital tool (*Nspire*) to the milieu. This might guide students to arrive more quickly at the mathematical task of designing a suitable piecewise defined function. This second part would be described as a vertical mathematization, though technical challenges limited the realization of this. From RME-perspective, a clear cut is needed between the two activities. Both activities are rich enough to fill a whole lesson and could stand on their own.

From the practice report, it is clear that most students arrive at the informal use of piecewise defined functions. The situation of the skiing slope might not in itself force that to happen, but the added A3 grid paper with an empty coordinate system and later also the table of functions and the available CAS-tool might guide students in this direction. Once the stage was properly set, passing the sharp line discussed above, students had potentially access to the means and goal of the activity. Here, the conditions students set for themselves invited them to move outside the realm of functions defined by a single elementary function. This brings us to a third reason for teachers to set a sharp line: it is important that students set their conditions for the slope before they try to model them mathematically, otherwise they may be inclined to restrict themselves to conditions that can be satisfied by a single function, as it happened for some students struggling with *Nspire*. It seems that the teachers are well aware of drawing on contexts "begging to be mathematized" by piecewise functions, but they could have considered in more detail the horizontal and vertical aspects of this mathematization process. One may even say that the second lesson design blurs these further.

Discussion and conclusion

We have explored two cases of how teachers used RME and TDS when engaging in LS in order to determine if they have developed a combined and consistent use of the theoretical frameworks. Moreover, we have outlined further potential of combining these two theoretical perspectives for teachers' designing and reflecting on their mathematics lessons.

As pointed out by Choy (2016), LS requires teachers to make design choices and assume several responsibilities in order to gain value from LS as professional development. For the two cases, some of those choices were inspired by designs introduced to them in the MERIA project—built on TDS and RME—much as the teachers in the study by Peters (2016) elaborated on examples they were given in the first introduction to RME. Teachers implemented the ski-slope as a realistic and meaningful context for piecewise defined functions, adapted from a context of slide of the MERIA-project, and other teachers adopted the use of a clothing line for students to share their work in lesson phases for formulation and validation—albeit with a varying degree of explicit theoretical reasoning.

The two cases showed that, to a limited extent, TDS and RME provided the teachers with vocabulary to describe situations and variables in their lesson designs and to focus on aspects that are crucial to students' learning. For instance, in the reports, they referred to the appropriateness of the milieu, richness of the problem situations and the time needed for validation and institutionalization. Certainly, teachers used these theoretical tools more liberally than researchers usually do. As noted by Pang and Ling (2012), "one of the advantages of using a theory seems to be that teachers get better at using it every time they put it to the test". As a consequence of the progressive internalization and understanding of TDS and RME, the teachers we observed gave more and more explicit attention to theoretical aspects of the lesson design, e.g. phases in TDS from problem devolution to institutionalization of intended learning goals. It supports their ability to notice crucial elements both in planning a lesson and while reflecting on how the planning affected students' learning.

In both cases, TDS has (as presented in MERIA) focused teachers' attentions on the importance of an adidactic action phase. TDS promoted explicit discussion about what to do and not to do in that phase and about the time needed for formulation, validation and institutionalization (building on the students' work). RME inspired teachers to engage students in "real" problems—sometimes from known contexts. The role of meaningful and relevant contexts in mathematics education was clearly visible in our two cases and, more generally, the research lessons developed in TIME. This deepened teachers' attention to the context and its influences on what students are able to do and learn during the lesson.

It takes time for teachers to adopt and incorporate elements from theories (like RME and TDS) into their process of design. As mentioned, the crucial role of validation situations only progressively appeared as a focus of the teachers. At first teachers tended to focus on enabling successful formulation situations, while validating, generalizing and connecting mathematical findings was mostly postponed to a follow-up lesson. In later lesson studies, validation and institutionalization became main foci for some of the teams.

Combining RME and TDS supported teachers in better articulating and understanding lesson design and classroom processes within the framework of LS. The case studies show both potential and actual benefits of teachers' engaging in LS that involves this combination of theoretical perspectives, where it can support their planning, observation and reflection on how their design choices affected students' learning in a given research lesson.

For us as researchers, combining the two theories helped us to identify elements of the lessons that we might otherwise have overlooked while preparing and analyzing the lessons with the teachers. Thus compared to previous studies (Bahn and Winsløw, 2019; Peters, 2016; Miyakawa and Winsløw, 2009), the collaboration enriched our understanding of mathematics education practices, as it enabled this particular combination of two otherwise separate theoretical perspectives. From a networking theories perspective, we see in particular a potential of the two theories complementing each other as supports for the central elements of LS.

We are aware that our findings are based upon the work of teachers who were immersed in two European projects (MERIA and TIME). Probably, this led teachers to adopt more advanced theoretical tools than was found in previous studies (e.g. Bahn and Winsløw, 2019). The projects provided a long-term professional development trajectory that goes beyond what teachers may commonly be offered. The question remains: to what extent is the cooperation with researchers necessary to support teachers' explicit and accurate work with theories? Is it possible to scale up this way of working without the intensive involvement of researchers that such projects may enable? These questions require further study and more long-term observation of the participating teachers.

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